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MULTIPLE SOLUTIONS FOR SCHRÖDINGER–POISSON SYSTEMS WITH CRITICAL NONLOCAL TERM

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Abstract. This paper is concerned with the existence of positive bound state solutions for Schrödinger–Poisson systems with critical nonlocal term:

$$(\mathcal{P})\qquad \begin{cases} -\Delta u = \phi|u|^3u + \lambda Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3. \end{cases}$$

Under certain assumptions on Q and λ, we prove that $(\mathcal{P})$ has multiple positive bound state solutions by decomposition the Nehari manifold and fine estimates.

1. Introduction and main results

In the last two decades the following Schrödinger–Poisson systems

$$(1.1)\qquad \begin{cases} -\Delta u + V(x)u + \phi u = |u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \varepsilon|u|^2 & \text{in } \mathbb{R}^3, \end{cases}$$

have been intensively studied by a lot of researchers, due to the fact that solutions $(u(x), \phi(x))$ of (1.1) correspond to standing wave solutions $(e^{-i\lambda t}u(x), \phi(x))$ of

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the time-dependent systems

$$\begin{cases} i\,\dfrac{\partial \psi}{\partial t} = -\Delta\psi + \widetilde{V}(x)\psi + \psi\phi - |\psi|^{q-2}\psi, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ -\Delta\phi = \varepsilon|\psi|^2, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \end{cases}$$

where i is the imaginary unit, $\mathbb{R}^+ = [0,+\infty)$, $\widetilde{V} = V + \lambda$ and ε takes value $+1$ or -1, depending whether the interaction between the particles is repulsive or attractive. Systems of this type stem from many physical problems, especially in quantum mechanics and semiconductor theory [7], [21], [24]. In particular, (1.1) was introduced by Benci and Fortunato in [7] as a model describing standing waves for the nonlinear Schrödinger equations interacting with an unknown electrostatic field. When the potential V is radially symmetric or even a positive constant, many papers have been devoted to studying existence and multiplicity of nontrivial solutions of (1.1) under various assumptions on the nonlinearities (see e.g. [3], [4], [11], [15], [17], [26]). In such a case, one can search solutions in the subspace of $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ which consists of radially symmetric functions. In the case that the potential V is not radially symmetric, (1.1) was also widely investigated (see e.g. [6], [10], [16], [27], [30], [32]).

Recently Schrödinger–Poisson systems with critical nonlocal term have been paid much attention to. For example, in [5] Azzollini and Avenia had obtained Brezis-Nirenberg type's results of the system:

$$\begin{cases} -\Delta u + \lambda u = q\phi|u|^3u & \text{in } B_r, \\ -\Delta\phi = q|u|^5 & \text{in } B_r, \\ u = \phi = 0 & \text{on } \partial B_r. \end{cases}$$

In [23] Liu had studied periodic solutions of the system:

$$\begin{cases} -\Delta u + V(x)u - K(x)\phi|u|^3u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)|u|^5 & \text{in } \mathbb{R}^3. \end{cases}$$

The existence, nonexistence and the multiplicity of positive radially symmetric solutions to the system

$$\begin{cases} -\Delta u + u + \lambda\phi|u|^3u = \mu|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3, \end{cases}$$

were studied by variational methods in [19].

Partially motivated by the works of [5], [18], [19], [23], [27], we are concerned with the following Schrödinger–Poisson system with critical nonlocal term:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u = \phi|u|^3u + \lambda Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3. \end{cases}$$

Such a system is related to the well-known Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,$$

which was introduced as an approximation to the Hartree–Fock theory of one component plasma in [20]. Here, $I_\alpha\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ denotes the Riesz potential, $1 < \alpha < N$ and $p > 1$. A quick and comprehensive understanding of Choquard equation is available in [25]. The conditions of Q we may use in this paper are as follows:

(Q_1) $Q \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}) \cap L^{6/(6-q)}(\mathbb{R}^3)$ for $1 < q < 2$;
(Q_2) for some $x_0 \in \mathbb{R}^3$ we have $Q(x_0) > 0$;
(Q_3) for all $x \in \mathbb{R}^3$ we have $Q(x) \geq 0$.

Now let us state our main results:

THEOREM 1.1. *Assume* (Q_1) *and* (Q_2) *hold. Then there exists a constant*

$$\lambda^* = \frac{8(2-q)^{(2-q)/8} S^{(6-q)/4}}{(10-q)^{(10-q)/8} |Q^+|_{6/(6-q)}}$$

such that for all $\lambda \in (0, \lambda^*)$ *the problem* $(\mathcal{P})$ *possesses a positive ground state solution.*

THEOREM 1.2. *Assume* (Q_1)–(Q_3) *hold. Then the problem* $(\mathcal{P})$ *possesses two positive solutions if* $\lambda \in (0, q(12-q)\lambda^*/20)$.

REMARK 1.3. Our conclusions have generalized the classical results of classical Schrödinger equation with combined concave and convex nonlinearities in [1] to Schrödinger–Poisson systems with nonlocal critical term. Furthermore, we improved those results because we can give a concrete estimate of λ^* and Q may change sign (see Theorem 1.1). On the other hand, we complete the results of [19], [27] in the sense that we study Schrödinger–Poisson systems with a sublinear perturbation.

The paper is organized as follows. In Section 2, we describe the notations and preliminaries. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2. Notations and preliminaries

Throughout the paper we will use the following notations:

- $B_r(z)$ denotes the open ball centered at z with the radius r and $B_r^c(z)$ the outer ball. In particular, $B_r := B_r(0)$ and $B_r^c = B_r(0)^c$ if $z = 0$;
- the domain for an integral is $\mathbb{R}^3$ without special explanation and we omit it;

- let H be the usual Sobolev space $D^{1,2}(\mathbb{R}^3)$ endowed with its standard scalar product and norm

$$(u,v)_H = \int \nabla u \cdot \nabla v, \quad \|u\| = \left(\int |\nabla u|^2 \right)^{1/2};$$

- for any $s \in [1,+\infty]$, $|\cdot|_s$ denotes the usual norm of the Lebesgue space $L^s(\mathbb{R}^3)$;
- $u^+ := \max\{u,0\}$ and $u^- := \min\{u,0\}$;
- S denotes the best constant of the embedding: $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, i.e.

$$S = \inf_{H\setminus\{0\}} \frac{\displaystyle\int |\nabla u|^2}{\left(\displaystyle\int |u|^6\right)^{1/3}};$$

- C, C_i, $i \in \mathbb{N}$ are intrinsic positive constants which can change from line to line;
- $O(1)$ and $o(1)$ denote the bounded and vanishing quantities as $n \to \infty$ or $\varepsilon \to 0^+$, respectively.

