Notre Dame Journal of Formal Logic
Volume 30, Number 4, Fall 1989

Near Coherence of Filters III: A Simplified Consistency Proof

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Abstract In the model obtained from a model of the continuum hypothesis by iterating rational perfect set forcing $\aleph_2$ times with countable supports, every two nonprincipal ultrafilters on ω have a common image under a finite-to-one function.

The principle of near coherence of filters (NCF) asserts that, for any two nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on the set ω of natural numbers, there exists a finite-to-one function $f: \omega \to \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$. This principle was introduced and studied in [1], and its consistency relative to ZFC was proved in [2]. Because [2] also contains the consistency proof for another statement (the existence of simple P_κ-points for two different κ), the model of NCF presented there was chosen to maximize the similarity of the two proofs. Although this approach is quite efficient for proving the consistency of both statements, there is a simpler consistency proof for NCF alone. The purpose of this paper is to present this proof.

By *rational perfect set forcing*, we mean the forcing introduced by Miller in [3]; a definition is given below.

Theorem NCF *holds in the model obtained from a model of the continuum hypothesis by iterating rational perfect set forcing* $\aleph_2$ *times with countable supports.*

The proof to be presented here can be viewed as the result of deleting, from the proof in [2], all references to (what is there called) depth. The observation that the consistency proof for NCF survives this deletion was made by Shelah shortly after he found the proof in [2]. Blass noticed that the resulting forcing was equivalent to Miller's rational perfect set forcing.

The substitution of rational perfect set forcing for the forcing used in [2]

Received May 5, 1986

considerably simplifies the analysis in Sections 2 and 3 of [2]. On the other hand, the parts of [2] that deal not with the specific forcing at hand but with general properties of iterated proper forcing, particularly Section 4, do not benefit at all from this substitution. Thus, a self-contained presentation of the new proof would include a verbatim transcription of these parts of the old proof. To avoid such unnecessary repetition, we simply quote here the general facts about proper forcing and its iteration that we shall need.

Lemma 1 ([5], p. 81) *Let G be a V-generic subset of a proper notion of forcing. If $X \in V[G]$ is a countable (in $V[G]$) subset of V, then $X \subseteq Y$ for some countable (in V) set $Y \in V$. In particular, $\aleph_1$ is absolute between V and $V[G]$.*

For the next four lemmas, let $\langle P_\alpha, \boldsymbol{Q}_\alpha : \alpha < \lambda \rangle$ be a countable support iteration of proper forcing with limit P_λ. That is,

P_0 is the trivial notion of forcing (a singleton);
$P_{\alpha+1} = P_\alpha * \boldsymbol{Q}_\alpha$ for all $\alpha < \lambda$;
P_β = the direct limit of $(P_\alpha)_{\alpha<\beta}$ for all $\beta \leq \lambda$ of uncountable cofinality;
P_β = the inverse limit of $(P_\alpha)_{\alpha<\beta}$ for all $\beta \leq \lambda$ of countable cofinality;
P_α forces "$\boldsymbol{Q}_\alpha$ is a proper notion of forcing" for all $\alpha < \lambda$.

Let G be a V-generic subset of P_λ. For each $\alpha \leq \lambda$, we write G_α for the V-generic subset $G \cap P_\alpha$ of P_α.

Lemma 2 ([5], p. 90) *P_α is proper for all $\alpha \leq \lambda$.*

Lemma 3 ([2], Lemma 5.10) *Let $\mathfrak{F} \in V[G]$ be a family of reals. There is an $\aleph_1$-closed unbounded set of ordinals $\alpha < \aleph_2$ for which $\mathfrak{F} \cap V[G_\alpha] \in V[G_\alpha]$.*

Lemma 4 ([5], p. 96) *Suppose that the continuum hypothesis holds in V, that $\lambda = \aleph_2$, and that, for each $\alpha < \aleph_2$, P_α forces that $|\boldsymbol{Q}_\alpha| \leq 2^{\aleph_0}$. Then, for each $\alpha < \aleph_2$, P_α has a dense set of cardinality $\leq \aleph_1$, $P_{\aleph_2}$ satisfies the $\aleph_2$-chain condition, and (therefore) cardinals are absolute between V and $V[G]$.*

(The comments after (5.4) in [2] explain how to handle some technical differences between our version of this lemma and the version in [5].)

