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A CLASS OF DELAY EVOLUTION HEMIVARIATIONAL INEQUALITIES AND OPTIMAL FEEDBACK CONTROLS

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Abstract. In this paper, we study the feedback optimal control for a class of evolution hemivariational inequalities with delay. First, we obtain the existence of feasible pairs by applying the Cesari property, the Filippov theorem, the properties of Clarke subdifferential and a fixed point theorem for multivalued maps. Next, the results of optimal feedback control pairs and time optimal control for delay evolution hemivariational inequalities are presented under sufficient conditions. Finally, an example is included to illustrate our main results.

1. Introduction

Hemivariational inequalities were introduced to deal with the mechanical problems with nonsmooth and nonconvex energy superpotentials (see [31], [32]). It is an efficient tool in mathematical models to describe the antiplane shear deformations of a piezoelectric cylinder in frictional contact with a foundation,

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and to describe the frictional contact between piezoelectric body and electrically conductive foundation (see [28], [29]). In recent years, as the control theory is an important area of application oriented mathematics which deals with the design and analysis of control systems, many researchers have paid increasing attention to the control problems for hemivariational inequalities. In particular, Haslinger and Panagiotopoulos [8] showed the existence of optimal control pairs for a class of coercive hemivariational inequalities. Migórski and Ochal [27] considered the optimal control problems for the parabolic hemivariational inequalities. J.Y. Park and S.H. Park [33], [34] proved the existence of optimal control pairs to the hyperbolic systems. In [40], [41], Tolstonogov considered the optimal control problems for subdifferential type differential inclusions. More results on hemivariational inequalities can be found in [10], [19]–[23], [26] and the references therein.

In addition, control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control action [6], [25]. Optimal feedback control became one of the main problems in modern control theory (see [7]). More precisely, Lin et al. [18] considered the optimal feedback control for dynamical systems with state constraints. Li and Yong [17] investigated the optimal feedback control for evolution equations. Moreover, optimal feedback control of semilinear evolution equations in Banach spaces was studied in [12] and [43]. Wang et al. [42] proved the existence of optimal feedback control for semilinear fractional evolution equations; however, optimal feedback control for evolution hemivariational inequalities has not been investigated yet and there are still many interesting ideas and unanswered questions to be investigated.

Furthermore, in many practical cases, the processes to be optimized can no longer be adequately modeled by control problems; instead, delays have to be employed for their description. For instance, Klamka [13], [14] studied the stochastic controllability of systems with delays. Ren et al. [36] studied the controllability of impulsive neutral stochastic differential inclusions with infinite delay. Kumar and Sukavanam [16] considered fractional order semilinear control systems with bounded delay. Zhou and Wang [46] considered the optimal feedback control for linear systems with input delays. Relevant results regarding the control systems with delay can be found in [11], [15], [37]–[39], [44] and the references therein.

Motivated by previously mentioned works, it is necessary and important to study the optimal control problems for delay evolution hemivariational inequalities and to develop more results for delayed optimal controls. The main objective of this paper is to investigate the optimal feedback controls for the following evolution control systems described by hemivariational inequalities with fixed delay:

$$
\begin{cases}
\langle -x'(t) + Ax(t) + g(t, x(t), x(t-r), u(t)), v\rangle_H \\
\qquad\qquad\qquad\qquad + F^0(t, x(t); v) \geq 0 & \text{for } t \in J = [0, b], \\
 & \text{for all } v \in H, \\
u(t) \in U(t, x(t)), \\
x(t) = \varphi(t) & \text{for all } t \in [-r, 0],
\end{cases}
\tag{1.1}
$$

where $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a separable Hilbert space $(H, \|\cdot\|)$. The notation $F^0(t, \cdot\,; \cdot\,)$ denotes the generalized Clarke subdifferential of a locally Lipschitz function $F(t, \cdot\,)\colon H \to \mathbb{R}$ (see [5]). Let V be a separable reflexive Banach space, $u(\cdot\,)$ take values in V, $g\colon [0, b] \times H \times H \times V \to H$ be a nonlinear function. The multimap $U\colon [0, b] \times H \to P(V)$ is a feedback control function and φ is a given function in $C([-r, 0], H)$. Moreover, in our discussion, the cost functional over the family of admissible control pair (x, u) is given by

$$
\mathcal{J}(x, u) = \int_0^b \mathcal{L}(t, x(t), x(t-r), u(t))\, dt.
$$

The rest of this paper is organized as follows. In Section 2, we provide some preliminary facts that we need in the sequel. In Section 3, we establish a general existence theorem of feasible pairs for hemivariational inequalities related to system (1.1). Section 4 focuses on the existence of optimal feedback control pairs for the considered problem. In Section 5 we establish results for time optimal control. Finally, in Section 6, an example is given to illustrate our main results.

2. Preliminaries

In this section, we recall some notations and preliminary results. Firstly, let us recall some basic definitions and results for multivalued maps from [1, 9]. Let E be a Banach space with the norm $\|\cdot\|$, E^* denote its dual and $\langle\,\cdot\,,\cdot\,\rangle$ be the duality pairing of E^* and E.

Definition 2.1 ([9]). Given a multivalued map $F\colon E \to 2^E\backslash\{\emptyset\} = P(E)$, we say

(a) F is (closed) convex valued if $F(x)$ is (closed) convex for all $x \in E$.
(b) F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in E for all $B \in P_b(E) = \{\Omega \subseteq E : \Omega \text{ is a nonempty bounded set}\}$ $\Big($i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty\Big)$.

(c) F is upper semicontinuous (u.s.c.) at $x_0 \in E$, if for every open set $U \subset E$ such that $F(x_0) \subset U$, there exists a neighborhood V of x_0 such that $F(V) \subseteq U$. We say F is u.s.c. if F is u.s.c. at every $x_0 \in E$.

(d) F is completely continuous if $F(B)$ is relatively compact for every $B \in P_b(E)$. If F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has a closed graph (i.e. if $x_n \to x$, $y_n \to y$, then $y_n \in F(x_n)$ implies $y \in F(x)$).

(e) F has a fixed point if there is an $x \in E$ such that $x \in F(x)$.

Definition 2.2 ([17, Definition 4.1]). Let E be a Banach space and Z be a metric space. Let $F\colon Z \to 2^E$ be a multifunction. We say F possesses the Cesari property at $x_0 \in Z$, if

$$\bigcap_{\varepsilon>0} \overline{\operatorname{co}}\, F(O_\varepsilon(x_0)) = F(x_0),$$

where $\overline{\operatorname{co}}\, D$ is the closed convex hull of D, $O_\varepsilon(x_0) = \{y \in Z : \|y - x_0\| \le \varepsilon\}$ is the ball centered at x_0 with radius $\varepsilon > 0$. If F has the Cesari property at every point $x \in Q \subset Z$, we simply say that F has the Cesari property on Q.

