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Enumerations of Turing Ideals with Applications

DAVID MARKER*

Abstract We examine enumerations of ideals in the Turing degrees and give several applications to the model theory of first- and second-order arithmetic.

A *Turing ideal* is a collection of subsets of ω closed under Turing reducibility and join. If I is a countable Turing ideal we say that E is an *enumeration* of I if and only if $I = \{E_n : n \in \omega\}$ where $E_n = \{m : \langle n,m\rangle \in E\}$. Enumerations of Turing ideals play an important role in the study of degrees coding recursively saturated models of Peano Arithmetic (see [5]). Our goal in this paper is to point out some simple facts about enumerations of Turing ideals and examine their consequences in the model theory of first- and second-order arithmetic.

1 ω-incompleteness theorems We consider three subsystems of second-order arithmetic, RCA_0, ACA_0, and WKL_0. RCA_0 is axiomatized by P^- (Peano Arithmetic without the induction axioms), the axiom of extensionality and the schemas of Σ^0_1-induction and recursive comprehension. WKL_0 is obtained from RCA_0 by adding an axiom saying every infinite subtree of $2^{<\omega}$ has an infinite path. ACA_0 is obtained from RCA_0 by adding the schema of arithmetic comprehension. For further information on these theories the reader should consult [7].

Our recent interest in this subject was motivated by considering the following incompleteness theorem of Steel.

Theorem 1.1 (Steel [9]) *Let T be an ω-consistent arithmetic extension of ACA_0. There is an ω-model* $\mathbf{M}$ *of T such that* $\mathbf{M} \models$ *"there is no ω-model of T".*

Thus even in ω-logic T does not prove its own ω-consistency. Steel's result is actually much stronger. Suppose $\mathbf{M} = (\omega, X)$ and $\mathbf{N} = (\omega, Y)$ are models of RCA_0. We say that $\mathbf{M} \gg \mathbf{N}$ if and only if there is $E \in X$ an enumeration of Y. If $\mathbf{M} \models$ "there is an ω-model of T", then there is an $E \in X$ such that $\mathbf{N} =$

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$(\omega, \{E_n : n \in \omega\})$ is a model of T. Clearly $\mathbf{M} \gg \mathbf{N}$. Steel's result shows that the collection of models of T has $\gg$ minimal elements. In fact he proved the stronger result that the collection of models of T is well founded under $\gg$. This is reminiscent of Kreisel's version (see [8]) of the second incompleteness theorem.

The situation is dramatically different if we weaken T.

Definition $S \subset \mathcal{P}(\omega)$ is a *Scott Set* if and only if
(1) S is a Turing ideal
(2) If $T \subset 2^{<\omega}$ is an infinite tree recursive in an element of S, then T has an infinite path recursive in an element of S.

The second requirement is equivalent to the requirement that every consistent theory in S has a completion in S.

It is easy to see that the Scott sets are exactly the ω-models of WKL_0. In [6] Scott showed that if T is a complete extension of Peano Arithmetic then $\mathrm{Rep}(T) = \{\{n : \phi(n) \in T\} : \phi \text{ a formula}\}$ is a Scott set.

Proposition 1.2 *If* $\mathbf{M}$ *is an* ω*-model of* WKL_0*, then there is* $\mathbf{N}$*, an* ω*-model of* WKL_0 *with* $\mathbf{M} \gg \mathbf{N}$*. Thus* WKL_0 *proves (in* ω*-logic) its own* ω*-consistency. In particular, below any model there is an infinite descending* $\gg$*-chain.*

Proof: Let $\mathbf{M} = (\omega, S)$, where S is a Scott set. S contains T, a completion of Peano Arithmetic. But then $\mathrm{Rep}(T)$ is a Scott set and S contains an enumeration of $\mathrm{Rep}(T)$.

For RCA_0 it is easy to see that the ω-models are exactly the Turing ideals.

Lemma 1.3 *If* E *is an enumeration of a Turing ideal* I*, then* $E \notin I$.

Proof: If $E \in I$, consider m such that $E_m = \{n : n \notin E_n\}$.