First of all, for any fixed $u \in H$, the second equation of (1.1) is a Poisson equation which is uniquely solvable. Then the system can be reduced to a single elliptic equation with a nonlocal term. The idea of this reduction method is originally due to Benci and Fortunato [7] and now is a basic strategy of studying the Schrödinger–Poisson system. For each $u \in H$, define the linear operator $T_u \colon H \to \mathbb{R}$ by

$$T_u(v) = \int |u|^5 v.$$

The Hölder inequality and Sobolev inequality lead to

$$|T_u(v)| \le |u|_6^5 |v|_6 \le S^{-1/2} |u|_6^5 \|v\|,$$

which implies that T_u is continuous. Then by Riesz representation theorem [8] it follows that for each $u \in H$, there is a unique $\phi_u \in H$ such that

$$-\Delta\phi = |u|^5 \quad \text{in } \mathbb{R}^3$$

and ϕ_u is continuous with respect to u. Moreover, by the Calderon–Zygmund inequality [14, Theorem 9.9] and the Sobolev embedding theorem we see that ϕ_u can be written as

$$\phi_u(x) = \frac{1}{4\pi} \int \frac{|u|^5(y)}{|x-y|}\, dy. \tag{2.1}$$

Hence, we can define the energy functional corresponding to the problem $(\mathcal{P})$ as

$$I(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{10}\int \phi_u |u|^5 - \frac{\lambda}{q}\int Q(x)|u|^q, \quad u \in H.$$

More properties of ϕ_u are summarized in the next lemma.

LEMMA 2.1. *For every $u \in H$, we have the following conclusions*:

(a) $\phi_u \geq 0$;

(b) $\|\phi_u\|^2 = \int \phi_u |u|^5 \leq S^{-1/2}\|\phi_u\||u|_6^5 \leq S^{-6}\|u\|^{10}$;

(c) *for any* $t > 0$, $\phi_{tu} = t^5\phi_u$;

(d) *if* $u_n \rightharpoonup u$ *weakly in* H *and* $u_n \to u$ *almost everywhere on* $\mathbb{R}^3$, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *in* H;

(e) *if* u *is radial, then* ϕ_u *is also radial.*

PROOF. The proofs are easy to obtain by (2.1). □

The following Hardy–Littlewood–Sobolev inequality [20], [22], [25] is vital for convolution.

LEMMA 2.2. *Let* $0 < \alpha < N$, $p, q > 1$ *and* $1 < r < s < \infty$ *be such that*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\alpha}{N}, \qquad \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

(a) *For any* $f \in L^p(\mathbb{R}^N)$ *and* $g \in L^q(\mathbb{R}^N)$, *one has*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)g(y)|}{|x-y|^{N-\alpha}}\,dx\,dy \leq C(N,\alpha,p)|f|_{L^p(\mathbb{R}^N)}|g|_{L^q(\mathbb{R}^N)}.$$

(b) *For any* $f \in L^r(\mathbb{R}^N)$, *one has*

$$\left|\frac{1}{|\cdot|^{N-\alpha}} * f\right|_{L^s(\mathbb{R}^N)} \leq C(N,\alpha,r)|f|_{L^r(\mathbb{R}^N)}.$$

In particular, in our case we have for all $u \in H$

$$\left(\frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u|^5(x)|u|^5(y)}{|x-y|}\,dx\,dy\right)^{1/5} \leq S^{-6/5}\int |\nabla u|^2$$

and

$$|\phi_u|_6 \leq S^{-1}|u|_6^5.$$

According to Lemma 2.2, we see $I \in \mathcal{C}^1(H)$. Hence, in order to solve equation $(\mathcal{P})$ we only need to find critical points of I and powerful variational methods can be applied.

It is useful to note that a counterpart of the Brezis–Lieb lemma holds for Riesz potential (see [18, Lemma 2.2] or [25]):

LEMMA 2.3. *If* $u_n \rightharpoonup u$ *weakly in* H, *then going to a subsequence, if necessary, we have*:

$$\begin{gathered} 2|u_n|^5 - |u_n - u|^5 - |u|^5 \to 0 \quad \text{in } L^{6/5}(\mathbb{R}^3), \\ \phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0 \quad \text{in } H, \\ \int \phi_{u_n}|u_n|^5 - \int \phi_{u_n-u}|u_n - u|^5 - \int \phi_u |u|^5 \to 0, \end{gathered} \tag{2.2}$$

and

$$\int \phi_{u_n}|u_n|^3 u_n\eta \to \int \phi_u|u|^3 u\eta \quad \text{for any } \eta \in \mathcal{C}_0^\infty(\mathbb{R}^3). \tag{2.3}$$

By (c) of Lemma 2.1 it is easy to see that for each $u \neq 0$ we have $\lim_{t\to+\infty} I(tu) = -\infty$. Hence, we obtain

$$\inf_{u\in H} I(u) = -\infty.$$

It is useful to make use of well-known Nehari manifold:

$$\mathcal{N} = \{u \in H\backslash\{0\} : \langle I'(u), u\rangle := N(u) = 0\}.$$

Adopting the method used in [27], [29], $\mathcal{N}$ can be decomposed to three mutually disjoint parts:

$$\begin{aligned}
\mathcal{N}^+ &= \{u \in \mathcal{N} \mid \langle N'(u), u\rangle > 0\},\\
\mathcal{N}^0 &= \{u \in \mathcal{N} \mid \langle N'(u), u\rangle = 0\},\\
\mathcal{N}^- &= \{u \in \mathcal{N} \mid \langle N'(u), u\rangle < 0\}.
\end{aligned}$$

In order to prove our main results, we introduce the following "limit problem":

$$\begin{cases} -\Delta u = \phi|u|^3 u & \text{in } \mathbb{R}^3\\ -\Delta\phi = |u|^5 & \text{in } \mathbb{R}^3. \end{cases} \tag{$\mathcal{P}_\infty$}$$

