

**STRONG SOLUTIONS IN L^2 FRAMEWORK
FOR FLUID-RIGID BODY INTERACTION PROBLEM.
MIXED CASE**

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Dedicated to the memory of Professor Marek Burnat

ABSTRACT. The paper deals with the problem describing the motion of a rigid body inside a viscous incompressible fluid when the mixed boundary conditions are considered. At the fluid–rigid body interface the slip Navier boundary condition is prescribed, having the continuity of velocity just in the normal component, and the Dirichlet condition is given on the boundary of the fluid domain. The existence and uniqueness of the local strong solution is proven by the local transformation and the fixed point argument.

1. Introduction

In this paper we investigate the motion of a rigid body inside a viscous incompressible fluid when mixed boundary conditions are considered. The fluid and the body occupy a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ ($d = 2$ or 3).

In order to describe our approach, let us denote by $\mathcal{B}(t) \subset \mathcal{O}$ a bounded domain occupied by the rigid body and a domain filled by the fluid by $\mathcal{F}(t) = \mathcal{O} \setminus \overline{\mathcal{B}(t)}$ at a time moment $t \in \mathbb{R}^+$. Assuming that the initial position $\mathcal{B}(0)$ of

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the rigid body is prescribed, for simplicity of notation we denote $\mathcal{B}_0 = \mathcal{B}(0)$ and $\mathcal{F}_0 = \mathcal{F}(0)$. The interface between the body and the fluid is denoted by $\partial\mathcal{B}(t)$, the normal vector to the boundary is denoted by $\mathbf{n}(t)$ and it is pointing outside \mathcal{O} and inside $\mathcal{B}(t)$. We write

$$\begin{aligned}\mathcal{Q}_{\mathcal{F}(t)} &= \{(t, \mathbf{x}) \in \mathbb{R}^{1+d} : t \in \mathbb{R}^+, \mathbf{x} \in \mathcal{F}(t)\}, \\ \mathcal{Q}_{\partial\mathcal{B}(t)} &= \{(t, \mathbf{x}) \in \mathbb{R}^{1+d} : t \in \mathbb{R}^+, \mathbf{x} \in \partial\mathcal{B}(t)\}.\end{aligned}$$

The fluid motion is governed by the equations

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u}_{\mathcal{F}} + \operatorname{div} \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}}) + (\mathbf{u}_{\mathcal{F}} \cdot \nabla) \mathbf{u}_{\mathcal{F}} = \mathbf{f}_0, \\ \operatorname{div} \mathbf{u}_{\mathcal{F}} = 0 & \text{in } \mathcal{Q}_{\mathcal{F}(t)}, \\ \mathbf{u}_{\mathcal{F}} = \mathbf{0} & \text{on } \partial\mathcal{O} \times \mathbb{R}^+, \\ (\mathbf{u}_{\mathcal{F}} - \mathbf{u}_{\mathcal{B}}) \cdot \mathbf{n} = 0, \\ 2\mu \mathbb{D}(\mathbf{u}_{\mathcal{F}}) \times \mathbf{n} = -\beta(\mathbf{u}_{\mathcal{F}} - \mathbf{u}_{\mathcal{B}}) \times \mathbf{n} & \text{on } \mathcal{Q}_{\partial\mathcal{B}(t)}, \\ \mathbf{u}_{\mathcal{F}}(0) = \mathbf{u}_0 & \text{in } \mathcal{F}_0, \end{cases}$$

where $\mathbf{u}_{\mathcal{F}}$ and $p_{\mathcal{F}}$ denote the velocity and the pressure of the fluid and $\mathbf{u}_{\mathcal{B}}$ is the full velocity of the rigid body. We recall that the rate of the strain tensor of the fluid and its stress tensor are defined by

$$\mathbb{D}(\mathbf{u}_{\mathcal{F}}) = \frac{1}{2}(\nabla \mathbf{u}_{\mathcal{F}} + (\nabla \mathbf{u}_{\mathcal{F}})^T) \quad \text{and} \quad \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}}) = 2\mu \mathbb{D}(\mathbf{u}_{\mathcal{F}}) - p_{\mathcal{F}} \mathbb{I},$$

with $\mu > 0$ being the viscosity of the fluid, and $\beta > 0$ is the slip length.

The fluid equations are coupled to the following balance equations for the translation velocity $\boldsymbol{\eta}$ and the angular velocity $\boldsymbol{\omega}$ of the body,

$$(1.2) \quad \begin{cases} m \boldsymbol{\eta}'(t) + \int_{\partial\mathcal{B}(t)} \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}})(t, \mathbf{x}) \mathbf{n}(t, \mathbf{x}) \, d\boldsymbol{\sigma} = \mathbf{f}_1(t), \\ (\mathbf{J} \boldsymbol{\omega})'(t) + \int_{\partial\mathcal{B}(t)} (\mathbf{x} - \mathbf{x}_c(t)) \times \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}})(t, \mathbf{x}) \mathbf{n}(t, \mathbf{x}) \, d\boldsymbol{\sigma} = \mathbf{f}_2(t), \\ \boldsymbol{\eta}(0) = \boldsymbol{\eta}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \end{cases} \quad \text{for } t \in \mathbb{R}^+,$$

where $m = \rho_{\mathcal{B}} |\mathcal{B}_0|$ and $\rho_{\mathcal{B}}$ are the mass and the constant density of the body, \mathbf{x}_c is the position of its center of gravity,

$$\mathbf{J} = \rho_{\mathcal{B}} \int_{\mathcal{B}(t)} (|\mathbf{x} - \mathbf{x}_c(t)|^2 \mathbb{I} - (\mathbf{x} - \mathbf{x}_c(t)) \otimes (\mathbf{x} - \mathbf{x}_c(t))) \, d\mathbf{x}$$

is the matrix of the inertia moments of the body $\mathcal{B}(t)$. The full velocity of the rigid body is given by

$$\mathbf{u}_{\mathcal{B}}(t, \mathbf{x}) = \boldsymbol{\eta}(t, \mathbf{x}) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_c(t)).$$

The functions \mathbf{f}_0 and \mathbf{f}_1 , \mathbf{f}_2 denote the external force and the torques, respectively.

Let us mention that the problem of the motion of one or several rigid bodies in a viscous fluid filling a bounded domain was investigated by several authors [2]–[4], [11]. In all articles mentioned a non-slip boundary condition has been considered on the boundaries of the bodies and of the domain. Hesla [9] and Hillairet [10] have shown that this condition gives a very paradoxical result of no collisions between the bodies and the boundary of the domain.

Our article is devoted to the problem of the motion of the rigid body in the viscous fluid when a slippage is allowed at the fluid-body interface $\partial\mathcal{B}(t)$ and a Dirichlet boundary condition on $\partial\mathcal{O}$. The slippage is prescribed by the Navier boundary condition, having only the continuity of velocity just in the normal component. We stress that taking into account slip boundary condition at the interface is very natural within this model, since the classical Dirichlet boundary condition leads to unrealistic collision behaviour between the solid and the domain boundary. Nevertheless, due to the slip condition, the velocity field is discontinuous across the fluid-solid interface. This makes many aspects of the theory of weak solutions for Dirichlet conditions inappropriate. It is worth noting that the case of bounded fluid domain \mathcal{O} furnishes additional difficulty of possible contacts of body and wall. For this reason, the body needs to start at some distance from the boundary. Furthermore the lifespan of the solution has to be restricted to a time interval in which no contacts occur.