Now, we introduce the definition of the generalized gradient of Clarke for a locally Lipschitz function $h\colon E \to \mathbb{R}$ (see [5]). The generalized directional derivative (in the sense of Clarke) of h at x in the direction v, denoted by $h^0(x;v)$, is defined by

$$h^0(x;v) = \limsup_{y\to x,\, t\downarrow 0} \frac{h(y+tv) - h(y)}{t}.$$

The Clarke subdifferential or the generalized gradient of h at x, denoted $\partial h(x)$, is the subset of E^* defined by

$$\partial h(x) = \{y \in E^* : h^0(x;v) \ge \langle y, v\rangle, \text{ for all } v \in E\}.$$

Lemma 2.3 ([5], [30, Proposition 3.23]). *If $h\colon E \to \mathbb{R}$ is a locally Lipschitz function on an open set $\Omega \subseteq E$, then*

(a) *for every $v \in E$, one has $h^0(x;v) = \max\{\langle y, v\rangle : y \in \partial h(x)\}$;*

(b) *for every $x \in \Omega$, $\partial h(x)$ is a nonempty, convex, weakly* compact subset of E^* and $\|y\|_{E^*} \le K_x$ for every $y \in \partial h(x)$ (where $K_x > 0$ is the Lipschitz constant of h near x);*

(c) *the multifunction $\Omega \ni x \mapsto \partial h(x) \subseteq E^*$ is u.s.c. from Ω into $E^*_{\omega^*}$ (where $E^*_{\omega^*}$ denotes the Banach space E^* furnished with the weak*-topology);*

(d) *the graph of ∂h is closed in $E \times E^*_{\omega^*}$ topology, i.e. if $\{x_n\} \subset \Omega$ and $\{y_n\} \subset E^*$ are sequences such that $y_n \in \partial h(x_n)$ and $x_n \to x$ in E, $y_n \to y$ weakly* in E^*, then $y \in \partial h(x)$;*

(e) *the function* $\Omega \times E \ni (x, v) \to h^0(x; v) \in \mathbb{R}$ *is u.s.c., i.e. for all* $x \in \Omega$, $v \in E$, $\{x_n\} \subset \Omega$, $\{v_n\} \subset E$ *such that* $x_n \to x$ *in* Ω *and* $v_n \to v$ *in* E, *we have* $\limsup_{n\to\infty} h^0(x_n; v_n) \le h^0(x; v)$.

We recall that the upper semicontinuous multimap G is said to be condensing if for any $B \in P_b(E)$ with $\beta(B) \neq 0$, we have $\beta(G(B)) < \beta(B)$, where β denotes the Kuratowski measure of noncompactness [3]. If G is upper semicontinuous and completely continuous, then G is condensing. At the end of this section, we recall the fixed point theorem which will be used to prove our existence results.

Theorem 2.4 ([4, Theorem 1.7], Martelli's fixed point theorem [24]). *Let E be a Banach space and let $G\colon E \to P(E)$ be a compact convex-valued, upper semicontinuous and condensing map. If the set*

$$\Omega = \{x \in E : \lambda x \in G(x) \ \textit{for some}\ \lambda > 1\}$$

is bounded, then G has a fixed point.

3. The existence of feasible pairs

In this section, we will study the existence of feasible pairs for system (1.1).

Firstly, we study the auxiliary problem described by hemivariational inequalities with delay:

$$(3.1)\quad \begin{cases} \langle -x'(t) + Ax(t) + g(t, x(t), x(t-r), u(t)), v\rangle \\ \qquad\qquad + F^0(t, x(t); v) \ge 0 & \text{for } t \in J, \text{ for all } v \in H, \\ x(t) = \varphi(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Here $u(\,\cdot\,)$ is given in $L^2(J, V)$ and V is a separable reflexive Banach space.

In fact, to investigate (3.1), we shall study the following evolution inclusion:

$$(3.2)\quad \begin{cases} x'(t) \in Ax(t) + g(t, x(t), x(t-r), u(t)) + \partial F(t, x(t)) & \text{for } t \in J, \\ x(t) = \varphi(t) & \text{for } t \in [-r, 0], \end{cases}$$

where ∂F denotes the generalized Clarke subdifferential of $F(t, \cdot\,)\colon H \to \mathbb{R}$.

We say that x is a solution to system (3.2) if there is $f \in L^2(J, H)$ such that $f(t) \in \partial F(t, x(t))$ for almost every $t \in J$ and

$$\begin{cases} x'(t) = Ax(t) + g(t, x(t), x(t-r), u(t)) + f(t) & \text{for } t \in J, \\ x(t) = \varphi(t) & \text{for } t \in [-r, 0]. \end{cases}$$

Then the above equation implies

$$\begin{cases} \langle -x'(t) + Ax(t) + g(t, x(t), x(t-r), u(t)), v\rangle + \langle f(t), v\rangle = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in J, \quad \text{for all } v \in H, \\ x(t) = \varphi(t) \qquad\qquad\qquad\qquad\quad \text{for } t \in [-r, 0]. \end{cases}$$

Since $f(t) \in \partial F(t, x(t))$ and $\langle f(t), v\rangle \leq F^0(t, x(t); v)$, it is easy to get (3.1). Thus, we will consider the differential inclusion system (3.2) in what follows. Moveover, we impose the following hypotheses:

(HA) $A\colon D(A) \subseteq H \to H$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ and the semigroup $T(t)$ is compact for $t > 0$.
By [35, Theorem 1.2.2], there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{\omega t} \leq Me^{\omega b} := C_0.$$

(HF) The function $F\colon J \times H \to \mathbb{R}$ satisfies the following hypotheses:
  (a) $F(\,\cdot\,, x)$ is measurable for all $x \in H$;
  (b) $F(t, \,\cdot\,)$ is locally Lipschitz continuous for almost every $t \in J$;
  (c) there exist a function $a \in L^2(J, \mathbb{R}^+)$ and a constant $c \geq 0$ such that

$$\|\partial F(t, x)\| = \sup\{\|f(t)\| : f(t) \in \partial F(t, x)\} \leq a(t) + c\|x\|$$

  for almost every $t \in J$ and all $x \in H$.

(HU) The feedback multimap $U\colon J \times H \to P(V)$ satisfies the following conditions:
  (a) there exist a $\phi \in L^2(J, \mathbb{R})$ and a constant $L_u > 0$ such that

$$\|U(t, x)\| = \sup\{\|u(t)\| : u(t) \in U(t, x)\} \leq \phi(t) + L_u\|x\|,\ (t, x) \in J \times H;$$

  (b) for almost every $t \in J$ and $x \in H$, the set $U(t, x)$ satisfies

$$\bigcap_{\varepsilon>0} \overline{\text{co}}\, U(O_\varepsilon(t, x)) = U(t, x).$$

(HG) The function $g\colon J \times H \times H \times V \to H$ is Borel measurable in (t, x, y, u) and continuous in (x, y, u). For $t \in J$, there exists a positive constant $M > 0$ such that

$$\|g(t, x, y, u)\| \leq M(1 + \|x\| + \|y\|), \quad \text{for } (t, x, y, u) \in J \times H \times H \times V.$$

Moreover, for almost every $t \in J$, the function g satisfies

$$\bigcap_{\varepsilon>0} \overline{\text{co}}\, g(t, O_\varepsilon(x), O_\varepsilon(y), U(O_\varepsilon(t, x)) = g(t, x, y, U(t, x)),$$

for $(t, x, y) \in J \times H \times H$.

Next, we define an operator $\mathcal{N}\colon L^2(J, H) \to 2^{L^2(J,H)}$ as

$$\mathcal{N}(x) = \{w \in L^2(J, H) : w(t) \in \partial F(t, x(t)) \text{ for a.e. } t \in J\} \quad \text{for } x \in L^2(J, H).$$

To obtain the existence results, we also need the following lemmas:

Lemma 3.1 ([20, Lemma 2.6]). *If the assumption* (HF) *holds, then for each $x \in L^2(J, H)$, the set $\mathcal{N}(x)$ is nonempty, convex and weakly compact.*

Lemma 3.2 ([26, Lemma 11]). *If* (HF) *holds and the operator* $\mathcal{N}$ *satisfies: if* $x_n \to x$ *in* $L^2(J,H)$, $w_n \to w$ *weakly in* $L^2(J,H)$ *and* $w_n \in \mathcal{N}(x_n)$, *then we have* $w \in \mathcal{N}(x)$.

Lemma 3.3 ([45, Lemma 3.2]). *Let* $T(t)$ *be a compact* C_0*-semigroup on the Banach space* E. *Then, for any* $p > 1$, *the operator*

$$S(g(\,\cdot\,)) = \int_0^{\cdot} T(\,\cdot\, - s)g(s)\,ds, \quad \text{for all } g(\,\cdot\,) \in L^p(J,E), \tag{3.3}$$

is a compact operator from $L^p(J,E)$ *to* $C(J,E)$.