It is well known that the best embedding constant S is achieved at the function

$$U(\,\cdot\,) = \frac{3^{1/4}}{(1+|\cdot|^2)^{1/2}}$$

(see [28]). Furthermore, U is a ground state solution of the equation:

$$-\Delta u = |u|^4 u \quad \text{in } \mathbb{R}^3. \tag{2.4}$$

Theorem 2.4. *All positive solutions of* $(\mathcal{P}_\infty)$ *have the form*

$$u(\,\cdot\,) = \phi(\,\cdot\,) = \varepsilon^{-1/2} U\bigg(\frac{\cdot - \xi}{\varepsilon}\bigg) \quad \text{for some } \xi \in \mathbb{R}^3 \text{ and } \varepsilon > 0.$$

Proof. Let u and ϕ be a pair of positive solution to $(\mathcal{P}_\infty)$. Then we have

$$-\Delta(u-\phi) = (\phi - u)|u|^4 \quad \text{in } \mathbb{R}^3.$$

Multiplying the equation by $u-\phi$, and integrating by part, we obtain

$$\int |\nabla(u-\phi)|^2 + \int |u-\phi|^2|u|^4 = 0.$$

Whence, we can conclude $u = \phi$. Furthermore, u satisfies the equation (2.4). Then, by standard argument, we can complete the proof. □

3. Proof of Theorem 1.1

LEMMA 3.1. *I is coercive on $\mathcal{N}$.*

PROOF. For any $u \in \mathcal{N}$, by the Hölder and Sobolev inequalities we can calculate

$$\begin{aligned} I|_{\mathcal{N}}(u) &= \frac{2}{5}\|u\|^2 - \left(\frac{1}{q}-\frac{1}{10}\right)\lambda \int Q(x)|u|^q \\ &\ge \frac{2}{5}\|u\|^2 - \left(\frac{1}{q}-\frac{1}{10}\right)\lambda |Q^+|_{6/(6-q)}|u|_6^q \\ &\ge \frac{2}{5}\|u\|^2 - \left(\frac{1}{q}-\frac{1}{10}\right)\lambda S^{-q/2}|Q^+|_{6/(6-q)}\|u\|^q. \end{aligned} \tag{3.1}$$

Hence, we draw the desired conclusion from $1 < q < 2$. Moreover, setting $D = (1/5q)(2-q)2^{(4+q)/(q-2)}(10-q)^{2/(2-q)}S^{-q/(2-q)}$ then we have

$$I|_{\mathcal{N}}(u) \ge -D[\lambda|Q^+|_{6/(6-q)}]^{2/(2-q)}. \tag{3.2}$$

□

LEMMA 3.2. *There exists $\lambda^* > 0$ such that for each $u \in H$ with $\int Q|u|^q > 0$ and $\lambda \in (0,\lambda^*)$ there exists a unique pair $t^\pm$ such that $t^\pm u \in \mathcal{N}^\pm$. Moreover, we have*

$$0 < t^+ < t^* := \left(\frac{\lambda(2-q)\displaystyle\int Q(x)|u|^q}{8\displaystyle\int \phi_u|u|^5}\right)^{1/(10-q)} < t^- < +\infty.$$

PROOF. Set $h(t) = I(tu)$ for $0 \le t < +\infty$. Then we can compute

$$h'(t) = t\left(\|u\|^2 - t^8\int \phi_u|u|^5 - \lambda t^{q-2}\int Q(x)|u|^q\right) := tg(t).$$

It is easy to see

$$g \in \mathcal{C}^2,\ g''(t) < 0 \quad \text{for } t \in (0,+\infty) \quad \text{and} \quad \lim_{t\to 0^+} g(t) = -\infty = \lim_{t\to+\infty} g(t).$$

Hence, we only need to prove $\max\limits_{t\in(0,+\infty)} g(t) > 0$. But we can calculate

$$\begin{aligned} \max_{t\in(0,+\infty)} g(t) &= g(t^*) \\ &= \|u\|^2 - \frac{10-q}{2-q}\left(\frac{2-q}{8}\right)^{8/(10-q)}\lambda^{8/(10-q)} \\ &\quad\times\left(\int \phi_u|u|^5\right)^{(2-q)/(10-q)}\left(\int Q(x)|u|^q\right)^{8/(10-q)} \\ &\ge \|u\|^2 - \frac{10-q}{2-q}\left(\frac{2-q}{8}\right)^{8/(10-q)}\lambda^{8/(10-q)}\big(S^{-6}\|u\|^{10}\big)^{(2-q)/(10-q)} \\ &\quad\times\big(|Q^+|_{6/(6-q)}S^{-q/2}\|u\|^q\big)^{8/(10-q)} \end{aligned}$$

$$= \|u\|^2 \big(1 - (10-q)(2-q)^{(q-2)/(10-q)} 8^{-8/(10-q)} \lambda^{8/(10-q)} \times S^{-(12-2q)/(10-q)} |Q^+|_{6/(6-q)}^{8/(10-q)}\big).$$

Then we complete the proof if we set

$$\lambda^* = \frac{8(2-q)^{(2-q)/8} S^{(6-q)/4}}{(10-q)^{(10-q)/8} |Q^+|_{6/(6-q)}}. \tag{3.3}$$

□

Lemma 3.3. *$\mathcal{N}^0 = \emptyset$ if $\lambda \in (0, \lambda^*)$.*

Proof. If $u \in \mathcal{N}^0$, then we have $N(u) = 0$ and $\langle N'(u), u\rangle = 0$. Thus, we can obtain

$$\lambda = \frac{8\|u\|^2}{(10-q)\displaystyle\int Q(x)|u|^q}$$

and

$$(2-q)\|u\|^2 = (10-q)\int \phi_u |u|^5. \tag{3.4}$$

According to the Hardy–Littlewood–Sobolev inequality, Lemma 2.2 and (3.4) we can obtain

$$\|u\| \geq \left(\frac{2-q}{10-q}\right)^{1/8} S^{3/4}.$$

Furthermore, by the Hölder inequality, Sobolev inequality and (3.3) we have

$$\lambda \geq \frac{8\|u\|^2}{(10-q)|Q^+|_{6/(6-q)}|u|_6^q} \geq \frac{8S^{q/2}\|u\|^{2-q}}{(10-q)|Q^+|_{6/(6-q)}} \geq \lambda^*.$$