To our knowledge the first solvability result was obtained by Neustupa and Penel [15], [16] in a particular situation, where they considered a prescribed collision of a ball with a wall, when the slippage was allowed on both boundaries. Their pioneer result shows that the slip boundary condition cleans the no-collision paradox. Recently Gérard–Varet, Hillairet [6] have proved a local-in-time existence result (up to collisions). The authors of [7] have investigated the free fall of a sphere above a wall, that is when the boundaries are C^∞ -smooth, in a viscous incompressible fluid in two different situations: *mixed case*: the Navier boundary condition is prescribed on the boundary of the body and the non-slip boundary condition on the boundary of the domain; *slip case*: the Navier boundary conditions are prescribed on both boundaries, i.e. of the body and of the domain. The result of them is interesting, saying that in the *mixed case* the sphere never touches the wall and in the *slip case* the sphere reaches the wall during a finite time period.

Recently, the global existence result for a weak solution was proven in the mixed case, see [1], even if the collisions of the body with the boundary of domain occur in a finite time under a lower regularity of the body and domain than [7]. Our article deals with the strong solution of the mixed case. The existence of strong solution was studied by Takahashi, and Tucsnak [18], [19] in the no-slip boundary conditions and in the slip case by Wang [20] in the 2D case.

The plan of the paper is as follows. In Section 2 we introduce the local transformation as in Inoue and Wakimoto [12], we define the functional framework at the basis of our work, we recall also the main result of this work. Next in Section 3 we prove the existence of solution to the linearised problem, we consider the non linear problem and we prove the existence of solution using a fixed point argument.

2. Preliminaries

2.1. Local transformation. Since the domain depends on the motion of the rigid body, we transform the problem to a fixed domain. There are at least two possibilities for this transform: the global transformation (cf. [5], [8]) is linear, meaning that the whole space is rigidly rotated and shifted back to its original position at every time $t > 0$. A fundamental difficulty of this approach is that the transformed problem in case of the exterior domain brings additional terms which are not local perturbation to parabolic equations and completely change the character of equations. The second one (cf. [12]) is characterized by a non-linear local change of coordinates which only acts in a suitable bounded neighbourhood of the obstacle. The advantage of the later transform is that it preserves the solenoidal condition on the fluid velocity, doesn't change the regularity of the solutions. However the rigid body equations change to become non-linear. Our analysis is based on the second approach. We define the local transformation introduced by Inoue and Wakimoto [12].

Let $\delta(t) = \text{dist}(\mathcal{B}(t), \partial\mathcal{O})$. We fix δ_0 such that $\delta(t) > \delta_0$ and define a C^∞ -smooth solenoidal velocity field $\mathbf{\Lambda} = \mathbf{\Lambda}(t, \mathbf{x})$, defined for $t \in \mathbb{R}^+$, $\mathbf{x} \in \mathcal{O}$, satisfying

$$\mathbf{\Lambda}(t, \mathbf{x}) = \begin{cases} 0 & \text{in the } \delta_0/4 \text{ neighbourhood of } \partial\mathcal{O}, \\ \boldsymbol{\eta}(t) + \boldsymbol{\omega}(t) \times (\mathbf{x} - \mathbf{x}_c(t)) & \text{in the } \delta_0/4 \text{ neighbourhood of } \mathcal{B}(t). \end{cases}$$

Then the flow $\mathbf{X}(t): \mathcal{O} \rightarrow \mathcal{O}$ is defined as the solution of the system

$$(2.1) \quad \frac{d}{dt} \mathbf{X}(t, \mathbf{y}) = \mathbf{\Lambda}(t, \mathbf{X}(t, \mathbf{y})), \quad \mathbf{X}(0, \mathbf{y}) = \mathbf{y}, \quad \text{for all } \mathbf{y} \in \mathcal{O}.$$

From the results of Takahashi [18, Lemma 4.2] it follows that (2.1) has a unique solution. Moreover, the mapping \mathbf{X} is a C^∞ diffeomorphism for \mathcal{O} and itself and a diffeomorphism from \mathcal{F}_0 onto $\mathcal{F}(t)$ such that the derivatives

$$\frac{\partial^{i+\alpha_j} \mathbf{X}(t, \mathbf{y})}{\partial t^i \partial y_j^{\alpha_j}}, \quad i \leq 1, \text{ for all } \alpha_j \geq 0, j = 1, \dots, d,$$

exist and are continuous.

Further, denoting \mathbf{Y} as the inverse of \mathbf{X} from [18, Lemma 4.2] it follows that \mathbf{Y} has also all continuous derivatives

$$\frac{\partial^{i+\alpha_j} \mathbf{Y}(t, \mathbf{x})}{\partial t^i \partial x_j^{\alpha_j}}, \quad i \leq 1, \text{ for all } \alpha_j \geq 0, j = 1, \dots, d.$$

Now we introduce the new unknown functions, defined for $t \in \mathbb{R}^+$ and $\mathbf{y} \in \mathcal{O}$,

$$\begin{aligned} \tilde{\mathbf{u}}_{\mathcal{F}}(t, \mathbf{y}) &= \mathcal{J}_{\mathbf{Y}}(t, \mathbf{X}(t, \mathbf{y})) \mathbf{u}_{\mathcal{F}}(t, \mathbf{X}(t, \mathbf{y})), \\ \tilde{p}_{\mathcal{F}}(t, \mathbf{y}) &= p_{\mathcal{F}}(t, \mathbf{X}(t, \mathbf{y})), \\ \mathcal{T}(\tilde{\mathbf{u}}_{\mathcal{F}}(t, \mathbf{y}), \tilde{p}_{\mathcal{F}}(t, \mathbf{y})) &= \mathbf{Q}^T(t) \mathbb{T}(\mathbf{Q}(t) \tilde{\mathbf{u}}_{\mathcal{F}}(t, \mathbf{y}), \tilde{p}_{\mathcal{F}}(t, \mathbf{y})) \mathbf{Q}(t), \\ \tilde{\mathbf{f}}_0(t, \mathbf{y}) &= \mathcal{J}_{\mathbf{Y}}(t, \mathbf{X}(t, \mathbf{y})) \mathbf{f}_0(t, \mathbf{X}(t, \mathbf{y})), \\ \tilde{\boldsymbol{\omega}}(t) &= \mathbf{Q}^T(t) \boldsymbol{\omega}(t), \quad \tilde{\boldsymbol{\eta}}(t) = \mathbf{Q}^T(t) \boldsymbol{\eta}(t), \\ \tilde{\mathbf{f}}_1(t) &= \mathbf{Q}^T(t) \mathbf{f}_1(t), \quad \tilde{\mathbf{f}}_2(t) = \mathbf{Q}^T(t) \mathbf{f}_2(t), \end{aligned}$$