A nonprincipal ultrafilter $\mathfrak{U}$ on ω is called a *P-point* if, whenever $X_n \in \mathfrak{U}$ for all $n \in \omega$, there is a $Y \in \mathfrak{U}$ such that Y is almost included in each X_n, i.e., $Y - X_n$ is finite. A classical result of Rudin [4] is that the continuum hypothesis implies the existence of P-points.

Lemma 5 ([2], Theorem 4.1) *If $\mathfrak{U}$ is a* P*-point (in V), if λ is a limit ordinal, and if, for each $\alpha < \lambda$, P_α forces "$\mathfrak{U}$ generates a* P*-point", then P_λ also forces "$\mathfrak{U}$ generates a* P*-point".*

This completes the list of general facts about proper forcing that we shall need. We now turn to the specific forcing that we shall iterate to get a model of NCF, Miller's rational perfect set forcing.

Let p be a tree of finite subsets of ω; that is, $\varnothing \in p$ and if $a \in p$ then every initial segment of a is also in p. These initial segments are the predecessors of a in p. A node $a \in p$ is said to be *infinitely branching* in p, if there are infinitely many n such that $a \cup \{n\} \in p$, or equivalently, if a has infinitely many immediate successors in p. The tree p is said to be *superperfect* if every node $a \in p$ has an infinitely branching successor. The superperfect trees are the forcing conditions in a notion of forcing Q, ordered so that the extensions of a condition are its superperfect subtrees. (Miller's definition in [3] involved trees of finite sequences rather than finite sets, but there are obvious isomorphisms that replace sequences by strictly increasing sequences and then by the sets that they enumerate.)

We shall need three methods of constructing extensions of a given condition p. The first is to select a node $a \in p$ and to form the subtree p/a of all nodes comparable with a. Clearly p/a is superperfect. This method of extension suffices to show that the set

$$\{p \in Q \mid p \text{ contains only one node of cardinality } n\}$$

is dense in Q. It follows that, if G is a V-generic subset of Q, then the intersection of all the trees $p \in G$ is a single path through the tree of all finite subsets of ω, so it defines an infinite subset $\bar{G}$ of ω. We write $f_G(n)$ for the cardinality of $\bar{G} \cap n$; thus, $f_G : \omega \to \omega$ is constant precisely on the intervals into which the members of G divide ω.

The second way to extend a given condition p is to select, for each infinitely branching node $a \in p$, an infinite subset X_a of $\{n \in \omega \mid a \cup \{n\} \in p \text{ and } n > \max(a)\}$ and to throw away all the immediate successors $a \cup \{n\}$ of a with $n \notin X_a$. The resulting subtree of p is

$$q = \{b \in p \mid \text{for every proper initial segment } a \text{ of } b, \text{ if } a \text{ is infinitely branching in } p, \text{ then } \min(b - a) \in X_a\}.$$

It is easy to verify that q is superperfect and is therefore an extension of p. We refer to this construction as *thinning* p with the sets X_a.

The third method of building an extension q of p is called *fusion* and is really a meta-method, a way of combining many extension processes into one. It proceeds as follows. Select an infinitely branching node $a \in p$. Throw away all nodes incomparable with a (so q will be a subtree of p/a), but put into q the node a, all its predecessors, and all its immediate successors in p. For each of these immediate successors $a \cup \{n\}$, select an extension p_n of $p/(a \cup \{n\})$ and select an infinitely branching node $a_n \in p_n$. Throw away all nodes still present that are in no p_n/a_n (so q will be included in the union of the p_n/a_n, a union which includes the nodes already put into q), and put into q all the nodes a_n, all their predecessors, and all their immediate successors in the corresponding p_n. For each of these immediate successors $a_n \cup \{m\}$, select an extension p_{nm} of $p_n/(a_n \cup \{m\})$, select an infinitely branching node $a_{nm} \in p_{nm}$, and proceed as before. Repeating this process ω times yields a tree q which is superperfect and therefore an extension of p. The infinitely branching nodes of q are precisely the nodes a_s (where s is a finite sequence of subscripts) selected at the various stages of the construction. For each immediate successor $a_s \cup \{k\}$ of such a node in q, the condition $q/(a_s \cup \{k\})$ is an extension of the p_{sk} chosen during the construction.