□

Lemma 3.4. *$\mathcal{N}^+$ is bounded in H.*

Proof. By $\langle I'(u), u\rangle = 0$ and $\langle N'(u), u\rangle > 0$ we can calculate

$$8\|u\|^2 < (10-q)\lambda \int Q(x)|u|^q \leq (10-q)S^{-q/2}\lambda |Q^+|_{6/(6-q)}\|u\|^q.$$

Hence, by $1 < q < 2$, we obtain

$$\|u\| \leq \left(\frac{10-q}{8S^{q/2}}\right)^{1/(2-q)} [\lambda |Q^+|_{6/(6-q)}]^{1/(2-q)}.$$

□

Lemma 3.5. *There exists $\delta > 0$ such that $\|u\| \geq \delta$ for all $u \in \mathcal{N}^-$.*

Proof. From $u \in \mathcal{N}^-$ we have $N(u) = 0$ and $\langle N'(u), u\rangle < 0$. Whence, we can derive

$$(2-q)\|u\|^2 < (10-q)\int \phi_u |u|^5.$$

Furthermore, by the Hardy–Littlewood–Sobolev inequality, Lemma 2.2 we see

$$\|u\| \geq \left(\frac{2-q}{10-q}\right)^{1/8} S^{3/4} := \delta. \tag{3.5}$$

□

From Lemma 3.2 and (Q$_1$)–(Q$_2$) we see $\mathcal{N}^{\pm} \neq \emptyset$ if $\lambda \in (0, \lambda^*)$. Hence, we can consider $c^{\pm} := \inf_{\mathcal{N}^{\pm}} I(u)$. By Lemma 3.1 it is easy to see $c^{\pm} > -\infty$. Furthermore, we have the following estimates.

LEMMA 3.6. *For all $\lambda \in (0, \lambda^*)$ we have $c^+ < 0$ and $c^+ \leq c^-$. Furthermore, if $\lambda \in (0, q\lambda^*/2)$ then we have $c^- \geq 0$.*

PROOF. By (Q_1) and (Q_2) we can pick certain $u \in H$ such that

$$\int Q(x)|u|^q > 0.$$

Then, by Lemma 3.2, there exists $t^+ > 0$ such that $t^+ u \in \mathcal{N}^+$. Hence, we have $I(t^+ u) < 0$ from the proof of Lemma 3.2. As a consequence, we see $c^+ < 0$. On the other hand, for each $u \in \mathcal{N}^-$, if $\int Q(x)|u|^q > 0$, then by Lemma 3.2, we have

$$I(u) = \max_{t^+ \leq t < +\infty} I(tu) > I(t^+ u) \geq c^+.$$

Whereas, if $\int Q(x)|u|^q \leq 0$ it is easy to see, for any $\lambda > 0$,

$$I(u) = \frac{2}{5}\|u\|^2 - \left(\frac{1}{q} - \frac{1}{10}\right)\lambda \int Q(x)|u|^q > 0 > c^+.$$

Furthermore, if $\lambda \in (0, q\lambda^*/2)$ then by (3.1) and (3.5) we can obtain the desired result. $\square$

LEMMA 3.7. *If $\lambda \in (0, \lambda^*)$ then there exists a sequence $\{u_n\} \subset \mathcal{N}^+$ (resp. $\mathcal{N}^-$) such that*:

(a) $I(u_n) = c^+ (\textit{resp. } c^-) + o(1)$;
(b) $I'(u_n) = o(1)$.

PROOF. By the well-known Ekeland variational principle [12] we see that there exists a sequence $\{u_n\} \subset \mathcal{N}^+$ such that

(i) $I(u_n) < c^+ + 1/n$;
(ii) for any $v \in \mathcal{N}^+$ we have

$$I(v) - I(u_n) \geq -\frac{1}{n}\|v - u_n\|.$$

Now fix n. Let us consider the function $F\colon \mathbb{R} \times H \to \mathbb{R}$ defined as

$$F(t, v) = t^2\|(u_n + v)\|^2 - t^{10}\int \phi_{u_n+v}|u_n + v|^5 - t^q\lambda \int Q(x)|u_n + v|^q.$$

It is easy to see

$$F(1, 0) = N(u_n) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}\bigg|_{(1,0)} = \langle N'(u_n), u_n \rangle > 0.$$

Hence, by implicit function theorem there exist $\varepsilon_n > 0$ and a $\mathcal{C}^1$ function t_n mapping from $B^H_{\varepsilon_n} := \{u \in H : \|u\| < \varepsilon_n\}$ to $\mathbb{R}$ such that

$$t_n(0) = 1, \quad F(t_n(v), v) = 0 \quad \text{and} \quad \langle t_n'(0), \xi\rangle = -\frac{\langle N'(u_n), \xi\rangle}{\langle N'(u_n), u_n\rangle}.$$

As a consequence, we can assume $t_n(v)(u_n + v) \in \mathcal{N}^+$. Then we can obtain

$$I(t_n(v)(u_n + v)) - I(u_n) \geq -\frac{1}{n}\|t_n(v)(u_n + v) - u_n\|.$$

Noting

$$t_n(v) - 1 = t_n(v) - t_n(0) = \int_0^1 \langle t_n'(sv), v\rangle\, ds,$$

since $t_n \in \mathcal{C}^1$ by reselecting a proper ε_n, we can derive

$$\|t_n(v)(u_n + v) - u_n\| = \|(t_n(v) - 1)u_n + t_n(v)v\| \leq C(1 + \|t_n'(0)\|)\|v\|.$$

Whence, we obtain

$$I(t_n(v)(u_n + v)) - I(u_n) \geq -\frac{C}{n}(1 + \|t_n'(0)\|)\|v\|, \quad \text{for all } v \in B^H_{\varepsilon_n}.$$