where $\mathcal{J}_{\mathbf{Y}}(t, \mathbf{x}) = (\partial Y_i(t, \mathbf{x}) / \partial x_j)$ and $\mathbf{Q}(t) \in \text{SO}(3)$ is a rotation matrix associated with the rigid body angular velocity $\boldsymbol{\omega}$. The transformed normal $\tilde{\mathbf{n}}$ on $\partial \mathcal{B}_0$ satisfies $\tilde{\mathbf{n}} = \mathbf{Q}^T(t) \mathbf{n}(t)$. The transformed inertia tensor $\mathbf{I} = \mathbf{Q}^T(t) \mathbf{J}(t) \mathbf{Q}(t)$ no longer depend on time. Furthermore the transformed total force and torque on the rigid body are given by

$$\begin{aligned} \int_{\partial \mathcal{B}(t)} \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}}) \mathbf{n}(t) d\boldsymbol{\sigma} &= \mathbf{Q} \int_{\partial \mathcal{B}_0} \mathcal{T}(\tilde{\mathbf{u}}_{\mathcal{F}}, \tilde{p}_{\mathcal{F}}) \tilde{\mathbf{n}} d\boldsymbol{\sigma}(\mathbf{y}), \\ \int_{\partial \mathcal{B}(t)} (\mathbf{x} - \mathbf{x}_c(t)) \times \mathbb{T}(\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}}) \mathbf{n}(t) d\boldsymbol{\sigma} &= \mathbf{Q} \int_{\partial \mathcal{B}_0} \mathbf{y} \times \mathcal{T}(\tilde{\mathbf{u}}_{\mathcal{F}}, \tilde{p}_{\mathcal{F}}) \tilde{\mathbf{n}} d\boldsymbol{\sigma}(\mathbf{y}). \end{aligned}$$

Thus for some $T > 0$, that will be founded later on, the new unknowns $\tilde{\mathbf{u}}_{\mathcal{F}}$, $\tilde{p}_{\mathcal{F}}$ and $\tilde{\boldsymbol{\eta}}$, $\tilde{\boldsymbol{\omega}}$, defined on the cylindrical domains $(0, T) \times \mathcal{F}_0$ and $(0, T) \times \mathcal{B}_0$, satisfy the following system of equations

$$(2.2) \quad \left\{ \begin{array}{ll} \partial_t \tilde{\mathbf{u}}_{\mathcal{F}} + (\mathcal{M} - \mu \mathcal{L}) \tilde{\mathbf{u}}_{\mathcal{F}} + \mathcal{N} \tilde{\mathbf{u}}_{\mathcal{F}} + \mathcal{G} \tilde{p}_{\mathcal{F}} = \tilde{\mathbf{f}}_0, & \text{in } (0, T) \times \mathcal{F}_0, \\ \operatorname{div} \tilde{\mathbf{u}}_{\mathcal{F}} = 0 & \text{in } (0, T) \times \mathcal{F}_0, \\ \tilde{\mathbf{u}}_{\mathcal{F}} = \mathbf{0} & \text{on } (0, T) \times \partial \mathcal{O}, \\ \tilde{\mathbf{u}}_{\mathcal{F}}(0) = \mathbf{u}_0 & \text{in } \mathcal{F}_0, \\ (\tilde{\mathbf{u}}_{\mathcal{F}} - \tilde{\mathbf{u}}_{\mathcal{B}}) \cdot \tilde{\mathbf{n}} = \mathbf{0}, & \\ 2\mu [\mathbb{D}(\tilde{\mathbf{u}}_{\mathcal{F}}) \tilde{\mathbf{n}}] \times \tilde{\mathbf{n}} = -\beta (\tilde{\mathbf{u}}_{\mathcal{F}} - \tilde{\mathbf{u}}_{\mathcal{B}}) \times \tilde{\mathbf{n}} & \text{on } (0, T) \times \partial \mathcal{B}_0, \\ m \tilde{\boldsymbol{\eta}}' - m (\tilde{\boldsymbol{\omega}} \times \tilde{\boldsymbol{\eta}}) + \int_{\partial \mathcal{B}_0} \mathcal{T}(\tilde{\mathbf{u}}_{\mathcal{F}}, \tilde{p}_{\mathcal{F}}) \tilde{\mathbf{n}} d\boldsymbol{\sigma} = \tilde{\mathbf{f}}_1(t), & \\ \mathbf{I} \tilde{\boldsymbol{\omega}}' - \tilde{\boldsymbol{\omega}} \times (\mathbf{I} \tilde{\boldsymbol{\omega}}) + \int_{\partial \mathcal{B}_0} \mathbf{y} \times \mathcal{T}(\tilde{\mathbf{u}}_{\mathcal{F}}, \tilde{p}_{\mathcal{F}}) \tilde{\mathbf{n}} d\boldsymbol{\sigma} = \tilde{\mathbf{f}}_2(t), & \text{for } t \in (0, T), \\ \tilde{\boldsymbol{\eta}}(0) = \boldsymbol{\eta}_0, \quad \tilde{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0, & \end{array} \right.$$

with $\tilde{\mathbf{u}}_{\mathcal{B}} = \tilde{\boldsymbol{\eta}} + \tilde{\boldsymbol{\omega}} \times \mathbf{y}$ and the convection term is transformed into

$$(\mathcal{N}\mathbf{u})_i = \sum_{j=1}^d u_j \partial_j u_i + \sum_{j,k=1}^d \Gamma_{jk}^i u_j u_k, \quad i = 1, \dots, d.$$

The transformed time derivative $\mathcal{M}\mathbf{u}$ and the gradient $\mathcal{G}p$ are calculated by

$$(\mathcal{M}\mathbf{u})_i = \sum_{j=1}^d \dot{Y}_j \partial_j u_i + \sum_{j,k=1}^d (\Gamma_{jk}^i \dot{Y}_k + (\partial_k Y_i)(\partial_j \dot{X}_k)) u_j, \quad (\mathcal{G}p)_i = \sum_{j=1}^d g^{ij} \partial_j p.$$

Moreover, the operator \mathcal{L} denotes the transformed Laplace operator, having the components

$$\begin{aligned} (\mathcal{L}\mathbf{u})_i = & \sum_{j,k=1}^d \partial_j (g^{jk} \partial u_i) + 2 \sum_{j,k,l=1}^d g^{kl} \Gamma_{jk}^i \partial_l u_j \\ & + \sum_{j,k,l=1}^d \left(\partial_k (g^{kl} \Gamma_{kl}^i) + \sum_{m=1}^n g^{kl} \Gamma_{jl}^m \Gamma_{km}^i \right) u_j. \end{aligned}$$

The coefficients are given by the metric covariant tensor $g_{ij} = X_{k,i} X_{k,j}$, the metric contra-variant tensor $g^{ij} = Y_{i,k} Y_{j,k}$ and the Christoffel symbols

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l}).$$

It is easy to observe that in particular it holds $\Gamma_{ij}^k = Y_{k,l} X_{l,ij}$. As described in [12], problem (1.1)–(1.2) is equivalent to problem (2.2) and a solution to the transformed problem (2.2) yields a solution to the initial problem (1.1)–(1.2).