Definition 3.4. For a given $u \in L^2(J,V)$, a function $x \in C([-r,b],H)$ is a mild solution to system (3.1) on $[-r,b]$, if $x(t) = \varphi(t)$ for $t \in [-r,0]$, and there exists an $f \in L^2(J,H)$ such that $f(t) \in \partial F(t,x(t))$ for almost every $t \in J$ and

$$x(t) = T(t)\varphi(0) + \int_0^t T(t-s)[g(s,x(s),x(s-r),u(s)) + f(s)]\,ds, \quad \text{for } t \in J.$$

Now, to obtain the feasible pair of (1.1), we first proof the existence of mild solutions to system (3.1).

Theorem 3.5. *For given* $u \in L^2(J,V)$ *and* $\varphi \in C([-r,0],H)$, *if the hypotheses* (HA), (HF) *and* (HG) *are satisfied, then* (3.1) *has at least one mild solution* $x \in C([-r,b],H)$.

Proof. For convenience, let

$$B_l = \{x \in C([-r,b],H) : \|x\|_{C([-r,b],H)} \le l\}, \quad l > 0.$$

For $x \in C([-r,b],H)$ and by Lemma 3.1, define a multivalued map

$$\mathcal{F} \colon C([-r,b],H) \to 2^{C([-r,b],H)}$$

as follows:

$$\begin{aligned}\mathcal{F}(x) = \Big\{\mu \in C([-r,b],H) : \mu(t) = \varphi(t),\ t \in [-r,0],\ \text{and}\\ \mu(t) = T(t)\varphi(0) + \int_0^t T(t-s)g(s,x(s),x(s-r),u(s))\,ds\\ + \int_0^t T(t-s)f(s)\,ds,\ t \in [0,b],\ f \in \mathcal{N}(x)\Big\}.\end{aligned}$$

It is clear that the existence of a solution to (3.1) is reduced to finding a fixed point of $\mathcal{F}$. We will show that $\mathcal{F}$ satisfies all the conditions of Theorem 2.4. To complete the proof, we divide it into six steps.

Step 1. $\mathcal{F}(x)$ is convex for each $x \in C([-r,b],H)$.

By Lemma 3.1, the set $\mathcal{N}(x)$ is convex. Hence, if $f_1, f_2 \in \mathcal{N}(x)$, then $\lambda f_1 + (1-\lambda)f_2 \in \mathcal{N}(x)$ for all $\lambda \in (0,1)$, which implies that $\mathcal{F}(x)$ is convex.

Step 2. $\mathcal{F}(B_l)$ is a bounded subset of $C([-r,b],H)$.

Obviously, B_l is a bounded, closed and convex set of $C([-r,b],H)$. We claim that there exists an $\ell > 0$ such that $\|\mu\|_{C([-r,b],H)} \leq \ell$ for each $\mu \in \mathcal{F}(x)$, $x \in B_l$.

In fact, if $\mu \in \mathcal{F}(x)$, then there exists an $f \in \mathcal{N}(x)$ such that

$$\mu(t) = T(t)\varphi(0) + \int_0^t T(t-s)[f(s) + g(s,x(s),x(s-r),u(s))]\,ds, \quad t \in J. \tag{3.4}$$

From (HF), (HG) and the Hölder inequality, we have, for $t \in J$,

$$\begin{aligned}\|\mu(t)\| &\leq \|T(t)\varphi(0)\| + \int_0^t \|T(t-s)[f(s) + g(s,x(s),x(s-r),u(s))]\|\,ds \\ &\leq C_0\|\varphi(0)\| + C_0\int_0^t [a(s) + cl + M(1+2l)]\,ds \\ &\leq C_0\big[\|\varphi(0)\| + \|a\|_{L^2(J,R^+)}\sqrt{b} + clb + Mb(1+2l)\big] := \ell_0.\end{aligned}$$

Thus $\|\mu\|_{C([-r,b],H)} \leq \max\{\|\varphi\|_{C([-r,0],H)}, \ell_0\} := \ell$, which implies $\mathcal{F}(B_l)$ is bounded in $C([-r,b],H)$.

Step 3. $\{\mathcal{F}(x) : x \in B_l\}$ is equicontinuous.

Firstly, for each $x \in B_l$, $\mu \in \mathcal{F}(x)$, there exists an $f \in \mathcal{N}(x)$ such that (3.4) holds. For $\tau_1, \tau_2 \in [-r,0]$, it is easy to see that $\|\mu(\tau_2) - \mu(\tau_1)\|$ tends to zero as $|\tau_2 - \tau_1| \to 0$. For $0 < \tau_1 < \tau_2 \leq b$ and $\varepsilon > 0$ small enough, we have

$$\begin{aligned}\|\mu(\tau_2) - \mu(\tau_1)\| &\leq \|[T(\tau_2) - T(\tau_1)]\varphi(0)\| \\ &\quad + \int_0^{\tau_1-\varepsilon} \|[T(\tau_2-s) - T(\tau_1-s)][f(s) + g(s,x(s),x(s-r),u(s))]\|\,ds \\ &\quad + \int_{\tau_1-\varepsilon}^{\tau_1} \|[T(\tau_2-s) - T(\tau_1-s)][f(s) + g(s,x(s),x(s-r),u(s))]\|\,ds \\ &\quad + \int_{\tau_1}^{\tau_2} \|T(\tau_2-s)[f(s) + g(s,x(s),x(s-r),u(s))]\|\,ds \\ &\leq \|[T(\tau_2) - T(\tau_1)]\varphi(0)\| \\ &\quad + \int_0^{\tau_1-\varepsilon} \|T(\tau_2-s) - T(\tau_1-s)\|[a(s) + cl + M(1+2l)]\,ds \\ &\quad + 2C_0\int_{\tau_1-\varepsilon}^{\tau_1} [a(s) + cl + M(1+2l)]\,ds \\ &\quad + C_0\int_{\tau_1}^{\tau_2} [a(s) + cl + M(1+2l)]\,ds \\ &\leq \|[T(\tau_2) - T(\tau_1)]\varphi(0)\| \\ &\quad + \sup_{s\in[0,\tau_1-\varepsilon]} \|T(\tau_2-s) - T(\tau_1-s)\|[\|a\|_{L^2(J,R^+)}\sqrt{b} \\ &\quad + M(1+2l)b + clb] + C_0\|a\|_{L^2(J,R^+)}(2\sqrt{\varepsilon} + \sqrt{\tau_2-\tau_1}) \\ &\quad + C_0M(1+2l)(2\varepsilon + (\tau_2-\tau_1)).\end{aligned} \tag{3.5}$$

Since the compactness of $T(t)(t > 0)$ implies the continuity of $T(t)$ $(t > 0)$ on the uniform operator topology (cf. [35, Theorem 2.3.2]), we can see that the right-hand side of (3.5) tends to zero independently of $x \in B_l$ as $\tau_2 \to \tau_1$ and $\varepsilon \to 0$.

Similarly, for $\tau_1 = 0$ and $0 < \tau_2 \le b$, we may prove that $\|\mu(\tau_2) - x_0\|$ tends to zero independently of $x \in B_l$ as $\tau_2 \to 0$.

Hence, by the above arguments, $\{\mathcal{F}(x) : x \in B_l\}$ is an equicontinuous family of functions in $C([-r, b], H)$.

Step 4. $\mathcal{F}$ is a compact multivalued map.