On the other hand, by $I \in \mathcal{C}^1$ and $t_n \in \mathcal{C}^1$, we can calculate

$$\begin{aligned} I(t_n(v)(u_n + v)) - I(u_n) &= \int_0^1 \frac{dI(t_n(sv)(u_n + sv))}{ds}\, ds \\ &= \int_0^1 \langle I'(t_n(sv)(u_n + sv)), \langle t_n'(sv), v\rangle(u_n + sv) + t_n(sv)v\rangle\, ds \\ &= \langle I'(u_n), v\rangle + (t_n(sv) - 1)\int_0^1 \langle I'(t_n(sv)(u_n + sv)), v\rangle\, ds \\ &\qquad + \int_0^1 \langle I'(t_n(sv)(u_n + sv)) - I'(u_n), v\rangle\, ds. \end{aligned}$$

Hence, since t_n and I' is continuous, I' maps a bounded set into a bounded set and $\{u_n\}_n$ is bounded we can obtain for certain $\varepsilon_n > 0$

$$I(t_n(v)(u_n + v)) - I(u_n) \leq \langle I'(u_n), v\rangle + \frac{C}{n}\|v\|, \quad \text{for all } v \in B^H_{\varepsilon_n}.$$

Therefore, we can conclude for some $\varepsilon_n > 0$

$$\langle I'(u_n), v\rangle \leq \frac{C}{n}(1 + \|t_n'(0)\|)\|v\|, \quad \text{for all } v \in H,\ \|v\| < \varepsilon_n,$$

which means $I'(u_n) \to 0$ as $n \to +\infty$ if $\{t_n'(0)\}_n$ is bounded.

In the following we show $\{t_n'(0)\}_n$ is bounded. On one hand, by the boundedness of $\{u_n\}$ and N' mapping a bounded set into a bounded set we have

$$|\langle N'(u_n), \xi\rangle| \leq C\|\xi\|.$$

On the other hand, we will claim $\liminf_n |\langle N'(u_n), u_n\rangle| > 0$. Then we can conclude

$$\|t'_n(0)\| \leq \sup_{\|\xi\|\leq 1} \frac{\|N'(u_n)\|\|\xi\|}{|\langle N'(u_n), u_n\rangle|} \leq C.$$

At first, by $c^+ < 0$ and $I(u_n) = c^+ + o(1)$ it is easy to obtain $\liminf_n \|u_n\| > 0$. Then, by contradiction, up to a subsequence we assume $\lim_n \langle N'(u_n), u_n\rangle = 0$. Arguing as the proof of Lemma 3.3 we can derive

$$\liminf_n \|u_n\| \geq \left(\frac{2-q}{10-q}\right)^{1/8} S^{3/4} = \delta \quad \text{and} \quad \lambda \geq \lambda^*.$$

However, we have assumed $\lambda \in (0, \lambda^*)$, a contradiction. □

LEMMA 3.8. *c^+ is attained on $\mathcal{N}^+$ if $\lambda \in (0, \lambda^*)$. Furthermore, the problem $(\mathcal{P})$ has a positive ground state solution.*

PROOF. Now let $\{u_n\} \subset \mathcal{N}^+$ be a $(PS)_{c^+}$ sequence of I in Lemma 3.7. Then $\{u_n\}$ is bounded since I is coercive on $\mathcal{N}$ by Lemma 3.1. Therefore, going to a subsequence if necessary we can assume

$$\begin{aligned} &u_n \rightharpoonup u && \text{in } H, \\ &u_n \to u && \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3) \text{ for all } 1 \leq s < 6, \\ &u_n \to u && \text{a.e. on } \mathbb{R}^3. \end{aligned} \tag{3.6}$$

Hence, by (2.3), (3.6) and the definition of weak convergence we can derive $I'(u) = 0$. Now we set $v_n = u_n - u$. Then we see $v_n \rightharpoonup 0$ weakly in H. In the following we prove $v_n \to 0$ strongly in H. We observe that by (Q_1) one can show

$$\lim_n \int Q(x)|u_n|^q = \int Q(x)|u|^q.$$

In fact, by the Hölder inequality and boundedness of $\{u_n\}$ we have

$$\int |Q(x)|\big||u_n|^q - |u|^q\big| \leq C|Q|_{L^\infty(B_R)} \int_{B_R} \big||u_n|^q - |u|^q\big| + C|Q|_{L^{6/(6-q)}(B_R^c)}.$$

For all $\varepsilon > 0$, by $Q \in L^{6/(6-p)}(\mathbb{R}^3)$ we can choose certain R large enough such that

$$C|Q|_{L^{6/(6-q)}(B_R^c)} < \frac{\varepsilon}{2}.$$

Then, by (3.6), there exists $N > 0$ such that for all $n > N$ we have

$$C|Q|_{L^\infty(B_R)} \int_{B_R} \big||u_n|^q - |u|^q\big| < \frac{\varepsilon}{2}.$$

Hence, by the Brezis–Lieb Lemma (Lemma 2.3) we can conclude

$$c^+ - I(u) + o(1) = \frac{1}{2}\|v_n\|^2 - \frac{1}{10}\int \phi_{v_n}|v_n|^5$$

and

$$\|v_n\|^2 - \int \phi_{v_n} |v_n|^5 = o(1).$$

By Lemma 2.1 (b), we also have

$$\int \phi_{v_n} |v_n|^5 \leq S^{-1/2} \|\phi_{v_n}\| |v_n|_6^5 \leq S^{-6} \|v_n\|^{10}.$$

Therefore, we infer that either $\|v_n\| = o(1)$ or $\|v_n\| \geq S^{3/4} + o(1)$. If the latter occurs, we have

$$c^+ - I(u) \geq \frac{2}{5} S^{3/2}. \tag{3.7}$$

At first, we can claim $u \in \mathcal{N}$ since we have obtained $N(u) = 0$ and we can obtain $u \neq 0$ from (3.7) and Lemma 3.6. In the following we show $u \in \mathcal{N}^+$. By contradiction, we assume $\langle N'(u), u\rangle \leq 0$. By the Brezis–Lieb Lemma (Lemma 2.3) again we can conclude that v_n satisfies

$$\|v_n\|^2 = \int \phi_{v_n} |v_n|^5 + o(1),$$

$$(2-q)\|v_n\|^2 \geq (10-q) \int \phi_{v_n} |v_n|^5 + o(1).$$

Hence, we can get $\|v_n\| = o(1)$, a contradiction.