2.2. Function spaces and the main theorem. In the sequel we use the following function spaces, defined on the moving domain $(0, T) \times \mathcal{F}(t)$,

$$\begin{aligned} L^2(0, T; H^2(\mathcal{F}(t))), & \quad C([0, T]; H^1(\mathcal{F}(t))), \\ H^1(0, T; L^2(\mathcal{F}(t))), & \quad L^2(0, T; H^1(\mathcal{F}(t))). \end{aligned}$$

If we consider $\mathbf{U}_{\mathcal{F}}(t, \mathbf{y}): \mathcal{F}_0 \rightarrow \mathbb{R}^d$, which is calculated as

$$\mathbf{U}_{\mathcal{F}}(t, \mathbf{y}) = \mathbf{u}_{\mathcal{F}}(t, \mathbf{X}(t, \mathbf{y})) \quad \text{for any function } \mathbf{u}_{\mathcal{F}}(t, \cdot): \mathcal{F}(t) \rightarrow \mathbb{R}^d,$$

then above mentioned function spaces can be redefined in the fixed domain $(0, T) \times \mathcal{F}_0$. For instance

$$L^2(0, T; H^2(\mathcal{F}_0)) = \{\mathbf{U}_{\mathcal{F}} : \mathbf{u}_{\mathcal{F}} \in L^2(0, T; H^2(\mathcal{F}(t)))\}.$$

Now we can formulate the main result.

THEOREM 2.1. *Suppose that $\overline{\mathcal{B}_0} \subset \mathcal{O}$ and*

$$(2.3) \quad \begin{aligned} \mathbf{u}_0 \in H^1(\mathcal{F}_0), \quad \mathbf{u}_{\mathcal{B},0} = \boldsymbol{\eta}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_c(0)) \in H^1(\mathcal{B}_0), \\ \mathbf{f}_0 \in L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathcal{F}_0)), \quad \mathbf{f}_1, \mathbf{f}_2 \in L^2_{\text{loc}}(\mathbb{R}^+), \end{aligned}$$

that satisfy

$$(\mathbf{u}_0 - \mathbf{u}_{\mathcal{B},0}) \cdot \mathbf{n}|_{\partial\mathcal{B}_0} = 0, \quad \mathbf{u}_0|_{\partial\mathcal{O}} = \mathbf{0}, \quad \operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \mathcal{F}_0.$$

Then there exists $T_0 > 0$ such that (1.1)–(1.2) has a unique solution which satisfies, for all $T < T_0$,

$$\mathbf{u}_{\mathcal{F}}, p_{\mathcal{F}}, \boldsymbol{\eta}(t), \boldsymbol{\omega}(t) \in \mathcal{U}_T(\mathcal{F}(t)) \times L^2(0, T; H^1(\mathcal{F}(t))) \times H^1(0, T) \times H^1(0, T),$$

where $\mathcal{U}_T(\mathcal{F}(t)) = L^2(0, T; H^2(\mathcal{F}(t))) \cap C(0, T; H^1(\mathcal{F}(t))) \cap H^1(0, T; L^2(\mathcal{F}(t)))$.

3. Strong solution

3.1. Stokes problem. We will consider the following linearized system, which couples Stokes type equations and linear ordinary differential equations

$$(3.1) \quad \left\{ \begin{array}{ll} \partial_t \mathbf{z}_{\mathcal{F}} - \mu \Delta \mathbf{z}_{\mathcal{F}} + \nabla q_{\mathcal{F}} = \mathbf{F}_0, & \operatorname{div} \mathbf{z}_{\mathcal{F}} = 0 \quad \text{in } (0, T) \times \mathcal{F}_0, \\ \mathbf{z}_{\mathcal{F}} = \mathbf{0} & \text{on } (0, T) \times \partial\mathcal{O}, \\ \mathbf{z}_{\mathcal{F}}(0) = \mathbf{u}_0 & \text{in } \mathcal{F}_0, \\ (\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \cdot \tilde{\mathbf{n}} = 0, & \\ 2\mu[\mathbb{D}(\mathbf{z}_{\mathcal{F}})\tilde{\mathbf{n}}] \times \tilde{\mathbf{n}} = -\beta(\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \times \tilde{\mathbf{n}} & \text{on } (0, T) \times \partial\mathcal{B}_0, \\ m \boldsymbol{\xi}' + \int_{\partial\mathcal{B}_0} \mathbb{T}(\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}})\tilde{\mathbf{n}} \, d\boldsymbol{\sigma} = \mathbf{F}_1, & \\ \mathbf{I} \mathbf{w}' + \int_{\partial\mathcal{B}_0} \mathbf{y} \times \mathbb{T}(\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}})\tilde{\mathbf{n}} \, d\boldsymbol{\sigma} = \mathbf{F}_2 & \text{for } t \in (0, T), \\ \boldsymbol{\xi}(0) = \boldsymbol{\eta}_0, \quad \mathbf{w}(0) = \boldsymbol{\omega}_0, & \end{array} \right.$$

with $\mathbf{z}_{\mathcal{B}} = \boldsymbol{\xi} + \mathbf{w} \times \mathbf{y}$.

Let us recall a well-known result (see Kato [13], [14]).

PROPOSITION 3.1. *Let H be a Hilbert space. Let $\mathbb{A}: D(\mathbb{A}) \rightarrow H$ be a self adjoint and accretive operator. If $\mathbf{F} \in L^2(0, T; H)$, $\mathbf{u}_0 \in D(\mathbb{A}^{1/2})$, then the problem*

$$\mathbf{u}' + \mathbb{A}\mathbf{u} = \mathbf{F}, \quad \mathbf{u}(0) = \mathbf{u}_0,$$

has a unique solution $\mathbf{u} \in L^2(0, T; D(\mathbb{A})) \cap C([0, T]; D(\mathbb{A}^{1/2})) \cap H^1(0, T; H)$, which satisfies

$$\begin{aligned} \|\mathbf{u}\|_{L^2(0, T; D(\mathbb{A}))} + \|\mathbf{u}\|_{C([0, T]; D(\mathbb{A}^{1/2}))} + \|\mathbf{u}\|_{H^1(0, T; H)} \\ \leq C(\|\mathbf{u}_0\|_{D(\mathbb{A}^{1/2})} + \|\mathbf{F}\|_{L^2(0, T; H)}) \end{aligned}$$

with a constant C depending on the operator \mathbb{A} and the time T . Moreover, the constant C is a non decreasing function of T .