If q is the condition just constructed by fusion, if r is any extension of q, and if $n \in \omega$, then r must contain one of the infinitely branching nodes a_s of q with s of length n. It must also contain $a_s \cup \{k\}$ for some k, and must therefore be compatible with $q/(a_s \cup \{k\})$, since $r/(a_s \cup \{k\})$ is a common extension. Thus, r is also compatible with p_{sk}.

If we were given a countable sequence $W_0, W_1, \ldots$ of maximal antichains (or predense sets) in Q, then, at the stage of the fusion construction where the conditions p_s are being chosen for s of length $n \geq 1$, we could choose each of these conditions to be an extension of a condition $w_s \in W_{n-1}$. Then, by the preceding paragraph, every extension of q is compatible with w_s for some s of length n; in other words, the countable subset

$$W'_{n-1} = \{w_s | s \text{ of length } n\}$$

of W_n is predense beyond q.

If all the arbitrary choices in this construction of q, with a prescribed condition p and prescribed W_n's, are made in some fixed manner, say in accordance with a specific well-ordering of (a sufficiently large piece of) the universe, then W'_n depends only on $W_0, W_1, \ldots, W_n$. List the countable set W'_n in an ω-sequence and write $F(p, W_0, \ldots, W_n, k)$ for its kth member.

Lemma 6 *The notion of forcing Q is proper.*

Proof: We verify the criterion for properness called $\mathrm{Con}_2(\lambda)$ in [5], p. 77, except that we dispense with the ordinal indexing used there. Suppose that s is a countable set containing all the natural numbers and closed under the function F defined above. Let $W_0, W_1, \ldots$ be all the predense subsets of Q that are members of s, and let p be any element of $Q \cap s$. The fusion construction above yields an extension q of p beyond which the sets W'_n are predense. The assumptions on s imply that $W'_n \subseteq W_n \cap s$, so each $W_n \cap s$ is predense beyond q. Since the collection of all s that satisfy these assumptions is closed and unbounded, $\mathrm{Con}_2(\lambda)$ holds.

Notice that our definition of fusion is such that, in the resulting tree q, the only nodes with more than one immediate successor are the a_s, which have infinitely many immediate successors. Thus, trees in which every branching node is infinitely branching are dense in Q and we may, whenever convenient, confine our attention to such trees. It will, in fact, be convenient to perform some additional normalizations on our trees, as follows.

We say that a superperfect tree p has *interval structure* if ω can be partitioned into (finite) intervals $[0, i_0), [i_0, i_1), [i_1, i_2), \ldots$ so that, if a node $a \in p$ has more than one immediate successor $a \cup \{n\} \in p$, then

(i) each interval $[i_k, i_{k+1})$ after the one containing $\max(a)$ contains exactly one n such that $a \cup \{n\} \in p$
(ii) every immediate successor $a \cup \{n\} \in p$ of a is as in (i); i.e., n is not in the same interval as $\max(a)$
(iii) each $a \cup \{n\}$ as above has an infinitely branching successor $b \in p$ with $\max(b)$ in the same interval $[i_k, i_{k+1})$ as n.

(The trivial problem that $\max(\varnothing)$ is undefined can be avoided by requiring $\varnothing$ to be a nonbranching node or by agreeing that every interval is "after the one containing $\max(\varnothing)$".) Notice that, in a tree with interval structure, every branching node is infinitely branching.

Lemma 7 *The set of superperfect trees with interval structure is dense in Q.*

Proof: Let p be any superperfect tree; we wish to extend it to one with interval structure. By a preliminary fusion, we can assume that every branching node of p is infinitely branching and that $\varnothing$ is not branching. We inductively define a sequence $(i_k)_{k\in\omega}$, which will provide the interval structure of an extension of p. Choose i_0 arbitrarily. After i_k is defined, choose i_{k+1} so large that, for each (infinitely) branching node a with $\max(a) < i_k$, there exist $n = n(a,k) \in \omega$ and $b = b(a,k) \in p$ such that

b is a branching node of p,
b is a successor of $a \cup \{n\}$ (so $a \cup \{n\} \in p$),
$i_k \leq n$, and
$\max(b) < i_{k+1}$.