Now by $u \in \mathcal{N}^+$, we have $c^+ - I(u) \leq 0$, a contradiction with (3.7). Therefore, we conclude that $u_n \to u$ strongly in H and u is a minimizer of c^+ on $\mathcal{N}^+$ by Lemma 3.3. At the same time, we see that u is a solution of $(\mathcal{P})$. To obtain a positive ground state solution of $(\mathcal{P})$ we only need to note that I is even and

$$c^+ = \inf_{v \in \mathcal{N}^+} I(v) = \inf_{v \in \mathcal{N}} I(v). \qquad \square$$

REMARK 3.9. Let u_λ be a minimizer of c^+. Then from the proof of Lemma 3.4 or (3.2), we can conclude $c^+ \to 0$ and $u_\lambda \to 0$ strongly in H as $\lambda \to 0^+$.

COROLLARY 3.10. *We have* $c^+ < c^-$ *if* $\lambda \in (0, \lambda^*)$.

PROOF. Note we have proved $c^+ \leq c^-$ in Lemma 3.6. Arguing by contradiction, we assume $c^+ = c^-$. By Lemma 3.7 we can obtain a sequence $\{u_n\} \subset \mathcal{N}^-$ such that $I(u_n) = c^- + o(1)$ and $I'(u_n) = o(1)$ as $n \to +\infty$. Repeating the argument of Lemma 3.8 we see that there exists $u \in H$ such that $I'(u) = 0$. Moreover, we have

$$\|v_n\| = o(1) \quad \text{or} \quad \|v_n\| \geq S^{3/4} + o(1).$$

We can rule out the latter case by (3.7) and $c^+ = c^-$. In fact, by Lemma 3.6 we see that for all $u \in H$ such that $I'(u) = 0$ we have $I(u) \geq c^+$. Then, from assumption $c^+ = c^-$, we see $c^- - I(u) \leq 0$, a contradiction with (3.7). Therefore, we conclude $u_n \to u$ strongly in H as $n \to +\infty$. Since $\lambda \in (0, \lambda^*)$ by Lemma 3.3

we have $u \in \mathcal{N}^-$. At last, by Lemma 3.2 we see that if $\int Q(x)|u|^q > 0$ then there exists $t^+ > 0$ such that $t^+u \in \mathcal{N}^+$ and

$$c^+ \le I(t^+u) < I(u) = c^- = c^+.$$

If $\int Q(x)|u|^q \le 0$ we have proved $c^- \ge 0 > c^+$ in Lemma 3.6. In both cases we derive a contradiction. □

4. Proof of Theorem 1.2

In order to find another solution of $(\mathcal{P})$, it is a natural way to prove c^- is attained. We have made great efforts without success because we can't obtain an effective estimate of c^-. Here we adopt the strategy in [1] and [9]. We find solution with the form $u_\lambda + v$, where u_λ is a positive ground state solution of $(\mathcal{P})$ which we have obtained in Theorem 1.1.

Above all, we need some regularity results about u_λ. Since u_λ satisfies the equation

$$-\Delta u = a(x)u + f(x),$$

where $a(x) = \phi_{u_\lambda} u_\lambda^3(x) \in L^{3/2}(\mathbb{R}^3)$ and $f(x) = Q(x)u_\lambda^{q-1}(x) \in L^{6/(q-1)}_{\mathrm{loc}}(\mathbb{R}^3)$, then by the Brezis–Kato type estimates (see [13, Lemma 5.5]) we can obtain $u_\lambda \in L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for all $s \in [1, +\infty)$. According to standard argument (see [13, Theorem 5.3] or [14]) we obtain $u_\lambda \in \mathcal{C}^\alpha_{\mathrm{loc}}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$. Then, by (Q_3) and strong maximal principle ([14]), we see $u_\lambda(x) > 0$ for all $x \in \mathbb{R}^3$.

By (Q_1)–(Q_2) and continuity of u_λ there exists $\rho > 0$ such that

$$0 < \frac{1}{2}\, Q(x_0) \le Q(x) \le \frac{3}{2}\, Q(x_0) \quad \text{and} \quad 0 < \frac{1}{2}\, u_\lambda(x_0) \le u_\lambda(x) \le \frac{3}{2}\, u_\lambda(x_0)$$

for all $x \in B_{2\rho}(x_0)$. Let η be a standard cut-off function supported in the closure of the ball $B_{2\rho}(x_0)$. Then we consider the function

$$v_\varepsilon(\,\cdot\,) = \eta(\,\cdot\,)\varepsilon^{-1/2}\, U\biggl(\frac{\cdot - x_0}{\varepsilon}\biggr) = \eta(\,\cdot\,)\, \frac{3^{1/4}\varepsilon^{1/2}}{(\varepsilon^2 + |\,\cdot - x_0|^2)^{1/2}}. \tag{4.1}$$

It is easy to see that $\int Q(x)|v_\varepsilon|^q > 0$. Thanks to the classical estimates due to Brezis and Nirenberg [9], we have

$$\|v_\varepsilon\|^2 = S^{3/2} + O(\varepsilon), \qquad |v_\varepsilon|_6^6 = S^{3/2} + O(\varepsilon^3). \tag{4.2}$$

Moreover, we also have

$$\int Q(x)|v_\varepsilon|^s = \begin{cases} O(\varepsilon^{s/2}) & \text{if } s \in [1,3), \\ O(\varepsilon^{3/2}|\ln \varepsilon|) + O(\varepsilon^{3/2}) & \text{if } s = 3, \\ O(\varepsilon^{3-s/2}) + O(\varepsilon^{s/2}) & \text{if } s = 4, 5. \end{cases} \tag{4.3}$$

Now we show

$$\int \phi_{v_\varepsilon} |v_\varepsilon|^5 = S^{3/2} + O(\varepsilon). \tag{4.4}$$

On one hand, by (2.1), (4.1) and Theorem 2.4, it is easy to see

$$\int \phi_{v_\varepsilon}|v_\varepsilon|^5 \le \int \phi_U U^5 = S^{3/2} \le S^{3/2} + O(\varepsilon).$$

On the other hand, from the Poisson equation $-\Delta\phi_{v_\varepsilon} = |v_\varepsilon|^5$ and Cauchy's inequality we have

$$\int |v_\varepsilon|^6 = \int \nabla\phi_{v_\varepsilon} \cdot \nabla v_\varepsilon \le \frac{1}{2}\|v_\varepsilon\|^2 + \frac{1}{2}\|\phi_{v_\varepsilon}\|^2 = \frac{1}{2}\|v_\varepsilon\|^2 + \frac{1}{2}\int \phi_{v_\varepsilon}|v_\varepsilon|^5.$$

Hence, we get

$$\int \phi_{v_\varepsilon}|v_\varepsilon|^5 \ge 2\int |v_\varepsilon|^6 - \|v_\varepsilon\|^2.$$

By estimates (4.2) we derive

$$\int \phi_{v_\varepsilon}|v_\varepsilon|^5 \ge S^{3/2} - O(\varepsilon).$$

We also need to estimate $\int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4$.