Let us define the functional spaces

$$\mathcal{H} = \{\phi \in L^2(\mathcal{O}) : \operatorname{div} \phi = 0 \text{ in } \mathcal{O}, \phi|_{\mathcal{F}_0} = \phi_{\mathcal{F}} \in \mathcal{D}'(\mathcal{F}_0), \phi|_{\mathcal{B}_0} = \phi_{\mathcal{B}} \in \mathcal{R}\},$$

$$\mathcal{V} = \{\phi \in \mathcal{H} : \phi_{\mathcal{F}} \in H^1(\mathcal{F}_0), \phi_{\mathcal{F}}|_{\partial\mathcal{O}} = 0, (\phi_{\mathcal{F}} - \phi_{\mathcal{B}}) \cdot \tilde{\mathbf{n}}|_{\partial\mathcal{B}_0} = 0\},$$

where $\mathcal{R} = \{\phi : \phi(\mathbf{y}) = \boldsymbol{\xi}_{\phi} + \mathbf{w}_{\phi} \times \mathbf{y} \text{ with } \boldsymbol{\xi}_{\phi}, \mathbf{w}_{\phi} \in \mathbb{R}^d\}$. For $\mathbf{u}, \mathbf{v} \in \mathcal{H}$ we define the inner product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{F}_0} \mathbf{u}_{\mathcal{F}} \cdot \mathbf{v}_{\mathcal{F}} \, d\mathbf{y} + \int_{\mathcal{B}_0} \rho_{\mathcal{B}} \mathbf{u}_{\mathcal{B}} \cdot \mathbf{v}_{\mathcal{B}} \, d\mathbf{y},$$

which equals to

$$(3.2) \quad (\mathbf{u}, \mathbf{v}) = \int_{\mathcal{F}_0} \mathbf{u}_{\mathcal{F}} \cdot \mathbf{v}_{\mathcal{F}} \, d\mathbf{y} + m \boldsymbol{\xi}_{\mathbf{u}_{\mathcal{B}}} \cdot \boldsymbol{\xi}_{\mathbf{v}_{\mathcal{B}}} + (\mathbf{I} \mathbf{w}_{\mathbf{u}_{\mathcal{B}}}) \cdot \mathbf{w}_{\mathbf{v}_{\mathcal{B}}}.$$

Let us denote

$$\mathcal{A}\mathbf{z}(\mathbf{y}) = \begin{cases} -\mu \Delta \mathbf{z}_{\mathcal{F}}(\mathbf{y}), & \mathbf{y} \in \mathcal{F}_0, \\ \frac{2\mu}{m} \int_{\partial\mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma} + \left(2\mu \mathbf{I}^{-1} \int_{\partial\mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} \times \mathbf{y} \, d\boldsymbol{\sigma} \right) \times \mathbf{y}, & \mathbf{y} \in \mathcal{B}_0, \end{cases}$$

and define the operator

$$(3.3) \quad \mathcal{A}\mathbf{z} = \mathbb{P}\mathcal{A}\mathbf{z} \quad \text{for any } \mathbf{z} \in D(A),$$

where $\mathbb{P}: L^2(\mathcal{O}) \rightarrow \mathcal{H}$ is the orthogonal projector on \mathcal{H} in $L^2(\mathcal{O})$ and the domain of the operator of A is defined by

$$D(A) = \{\phi \in \mathcal{H} : \phi_{\mathcal{F}} \in H^2(\mathcal{F}_0), \phi_{\mathcal{F}}|_{\partial\mathcal{O}} = 0, (\phi_{\mathcal{F}} - \phi_{\mathcal{B}}) \cdot \tilde{\mathbf{n}}|_{\partial\mathcal{B}_0} = 0, \\ 2\mu(\mathbb{D}(\phi_{\mathcal{F}}) \cdot \tilde{\mathbf{n}}) \times \tilde{\mathbf{n}}|_{\partial\mathcal{B}_0} = -\beta(\phi_{\mathcal{F}} - \phi_{\mathcal{B}}) \times \tilde{\mathbf{n}}|_{\partial\mathcal{B}_0}\},$$

PROPOSITION 3.2. *The operator A defined by (3.3) is self adjoint and positive. Consequently A is a generator of contraction analytic semi-group in \mathcal{H} . Moreover, there exists a constant $C > 0$, such that for any $\mathbf{z} \in D(A)$ we have*

$$\|\mathbf{z}_{\mathcal{F}}\|_{H^2(\mathcal{F}_0)} + \|\mathbf{z}_{\mathcal{B}}\|_{H^2(\mathcal{B}_0)} \leq C \|(\mathbb{I} + A)\mathbf{z}\|_{L^2(\mathcal{O})}.$$

PROOF. (1) *A is symmetric.* Let $\mathbf{z}, \mathbf{v} \in D(A)$. Then the integration by parts used twice gives that

$$(\mathcal{A}\mathbf{z}, \mathbf{v}) = 2\mu \int_{\mathcal{F}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) : \mathbb{D}(\mathbf{v}_{\mathcal{F}}) \, d\mathbf{y} \\ + \beta \int_{\partial\mathcal{B}_0} [(\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \times \tilde{\mathbf{n}}] \cdot [(\mathbf{v}_{\mathcal{F}} - \mathbf{v}_{\mathcal{B}}) \times \tilde{\mathbf{n}}] \, d\boldsymbol{\sigma} = (\mathbf{z}, \mathcal{A}\mathbf{v}).$$

Hence A is a symmetric operator.

(2) *A is positive.* From (1) we have that

$$(\mathcal{A}\mathbf{z}, \mathbf{z}) = 2\mu \|\mathbb{D}(\mathbf{z}_{\mathcal{F}})\|_{L^2(\mathcal{F}_0)}^2 + \beta \int_{\partial\mathcal{B}_0} |\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}|^2 \, d\boldsymbol{\sigma} \quad \text{for any } \mathbf{z} \in D(A).$$

Thus A is a positive operator.

(3) *A is self-adjoint.* In order to prove that A is self adjoint, it suffices to prove that the operator $\mathbb{I} + A: D(A) \rightarrow \mathcal{H}$ is surjective.

First, let us note that the solution $\mathbf{z} \in D(A)$ of the problem $(\mathbb{I} + A)\mathbf{z} = \mathbf{F} \in \mathcal{H}$ in the weak formulation satisfies the integral equality

$$(\mathbf{z}, \mathbf{v}) + (A\mathbf{z}, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) \quad \text{for any } \mathbf{v} \in \mathcal{V},$$

that is, for any $\mathbf{v} \in \mathcal{V}$,

$$(\mathbf{z}, \mathbf{v}) + 2\mu \int_{\mathcal{F}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) : \mathbb{D}(\mathbf{v}_{\mathcal{F}}) d\mathbf{y} + \beta \int_{\partial\mathcal{B}_0} (\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \cdot (\mathbf{v}_{\mathcal{F}} - \mathbf{v}_{\mathcal{B}}) d\boldsymbol{\sigma} = (\mathbf{F}, \mathbf{v}).$$

