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Actuality and Quantification

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Abstract A natural deduction system of quantified modal logic (S5) with an actuality operator and "rigid" quantifiers (ranging, at every world, over the domain of the actual world) is described and proved to be complete. Its motivation and relation to other systems are discussed.

1 The language Predicates. One logical predicate: E!, "exists". Individual constants if you want, though for simplicity I'll ignore them (constants thought of as formalizing names or other "rigid designators" ought to behave like the free variables). Individual "parameters" (free variables): $u, v, \ldots$. Individual bound variables: $x, y, \ldots$ (I follow the conventions of Thomason [16] here). Truth functional connectives: &, $\vee$, $\supset$, $\sim$. Modal operators: $\Box$ (necessity), $\Diamond$ (possibility), $\bigcirc$ (actuality). Ordinary quantifiers: $\forall$, $\exists$. "Actuality" quantifiers: $\forall^\circ$, $\exists^\circ$. The usual formation rules (bound variables never occurring free).

2 Semantics A *model* is a quadruple $M = \langle W, @, D, I \rangle$ where

W is a set (of "worlds"),

$@ \in W$ ($@$ is "the actual world"),

D is a function assigning to each $w \in W$ a (not necessarily nonempty) set as its domain, and

I is an interpretation function assigning to each n-adic predicate a function assigning to each $w \in W$ a set of n-tuples drawn from $\bigcup_{v \in W} D(v)$, with the condition that $[I(\mathrm{E!})](w) = D(w)$.

Note that, corresponding to various intuitive readings of the predicates of the formal language, and to various metaphysical positions, we might want to impose further conditions on the interpretation function; these will often validate extensions of the logic described below.

An *assignment* for m is a partial function from the individual parameters into

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$\bigcup_{w \in W} D(w)$ (*partial* function in order to avoid validating $\Diamond E!u$, which is not derivable in the system described. If you want motivation, think of some parameters as formalizing names from fiction.)

We define in the first instance: *Truth in a model, at a world, on an assignment.* Truth in a model at a world is Truth at that world in that model on every assignment. Truth in a model is Truth at the actual world in the model. Validity is Truth in every model; validity of an argument is validity of its associated conditional.

Base clause of recursion: An atomic formula, $F(u_1, \ldots, u_n)$, where F is an n-adic predicate, is True(M, w, α) if and only if

(i) $\alpha(u_1), \ldots, \alpha(u_n)$ are all defined, and
(ii) $\langle \alpha(u_1), \ldots, \alpha(u_n) \rangle \in [I(F)]\,(w)$.

Recursion clauses:

For truth functional compounds: Standard.

For modal and actuality operators:

$\Box A$ is True(M, w, α) iff A is True(M, w', α) for every $w' \in W$,
$\Diamond A$ is True(M, w, α) iff A is True(M, w', α) for at least one $w' \in W$,
$\bigcirc A$ is True(M, w, α) iff A is True$(M, @, \alpha)$.

For ordinary quantifiers:

$\forall x A\ (x/u)$ is True(M, w, α) iff A is True(M, w, β) for every assignment β such that

(i) $\beta(v) \cong \alpha(v)$ for every parameter $v \neq u$,
(ii) $\beta(u)$ is defined, and
(iii) $\beta(u) \in D(w)$.

$\exists x A\ (x/u)$ is True(M, w, α) iff A is True(M, w, β) for at least one assignment β such that

(i) $\beta(v) \cong \alpha(v)$ for every parameter $v \neq u$,
(ii) $\beta(u)$ is defined, and
(iii) $\beta(u) \in \mathrm{D}(w)$.

For actuality quantifiers:

$\forall^\circ x A (x/u)$ is True(M, w, α) iff A is True(M, w, β) for every assignment β such that

(i) $\beta(v) \cong \alpha(v)$ for every parameter $v \neq u$,
(ii) $\beta(u)$ is defined, and
(iii) $\beta(u) \in \mathrm{D}(@)$.