Since only finitely many a have $\max(a) < i_k$, and since p is superperfect, it is clear that such an i_{k+1} can be found. Finally, the desired extension of p, having interval structure given by these i_k's, is obtained by thinning p with $X_a = \{n(a,k) | \max(a) < i_k\}$; i.e.,

$$q = \{b \in p | \text{ for every proper initial segment } a \text{ of } b, \text{ if } a \text{ is branching in } p \text{ then } \min(b-a) = n(a,k) \text{ for some } k \text{ (with } \max(a) < i_k)\}.$$

If p has interval structure given by intervals $I_k = [i_{k-1}, i_k)$ (where i_{-1} is 0) and if we choose any subsequence of this sequence of intervals, say $J_k = I_{n_k}$, then we can thin p with $X_a = \{n | a \cup \{n\} \in p \text{ and } n \in J_k \text{ for some } k\}$ to obtain an extension q of p in which every node is a subset of the union of the J_k's and the first branching node a_0. The J_k's fail to provide an interval structure for q only because they are not in general adjacent in ω; if we expand them to adjacent intervals, for example by adding to each J_k the intervals I_m between J_k and J_{k+1}, then we obtain an interval structure for q.

For the next lemma, recall that if G is a generic subset of Q, then $\bar{G}$ is the infinite subset of ω determined by G (namely, the union of the nodes that are in every $p \in G$), and $f_G : \omega \to \omega$ is constant with value n on the nth of the intervals into which G divides ω.

Lemma 8 *Let X be an infinite subset of ω and let $\mathcal{U}$ be a nonprincipal ultrafilter on ω, both in V. Let G be a V-generic subset of Q. Then there exists a $Y \in \mathcal{U}$ with $f_G(Y) \subseteq f_G(X)$.*

Proof: Let $X, \mathcal{U}$, and a condition $p \in Q$ be given. We shall find a $Y \in \mathcal{U}$ and an extension q of p forcing that $f_G(Y) \subseteq f_G(X)$, i.e., that if $m < n$ are members of $\bar{G}$ and the interval $(m,n]$ meets Y, then it also meets X. This will clearly suffice to prove the lemma. By Lemma 7, we may assume that p has interval structure, and by the construction following that lemma we may assume that each of these intervals I_k meets X. $\mathcal{U}$ must contain either the union of the even-numbered intervals or the union of the odd-numbered ones; let Y_0 be whichever of these two unions belongs to $\mathcal{U}$. Applying once more the thinning construction given after Lemma 7, extend p to a condition q such that every node in q is a subset of $a_0 \cup (\omega - Y_0)$, where a_0 is the first branching node of q. Let

$Y \in \mathcal{U}$ be obtained by removing from Y_0 all (finitely many) intervals I_k up to and including the one containing $\max(a_0)$. Since q forces "$\bar{G} \subseteq a_0 \cup (\omega - Y_0)$", it also forces that, if $m < n$ are in $\bar{G}$ and $(m,n]$ meets Y then $(m,n]$ contains a whole interval $I_k \subseteq Y$, and therefore meets X.

If $\mathcal{U}$ is an ultrafilter in V, then we write $\bar{\mathcal{U}}$ for the filter generated by $\mathcal{U}$ in some generic extension of V. The context will always make it clear which extension is meant. In general, $\bar{\mathcal{U}}$ may or may not be an ultrafilter in the extension.

Corollary 1 *If $\mathcal{U}$ and $\mathcal{U}'$ are nonprincipal ultrafilters on ω in V and if G is a V-generic subset of Q, then $f_G(\bar{\mathcal{U}}) = f_G(\bar{\mathcal{U}}')$.*

Proof: $f_G(\bar{\mathcal{U}}')$ is generated by the sets $f_G(X)$ for $X \in \mathcal{U}'$. By Lemma 8, every such set has a subset in $f_G(\bar{\mathcal{U}})$. So $f_G(\bar{\mathcal{U}}') \subseteq f_G(\bar{\mathcal{U}})$, and the reverse inclusion follows by symmetry.

Lemma 9 ([3], Claim 2.4) *If the set of infinitely branching nodes of a tree $p \in Q$ is partitioned into two pieces, then p has an extension q all of whose infinitely branching nodes are in the same piece.*

Proof: We attempt a fusion construction in which the nodes a_s, chosen to become the infinitely branching nodes of q, are all in the first piece. If we succeed, the fusion produces the desired q. If we fail, it is because at some stage of the construction the tree $p_s/(a_s \cup \{n\})$, from which an a_{sn} had to be chosen, had no infinitely branching node in the first piece. Then this $p_s/(a_s \cup \{n\})$ is the desired q.