Let us define the bilinear form $a: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ by

$$(3.4) \quad a(\mathbf{z}, \mathbf{v}) = (\mathbf{z}, \mathbf{v}) + 2\mu \int_{\mathcal{F}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) : \mathbb{D}(\mathbf{v}_{\mathcal{F}}) d\mathbf{y} + \beta \int_{\partial\mathcal{B}_0} (\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \cdot (\mathbf{v}_{\mathcal{F}} - \mathbf{v}_{\mathcal{B}}) d\boldsymbol{\sigma}$$

for any $\mathbf{z}, \mathbf{v} \in \mathcal{V}$. Using the positivity of the operator A , we easily check that a is a bilinear continuous coercive form on \mathcal{V} . Furthermore the mapping $\mathbf{v} \rightarrow (\mathbf{F}, \mathbf{v})$ is a continuous linear form on \mathcal{V} . Therefore the Lax–Milgram theorem implies the existence of a unique solution $\mathbf{z} \in \mathcal{V}$ of the problem (3.4). Using [17] we deduce that there exists $q_{\mathcal{F}} \in \mathcal{D}'(\mathcal{F}_0)$, such that

$$\mathbf{z}_{\mathcal{F}} - \mu \Delta \mathbf{z}_{\mathcal{F}} + \nabla q_{\mathcal{F}} = \mathbf{F} \quad \text{in } \mathcal{D}'(\mathcal{F}_0).$$

In addition, $\mathbf{z}_{\mathcal{F}}$ is a unique weak solution of the system

$$\begin{cases} \mathbf{z}_{\mathcal{F}} - \mu \Delta \mathbf{z}_{\mathcal{F}} + \nabla q_{\mathcal{F}} = \mathbf{F}, & \operatorname{div} \mathbf{z}_{\mathcal{F}} = 0 & \text{in } \mathcal{F}_0, \\ (\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \cdot \tilde{\mathbf{n}} = 0, & 2\mu [\mathbb{D}(\mathbf{z}_{\mathcal{F}})\tilde{\mathbf{n}}] \times \tilde{\mathbf{n}} = -\beta(\mathbf{z}_{\mathcal{F}} - \mathbf{z}_{\mathcal{B}}) \times \tilde{\mathbf{n}} & \text{on } \partial\mathcal{B}_0, \\ \mathbf{z}_{\mathcal{F}} = \mathbf{0} & & \text{on } \partial, \mathcal{O} \end{cases}$$

and it satisfies the estimate

$$\|\mathbf{z}_{\mathcal{F}}\|_{H^2(\mathcal{F}_0)} \leq C (\|\mathbf{F}\|_{L^2(\mathcal{F}_0)} + \|\mathbf{z}_{\mathcal{B}}\|_{H^{3/2}(\partial\mathcal{B}_0)}).$$

On the other hand, since $\mathbf{z}_{\mathcal{B}} \in \mathcal{R}$, there exist two vectors $\boldsymbol{\xi}, \mathbf{w} \in \mathbb{R}^d$, such that $\mathbf{z}_{\mathcal{B}} = \boldsymbol{\xi} + \mathbf{w} \times \mathbf{y}$ in \mathcal{B}_0 , that gives

$$\|\mathbf{z}_{\mathcal{B}}\|_{H^2(\mathcal{B}_0)} \leq C \|\mathbf{F}\|_{L^2(\mathcal{O})}.$$

Hence we conclude that

$$\|\mathbf{z}_{\mathcal{F}}\|_{H^2(\mathcal{F}_0)} + \|\mathbf{z}_{\mathcal{B}}\|_{H^2(\mathcal{B}_0)} \leq C \|(\mathbb{I} + A)\mathbf{z}\|_{L^2(\mathcal{O})}. \quad \square$$

Now we are in a position to prove the following result for the linearised fluid-structure problem (3.1).

PROPOSITION 3.3. *Let $T > 0$. If*

$$\tilde{\mathbf{u}}_0 = (\tilde{\mathbf{u}}_{\mathcal{F},0}, \tilde{\mathbf{u}}_{\mathcal{B},0}) \in \mathcal{V}, \quad \mathbf{F}_0 \in L^2(0, T; L^2(\mathcal{F}_0)) \quad \text{and} \quad \mathbf{F}_1, \mathbf{F}_2 \in L^2(0, T),$$

then problem (3.1) has a unique solution on $[0, T]$, that satisfies a priori estimate

$$(3.5) \quad \|\mathbf{z}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|\nabla q_{\mathcal{F}}\|_{L^2(0,T;L^2(\mathcal{O}))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + \|\mathbf{w}\|_{H^1(0,T)} \\ \leq C(\|(\mathbf{F}_1, \mathbf{F}_2)\|_{L^2(0,T)} + \|\mathbf{F}_0\|_{L^2(0,T;L^2(\mathcal{F}_0))} + \|\tilde{\mathbf{u}}_{\mathcal{B},0}\|_{H^1(\mathcal{B}_0)} + \|\tilde{\mathbf{u}}_0\|_{H^1(\mathcal{F}_0)}),$$

where C is a non-decreasing function of T .

PROOF. We follow Wang *verbatim* [20]. The difference between Wang's problem and our problem is that, Wang considered slip boundary conditions on both boundaries and we consider the Mixed case. Moreover, in [20] only 2D case is investigated. We consider 3D case. For completeness, we will give the principal part of the proof.

We will show that the linearized fluid-solid problem (3.1) can be written in the form

$$(3.6) \quad \partial_t \mathbf{z} + A\mathbf{z} = \mathbf{F}, \quad \mathbf{z}(0) = \tilde{\mathbf{u}}_0,$$

where $\mathbf{z} = \mathbf{z}_{\mathcal{F}}1_{\mathcal{F}_0} + \mathbf{z}_{\mathcal{B}}1_{\mathcal{B}_0}$, $\tilde{\mathbf{u}}_0 = \mathbf{z}_{\mathcal{F}}(0)1_{\mathcal{F}_0} + \mathbf{z}_{\mathcal{B}}(0)1_{\mathcal{B}_0}$ and

$$\mathbf{F} = \mathbb{P} \left(\mathbf{F}_0 1_{\mathcal{F}_0} + \left(\frac{\mathbf{F}_1}{m} + \mathbf{I}^{-1} \mathbf{F}_2 \times \mathbf{y} \right) 1_{\mathcal{B}_0} \right).$$

By Proposition 3.2, the fluid-solid operator $A: D(A) \rightarrow \mathcal{H}$ is a positive self adjoint operator. Thus by Proposition 3.1, the problem (3.1) has a unique solution

$$\mathbf{z} \in L^2(0, T; D(A)) \cap C([0, T]; D(A^{1/2})) \cap H^1(0, T; \mathcal{H}).$$

Recall that the norm of $D(A^{1/2})$ is equivalent to the norm of \mathcal{V} .