$\exists^\circ x A\ (x/u)$ is True(M, w, α) iff A is True(M, w, β) for at least one assignment β such that

(i) $\beta(v) \cong \alpha(v)$ for every parameter $v \neq u$,
(ii) $\beta(u)$ is defined, and
(iii) $\beta(u) \in D(@)$.

($\cong$ means that either both terms are undefined or both are defined and have the same value.)

Note that, whereas at each world the ordinary quantifiers range over the domain of that world, the actuality quantifiers range at each world over the domain of the actual world. (In much of the literature on the subject, what I have called the *ordinary* quantifiers are called *actualist* quantifiers. For obvious reasons, I avoid this terminology; I have sometimes called them *world-restricted* quantifiers.)

3 Motivation The language of quantified modal logic with only the ordinary quantifiers lacks expressive power; the addition of either the actuality operator or the actuality quantifiers allows the formulation of sentences that have no equivalents in the unenriched language. Furthermore—this is what makes the purely formal semantic fact interesting—some such sentences seem to be appropriate formalizations of apparently meaningful natural language sentences whose meanings and logical relations would seem, pre-analytically, to form part of the proper subject matter of modal logic. So that, if modal logic is regarded as a tool for semantics and/or conceptual analysis, rather than as a purely mathematical study, it *ought* to consider languages with this added expressive power. The operator and new quantifiers are both motivated, for there are sentences such as

$$\Diamond(Q \;\&\; \forall x(Fx \supset \bigcirc Fx))$$

(where Q is a O-adic predicate, i.e., a sentence letter) formulable using the actuality operator, for which there are no equivalents using only the actuality quantifiers, and also sentences such as

$$\Diamond(P \;\&\; \forall^{\circ}x\mathrm{E}!x)$$

of which the opposite is true. The first might be used in formalizing something like "The truth of Q wouldn't entail that anything was F other than the things which are actually F"; the second something like "The truth of P wouldn't entail the nonexistence of any actual object".

4 Previous partial completeness results In this section I will cite mainly my own work, not because I claim great originality, but because I know it better than I do the rest of the literature.

(A) Propositional modal logic (S5) with actuality My 1978 note [8] defines and proves complete a Fitch-style natural deduction (cf. Fitch [4] and Thomason [16]) system for Propositional S5 with an actuality operator. With a few later simplifications, we may state its rules as follows.

Define a formula to be *strict* iff it is a truth-functional compound of formulas beginning with modal or actuality operators. (NB: in extending the system to languages with quantifiers, it is important to keep to this definition of strictness. Compounding of strict formulas with truth-functional *propositional* connectives makes strict formulas; quantification of strict formulas doesn't. Thus the notion of a strict formula is not the same as the usual notion of a modally closed formula.) In addition to the familiar subproofs for the $\vee$-, $\supset$-, and $\sim$-rules, we have *strict* subproofs; nonstrict formulas may not be reiterated into strict subproofs. A strict subproof used in an application of the $\Box$I or $\Diamond$E rules is called a *modal* subproof; one used in an application of $\bigcirc$ sub is an *actuality* subproof.

We define, inductively, two kinds of (sub)proof, the *A-proof* and the *F-proof* (mnemonic: *F*oreign). Intuitively, the formulas occurring as items of an A-proof are thought of as being supposed true at the actual world, those in an F-proof as being supposed true in foreign worlds. *Base:* (i) the main proof is an A-proof, (ii) modal proofs are F-proofs, and (iii) actuality proofs are A-proofs. *Recursion:* a nonstrict subproof is of the same kind as the proof of which it is an item.

We assume rules for the truth-functional connectives; the (two-valued) rules of Fitch's textbook or the rules of Thomason's will do.

$\Box$I: $\Box A$ is a direct consequence of a categorical (hypothesisless) strict subproof with A as an item.

$\Box$E: A is a direct consequence of $\Box A$.