Let $t \in [-r, b]$ be fixed. We will show the set $\Pi(t) = \{\mu(t) : \mu \in \mathcal{F}(B_l)\}$ is relatively compact in H. Clearly, for $t \in [-r, 0]$, $\Pi(t) = \{\varphi(t)\}$ is compact. So it is only necessary to consider $t > 0$. Let $0 < t \le b$ be fixed. For $x \in B_l$ and any $\mu \in \mathcal{F}(x)$, there exists an $f \in \mathcal{N}(x)$ such that

$$\mu(t) = T(t)\varphi(0) + \int_0^t T(t-s)[f(s) + g(s, x(s), x(s-r), u(s))]\, ds, \quad t \in J.$$

For each $\varepsilon \in (0, t)$, $t \in (0, b]$, and any $x \in B_l$, we define

$$\begin{aligned} \mu^\varepsilon(t) &= T(t)\varphi(0) + \int_0^{t-\varepsilon} T(t-s)[f(s) + g(s, x(s), x(s-r), u(s))]\, ds \\ &= T(t)\varphi(0) + T(\varepsilon)\int_0^{t-\varepsilon} T(t-s-\varepsilon)[f(s) + g(s, x(s), x(s-r), u(s))]\, ds. \end{aligned}$$

From the boundedness of $\int_0^{t-\varepsilon} T(t-s-\varepsilon)[f(s) + g(s, x(s), x(s-r), u(s))]\, ds$ and the compactness of $T(t)$ $(t > 0)$, we can know that the set $\Pi_\varepsilon(t) = \{\mu^\varepsilon(t) : \mu \in \mathcal{F}(B_l)\}$ is relatively compact in H for each $\varepsilon \in (0, t)$. Moreover, for every $\mu \in \mathcal{F}(x)$, we have

$$\begin{aligned} \|\mu(t) - \mu^\varepsilon(t)\| &\le C_0 \int_{t-\varepsilon}^t [a(s) + cl + M(1 + 2l)]\, ds \\ &\le C_0[\|a\|_{L^2(J,R^+)}\sqrt{\varepsilon} + (cl + M(1 + 2l))\varepsilon]. \end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$ $(t > 0)$. Thus the set $\Pi(t)$ $(t > 0)$ is also relatively compact in H. Hence, from Steps 2 and 3, $\mathcal{F}$ is a compact multivalued map by the generalized Ascoli–Arzela theorem. Moreover, by Definition 2.1 (d), we know that $\mathcal{F}$ is completely continuous.

Step 5. $\mathcal{F}$ has a closed graph.

Let $x_n \to x_*$ in $C([-r, b], H)$, $\mu_n \in \mathcal{F}(x_n)$ and $\mu_n \to \mu_*$ in $C([-r, b], H)$. We will show that $\mu_* \in \mathcal{F}(x_*)$.

Indeed, there exists $f_n \in \mathcal{N}(x_n)$ such that, for $t \in J$,

$$\begin{aligned}(3.6)\quad \mu_n(t) = T(t)\varphi(0) &+ \int_0^t T(t-s)f_n(s)\,ds \\ &+ \int_0^t T(t-s)g(s, x_n(s), x_n(s-r), u(s))\,ds.\end{aligned}$$

From (HF) (c), it is not difficult to show that $\{f_n\}_{n\geq 1}$ is bounded in $L^2(J,H)$. Hence, we may assume, passing to a subsequence if necessary, that

$$(3.7)\qquad f_n \to f_* \quad \text{weakly in } L^2(J,H).$$

It follows from (3.6), (3.7) and the compactness of $T(t)$ that

$$\begin{aligned}(3.8)\quad \mu_n(t) \to T(t)\varphi(0) &+ \int_0^t T(t-s)f_*(s)\,ds \\ &+ \int_0^t T(t-s)g(s, x_*(s), x_*(s-r), u(s))\,ds.\end{aligned}$$

Note that $\mu_n \to \mu_*$ in $C([-r,b],H)$ and $f_n \in \mathcal{N}(x_n)$. From Lemma 3.2 and (3.8), we obtain $f_* \in \mathcal{N}(x_*)$. Hence, $\mu_* \in \mathcal{F}(x_*)$, which implies $\mathcal{F}$ has a closed graph. By Proposition 3.3.12 (2) of [30], $\mathcal{F}$ is u.s.c.

Step 6. A priori estimate.

By Steps 1 and 4, we deduce $\mathcal{F}$ is a compact and convex multivalued map. By Steps 4 and 5, we deduce $\mathcal{F}$ is condensing since it is completely continuous and upper semicontinuous. According to Theorem 2.4, it remains to prove that the set

$$\Omega = \{x \in C([-r,b],H) : \lambda x \in \mathcal{F}(x),\ \lambda > 1\}$$

is bounded.

Let $x \in \Omega$, then there exists an $f \in \mathcal{N}(x)$ such that $x(t) = \varphi(t)$, $t \in [-r,0]$, and

$$x(t) = \lambda^{-1}T(t)\varphi(0) + \lambda^{-1}\int_0^t T(t-s)[f(s) + g(s, x(s), x(s-r), u(s))]\,ds,$$

for $t \in [0,b]$. Then by (HA), (HF) and (HG), for $t \in [0,b]$, we have

$$\begin{aligned}(3.9)\quad \|x(t)\| &\leq \|T(t)\varphi(0)\| + \bigg\| \int_0^t T(t-s)[f(s) + g(s, x(s), x(s-r), u(s))]\,ds \bigg\| \\ &\leq C_0\|\varphi(0)\| + C_0\int_0^t [a(s) + c\|x(s)\| + M(1 + \|x(s)\| + \|x(s-r)\|)]\,ds \\ &\leq \rho + C_0(c+2M)\int_0^t \|x(s)\|\,ds,\end{aligned}$$

where $\rho = C_0\big[\|\varphi(0)\| + \|a\|_{L^2(J,R^+)}\sqrt{b} + M(b + r\|\varphi\|_{C([-r,0],H)})\big]$. By (3.9) and the Gronwall inequality, we obtain

$$\|x(t)\| \leq \rho e^{C_0(c+2M)t} \leq \rho e^{C_0(c+2M)b} := \overline{\mathcal{M}}, \quad t \in [0,b].$$

Hence, $\|x\|_{C([-r,b],H)} = \sup_{t\in[-r,b]} \|x(t)\| \leq \max\{\|\varphi\|_{C([-r,0],H)}, \overline{\mathcal{M}}\}$, which implies the set Ω is bounded. By Theorem 2.4, $\mathcal{F}$ has a fixed point, i.e. (3.1) has at least one mild solution. □

DEFINITION 3.6. A pair (x,u) is said to be a feasible pair of control system (1.1) if x is a mild solution of (3.1) on $[-r,b]$ and u is a measurable function such that $u(t) \in U(t,x(t))$ for almost every $t \in J$.

Set

$$\mathcal{V}[0,b] = \{u : [0,b] \to V : u(\,\cdot\,) \text{ is measurable}\},$$
$$\mathcal{H}[0,b] = \{(x,u) \in C([-r,b],H) \times \mathcal{V}[0,b] : (x,u) \text{ is feasible pair of (1.1)}\}.$$

LEMMA 3.7 ([10, Lemma 3.2]). *Assume that the condition*(HF) *holds, then for almost every $t \in J$, the multimap $\partial F(t,\cdot)\colon H \to P(H)$ has the Cesari property, i.e.*

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}\,\partial F(t, O_\varepsilon(x)) = \partial F(t,x), \quad x \in H,$$

where $O_\varepsilon(x) = \{y \in H : \|y-x\| \leq \varepsilon\}$ denotes the ball centered at x with the radius $\varepsilon > 0$.

Next, we establish the existence of feasible pairs.