Since $\mathbf{z} \in H^1(0, T; \mathcal{H})$, there exist two vector functions $\boldsymbol{\xi}, \mathbf{w} \in H^1(0, T)$, such that

$$\mathbf{z}_{\mathcal{B}}(t, \mathbf{y}) = \boldsymbol{\xi}(t) + \mathbf{w}(t) \times \mathbf{y} \quad \text{for any } \mathbf{y} \in \mathcal{B}_0.$$

If we take the inner product (3.2) of equality (3.6)₁ and $\phi \in \mathcal{H}$, we get

$$(3.7) \quad \int_{\mathcal{F}_0} \mathbf{z}'_{\mathcal{F}} \cdot \phi_{\mathcal{F}} d\mathbf{y} + m \left(\boldsymbol{\xi}' - \frac{\mathbf{F}_1}{m} \right) \cdot \boldsymbol{\xi}_{\phi} + \mathbf{I}(\mathbf{w}' - \mathbf{I}^{-1} \mathbf{F}_2) \cdot \mathbf{w}_{\phi} \\ - \int_{\mathcal{F}_0} \mu \Delta \mathbf{z}_{\mathcal{F}} \cdot \phi_{\mathcal{F}} d\mathbf{y} + 2\mu \int_{\partial \mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} d\boldsymbol{\sigma} \cdot \boldsymbol{\xi}_{\phi} \\ + 2\mu \left(\int_{\partial \mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} \times \mathbf{y} d\boldsymbol{\sigma} \right) \cdot \mathbf{w}_{\phi} = \int_{\mathcal{F}_0} \mathbb{P} \mathbf{F}_0 \cdot \phi_{\mathcal{F}} d\mathbf{y}.$$

Considering test functions $\phi \in \mathcal{H}$, such that $\phi_{\mathcal{B}} = \mathbf{0}$, we obtain that there exists a function $q_{\mathcal{F}} \in L^2(0, T; H^1(\mathcal{F}_0))$ satisfying the equation

$$\mathbf{z}'_{\mathcal{F}} - \mu \Delta \mathbf{z}_{\mathcal{F}} + \nabla q_{\mathcal{F}} = \mathbf{F}_0 \quad \text{in } \mathcal{F}_0.$$

Thus, for arbitrary $\phi \in \mathcal{H}$, we have

$$\int_{\mathcal{F}_0} (\mathbf{z}'_{\mathcal{F}} - \mu \Delta \mathbf{z}_{\mathcal{F}} - \mathbf{F}_0) \cdot \phi_{\mathcal{F}} d\mathbf{y} = - \int_{\partial \mathcal{B}_0} q_{\mathcal{F}} \phi_{\mathcal{F}} \cdot \tilde{\mathbf{n}} d\boldsymbol{\sigma}.$$

Substituting this equality into (3.7), we obtain that

$$\begin{aligned} m \left(\boldsymbol{\xi}'(t) - \frac{\mathbf{F}_1}{m} \right) \cdot \boldsymbol{\xi}_\phi + \mathbf{I}(\mathbf{w}'_{\mathcal{B}} - \mathbf{I}^{-1}\mathbf{F}_2) \cdot \mathbf{w}_\phi + 2\mu \int_{\partial\mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma} \cdot \boldsymbol{\xi}_\phi \\ + 2\mu \left(\int_{\partial\mathcal{B}_0} \mathbb{D}(\mathbf{z}_{\mathcal{F}}) \tilde{\mathbf{n}} \times \mathbf{y} \, d\boldsymbol{\sigma} \right) \cdot \mathbf{w}_\phi = \int_{\partial\mathcal{B}_0} q_{\mathcal{F}} \phi_{\mathcal{F}} \cdot \tilde{\mathbf{n}} \, d\boldsymbol{\sigma}. \end{aligned}$$

Since the function ϕ is divergence free, we have $(\phi_{\mathcal{F}} - \phi_{\mathcal{B}}) \cdot \tilde{\mathbf{n}}|_{\partial\mathcal{B}_0} = 0$. As a consequence we obtain that

$$\begin{aligned} m \boldsymbol{\xi}'(t) + \int_{\partial\mathcal{B}_0} (2\mu \mathbb{D}(\mathbf{z}_{\mathcal{F}}) - q_{\mathcal{F}} \mathbb{I}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma} &= \mathbf{F}_1, \\ \mathbf{I} \mathbf{w}'(t) + \int_{\partial\mathcal{B}_0} (2\mu \mathbb{D}(\mathbf{z}_{\mathcal{F}}) - q_{\mathcal{F}} \mathbb{I}) \tilde{\mathbf{n}} \times \mathbf{y} \, d\boldsymbol{\sigma} &= \mathbf{F}_2. \end{aligned}$$

Therefore a problem (3.6) is equivalent to a problem (3.1). Finally Propositions 3.1 and 3.2 imply the uniqueness of the solution $(\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}}, \boldsymbol{\xi}, \mathbf{w})$, that satisfies estimate (3.5). \square

3.2. Nonlinear case (Proof of Theorem 2.1). In this section we show Theorem 2.1. To do it we prove existence and uniqueness results for the modified system (2.2). The proof is based on the fixed point argument. Let us define

$$\mathcal{P}: (\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) \rightarrow (\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}}, \boldsymbol{\xi}, \mathbf{w}),$$

which maps $\mathcal{U}_T(\mathcal{F}_0) \times L^2(0, T; H^1(\mathcal{F}_0)) \times H^1(0, T) \times H^1(0, T)$ into itself. Functions $(\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}}, \boldsymbol{\xi}, \mathbf{w}) = \mathcal{P}(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}})$ are the solution of the linear system (3.1) with

$$\begin{aligned} \mathbf{F}_0 &= \mathbf{F}_0(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) = -(\mathcal{M} - \mu\mathcal{L} + \mu\Delta)\widehat{\mathbf{z}}_{\mathcal{F}} + (\nabla - \mathcal{G})\widehat{q}_{\mathcal{F}} - \mathcal{N}\widehat{\mathbf{z}}_{\mathcal{F}} + \widetilde{\mathbf{f}}_0, \\ \mathbf{F}_1 &= \mathbf{F}_1(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) = \widetilde{\mathbf{f}}_1 + m(\widehat{\mathbf{w}} \times \widehat{\boldsymbol{\xi}}) \\ &\quad + \int_{\partial\mathcal{B}_0} \mathbb{T}(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma} - \int_{\partial\mathcal{B}_0} \mathcal{T}(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma}, \\ \mathbf{F}_2 &= \mathbf{F}_2(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) = \widetilde{\mathbf{f}}_2 + \widehat{\mathbf{w}} \times (\mathbf{I}\widehat{\mathbf{w}}) \\ &\quad + \int_{\partial\mathcal{B}_0} \mathbf{y} \times \mathbb{T}(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma} - \int_{\partial\mathcal{B}_0} \mathbf{y} \times \mathcal{T}(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}) \tilde{\mathbf{n}} \, d\boldsymbol{\sigma}. \end{aligned}$$

For some $R > 0$ we define the set

$$\begin{aligned} K &= \{(\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) \in \mathcal{U}_T(\mathcal{F}_0) \times L^2(0, T; H^1(\mathcal{F}_0)) \times H^1(0, T) \times H^1(0, T) : \\ &\quad \|\widehat{\mathbf{z}}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|\widehat{q}_{\mathcal{F}}\|_{L^2(0, T; H^1(\mathcal{F}_0))} + \|\widehat{\boldsymbol{\xi}}\|_{H^1(0, T)} + \|\widehat{\mathbf{w}}\|_{H^1(0, T)} \leq R\}. \end{aligned}$$