$\Diamond$I: $\Diamond A$ is a direct consequence of A.

$\Diamond$E: Where B is a *strict* formula, B is a direct consequence of $\Diamond A$ together with a strict subproof having A as its only hypothesis and containing B as an item.

$\bigcirc$sub: $\bigcirc A$ is a direct consequence of a categorical strict subproof having A as an item.

$\bigcirc$I: (NB: this rule may only be used in A-proofs) $\bigcirc A$ is a direct consequence of A.

$\bigcirc$E: (NB: this rule may only be used in A-proofs) A is a direct consequence of $\bigcirc A$.

The system is a formulation of what might be called the A-logic of the actuality operator. It is sound and complete with regard to validity as defined above: truth (or, for arguments, truth-preservation) at the actual world. One may also define an auxiliary logic, the F-logic of the actuality operator: the logic corresponding to a deviant definition of validity as truth (truth-preservation) at every world of every model. A natural deduction formulation of the F-logic is obtained basically by changing the inductive definition of A-proofs and F-proofs so as to make the main proof an F-proof (for another technical change needed, see below). The F-logic has an important role in Henkin-style completeness proofs for this logic and its quantified extensions. In such a proof we define a model, taking certain maximal consistent sets of formulas as starting points. The maximal consistent set destined to give rise to the @ of the model is maximal consistent with respect to the A-logic, but those sets of formulas giving us the other possible worlds are only maximal consistent with respect to the F-system.

Note that the formulation of the $\bigcirc$sub rule given is formally parallel to the $\Box$I rule. Since, however, formulas starting with $\bigcirc$ are strict, and $\bigcirc$E can be used in an A-proof, a form parallel to $\Diamond$E is a derivable rule of the system. The only touchy point is in connection with the use of *reductio* in F-proofs where an "actual contradiction" such as $\bigcirc(P \mathbin{\&} {\sim}P)$ has to count as a contradiction, and it is not possible to refute it without use of the rules only allowed in A-proofs. The trick is to derive its negation in the main proof (an A-proof by definition) and reiterate through to where needed. A formulation of the F-logic would have to have a second actuality subproof rule, or take ${\sim}\bigcirc(P \mathbin{\&} {\sim}P)$ as an axiom, or something similar.

Discussion The part of this system without the actuality operator and its rules is equivalent to Fitch's formulation of S5 in [5]. One of the advantages of defining a class of strict formulas and allowing all but only strict formulas to be reiterated into strict subproofs is that the analogies between the modal rules and quantifier rules are enhanced. In particular, this formulation of S5, unlike many other natural deduction formulations of modal logics, allows the derivation of the necessity rules from the possibility rules when necessity is taken as defined, and the derivation of the interdefinitions of necessity and possibility when both are taken as primitive. Fitch's version of natural deduction is, for nonmodal logic, hardly more than a notational variant of Gentzen's N-systems: derivations in one sort of natural deduction system can be transformed into derivations in the other in a purely routine, scissors-and-paste, way. (Fitch's notation does, unlike Gentzen's, make explicit which assumptions are discharged at any potentially assumption-discharging inference.) When we move from quantification theory to modal logic, however, and let the restrictions on nonstrict formulas take a form parallel to the restrictions on formulas containing the *Eigenvariable* of a quantifier inference, Fitch's version is a bit more flexible. Reiteration can be used to avoid some Cuts; compare, for example, the proof of $P \supset \Box\Diamond P$ in our Fitch-style version with that in a Gentzen-style "NS5", or with the proof in the system of Corcoran and Weaver [1]: by reiterating $\Diamond P$ into the strict subproof, a Fitch-style proof can be given in five lines, whereas it is necessary in the other systems to derive $\Diamond P \supset \Box\Diamond P$ (which will be a *maximum formula* of the proofs in Prawitz's sense), and to infer the desired thesis from it. (S5 has a simple and intuitive possible world semantics and pleasant syntactic properties such as the equivalence of every formula to one in a simple normal form, so it comes as a surprise that its proof theory is as convoluted as it is. As Sylvan, and doubtless many others, have pointed out, cut elimination/normalization *fails* for the most natural formulations; to obtain it Sato [14] was forced to consider a quite baroque sequent calculus. For an indication of what *can* be done with Fitch-style rules, see Fitting [6].)