THEOREM 3.8. *If the hypotheses* (HA)*,* (HF)*,* (HG) *and* (HU) *are satisfied, then the set of feasible pairs $\mathcal{H}[0,b]$ is nonempty.*

PROOF. Take any integer number $k \geq 1$, let $t_j = jb/k$, $0 \leq j \leq k-1$. We suppose

$$u^k(t) = \sum_{j=0}^{k-1} \nu_j \chi_{[t_j,t_{j+1})}(t), \quad t \in J,$$

where $\chi_{[t_j,t_{j+1})}$ is the characteristic function of interval $[t_j,t_{j+1})$ and the sequence $\{\nu_j\}$ is constructed as follows.

Firstly, we take $\nu_0 \in U(0,x^0)$, $x^0 = \varphi(0)$. By Theorem 3.5, there exists $x^k(\,\cdot\,)$ which satisfies $x^k(t) = \varphi(t)$ for $t \in [-r,0]$ and

$$x^k(t) = T(t)\varphi(0) + \int_0^t T(t-s)[f^k(s) + g(s,x^k(s),x^k(s-r),u^0(s))]\,ds,$$

for $t \in [0,t_1]$, where $f^k \in \mathcal{N}(x^k)$ and $u^0(s) = \nu_0$, $s \in [0,t_1]$. Then, take $\nu_1 \in U(t_1,x^k(t_1))$. We can repeat this procedure to obtain x^k on $[t_1,t_2]$, etc. By induction, we get the following equation:

$$\begin{cases} x^k(t) = T(t)\varphi(0) \\ \qquad + \displaystyle\int_0^t T(t-s)[f^k(s) + g(s,x^k(s),x^k(s-r),u^k(s))]\,ds \quad \text{for } t \in J, \\ u^k(t) \in U(t_j,x^k(t_j)) \quad \text{for } t \in [t_j,t_{j+1}),\ 0 \leq j \leq k-1, \end{cases} \tag{3.10}$$

where $f^k \in \mathcal{N}(x^k)$. By the proof of Theorem 3.5, there exists an $M_0 > 0$ such that

$$\|x^k\|_{C([-r,b],H)} \leq M_0.$$

Moreover, the conditions (HF), (HU) and (HG) imply that, there exist three constants M_1, M_2, $M_3 > 0$ such that

$$\|f^k\|_{L^2(J,H)} \leq M_1, \qquad \|u^k\|_{L^2(J,V)} \leq M_2,$$
$$\|g(\,\cdot\,, x^k(\,\cdot\,), x^k(\,\cdot\,-r), u^k(\,\cdot\,))\|_{L^2(J,H)} \leq M_3.$$

Since $L^2(J,H)$ and $L^2(J,V)$ are reflexive Banach spaces, there are subsequences of $\{f^k\}$, $\{u^k\}$ and $\{g(\,\cdot\,, x^k(\,\cdot\,), x^k(\,\cdot\,-r), u^k(\,\cdot\,))\}$, denoted again in the same way, such that

$$f^k \rightharpoonup \overline{f} \quad \text{in } L^2(J,H), \qquad u^k \rightharpoonup \overline{u} \quad \text{in } L^2(J,V), \tag{3.11}$$

$$g(\,\cdot\,, x^k(\,\cdot\,), x^k(\,\cdot\,-r), u^k(\,\cdot\,)) \rightharpoonup \overline{g}(\,\cdot\,) \quad \text{in } L^2(J,H), \tag{3.12}$$

for some $\overline{f}, \overline{g} \in L^2(J,H)$ and $\overline{u} \in L^2(J,V)$.

Next, by Lemma 3.3 and (3.10), we have

$$\int_0^t T(t-s)[f^k(s) + g(s, x^k(s), x^k(s-r), u^k(s))]\,ds \to \int_0^t T(t-s)[\overline{f}(s) + \overline{g}(s)]\,ds.$$

Hence, we denote

$$\overline{x}(t) = T(t)\varphi(0) + \int_0^t T(t-s)[\overline{f}(s) + \overline{g}(s)]\,ds.$$

Then, we have that $x^k(t) \to \overline{x}(t)$ uniformly on $t \in J$, which implies

$$x^k \to \overline{x} \quad \text{in } C([-r,b],H). \tag{3.13}$$

By (3.13), for any $\varepsilon > 0$, there exists $k_0 > 0$ such that

$$x^k(t) \in O_\varepsilon(\overline{x}(t)), \quad t \in J, \ k \geq k_0. \tag{3.14}$$

Furthermore, by the definition of $u^k(\,\cdot\,)$, for k large enough,

$$u^k(t) \in U(t_j, x^k(t_j)) \subset U(O_\varepsilon(t, \overline{x}(t))), \tag{3.15}$$

for all $t \in [t_j, t_{j+1})$, $0 \leq j \leq k-1$.

Secondly, by (3.11), (3.12) and the Mazur theorem ([17, Chapter 2, Corollary 2.8]), there exist $a_{il} \geq 0$, $b_{il} \geq 0$ and $c_{il} \geq 0$ with

$$\sum_{i\geq 0} a_{il} = \sum_{i\geq 0} b_{il} = \sum_{i\geq 0} c_{il} = 1$$

such that

$$\phi_l = \sum_{i\geq 1} a_{il} f^{i+l} \to \overline{f} \quad \text{in } L^2(J,H),$$
$$\psi_l = \sum_{i\geq 1} b_{il} u^{i+l} \to \overline{u} \quad \text{in } L^2(J,V),$$

$$\omega_l = \sum_{i\geq 1} c_{il} g(\,\cdot\,, x^{i+l}(\,\cdot\,), x^{i+l}(\,\cdot\, - r), u^{i+l}(\,\cdot\,)) \to \overline{g}(\,\cdot\,) \quad \text{in } L^2(J,H).$$

Then, there are subsequences of $\{\phi_l\}$, $\{\varphi_l\}$, $\{\omega_l\}$, without loss of generality still denoted as $\{\phi_l\}$, $\{\varphi_l\}$, $\{\omega_l\}$, such that

$$\begin{aligned} \phi_l(t) &\to \overline{f}(t) \quad \text{in } H, \text{ for a.e. } t \in J, \\ \varphi_l(t) &\to \overline{u}(t) \quad \text{in } V, \text{ for a.e. } t \in J, \\ \omega_l(t) &\to \overline{g}(t) \quad \text{in } H, \text{ for a.e. } t \in J. \end{aligned}$$

Hence, from (3.13) and (3.14), for l large enough,

$$\begin{aligned} &\phi_l(t) \in \operatorname{co} \partial F(t, O_\varepsilon(\overline{x}(t))), \quad \varphi_l(t) \in \operatorname{co} U(O_\varepsilon(t, \overline{x}(t))), \quad \text{for a.e. } t \in J, \\ &\omega_l(t) \in \operatorname{co} g(t, O_\varepsilon(\overline{x}(t)), O_\varepsilon(\overline{x}(t-r)), U(O_\varepsilon(t, \overline{x}(t))), \qquad \text{for a.e. } t \in J. \end{aligned}$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned} &\overline{f}(t) \in \overline{\operatorname{co}}\, \partial F(t, O_\varepsilon(\overline{x}(t))), \quad \overline{u}(t) \in \overline{\operatorname{co}}\, U(O_\varepsilon(t, \overline{x}(t))) \quad \text{for a.e. } t \in J, \\ &\overline{g}(t) \in \overline{\operatorname{co}}\, g(t, O_\varepsilon(\overline{x}(t)), O_\varepsilon(\overline{x}(t-r)), U(O_\varepsilon(t, \overline{x}(t))) \quad \text{for a.e. } t \in J. \end{aligned}$$

By (HU) and Lemma 3.7, we have

$$\overline{f}(t) \in \partial F(t, \overline{x}(t)), \quad \overline{u}(t) \in U(t, \overline{x}(t)) \quad \text{for a.e. } t \in J.$$