Corollary 1.4 *If Rec is the collection of recursive sets,* $(\omega, Rec) \models$ *"there is no* ω*-model of* RCA_0*".*

In Section 2 we show there are infinite descending chains of ω-models of $RCA_0 + \neg WKL_0$. In Section 3 we build other models of RCA_0+ "there is no ω-model of RCA_0."

2 Generic enumerations

Let $I \subset \mathcal{P}(\omega)$ be a countable Turing ideal. Let $\mathbf{P} = \{p : \omega \to I \mid \text{domain of } p \text{ is finite}\}$. If $G \subset \mathbf{P}$ is a reasonably generic filter then $g = \bigcup G$ is a function from ω onto I. Let $E \subset \omega$ be defined by $\langle n,m \rangle \in E$ if and only if $m \in g(n)$. E is a generic enumeration of I.

Let $T \subset 2^{<\omega}$ be a tree with no infinite path in I. We will show that suitably generic enumerations do not add paths to T. Let $D = \{p \in \mathbf{P} : p \Vdash \phi_e^E \text{ is not a path through } T\}$.

Claim *D is dense.*

Proof: Let $p \in \mathbf{P}$. We may assume that p determines $E_0, E_1 \ldots E_n$ and gives no information about the rest of E. Let E^p denote the portion of E determined by p. σ and τ will range over finite ways to extend E^p.

Case 1: $\exists \sigma \exists n \forall \tau \supset \sigma\, \phi_e^{E^p \cup \tau}(n)\uparrow$. In this case we choose $q \leq p$ such that $E^q \supset E^p \cup \sigma$. Clearly $q \Vdash \phi_e^E(n)\uparrow$.

Case 2: $\exists\sigma\exists n\forall m \leq n\, \phi_e^{E^{p\cup\sigma}}(n)\downarrow$, but $\langle\phi_e^{E^{p\cup\sigma}}(0)\ldots\phi_e^{E^{p\cup\sigma}}(n)\rangle \notin T$. In this case we can choose $q \leq p$ such that $E^q \supset E^{p\cup\sigma}$. Clearly $q \mid\vdash \phi_e^E$ is not a path through T.

Case 3: Otherwise. We build $\sigma_0 = \varnothing \subset \sigma_1 \subset \sigma_2 \ldots$ as follows: Given σ_n we search through extensions until we find σ_{n+1} such that $\phi_e^{E^{p\cup\sigma_{n+1}}}(n)\downarrow$. Such a sequence may be built recursively in E^p. Clearly for each n, $\langle\phi_e^{E^{p\cup\sigma_1}}(0),\ldots,$ $\phi_e^{E^{p\cup\sigma_{n+1}}}(n)\rangle \in T$. Thus we have found a path through T recursive in E^p. But $E^p \in I$, a contradiction.

For $x \notin I$ let $D^x = \{p \in \mathbf{P} : p \mid\vdash \phi_e^E \neq x\}$. Similar arguments show that D^x is dense. Thus we can prove the following:

Lemma 2.1 *If $I \subset \mathcal{P}(\omega)$ is a countable ideal, $\mathfrak{T}$ is a countable collection of trees without paths in I and $J \subset \mathcal{P}(\omega)$ is a countable set disjoint from I, then there is E an enumeration of I such that for all $x \in J$, $x \not\leq_T E$, and for all $T \in \mathfrak{T}$, T has no path recursive in E.*

Corollary 2.2 *If $\mathbf{M}$ is a countable ω-model of RCA_0 there is $\mathbf{N} \supset \mathbf{M}$, a proper extension such that $\mathbf{N}$ is also an ω-model of RCA_0 and every tree in $\mathbf{M}$ with no path in $\mathbf{M}$ still has no path in $\mathbf{N}$.*

Proof: Let E be a suitably generic enumeration of $\mathbf{M}$. Let $\mathbf{N}$ be the sets recursive in E.

Thus 2.1 allows us to build models where every nontrivial instance of WKL_0 fails. (In a similar way techniques from [2] allow us to build models of WKL_0 where every nontrivial instance of arithmetic comprehension fails.)