LEMMA 4.1. *For small* $\varepsilon > 0$ *we have*

$$C_1\varepsilon^{1/2} \le \int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4 \le C_2\varepsilon^{1/2}. \tag{4.5}$$

PROOF. On one hand, by (2.1) and (4.1) we can calculate

$$\begin{aligned}
\int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4 &= \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u_\lambda(x)}{|x-y|}\bigg(\frac{3^{1/4}\varepsilon^{1/2}\eta(x)}{(\varepsilon^2+|x-x_0|^2)^{1/2}}\bigg)^4\bigg(\frac{3^{1/4}\varepsilon^{1/2}\eta(y)}{(\varepsilon^2+|y-x_0|^2)^{1/2}}\bigg)^5 dx\,dy \\
&\ge C\varepsilon^{1/2}\int_{B_{\rho/\varepsilon}\times B_{\rho/\varepsilon}} \frac{1}{|x'-y'|(1+|x'|^2)^2(1+|y'|^2)^{5/2}}\,dx'\,dy' \\
&\ge C\varepsilon^{1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{|x'-y'|(1+|x'|^2)^{5/2}(1+|y'|^2)^{5/2}}\,dx'\,dy' \\
&= C\varepsilon^{1/2}\int \phi_U U^5 = C\varepsilon^{1/2}|U|_6^6 = C\varepsilon^{1/2}.
\end{aligned}$$

On the other hand, making similar argument, we have

$$\begin{aligned}
\int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4 &\le C\varepsilon^{1/2}\int_{B_{2\rho/\varepsilon}\times B_{2\rho/\varepsilon}} \frac{1}{|x'-y'|(1+|x'|^2)^2(1+|y'|^2)^{/2}}\,dx'\,dy' \\
&\le C\varepsilon^{1/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{U^4(x')U^4(y')}{|x'-y'|}\,dx'\,dy' \\
&\le C\varepsilon^{1/2}|U^4|_{6/5}^2 \quad \text{(by (a) of Lemma 2.2)} \\
&= C\varepsilon^{1/2}.
\end{aligned}$$

□

The following estimate is significantly important to obtain the compactness of a $(\mathrm{PS})_c$ sequence of I for $c < I(u_\lambda) + (2/5)S^{3/2}$.

LEMMA 4.2. *There exists* $\varepsilon_0 > 0$ *such that*

$$\sup_{t\ge 0} I(u_\lambda + tv_\varepsilon) < I(u_\lambda) + \frac{2}{5}S^{3/2} \quad \textit{for all } \varepsilon \in (0,\varepsilon_0).$$

PROOF. By (2.1) and direct computation we see that for all $t > 0$

$$\begin{aligned}\int \phi_{u_\lambda+tv_\varepsilon}(u_\lambda+tv_\varepsilon)^5 &= \frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(u_\lambda+tv_\varepsilon)^5(x)(u_\lambda+tv_\varepsilon)^5(y)}{|x-y|}\,dx\,dy\\ &= \int \phi_{u_\lambda}u_\lambda^5 + 10t\int \phi_{u_\lambda}u_\lambda^4 v_\varepsilon + \ldots + 10t^9\int \phi_{v_\varepsilon}u_\lambda v_\varepsilon^4 + t^{10}\int \phi_{v_\varepsilon}v_\varepsilon^5\\ &\geq \int \phi_{u_\lambda}u_\lambda^5 + 10t\int \phi_{u_\lambda}u_\lambda^4 v_\varepsilon + 10t^9\int \phi_{v_\varepsilon}u_\lambda v_\varepsilon^4 + t^{10}\int \phi_{v_\varepsilon}v_\varepsilon^5.\end{aligned}$$

Then, by the equation of u_λ and convexity of the function x^q, we can calculate

$$\begin{aligned}I(u_\lambda+tv_\varepsilon) &= \frac{1}{2}\|u_\lambda+tv_\varepsilon\|^2 - \frac{1}{10}\int \phi_{u_\lambda+tv_\varepsilon}(u_\lambda+tv_\varepsilon)^5 - \frac{\lambda}{q}\int Q(x)|u_\lambda+tv_\varepsilon|^q\\ &\leq I(u_\lambda) + \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{10}}{10}\int \phi_{v_\varepsilon}v_\varepsilon^5 - t^9\int \phi_{v_\varepsilon}u_\lambda v_\varepsilon^4\\ &\quad - \frac{\lambda}{q}\int Q(x)[(u_\lambda+tv_\varepsilon)^q - u_\lambda^q - tqu_\lambda^{q-1}v_\varepsilon]\\ &\leq I(u_\lambda) + \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{10}}{10}\int \phi_{v_\varepsilon}v_\varepsilon^5 - t^9\int \phi_{v_\varepsilon}u_\lambda v_\varepsilon^4.\end{aligned}$$

For $t \in [0, +\infty)$ define

$$h(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{10}}{10}\int \phi_{v_\varepsilon}v_\varepsilon^5 - t^9\int \phi_{v_\varepsilon}u_\lambda v_\varepsilon^4.$$

It is easy to see $h(0) = 0$, $h(t) > 0$ for small t and $\lim_{t\to+\infty} h(t) = -\infty$. Hence, there exists $t_\varepsilon > 0$ such that $h(t_\varepsilon) = \max_{t\geq 0} h(t)$. Moreover, by direct calculation, we see that h' only has one positive zero t_ε for ε small enough.