The following two lemmas are due to Miller ([3], Propositions 4.1 and 4.2]); we give a different proof, parallel to ([2], Theorem 3.3]).

Lemma 10 *If $\mathcal{U}$ is a* P*-point in V and G is a V-generic subset of Q, then the filter $\bar{\mathcal{U}}$ generated in $V[G]$ by $\mathcal{U}$ is an ultrafilter in $V[G]$.*

Proof: Let $\boldsymbol{A}$ be a name in the forcing language associated with Q, and let $p \in Q$ force that $\boldsymbol{A} \subseteq \omega$. We shall find a $B \in \mathcal{U}$ and an extension q of p such that either q forces B to be a subset of $\boldsymbol{A}$ or q forces B to be disjoint from $\boldsymbol{A}$; this will clearly suffice to prove the lemma. We shall extend p in several steps to obtain the desired q. (The argument takes place in V.)

We begin by performing a fusion construction on p, choosing the conditions p_{sn} at each stage so that, for each $j \leq n$, p_{sn} decides whether or not $j \in \boldsymbol{A}$. The resulting condition p' has the property that, for each of its (infinitely) branching nodes a, each immediate successor $a \cup \{n\} \in p'$, and each $j \leq n$, $p'/(a \cup \{n\})$ decides whether or not $j \in \boldsymbol{A}$. Notice that every extension of p' has the same property. By thinning p', we obtain p'' such that the decision by $p''/(a \cup \{n\})$ about whether or not $j \in \boldsymbol{A}$ depends only on a and j, not on n, once n is large enough. (Here "large enough" depends also on a and j.) For each branching node a of p'', set

$$A'(a) = \{j \mid \text{ for all sufficiently large } n \text{ such that } a \cup \{n\} \in p'',\ p''/(a \cup \{n\}) \text{ forces } j \in \boldsymbol{A}\}.$$

Partition the branching nodes a of p'' into two classes according as whether or not $A'(a) \in \mathcal{U}$, and apply Lemma 9 to extend p'' to p''' with all its infinitely branching nodes in the same class. We may assume that $A'(a) \in \mathcal{U}$ for all infinitely branching nodes a of p''', for the other case reduces to this one if we replace $\boldsymbol{A}$ with its complement. Notice that $A'(a)$ is unchanged when p'' is replaced by p''' (or any other extension of p'') in its definition, provided that a is an infinitely branching node of p''' (or of the other extension in question).

As $\mathcal{U}$ is a P-point, there is a $B \in \mathcal{U}$ such that $B - A'(a)$ is finite for every infinitely branching node a of p'''. By Lemma 7, we can extend p''' to a condition $p^{(4)}$ with interval structure, and by the discussion following that lemma, we can extend this condition further to a $p^{(5)}$ with interval structure $[0, i_0)$, $[i_0, i_1) \ldots$ and with the further property that, whenever a is a branching node of $p^{(5)}$ and $a \subseteq i_k$ (i.e., all members of a are $< i_k$), then

(i) $B - A'(a) \subseteq i_{k+1}$

(ii) if $a \cup \{n\} \in p^{(5)}$, $n \geq i_{k+1}$, and $j \in A'(a) \cap i_k$, then $p^{(5)}/(a \cup \{n\})$ forces $j \in \boldsymbol{A}$.

To see this, it suffices to choose inductively the i_k from among the endpoints of intervals involved in the interval structure of $p^{(4)}$, so that each interval for $p^{(5)}$ is a union of intervals for $p^{(4)}$. Once i_k is chosen, i_{k+1} can be chosen large enough to satisfy (i) and (ii) because only finitely many a's and j's are involved, each $B - A'(a)$ is finite (by definition of B), and all sufficiently large n are as desired in (ii) (by definition of $A'(a)$).

It will be convenient to assume that the first branching node a_0 of $p^{(5)}$ has $\max(a_0) < i_0$; this can be achieved by combining into a single interval all the intervals up to and including the one that contains $\max(a_0)$.