Lemma 2.1 allows us to build long increasing $\gg$ chains of models of $RCA_0 + \neg WKL_0$. A trick from descriptive set theory allows us to build infinite decreasing chains. This trick was first used by Harrison [1].

Suppose $(X,<)$ is a linear order with a least element 0_X such that every nonmaximal element of X has a unique successor. We say that a function $f: X \to \mathcal{P}(\omega)$ is an *X-enumeration sequence* if and only if:

(1) $f(0_X) = \varnothing$.
(2) For all $x \in X$, $f(x)$ is an enumeration of the Turing ideal I_x generated by $\{f(y) : y < x\}$.
(3) For all x, if T is a recursive tree with no recursive paths then T has no path recursive in $f(x)$.

Clearly if $x < y$ then $f(x) <_T f(y)$. Also, if x is the immediate successor of y then $I_x = \{z : z \leq_T f(y)\}$.

If α is a countable ordinal we can build an enumeration sequence $f: \alpha \to \mathcal{P}(\omega)$ by iterating Lemma 2.1.

Definition We define $O^+ = \{e \in \omega$:
(1) ϕ_e is total and codes $<_e$ a linear order of ω,
(2) $<_e$ has a least element 0_e,
(3) each nonmaximal element has a $<_e$ successor$\}$.

Definition $O = \{e \in O^+ : <_e \text{ is a well order}\}$.

O is *Kleene's* O, a complete Π_1^1 set. It is easy to see that O^+ is arithmetic and $O \subset O^+$.

Let $S = \{e \in O^+$: there is an enumeration sequence for $(\omega,<_e)\}$. S is clearly Σ_1^1. By the above remarks $O \subset S$. Thus since O is Π_1^1 and not Σ_1^1, there is $e \in S - O$. Let f be an enumeration sequence for $<_e$. Since $<_e$ is not well founded we can find $n_0, n_1, n_2, \ldots$ such that $n_{i+1} <_e n_i$. But then $I_{n_0} \gg I_{n_1} \gg I_{n_2} \ldots$, and each I_{n_i} is an ω-model of $RCA_0 + \neg WKL_0$. Thus we have shown:

Proposition 2.3 *There is an infinite descending $\gg$ sequence of models of $RCA_0 + \neg WKL_0$.*

We note two more applications of Lemma 2.1.

Proposition 2.4 *If I and J are distinct countable Turing ideals there is a degree* **d** *containing an enumeration of one but not the other.*

Proposition 2.4 has consequences for degrees coding models of Peano arithmetic. If $\mathbf{M} \models PA$ and $a \in \mathbf{M}$, let $r(a) = \{n \in \omega$: the nth prime divides $a\}$. Let $Re(\mathbf{M}) = \{r(a) : a \in \mathbf{M}\}$. $Re(M)$ is a Scott set. In [5] we showed that if **M** is recursively saturated, the set of degrees containing copies of the atomic diagram of **M** is exactly the set of degrees containing enumerations of $Re(\mathbf{M})$. The next corollary is now immediate.

Corollary 2.5 *Let* **M** *and* **N** *be countable recursively saturated models of Peano arithmetic such that* $Re(\mathbf{M}) \neq Re(\mathbf{N})$; *then there is a Turing degree containing the atomic diagram of one model but not the other.*

Note that we may have nonelementarily equivalent recursively saturated models of PA with the same Scott set. These models will have the same degrees containing copies of their diagrams.

3 Avoiding enumerations In this section we show how to extend a Turing ideal without adding enumerations of ideals for which we did not already have enumerations. We will do this by adding minimal upper bounds by perfect set forcing.

We say that E is a *sub-enumeration* of I if $I \subseteq \{E_n : n \in \omega\}$. We say that J is a *simple subideal* of I if $J = I$ or for some $x \in I$, $J \subseteq \{y : y \leq_T x\}$.