Now we give a fine estimate of t_ε. Define a function on $\mathbb{R}^4$ as

$$F(t,x,y,z) = t^2x - t^{10}y - t^9z.$$

By a direct calculation we have

$$F(1, S^{3/2}, S^{3/2}, 0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}\Big|_{(1,S^{3/2},S^{3/2},0)} = -8S^{3/2} < 0.$$

Thus, by the implicit function theorem, we see that there exists a continuous differentiable function $t = t(x,y,z)$ in a neighbourhood of $(S^{3/2}, S^{3/2}, 0)$ such that

$$F(t(x,y,z),x,y,z) = 0, \quad t(S^{3/2}, S^{3/2}, 0) = 1$$

and

$$\nabla t|_{(S^{3/2},S^{3/2},0)} = \frac{-1}{\dfrac{\partial F}{\partial t}}\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)\Big|_{(S^{3/2},S^{3/2},0)} = \frac{1}{8S^{3/2}}(1,-1,-1).$$

Therefore, by Taylor's expansion in a neighbourhood of $(S^{3/2}, S^{3/2}, 0)$, we have

$$t(x,y,z) = 1 + \frac{1}{8S^{3/2}}(1,-1,-1)\cdot(x - S^{3/2}, y - S^{3/2}, z - 0) \\ + o(1)\sqrt{(x - S^{3/2})^2 + (y - S^{3/2})^2 + z^2}.$$

As a consequence, if we set

$$x = \|v_\varepsilon\|^2, \qquad y = \int \phi_{v_\varepsilon|}v_\varepsilon|^5, \qquad z = 9\int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4,$$

then, by the estimates (4.2) and (4.5), we have

$$t_\varepsilon = 1 - O(\varepsilon^{1/2}) + o(\varepsilon^{1/2}) = 1 + O(\varepsilon^{1/2}). \tag{4.6}$$

On the other hand, by estimates (4.2) and (4.4) we can show that

$$\sup_{t\geq 0} \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{10}}{10}\int \phi_{v_\varepsilon}|v_\varepsilon|^5 = \frac{2}{5}\,(\|v_\varepsilon\|^2)^{5/4}\bigg(\int \phi_{v_\varepsilon}|v_\varepsilon|^5\bigg)^{-1/4} = \frac{2}{5}\,S^{3/2} + O(\varepsilon).$$

Furthermore, by (4.5) and (4.6) we can calculate

$$\sup_{t\geq 0} h(t) = h(t_\varepsilon) \leq \sup_{t\geq 0} \frac{t^2}{2}\,\|v_\varepsilon\|^2 - \frac{t^{10}}{10}\int \phi_{v_\varepsilon}|v_\varepsilon|^5 - C\int \phi_{v_\varepsilon} u_\lambda v_\varepsilon^4 \\ \leq \frac{2}{5}\,S^{3/2} + C_3\varepsilon - C_4\varepsilon^{1/2}.$$

Thus we conclude, for small ε,

$$\sup_{t\geq 0} I(u_\lambda + tv_\varepsilon) < I(u_\lambda) + \frac{2}{5}\,S^{3/2}. \qquad \square$$

Lemma 4.3. *I possesses the geometry structure of mountain pass as long as $\lambda \in (0, q(12-q)\lambda^*/20)$.*

Proof. Let δ be the constant in Lemma 3.5 and note that if $\lambda = 0$ then we have

$$I\big|_{\|u\|=\delta} \geq \frac{2(12-q)}{5(10-q)}\,\delta^2.$$

Hence, for all $\lambda \in (0, q(12-q)\lambda^*/20)$, we have

$$I\big|_{\|u\|=\delta} \geq \frac{\delta^q}{qS^{q/2}|Q^+|_{6/(6-q)}}\bigg[\frac{q(12-q)}{20}\lambda^* - \lambda\bigg] := b_\lambda > 0.$$

On one hand, since $\lambda \in (0, \lambda^*)$ by Lemma 3.4 we see $\|u_\lambda\| < \delta$. Noting u_λ is a minimizer of c^+ and $c^+ < 0$ we have $I(u_\lambda) < 0$.

On the other hand, we see $\lim_{t\to+\infty} I(u_\lambda + tv_\varepsilon) = -\infty$. Hence, there exists an element $e = u_\lambda + t_0 v_\varepsilon \in H$ for some $t_0 > 0$ such that $I(e) < 0$ and $\|e\| > \delta$. $\square$

Now we can define mountain pass level

$$c_{mp} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1]; H) : \gamma$ is a path connected u_λ and e in $H\}$.

PROOF OF THEOREM 1.2. Applying the deformation lemma without $(PS)_c$ condition([2], [31]) we see that there exists a $(\mathrm{PS})_{c_{mp}}$ sequence $\{u_n\}$ of I. By standard argument we see that $\{u_n\}$ is bounded in H. Then repeating the proof of Lemma 3.8 we see that there exists $u \in H$ satisfying formula (3.6) and $I'(u) = 0$. Moreover, if $u_n \nrightarrow u$ strongly in H then the inequality (3.7) holds, i.e.

$$c_{mp} - I(u) \geq \frac{2}{5} S^{3/2}.$$

Since u_λ is a ground state solution and $I(u_\lambda) = c^+ < 0$ we obtain

$$c_{mp} - I(u_\lambda) \geq c_{mp} - I(u) \geq \frac{2}{5} S^{3/2}.$$

However, making use of Lemma 4.2 we have

$$c_{mp} < I(u_\lambda) + \frac{2}{5} S^{3/2}.$$

Hence, we obtain a contradiction. Therefore, we conclude $u_n \to u$ strongly in H. Furthermore, by Lemma 4.3, we see that $c_{mp} \geq b_\lambda > 0$. As a consequence, we have $u \neq 0$ and $u \neq u_\lambda$. Hence, $(\mathcal{P})$ has at least two solutions: one is a ground state solution u_λ, the other is mountain pass solution u. To obtain a positive mountain pass solution, we only need to consider the modified functional

$$I^+(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{10}\int \phi_{u^+}|u^+|^5 - \frac{\lambda}{q}\int Q(x)|u^+|^q, \quad u \in H. \qquad \square$$

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