From (HG), we get

$$\overline{g}(t) \in \overline{\operatorname{co}}\, g(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t)) \quad \text{for a.e. } t \in J.$$

By (HU) and the Fillipov theorem [1], there exists an $\overline{u} \in \mathcal{V}[0,b]$ such that $\overline{u}(t) \in U(t, \overline{x}(t))$ for almost every $t \in J$, and

$$\overline{f}(t) \in \partial F(t, \overline{x}(t)), \quad \overline{g}(t) = g(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t)) \text{ for a.e. } t \in J.$$

Therefore, $(\overline{x}, \overline{u})$ is a feasible pair of control system (1.1). □

4. Existence of optimal feedback control pairs

In this section, we consider the Lagrange problem (P): find an admissible state feedback control pair (x^0, u^0) such that

$$\mathcal{J}(x^0, u^0) \leq \mathcal{J}(x,u) \quad \text{for all } (x,u) \in \mathcal{H}[0,b],$$

where

$$\mathcal{J}(x,u) = \int_0^b \mathcal{L}(t, x(t), x(t-r), u(t))\, dt. \tag{4.1}$$

To discuss the existence of optimal control pairs for problem (P), we need the following assumptions:

(HL) The functional $\mathcal{L}\colon J \times H \times H \times V \to \mathbb{R} \cup \{\infty\}$ satisfies:
  (a) $\mathcal{L}\colon J \times H \times H \times V \to \mathbb{R} \cup \{\infty\}$ is Borel measurable;

(b) for almost every $t \in J$, $\mathcal{L}(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $H \times H \times V$ and there is a constant $M_1 > 0$ such that

$$\mathcal{L}(t, x, y, u) \geq M_1, \quad \text{for all } (t, x, y, u) \in J \times H \times H \times V.$$

For $(t, x, y) \in J \times H \times H$, we define the set

$$\begin{aligned} \varepsilon(t, x, y) = \{&(z^0, z^1, z^2, z^3) \in \mathbb{R} \times H \times H \times V : \\ &z^0 \geq \mathcal{L}(t, x, y, z^3), z^1 \in \partial F(t, x),\ z^2 = g(t, x, y, z^3),\ z^3 \in U(t, x)\}. \end{aligned}$$

(HE) For almost all $t \in J$, the map $\varepsilon(t, \cdot, \cdot)\colon H \times H \to P(\mathbb{R} \times H \times H \times V)$ has the Cesari property, i.e.

$$\bigcap_{\delta > 0} \overline{\text{co}}\, \varepsilon(t, O_\delta(x, y)) = \varepsilon(t, x, y), \quad \text{for all } (x, y) \in H \times H,$$

where $O_\delta(x, y) = \{(x', y') \in H \times H | (\|x' - x\|^2 + \|y' - y\|^2)^{1/2} \leq \delta\}$.

Now, we can give the main result in this section.

Theorem 4.1. *Assume that all the hypotheses of Theorem* 3.8 *and the assumptions* (HL), (HE) *are satisfied. Then the Lagrange problem* (P) *admits at least one optimal feedback control pair.*

Proof. If $\inf\{\mathcal{J}(x, u) : (x, u) \in \mathcal{H}[0, b]\} = +\infty$, it is easy to see that there is nothing to prove. So we assume that $\inf\{\mathcal{J}(x, u) : (x, u) \in \mathcal{H}[0, b]\} = m < +\infty$. By condition (HL), we have $\mathcal{J}(x, u) \geq m > -\infty$. According to definition of infimum, there exists a minimizing sequence of feasible pair $\{(x^n, u^n)\} \subset \mathcal{H}[0, b]$ such that

$$\mathcal{J}(x^n, u^n) \to m \quad \text{as } n \to +\infty.$$

By the proof of Theorem 3.8, without loss of generality, we may assume that

$$\begin{gathered} f^n \rightharpoonup \overline{f} \quad \text{in } L^2(J, H), \qquad u^n \rightharpoonup \overline{u} \quad \text{in } L^2(J, V), \\ g(\cdot\,, x^n(\cdot), x^n(\cdot - r), u^n(\cdot)) \rightharpoonup \overline{g}(\cdot) \quad \text{in } L^2(J, H). \end{gathered}$$

By Lemma 3.3, we get

$$\begin{aligned} x^n(t) = T(t)\varphi(0) &+ \int_0^t T(t - s)[f^n(s) + g(s, x^n(s), x^n(s - r), u^n(s))]\, ds \\ &\to T(t)\varphi(0) + \int_0^t T(t - s)[\overline{f}(s) + \overline{g}(s)]\, ds := \overline{x}(t) \end{aligned}$$

uniformly in $t \in J$, i.e.

$$x^n \to \overline{x} \quad \text{in } C(J, H). \tag{4.2}$$

By using the Mazur theorem, there exist $a_{il} \geq 0$, $b_{il} \geq 0$ and $c_{il} \geq 0$ with

$$\sum_{i \geq 0} a_{il} = \sum_{i \geq 0} b_{il} = \sum_{i \geq 0} c_{il} = 1$$

such that

$$\phi_l = \sum_{i\geq 1} a_{il} f^{i+l} \to \overline{f} \quad \text{in } L^2(J,H), \qquad \psi_l = \sum_{i\geq 1} b_{il} u^{i+l} \to \overline{u} \quad \text{in } L^2(J,V),$$

$$\omega_l(\,\cdot\,) = \sum_{i\geq 1} c_{il} g(\,\cdot\,, x^{i+l}(\,\cdot\,), x^{i+l}(\,\cdot\, - r), u^{i+l}(\,\cdot\,)) \to \overline{g}(\,\cdot\,) \quad \text{in } L^2(J,H).$$

Let

$$\omega_l^0 = \sum_{k\geq 1} c_{kl} \mathcal{L}(\,\cdot\,, x^{k+l}(\,\cdot\,), x^{k+l}(\,\cdot\, - r), u^{k+l}(\,\cdot\,))$$

and $\mathcal{L}^0(t) = \varliminf_{l\to+\infty} \omega_l^0(t) \geq -M_1$ for almot every $t \in J$. For l large enough and any $\delta > 0$, we have

$$(\omega_l^0(t), \phi_l(t), \omega_l(t), \psi_l(t)) \in \varepsilon(t, O_\delta(\overline{x}(t), \overline{x}(t-r))).$$

Thus $(\mathcal{L}^0(t), \overline{f}(t), \overline{g}(t), \overline{u}(t)) \in \overline{\text{co}}\, \varepsilon(t, O_\delta(\overline{x}(t), \overline{x}(t-r)))$.

By assumption (HE), we get $(\mathcal{L}^0(t), \overline{f}(t), \overline{g}(t), \overline{u}(t)) \in \varepsilon(t, \overline{x}(t), \overline{x}(t-r))$ for almost every $t \in J$. This means that there exists an $u \in V$ such that

$$\begin{aligned}
\mathcal{L}^0(t) &\geq \mathcal{L}(t, \overline{x}(t), \overline{x}(t-r), u) && \text{for } t \in J,\\
\overline{f}(t) &\in \partial F(t, \overline{x}(t)) && \text{for a.e. } t \in J,\\
\overline{g}(t) &= g(t, \overline{x}(t), \overline{x}(t-r), u) && \text{for } t \in J,\\
u &\in U(t, \overline{x}(t)).
\end{aligned}$$

By the Filippov theorem [1] again, there exists a measurable selection $\overline{u}(\,\cdot\,)$ of $U(\,\cdot\,, \overline{x}(\,\cdot\,))$ such that

$$\begin{aligned}
\mathcal{L}^0(t) &\geq \mathcal{L}(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t))) && \text{for a.e. } t \in J,\\
\overline{f}(t) &\in \partial F(t, \overline{x}(t)) && \text{for a.e. } t \in J,\\
\overline{g}(t) &= g(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t)) && \text{for a.e. } t \in J.
\end{aligned}$$