Crossley and Humberstone [2] present an axiomatic formulation of what I have dubbed the F-logic. Their paper and mine [8] were written two or three years before their publication; our work was carried on in complete independence. Their paper is conceptually weak (for reasons discussed by Zalta [18]) in giving priority to the F-logic, but is technically richer than mine.

(B) Non-modal quantifier logic (universally free logic) In my dissertation [7] I show that, by adding existence premisses to the rules $\forall$E and $\Diamond$I, and existence hypotheses to the subproofs of the $\forall$I and $\exists$E rules of, say, the version of standard predicate logic in Thomason [16], one obtains a complete natural deduction for universally free logic (which, after all, is what one wants for the logic of world-restricted quantification in a system of quantified modal logic countenancing contingent existence). Depending on the application—on, e.g., whether one is construing variables as ranging over a broader or narrower category of entities, and on what sorts of conditions one wishes to allow atomic predicates to formalize—one may or may not wish to impose what Kit Fine calls the "falsehood convention". If one does, the appropriate move is to make $E!u$ a direct consequence of any atomic formula containing u. One may not, however, wish to: universally free logic is the obvious logic for *sortal* quantification, with E! interpreted as merely meaning "is of the appropriate sort".

Kathleen Johnson Wu's recently published natural deduction system for universally free logic [17] is very similar to, and in some ways more elegant than, mine. She avoids the use of an existence predicate; both for incorporation into a system of quantified modal logic and from the point of view that sees universally free logic as a dry run for a system of sortal quantification, however, the existence predicate seems desirable.

(C) Quantified modal logic with ordinary quantifiers If you simply combine in a single system all the rules from propositional logic together with all the rules for nonmodal quantifier logic, one of four things will happen:

(i) You'll get an incomplete system, and have to postulate some additional principles relating quantifiers and modality. (Example: if you want to use free logic for reasons unconnected with modality but wish to interpret the quantifiers as ranging over necessary existents, you'll have to jigger something to get the Barcan Principle and its converse.)
(ii) You'll get an unsound system, and have to place restrictions on one or more of the rules that aren't needed, or perhaps even formulable, with respect to the fragment of the language to which the rule is native. (Standard example: with unfree predicate logic, avoiding the Converse Barcan Principle if your semantics doesn't call for it.)
(iii) Some combination of (i) and (ii).
(iv) You'll get a sound and complete (relative to the desired semantics) system of quantified modal logic.

In [7] I prove that combining universally free logic (for ordinary quantifiers) with S5 and an actuality operator is one of the lucky type (iv) cases. The proof is a standard Henkin-style argument, stupefyingly complicated in its details but without conceptual novelty. (I also prove that adding rules for identity doesn't cause any problems, either.)

5 Rules for actuality quantifiers Ordinary, world-restricted, quantifiers are interpreted as ranging over *existents*; their logic is formalized by putting *existence* premisses/hypotheses into the familiar rules. Actuality quantifiers are interpreted as ranging over *things that actually exist* (actual existents); the obvious way to formalize their logic is to give a formally identical set of rules, but with *actual existence* premisses/hypotheses. Thus:

$\forall^\circ$I: $\forall^\circ x(A(x/u))$ is a direct consequence of a subproof, general with respect to u, having $\bigcirc E!u$ as its only hypothesis and having A as an item:

$$\begin{array}{l|l} & \vdots \\ & \quad \begin{array}{|l} \bigcirc\mathrm{E}!u \\ \hline \vdots \\ A \end{array} \\ & \forall^\circ x(A(x/u)) \\ & \vdots \end{array}$$

$\forall^\circ$**E**: $A(v/u)$ is a direct consequence of two premisses, $\forall^\circ x(A(x/u))$ and $\bigcirc\mathrm{E}!v$.