As the first step we show that $\mathcal{P}(K) \subset K$. We put C_0, B_0 constants that depends only on $T, \|\mathbf{u}_0\|_{H^1(\mathcal{F}_0)}, \|\mathbf{u}_{\mathcal{B}, 0}\|_{H^1(\mathcal{B}_0)}, \|\mathbf{f}_0\|_{L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathcal{F}_0))}, \|(\mathbf{f}_1, \mathbf{f}_2)\|_{L^2_{\text{loc}}(\mathbb{R}^+)}$

(see the regularity (2.3)). Moreover, C_0, B_0 are nondecreasing functions of T . Also C_0 is a nondecreasing function of R . Then Proposition 3.3 gives

$$\begin{aligned} & \|\mathbf{z}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|q_{\mathcal{F}}\|_{L^2(0,T;H^1(\mathcal{F}_0))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + \|\mathbf{w}\|_{H^1(0,T)} \\ & \leq C_0(\|(\mathbf{F}_1, \mathbf{F}_2)\|_{L^2(0,T)} + \|\mathbf{F}_0\|_{L^2(0,T;L^2(\mathcal{F}_0))} + 1). \end{aligned}$$

From [18] we have

$$\|\mathbf{F}_0\|_{L^2(0,T;L^2(\mathcal{F}_0))} + \|(\mathbf{F}_1, \mathbf{F}_2)\|_{L^2(0,T)} \leq C_0 T^{1/10} + B_0.$$

Therefore it follows that

$$\|\mathbf{z}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|q_{\mathcal{F}}\|_{L^2(0,T;H^1(\mathcal{F}_0))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + \|\mathbf{w}\|_{H^1(0,T)} \leq C_0 T^{1/10} + B_0.$$

Now, choosing R and T such that $4B_0 < R$ and $C_0(T)T^{1/10} < R/4$, we deduce that

$$C_0 T^{1/10} + B_0 < R \quad \text{and} \quad \mathcal{P}(K) \subset K.$$

In the second step we prove that \mathcal{P} is a contraction operator, when T is small enough and R is large enough. Let us define

$$(\mathbf{z}_{\mathcal{F}}^i, q_{\mathcal{F}}^i, \boldsymbol{\xi}^i, \mathbf{w}^i) = \mathcal{P}(\widehat{\mathbf{z}}_{\mathcal{F}}^i, \widehat{q}_{\mathcal{F}}^i, \widehat{\boldsymbol{\xi}}^i, \widehat{\mathbf{w}}^i) \quad \text{for } (\widehat{\mathbf{z}}_{\mathcal{F}}^i, \widehat{q}_{\mathcal{F}}^i, \widehat{\boldsymbol{\xi}}^i, \widehat{\mathbf{w}}^i) \in K, \quad i = 1, 2,$$

and calculate the differences

$$\begin{aligned} (\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}}, \boldsymbol{\xi}, \mathbf{w}) &= (\mathbf{z}_{\mathcal{F}}^1, q_{\mathcal{F}}^1, \boldsymbol{\xi}^1, \mathbf{w}^1) - (\mathbf{z}_{\mathcal{F}}^2, q_{\mathcal{F}}^2, \boldsymbol{\xi}^2, \mathbf{w}^2), \\ (\widehat{\mathbf{z}}_{\mathcal{F}}, \widehat{q}_{\mathcal{F}}, \widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}}) &= (\widehat{\mathbf{z}}_{\mathcal{F}}^1, \widehat{q}_{\mathcal{F}}^1, \widehat{\boldsymbol{\xi}}^1, \widehat{\mathbf{w}}^1) - (\widehat{\mathbf{z}}_{\mathcal{F}}^2, \widehat{q}_{\mathcal{F}}^2, \widehat{\boldsymbol{\xi}}^2, \widehat{\mathbf{w}}^2). \end{aligned}$$

Then the functions $(\mathbf{z}_{\mathcal{F}}, q_{\mathcal{F}}, \boldsymbol{\xi}, \mathbf{w})$ satisfy the system (3.1) with zero initial conditions, i.e.

$$\mathbf{z}_{\mathcal{F}}(0) = \mathbf{0} \quad \text{in } \mathcal{F}_0, \quad \boldsymbol{\xi}(0) = \mathbf{0}, \quad \mathbf{w}(0) = \mathbf{0}$$

and

$$\mathbf{F}_k = \mathbf{F}_k(\widehat{\mathbf{z}}_{\mathcal{F}}^1, \widehat{q}_{\mathcal{F}}^1, \widehat{\boldsymbol{\xi}}^1, \widehat{\mathbf{w}}^1) - \mathbf{F}_k(\widehat{\mathbf{z}}_{\mathcal{F}}^2, \widehat{q}_{\mathcal{F}}^2, \widehat{\boldsymbol{\xi}}^2, \widehat{\mathbf{w}}^2), \quad k = 0, 1, 2.$$

It is easy to check

$$\begin{aligned} & \|\mathbf{F}_0\|_{L^2(0,T;L^2(\mathcal{F}_0))} + \|(\mathbf{F}_1, \mathbf{F}_2)\|_{L^2(0,T)} \\ & \leq C_0 T^{1/10} (\|\widehat{\mathbf{z}}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|\widehat{q}_{\mathcal{F}}\|_{L^2(0,T;H^1(\mathcal{F}_0))} + \|(\widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}})\|_{H^1(0,T)}). \end{aligned}$$

Applying Proposition 3.3 we obtain

$$\begin{aligned} & \|\mathbf{z}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|q_{\mathcal{F}}\|_{L^2(0,T;H^1(\mathcal{F}_0))} + \|\boldsymbol{\xi}\|_{H^1(0,T)} + \|\mathbf{w}\|_{H^1(0,T)} \\ & \leq C_0 T^{1/10} (\|\widehat{\mathbf{z}}_{\mathcal{F}}\|_{\mathcal{U}_T(\mathcal{F}_0)} + \|\widehat{q}_{\mathcal{F}}\|_{L^2(0,T;H^1(\mathcal{F}_0))} + \|(\widehat{\boldsymbol{\xi}}, \widehat{\mathbf{w}})\|_{H^1(0,T)}). \end{aligned}$$

Thus, when T is small enough, \mathcal{P} is a contraction operator, such that the unique fixed point of \mathcal{P} is a unique solution $(\widetilde{\mathbf{u}}_{\mathcal{F}}, \widetilde{p}_{\mathcal{F}}, \widetilde{\boldsymbol{\eta}}, \widetilde{\boldsymbol{\omega}})$ of system (2.2) in K . For given two strong solutions of (2.2), there exists a large enough R , such that these solutions belong to the set K . Since the system (2.2) has a unique solution in K by the continuity argument we get that system (1.1)–(1.2) has a unique solution.

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