Furthermore, we have

$$\overline{x}(t) = T(t)\varphi(0) + \int_0^t T(t-s)[\overline{f}(s) + g(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t))]\, ds.$$

Therefore, $(\overline{x}, \overline{u}) \in \mathcal{H}[0,b]$. Finally, by Fatou's lemma, we obtain

$$\begin{aligned}
\int_0^b \mathcal{L}^0(t)\, dt &= \int_0^b \varliminf_{l\to+\infty} \omega_l^0(t)\, dt \leq \varliminf_{l\to+\infty} \int_0^b \omega_l^0(t)\, dt\\
&= \varliminf_{l\to+\infty} \int_0^b \sum_{k\geq 1} c_{kl} \mathcal{L}(t, x^{k+l}(t), x^{k+l}(t-r), u^{k+l}(t))\, dt\\
&= \varliminf_{l\to+\infty} \sum_{k\geq 1} c_{kl} \int_0^b \mathcal{L}(t, x^{k+l}(t), x^{k+l}(t-r), u^{k+l}(t))\, dt = m,
\end{aligned}$$

then

$$m \leq \mathcal{J}(\overline{x}, \overline{u}) = \int_0^b \mathcal{L}(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t))\, dt \leq m,$$

i.e.

$$\mathcal{J}(\overline{x}, \overline{u}) = \int_0^b \mathcal{L}(t, \overline{x}(t), \overline{x}(t-r), \overline{u}(t))\, dt = \inf_{(x,u)\in\mathcal{H}[0,b]} \mathcal{J}(x,u) = m.$$

Thus $(\overline{x}, \overline{u})$ is an optimal feedback control pair. □

5. Time optimal control results

In this section, we consider the results of time optimal control for the evolution control system (1.1).

Let $x^0, x^1 \in H$ be two different elements. For some $t > 0$, we suppose that there exists an admissible control u satisfying $x(t; u) = x^1$ and $x(0) = x^0 = \varphi(0)$. Let us define the transition time which is the first time t^u such that $x(t^u; u) = x^1$.

The optimal time is defined by low limit t^0 of t^u such that $x(t^u; u) = x^1$ for admissible control u. We say u^0 is the time optimal control if a feedback control $u^0(t) \in U(t, x(t; u^0))$ such that $x(t^0; u^0) = x^1$. It is sufficient to prove that the existence of the admissible control satisfies $x(t^0; u^0) = x^1$ with respect to $\{x^0, x^1\}$.

Now, we find a control which transfers the trajectory of the constraint system (1.1) from the initial data to the target in the shortest time. The main idea of the proof comes from [2], [11], [15].

THEOREM 5.1. *Assume that all the hypotheses of Theorem* 3.8 *hold. Then, there exists a time optimal control with respect to* $\{x^0, x^1\}$.

PROOF. Firstly, let $t^0 = \inf\{t : x(t; u) = x^1$, where (x, u) is a feasible pair of the system (1.1)$\}$. Then, there exists a monotone decreasing sequence $\{t^n\}$ such that $t^n \to t^0$ as $n \to \infty$. Assume that $u^n(t) \in U(t, x_n(t))$ is the corresponding feedback control such that

$$x_n(t; u^n) = \begin{cases} \varphi(t) & \text{for } t \in [-r, 0], \\ T(t)\varphi(0) + \displaystyle\int_0^t T(t-s) g(s, x_n(s), x_n(s-r), u^n(s))\, ds \\ \qquad\qquad + \displaystyle\int_0^t T(t-s) f^n(s)\, ds, & \text{for } f^n \in \mathcal{N}(x_n), t \in J, \end{cases} \tag{5.1}$$

satisfying $x_n(t^n; u^n) = x^1$, for $n = 1, 2, \ldots$

Notice that $x_n(\,\cdot\,; u^n) \in C([-r, b]; H)$. Since $u^n(t) \in U(t, x_n(t; u^n))$, $\{u^n\}$ is bounded in $L^2(J, V)$, by the reflexivity of $L^2(J, V)$, there exists a subsequence of $\{u^n\}$, relabeled as $\{u^n\}$, such that $u^n \rightharpoonup u^0$ in $L^2(J, V)$.

For every $t^n \in [0,b]$, we know that $x_n(t^n;u^n)$ can be rewritten as

$$
\begin{aligned}
x_n(t^n;u^n) = {} & T(t^n)\varphi(0) \\
& + \int_0^{t^0} T(t^n-s)[g(s,x_n(s),x_n(s-r),u^n(s)) + f^n(s)]\,ds \\
& + \int_{t^0}^{t^n} T(t^n-s)[g(s,x_n(s),x_n(s-r),u^n(s)) + f^n(s)]\,ds.
\end{aligned}
\tag{5.2}
$$

From the proof of Theorem 3.5, there exists a constant $C' > 0$ such that $\|x_n\|_{C([-r,b],H)} < C'$. Thus it follows from hypotheses (HG), (HF) and the Hölder inequality that

$$
\begin{aligned}
& \left\| \int_{t^0}^{t^n} T(t^n-s)[g(s,x_n(s),x_n(s-r),u^n(s)) + f^n(s)]\,ds \right\| \\
& \quad \le C_0 \int_{t^0}^{t^n} \|g(s,x_n(s),x_n(s-r),u^n(s)) + f^n(s)\|\,ds \\
& \quad \le C_0\big[M + (2M+c)C'\big]|t^n - t^0| + C_0\|a\|_{L^2(J,R^+)}\sqrt{|t^n-t^0|}.
\end{aligned}
$$

So, we conclude that the first and the third term of the right-hand side of (5.2) converge to $T(t^0)\varphi(0)$ and 0, respectively. So we focus on the second term. By (HG) and weak compactness of u^n and Lemma 3.3, we obtain

$$
\begin{aligned}
& \int_0^{t^0} T(t^n-s)[g(s,x_n(s),x_n(s-r),u^n(s)) + f^n(s)]\,ds \\
& \qquad \to \int_0^{t^0} T(t^0-s)[g(s,x_0(s),x_0(s-r),u^0(s)) + f^0(s)]\,ds,
\end{aligned}
$$

where $f^0 \in \mathcal{N}(x_0(\,\cdot\,))$. Hence, it follows

$$
x^1 = T(t^0)\varphi(0) + \int_0^{t^0} T(t^0-s)[g(s,x_0(s),x_0(s-r),u^0(s)) + f^0(s)]\,ds = x(t^0;u^0),
$$

where $f^0 \in \mathcal{N}(x_0(\,\cdot\,))$, that is, u^0 is the time optimal control, and $x(\,\cdot\,;u^0)$ is the trajectory corresponding to the control u^0. □