$\exists^\circ$**I**: $\exists^\circ x(A(x/u))$ is a direct consequence of two premisses, $A(v/u)$ and $\bigcirc\mathrm{E}!v$.

$\exists^\circ$**E**: Where B contains no occurrence of u, B is a direct consequence of two items, $\exists^\circ x(A(x/u))$ and a subproof, general with respect to u, having A and $\bigcirc\mathrm{E}!u$ as its only hypotheses and having B as an item:

$$\begin{array}{l|l} & \vdots \\ & \exists^\circ x(A(x/u)) \\ & \vdots \\ & u \begin{array}{|l} A \\ \bigcirc\mathrm{E}!u \\ \hline \vdots \\ B \end{array} \\ & \vdots \\ & B \\ & \vdots \end{array}$$

6 The completeness proof The completeness proof in [7] for the system without the actuality quantifiers is along the lines of the proofs in Hughes and Cresswell [9], defining certain saturated sets of formulas which become the possible worlds of a model. The building up of these sets is a bit complex: in order to ensure that, e.g., the new constants introduced in "saturating" one set occur in the right modalized formulas in other sets, I allowed alternating phases of

(i) Expansion to saturation,
(ii) Insertion of formulas (starting with $\Box$, $\Diamond$, or $\bigcirc$) that have appeared in one set into the others.

In extending the completeness proof to a system with the actuality quantifiers, it is sufficient (and necessary) to verify that formulas starting with actuality quantifiers imply enough formulas starting with modal or actuality operators to keep the books balanced. Considering cases, the only unobvious are the derivability of the Barcan Principle and its converse for the actuality quantifiers. (Since whatever world an actuality quantification is evaluated at, it always ranges over the same domain – $D(@)$ – it is obvious that BP and CnvBP are valid for actuality quantifiers.) The needed derivations follow.

CnvBP:

1	$\Box\forall^\circ x(A(x/u))$	
2	u │ $\mathrm{O}E!u$	hyp
3	│ $\Box\forall^\circ x(A(x/u))$	1, reit
4	│ $\Box$ │ $\mathrm{O}E!u$	2, reit
5	│ │ $\Box\forall^\circ x(A(x/u))$	3, reit
6	│ │ $\forall^\circ x(A(x/u))$	5, $\Box E$
7	│ │ A	6, 4, $\forall^\circ$E
8	│ $\Box A$	4–7, $\Box$I
9	$\forall^\circ x(\Box A(x/u))$	2–8, $\forall^\circ$I

As experience with quantified modal logic would lead one to suspect, the derivation of the Barcan Principle is a bit trickier than that of its converse.

BP:

1	$\forall^\circ x(\Box A(x/u))$	
2	$\Diamond\forall^\circ x(\Box A(x/u))$	1, $\Diamond$I
3	$\Box$ │ $\Diamond\forall^\circ x(\Box A(x/u))$	2, reit
4	│ u │ $\mathrm{O}E!u$	hyp
5	│ │ $\Diamond\forall^\circ x(\Box A(x/u))$	3, reit
6	│ │ $\Box$ │ $\forall^\circ x(\Box A(x/u))$	hyp
7	│ │ │ $\mathrm{O}E!u$	4, reit
8	│ │ │ $\Box A$	6, 7, $\forall^\circ$E
9	│ │ $\Box A$	5, 6–8, $\Diamond$E
10	│ │ A	9, $\Box$E
11	│ $\forall^\circ x(A(x/u))$	4–10, $\forall^\circ$I
12	$\Box\forall^\circ x(A(x/u))$	3–11, $\Box$I

There is an unpleasant roundaboutness to this derivation. At line (2), a formula is inferred by $\Diamond$I, while at (9) something is inferred *from* this formula by $\Diamond$E. But then, we have long known that S5 was proof-theoretically recalcitrant. This little detour through the $\Diamond$ is needed because (1) isn't suitably "wrapped" for reiteration into the strict subproof (3)–(11) where it is needed. This is an example of how S5 systems resist Prawitz-normalization.