Partition ω into four pieces, each containing every fourth interval $[i_k, i_{k+1})$; that is, the mth piece $(0 \leq m < 4)$ is the union of the $[i_k, i_k + 1)$ for $k \equiv m \pmod 4$. Being an ultrafilter, $\mathcal{U}$ must contain one of these pieces. Replacing B by its intersection with this piece, we can ensure that B meets only (at most) every fourth interval, while all our previous statements about B remain true. Similarly, we can ensure that B has no members smaller than i_2.

Thin $p^{(5)}$ to obtain an extension q (still with a_0 as its first branching node) all of whose nodes a have $a - a_0$ disjoint from the intervals $[i_k, i_{k+1})$ that meet B, as well as from the immediately preceding and following intervals $[i_{k-1}, i_k)$ and $[i_{k+1}, i_{k+2})$. Since B meets only every fourth interval, there are infinitely many intervals that $a - a_0$ is permitted to meet, so the construction after Lemma 7 yields such a q. (We continue to use the notation $[i_k, i_{k+1})$ for the intervals associated with $p^{(5)}$, not the larger ones associated with q.) We complete the proof by showing that q forces $B \subseteq \boldsymbol{A}$.

Suppose the contrary. Then there exist a $j \in B$ and an extension r of q such that r forces $j \notin \boldsymbol{A}$. Let $[i_k, i_{k+1})$ be the interval containing j. By our normalization of B two paragraphs ago, $k \geq 2$ and, for every node a of q, $a - a_0$ is disjoint from $[i_{k-1}, i_{k+2})$. Furthermore, by our earlier normalization of the i_k's, the first branching node a_0 of q (and of $p^{(5)}$) has $\max(a_0) < i_0 \leq i_{k-2}$, so in fact every node a of q is disjoint from $[i_{k-1}, i_{k+2})$.

Extending r to some r/b if necessary, we can arrange that the first branching node b_0 of r has $\max(b_0) \geq i_{k+2}$. Let a be the last branching node of q that is a predecessor of b_0 and has $\max(a) < i_{k+2}$. (a_0 is a predecessor of b_0 and $\max(a_0) < i_{k+2}$, so a exists.) Since a is disjoint from $[i_{k-1}, i_{k+2})$, it follows that $\max(a) < i_{k-1}$. In view of the interval structure of $p^{(5)}$, b_0 is a successor of $a \cup \{n\}$ for some $n \geq i_{k+2}$, for if n were smaller there would be a branching node a', between $a \cup \{n\}$ and b_0, with $\max(a')$ in the same interval as n, hence smaller than i_{k+2}, contrary to the choice of a.

Since $\max(a) < i_{k-1}$, requirement (i) (with k changed to $k-1$) in the definition of $p^{(5)}$ says that $B - A'(a) \subseteq i_k$. But $j \in B$ and $j \geq i_k$, so $j \in A'(a)$. Then, since $j < i_{k+1}$ and $n \geq i_{k+2}$, requirement (ii) (with k changed to $k+1$) in the definition of $p^{(5)}$ says that $p^{(5)}/(a \cup \{n\})$ forces $j \in \boldsymbol{A}$. This is absurd, because r is an extension of $p^{(5)}/(a \cup \{n\})$ and yet it forces $j \notin \boldsymbol{A}$. This contradiction completes the proof of Lemma 10.

Lemma 11 *Under the hypotheses of Lemma 10, $\bar{\mathcal{U}}$ is a* P*-point in $V[G]$.*

Proof: Let countably many sets $X_n \in \bar{\mathcal{U}}$ be given (in $V[G]$); we seek a $Y \in \bar{\mathcal{U}}$ almost included in every X_n. As each X_n has a subset in $\mathcal{U}$, we may as well suppose that $X_n \in \mathcal{U}$ for all n. By Lemmas 1 and 6, there is a countable (in V) family $\mathfrak{F} \in V$ that contains all the X_n's. As $\mathcal{U}$ is a P-point in V, it contains a Y that is almost included in every member of $\mathcal{U} \cap \mathfrak{F}$, hence in particular in every X_n.