6. An example

As an application of our main results, we consider a control system governed by the following parabolic boundary initial value problem:

$$
\begin{aligned}
& \int_0^b \left(1 + \int_0^\pi |x(t,y)|^2\,dy\right)^{1/2} dt \\
& \qquad + \int_0^b \int_0^\pi |u(t,y)|^2\,dy\,dt + \int_0^b \int_0^\pi |x(t-r,y)u(t,y)|\,dy\,dt \to \inf,
\end{aligned}
\tag{P$_1$}
$$

subject to the following heat equation:

$$
(6.1)\quad \begin{cases}
\dfrac{\partial}{\partial t}\, x(t,y) = \dfrac{\partial^2}{\partial y^2} x(t,y) + \displaystyle\int_0^\pi k(y,\eta)x(t-r,\eta)\,d\eta \\
\qquad + h_0(t,x(t,y)) + \mathcal{B}(t)u(t,y)\bigg(\displaystyle\int_0^b\int_0^\pi |u(s,\eta)|^2 d\eta\, ds\bigg)^{-1/2} + f(t,y) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } t\in(0,b),\ y\in(0,\pi), \\
f(t,y)\in\partial F(t,y,x(t,y)) \qquad\qquad\qquad \text{for a.e. } t\in(0,b), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y\in(0,\pi)=\Omega, \\
u(t,y)\in[h_1(t,x(t,y)),h_2(t,x(t,y))] \quad \text{for a.e. } t\in(0,b),\ y\in\Omega, \\
x(t,0)=x(t,\pi)=0 \qquad\qquad\qquad\qquad \text{for } t\in(0,b), \\
x(t,y)=\varphi(t,y) \qquad\qquad\qquad\qquad\quad \text{for } t\in[-r,0],\ y\in\Omega,
\end{cases}
$$

where $x(t,y)$ represents the temperature at the point $y\in(0,\pi)$ and time $t\in(0,b)$. $k\colon\Omega\times\Omega\to\mathbb{R}$ is a continuous function. For $i\in\{0,1,2\}$, h_i is continuous and the partial derivative $\partial h_i/\partial z\colon[0,b]\times\mathbb{R}\to\mathbb{R}$ is continuous, and there are positive constants N_i such that,

$$
\bigg|\frac{\partial h_i(t,z)}{\partial z}\bigg| \le N_i, \quad (t,z)\in[0,b]\times\mathbb{R}.
$$

The linear operator $\mathcal{B}\colon[0,b]\to\mathcal{L}(L^2(\Omega,\mathbb{R}))$ for $t\in[0,b]$, and $\mathcal{B}(t)\colon L^2(\Omega,\mathbb{R})\to L^2(\Omega,\mathbb{R})$ satisfies $\|\mathcal{B}(t)\|\le l_B$, $l_B>0$. Moreover, φ is continuous on $[-r,0]\times(0,\pi)$ and $\partial F(t,y,\theta)$ denotes the Clarke generalized gradient with respect to the last variable of a nonsmooth and nonconvex function $F\colon(0,b)\times(0,\pi)\times\mathbb{R}\to\mathbb{R}$, which is a locally Lipschitz in the third variable. The simple example of a function F which satisfies hypotheses (HF) (b) is $F(\theta)=\min\{f_1(\theta),f_2(\theta)\}$, where $f_i\colon\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are convex quadratic functions (cf. [30]).

Now, we set $H=V=L^2(0,\pi)$ and $Ax=x''$ with domain

$$
D(A)=\{x\in H : x, x' \text{ are absolutely continuous, } x''\in H,\ x(0)=x(\pi)=0\}.
$$

Then the operator A can be written as

$$
Ax=\sum_{n=1}^\infty(-n^2)\langle x,e_n\rangle e_n, \quad x\in D(A),
$$

where $e_n(y)=\sqrt{2/\pi}\sin(ny)$, $n=1,2,\ldots$, is an orthonormal base for H. It is well known that A generates a compact semigroup $T(t)(t>0)$ on H given by

$$
T(t)x=\sum_{n=1}^\infty e^{-n^2t}\langle x,e_n\rangle e_n, \quad x\in H.
$$

Now, define $x(t)(y)=x(t,y)$, $u(t)(y)=u(t,y)$, $U\colon[0,b]\times V\to P(V)$ and $g\colon[0,b]\times H\times H\times V\to H$ as follows:

$$
U(t,x(t))(y)=[h_1(t,x(t,y)),h_2(t,x(t,y))],
$$

$$g\big(t, x(t), x(t-r), u(t)\big)(y) = \int_0^\pi k(y,\eta)x(t-r,\eta)\,d\eta + h_0(t, x(t,y)) + \mathcal{B}(t)u(t,y)\bigg(\int_0^b \int_0^\pi |u(s,\eta)|^2\,d\eta\,ds\bigg)^{-1/2}.$$

It is not difficult to verify that problem (6.1) can be rewritten to the abstract of (1.1). For the particular case

$$\phi(t) = 2\pi(|h_1(t,0)| + |h_2(t,0)|)^2, \quad L_u = 2(N_1 + N_2),$$

$M = \sup\limits_{t\in(0,b)} h_0^2(t,0) + \max\{K, N_0^2, l_B^2\}$ with $K = \pi \sup\limits_{(\xi,\eta)\in\Omega\times\Omega} k^2(\xi,\eta)$, and

$$\begin{aligned}
\|U(t,x)\|^2_{L^2(0,\pi)} \le &\int_0^\pi [|h_1(t,x(y))| + |h_2(t,x(y))|]^2\,dy \\
\le & 2\int_0^\pi [|h_1(t,0)| + |h_2(t,0)|]^2\,dy \\
&+ 2\int_0^\pi (N_1+N_2)^2|x(y)|^2\,dy = \phi(t) + L_u\|x\|^2_{L^2(0,\pi)}, \\
\|g(t,x,z,u)\|^2_{L^2(0,\pi)} \le & 3\int_0^\pi \int_0^\pi k^2(y,\eta)z^2(\eta)\,d\eta\,dy + 3\int_0^\pi h_0^2(t,x(y))\,dy \\
&+ 3\int_0^\pi \|\mathcal{B}(t)\|^2u^2(y)\,dy\bigg(\int_0^\pi |u(\eta)|^2\,d\eta\bigg)^{-1} \\
\le & 3M(\|z\|^2_{L^2(0,\pi)} + \|x\|^2_{L^2(0,\pi)} + 1).
\end{aligned}$$

Furthermore, by the assumptions on h_1 and h_2, we can check the multimap U is the upper semicontinuous, convex, and closed valued. Then by Proposition 4.2 of [17], U has the Cesari property and satisfies hypotheses (HU). Also the function $g\colon [0,b]\times H\times H\times V \to H$ is a continuous function. By the fact that the composition of two upper semicontinuous multimaps is still a upper semicontinuous multimap. By a similar way, we can see that the hypotheses (HG) holds too. So, one can check that the conditions of Theorems 3.5 and 3.8 are satisfied. Then the problem (6.1) has a mild solution $x \in C([-r,b],H)$ and the set of feasible pair of (6.1) is nonempty.

Besides, define a functional $\hbar^0\colon L^2(0,\pi)\times L^2(0,\pi)\to\mathbb{R}$ as

$$\hbar^0(x,u) = \int_0^\pi |x(y)u(y)|\,dy,$$

thus the Lagrange problem (P_1) can be written as the problem (1.1) with the cost function

$$\mathcal{J}(x,u) = \int_0^b \left(\sqrt{1 + \|x(t)\|^2_{L^2(0,\pi)}} + \|u(t)\|^2_{L^2(0,\pi)} + \hbar^0(x(t-r),u(t))\right)dt.$$

Next, let $\mathcal{L}\colon [0,b]\times H\times H\times V \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ be a function defined by

$$\mathcal{L}(t,x_1,x_2,u) = \sqrt{1+\|x_1\|^2} + \|u\|^2 + \hbar^0(x_2,u)) \ge 1.$$

It is easy to see that $\mathcal{L}$ is a continuous function. Similarly to the above discussion, the hypotheses (HL) and (HE) are hold too. Summarizing the above, the assumptions of Theorem 4.1 are satisfied. Therefore, the problem (P_1) has at least one optimal control pair $(x,u) \in H \times V$.

7. Conclusions

This paper deals with the optimal feedback control of a class of control systems described by semilinear hemivariational inequalities with a fixed delay in state. The existence of a mild solution and feasible pairs for delay evolution hemivariational inequalities are shown and proved mainly by using fixed point theorem of multivalued maps, properties of the Clarke subdifferential, the Filippov theorem and the Mazur theorem etc. Under some natural assumptions, it is shown that the Lagrange problem admits at least one optimal pair of state control. The existence of the time optimal control is also obtained.

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