A *perfect tree* is a function $T: 2^{<\omega} \to 2^{<\omega}$ such that $\forall \sigma, \tau (\sigma \subset \tau \Rightarrow T(\sigma) \subset T(\tau))$ and $\forall \sigma T(\sigma 0)$ and $T(\sigma 1)$ are incomparable. We say $T \leq T'$ if $\forall \sigma (T(\sigma) \supseteq T'(\sigma))$. We say T is *A-pointed* if $T \leq_T A$ and $\forall f \in [T]\ A \leq_T f$. (Here $[T] = \{\bigcup T(f|n) : f \in 2^\omega)\}$.) Note that if $T' \leq T$, $T' \leq_T T$ and T is A-pointed, then T' is A-pointed.

Let I be a countable Turing ideal. Let **P** be the set of perfect trees which are A-pointed for some $A \in \mathbf{P}$. Forcing with **P** produces a minimal upper bound for I. Below we give a listing of dense sets which we could meet in the construction of a generic. The first four are standard (see [4]).

(1) Let $D_n^0 = \{T \in \mathbf{P} : |T(\langle\ \rangle)| \geq n\}$.
If $G \subset \mathbf{P}$ is filter meeting each D_n^0, then we build a generic real $f = \bigcup\{T(\langle\ \rangle) : T \in G\}$.

(2) For $A \in I$ let $D^1_{A,e} = \{T \in \mathbf{P} : T \mid\vdash \phi^f_e \neq A\}$.

(3) For $A \in I$ let $D^2_A = \{T \in \mathbf{P}$: for some $B \in I$ $B \geq_T A$ and T is B-pointed$\}$.
Meeting D^2_A and all $D^1_{A,e}$ for each $A \in I$ forces the generic real f to be strictly above I.

(4) Let $D^3_e = \{T \in \mathbf{P} : T \mid\vdash \phi^f_e$ is not total or $T \mid\vdash \phi^f_e \leq_T T$ or $T \mid\vdash f \leq_T \langle \phi^f_e, T\rangle\}$.
This is the usual main step of a minimal degree construction. As the ideas in the density argument for D^3_e will be useful below we outline them here. Let $T \in \mathbf{P}$. If there are n and σ such that for all $\tau \supset \sigma$ $\phi^{T(\tau)}_e(n)\uparrow$, we can find $T' \leq T$ such that $T' \mid\vdash \phi^f_e$ is not total. Otherwise we can find $T' \leq T$ such that for each n and each σ of length n, $\phi^{T'(\sigma)}_e(n)$ converges by stage $|T(\sigma)|$. Clearly $T' \mid\vdash \phi^f_e$ is total.
Assume that $T \mid\vdash \phi^f_e$ is total. We say that σ is an *e-splitting node* if there are $\tau_0, \tau_1 \supset \sigma$ and n such that $\phi^{T(\tau_0)}_e(n)\downarrow$, $\phi^{T(\tau_1)}_e(n)\downarrow$, but $\phi^{T(\tau_0)}_e(n) \neq \phi^{T(\tau_1)}_e(n)$.
If there is σ which is not an e-splitting node, we can find $T' \leq T$ and $g \leq_T T$ such that $T' \mid\vdash \phi^f_e = g$. If every node is e-splitting we can find $T' \leq T$ such that for all $\sigma, \sigma 0$ and $\sigma 1$ demonstrate that σ is e-splitting. In this case for any path h through T', h can be recursively reconstructed from T' and ϕ^h_e.

(5) Let $S \subset 2^{<\omega}$ be a tree with no paths in I. Let $D^4_{S,e} = \{T : T \mid\vdash \phi^f_e \notin [S]\}$.
Let $T \in \mathbf{P}$. As in case (4) we may assume that $T \mid\vdash \phi^e_f$ is total. Since S has no paths in I there is an n such that $\langle \phi^{T(0^n)}_e(0), \ldots \phi^{T(0^n)}_e(n)\rangle \notin S$. Let $T' \leq T$ with $T'(\langle\ \rangle) = T(0^n)$. Clearly $T' \mid\vdash \phi^f_e \notin [S]$.