Armed with all these lemmas, we are ready to prove the theorem. Assume the continuum hypothesis in the ground model V. Let $\langle P_\alpha, \boldsymbol{Q}_\alpha : \alpha < \aleph_2 \rangle$ be a countable support iteration in which, for each α, P_α forces "$\boldsymbol{Q}_\alpha$ is the set of superperfect trees ordered by inclusion". Let G be a V-generic subset of $P = P_{\aleph_2}$ = the direct limit of the P_α's. For each $\alpha < \aleph_2$, let G_α be the V-generic subset $G \cap P_\alpha$ of P_α, and let H_α be the $V[G_\alpha]$-generic subset of Q_α (the value in $V[G_\alpha]$ of $\boldsymbol{Q}_\alpha$) such that $G_\alpha * H_\alpha = G_{\alpha+1}$.

Since the continuum hypothesis holds in V, there is a P-point in V. By Lemmas 5 and 11, it generates a P-point in each $V[G_\alpha]$ and in $V[G]$. We write $\mathcal{U}_0$ for the one in $V[G]$, so, for each α, $\mathcal{U}_0 \cap V[G_\alpha]$ is the ultrafilter in $V[G_\alpha]$ generated by $\mathcal{U}_0 \cap V$.

Let $\mathcal{U}$ be an arbitrary nonprincipal ultrafilter on ω in $V[G]$. By Lemma 3, there is an $\aleph_1$-closed unbounded set of ordinals α such that $\mathcal{U} \cap V[G_\alpha] \in V[G_\alpha]$. For each such α, $\mathcal{U} \cap V[G_\alpha]$ is clearly a nonprincipal ultrafilter on ω in $V[G_\alpha]$. Applying Corollary 1, with $V[G_\alpha]$ as the ground model, H_α as the generic set, and $\mathcal{U} \cap V[G_\alpha]$ and $\mathcal{U}_0 \cap V[G_\alpha]$ as the two ultrafilters, we find that $f_\alpha(\overline{\mathcal{U} \cap V[G_\alpha]}) = f_\alpha(\overline{\mathcal{U}_0 \cap V[G_\alpha]})$, where f_α abbreviates f_{H_α} and where the bars mean "filter generated in $V[G_{\alpha+1}]$ by". Thus,

$$f_\alpha(\mathcal{U}) \supseteq f_\alpha(\overline{\mathcal{U} \cap V[G_\alpha]}) = f_\alpha(\overline{\mathcal{U}_0 \cap V[G_\alpha]}) \supseteq f_\alpha(\mathcal{U}_0 \cap V).$$

But $\mathcal{U}_0 \cap V$ generates $\mathcal{U}_0$, so $f_\alpha(\mathcal{U}) \supseteq f_\alpha(\mathcal{U}_0)$. Since $f_\alpha(\mathcal{U}_0)$ is an ultrafilter, it follows that $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{U}_0)$.

If $\mathcal{U}'$ is another nonprincipal ultrafilter on ω in $V[G]$, then it too satisfies $f_\alpha(\mathcal{U}') = f_\alpha(\mathcal{U}_0)$ for an $\aleph_1$-closed unbounded class of α's. Since any two $\aleph_1$-closed unbounded subsets of $\aleph_2$ intersect (here we use that $\aleph_2$ is preserved, by Lemma 4), there is an α that works for both $\mathcal{U}$ and $\mathcal{U}'$. So $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}')$. Since f_α is finite-to-one, this completes the verification of NCF in $V[G]$.

REFERENCES

[1] Blass, A., "Near coherence of filters, I," *Notre Dame Journal of Formal Logic*, vol. 27 (1986), pp. 579-591; "Near coherence of filters, II," *Transactions of the American Mathematical Society*, vol. 300 (1987), pp. 557-581.

[2] Blass, A. and S. Shelah, "There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$ points and the Rudin-Keisler ordering may be downward directed," *Annals of Pure and Applied Logic*, vol. 83 (1987), pp. 213-243.

[3] Miller, A., "Rational perfect set forcing," pp. 143-159 in *Axiomatic Set Theory*, edited by J. Baumgartner, D. A. Martin, and S. Shelah, Contemporary Mathematics 31, American Mathematical Society, Providence, Rhode Island, 1984.

[4] Rudin, W., "Homogeneity problems in the theory of Cech compactifications," *Duke Mathematical Journal*, vol. 23 (1956), pp. 409-419.

[5] Shelah, S., *Proper Forcing*, Springer Lecture Notes in Mathematics 940, Springer-Verlag, Berlin, 1982.

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