7 Why it doesn't matter Using formulas of the form $\Box E!u$ as "existential" premisses/hypotheses, we could in the same manner formulate rules for a sort of quantifier ranging over necessary existents. Since, however, the necessary existents of a model are a common subset of all the $D(w)$, definable by the formula $\Box E!u$, such quantifiers would not increase the expressive power of the language.

Using formulas of the form $\Diamond E!u$ as premisses/hypotheses, we can formulate rules for quantifiers ranging over all possible existents. These would be the familiar "possibilist" quantifiers, sometimes adopted as primitives, sometimes condemned as conceptually/metaphysically suspect. I find it rather pleasing that they can be seen, in their formal rules, as instances of a common pattern. (The quantified modal logic of counterfactual conditionals has not, to my knowledge, been much studied. Were it to be, the logic of quantifiers over "things that would have existed if . . ." might prove an interesting, and hairy, subject; I have no conjecture as to whether rules analogous to those considered above would suffice.)

Natural languages, however, seem a good deal richer than quantified modal logics. The operator $\bigcirc$ "exempts" the subformula it governs from the semantic effects of *all* the modal operators in whose scope it lies. Natural language actuality locutions are more flexible:

> "Smith thought that if Jones had been at top form, the winning score would have been higher than it actually was"

seems ambiguous to me. On one reading *actually* exempts the stuff it governs only from the inner intensional operator, *if Jones had been at top form* __, so the contrast is with the score Smith thought the winner made. On the other reading, *actually* exempts from *Smith thought that* __ as well, so the contrast is with Jones's real life score. In formalizing the modal notions expressible in natural language, then, a modal logic should have a variety of actuality operators of differing strengths. It should be possible to prefix a subformula occurring within the scope of n modal operators with any of n actuality operators, $\bigcirc_1, \ldots, \bigcirc_n$, with $\bigcirc_i$ exempting the stuff it governs from the innermost i of the nested modal operators governing it. A language with such operators, however, is equivalent, in its expressive power, to one with explicit, variable-binding quantification over possible worlds. (Hint: show that sentences of such a language have a normal form in which the $\bigcirc_i$ occur only before atomic formulas. Sentences in normal form can be reanalyzed, treating

$$\bigcirc_i F u_1, \ldots, u_n$$

as an $(n + 1)$-adic predication, modal operators as quantifiers, and the subscripts on the actuality operators as a notational variant of variables.) In an environment like that, however, the ordinary world-restricted quantifiers can be made to do the work of the actuality quantifiers and of the possibilist ones. At which point the question of the formalization of the logic of actuality quantifiers takes on the aspect of an "axiomatizability within a fragment" problem, and not one whose solution promises much conceptual insight.

There is a bit of philosophical bite to the result on the expressive power of modal languages with multiple actuality operators. Hilary Putnam in [13] speaks of the necessity operator as the only non-nominalistic notion needed for the modal-logical construal of mathematics, but the actual construction he sketches seems to presuppose a modal language of at least this expressive power. Given the power of this language, therefore, one might ask in what sense Putnam has provided an alternative to the "mathematical objects" view: possible worlds are surely no more innocent, ontologically, than sets. (Charles Parsons, whose work on modal set theory and modal number theory is a good deal more detailed than anything Putnam has published, and who does not claim to be *avoiding* ontology, has also found such languages necessary for the expression of some intuitive arguments for set-theoretic axioms, and studies them explicitly.)

NOTE

Readers interested in extensions of the ordinary language of quantified S5 will also want to consult the three papers by Harold Hodes in the *Journal of Philosophical Logic*, v. 13 (1984).

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