(6) Let J be a simple subideal of I with no subenumeration in I. Let $D^5_{J,e} = \{T \in \mathbf{P} : t \mid\vdash \phi^f_e$ is not a subenumeration of $J\}$.
Let $T \in \mathbf{P}$. As in case (4) we may assume that $T \mid\vdash \phi^f_e$ is total. If $J \subset \{D : D \leq_T A\}$ for some $A \in I$, then, by case (3), we may assume that T is A-pointed. We form a set E as follows. Let $e_0, e_1, \ldots$ be such that $\forall X \forall m \forall n\ \phi^X_{e_n}(m) = \phi^X_e(\langle n, m\rangle)$. Let $\langle\langle \sigma, n\rangle, m\rangle \in E \Leftrightarrow \exists \tau \supset \sigma \exists s[|\tau| \geq m \wedge \phi^{T(\tau)}(m) = 1$ by stage $s \wedge \forall \tau'((|\tau'| \leq |\tau| \wedge \tau' \supset \sigma \wedge \phi^{T(\tau')}_{e_n}(m)$ converges by stage $s) \Rightarrow \phi^{T(\tau')}_{e_n}(m) = 1)]$.
Since for all large enough $\tau \supset \sigma$ $\phi^{T(\tau)}_{e_n}(m)\downarrow$, E is recursive in T. Thus $E \in I$, so E is not a subenumeration of J. We can find $D \in J$ such that for all n $D \neq E_n$ and $D \leq_T T$. [If $J \subset \{C : C \leq_T A\}$, we use the fact that T is B-pointed where $A \leq_T B$. If $J = I$, we let $D = \{n : n \notin E_n\}$.]
We build $T' \leq T$. Let $T'(\langle\ \rangle) = T(\langle\ \rangle)$. Let $T'(\sigma i) = T(\tau i)$, where τ is the first node found such that $T(\tau) \supset T'(\sigma)$ and $\exists m\ \phi^{T(\tau)}_{e_{|\sigma|}}(m)\downarrow \neq D$. Clearly $T' \mid\vdash D \notin \{\phi^f_{e_n} : n \in \omega\}$.

(7) Let J be a simple subideal of I with no enumeration in I. Let $D^6_{J,e} = \{T \in \mathbf{P} : T \mid\vdash \phi^f_e$ is not an enumeration of $J\}$.
We build E as in case (6). If E is not a subenumeration of J, then we proceed as above. If not, then since E is not an enumeration, there are σ and n such that $E_{\langle\sigma,n\rangle} \notin I$. We define $T' \leq T$ by: $T'(\tau) = T(\sigma\tau)$. Since $E_{\langle\sigma,n\rangle}$ is infinite, there are no e_n-splittings below σ. Thus $T' \mid\vdash \phi^f_{e_n} = E_{\langle\sigma,n\rangle}$.

We give two applications of this method.

Proposition 3.1 *For any countable ideal I there is a minimal upperbound* $\mathbf{d}$ *which does not contain a subenumeration of I.*

This was proved in [3] in case I is the ideal of arithmetic sets. [In fact their proof works for countable jump ideals.]

Proposition 3.2 *There is an ω-model* $\mathbf{M}$ *of RCA_0 of power $\aleph_1$ such that* $\mathbf{M} \models$ *"there is no ω-model of RCA_0" and if $S \subset 2^{<\omega} \in \mathbf{M}$ is a tree with no path recursive in S, then S has no path in* $\mathbf{M}$.

Proof: We build a chain of countable ideals $\langle J_\alpha : \alpha < \omega_1 \rangle$. Let J_0 be the recursive sets. Given J_α force to build f_α a minimal upperbound for J_α such that if $E \leq_T f_\alpha$, then E is not an enumeration of any $J_\beta, \beta \leq \alpha$ and every tree in J_α with no path in J_α has no path recursive in f_α. Let $J_{\alpha+1} = \{A : A \leq_T f_\alpha\}$. If α is a limit, then $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$. Let $I = \bigcup J_\alpha$, and let $\mathbf{M} = (\omega, I)$. The proposition follows since the J_α are the only subideals of I.

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Department of Mathematics, Statistics and Computer Science
University of Illinois
Chicago, Illinois