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SINGULAR LEVELS AND TOPOLOGICAL INVARIANTS OF MORSE–BOTT FOLIATIONS ON NON-ORIENTABLE SURFACES

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Abstract. We investigate the classification of closed curves and eight curves of saddle points defined on non-orientable closed surfaces, up to an ambient homeomorphism. The classification obtained here is applied to Morse–Bott foliations on non-orientable closed surfaces in order to define a complete topological invariant.

1. Introduction

Global invariants of conjugated flows, foliations or functions by means of a homeomorphism and defined on surfaces have been investigated since a long time ago and with a wide variety of techniques and tools. For instance see [4], [17], [19], [20] and the references therein. In this paper studies how the topological aspects of the singularities can be used to define complete invariants. To be more precise we need to introduce some notations and definitions.

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Let $\Pi(q,m)$ be a closed, connected, compact surface of genus q with m holes, $\Sigma(g,m)$ an orientable surface of genus g ($g \geq 0$). $\Upsilon(h,m)$, will be a non-orientable surface with genus $h \geq 1$. The boundary of each one is a collection of m disjoint Jordan curves $(J_1,\ldots,J_m)$. In general, we can assume that $\Sigma(g,m)$ is a subset of $\mathbb{R}^3$ and $\Upsilon(h,m)$ is a subset of $\mathbb{R}^4$.

Definition 1.1. Let Λ_i be curves contained in the surface Π_i, we will say that Λ_1 and Λ_2 are ambiently homeomorphic, in short *ah* or $\Lambda_1 \mathrel{\mathring{=}} \Lambda_2$ if there exists a homeomorphism $h\colon \Pi_1 \to \Pi_2$ such that $h(\Lambda_1) = \Lambda_2$.

Two homeomorphic objects will be related by the symbol $\cong$ and an ambient homeomorphism equivalence by $\mathring{=}$.

Notice that the answer to an *ah*-equivalence problem depends on the answer to an extension problem. For this reason, this paper is organized as a succession of extension problems allowing *ah*-classifications, from simple curves to foliations. In Section 3 we begin with simple closed curves and in Section 4 we consider figure eight curves.

Definition 1.2. A separatrix eight $\mathfrak{B}$, figure eight curve or in short an eight, is the image of an immersion of S^1 into Σ, $\psi\colon S^1 \to \Sigma$, homeomorphic to two circumferences s_1 and s_2 glued by a point p, $\mathfrak{B} = s_1 \underset{p}{\cup} s_2$. A component of $\mathfrak{B}$ is one of the circumferences s_i.

These eight curves usually appear as level sets (points with the same image) of a real function $f\colon W^2 \to \mathbb{R}$ defined on a surface and also as invariant manifolds of saddle equilibrium points of a flow. The collection of all level sets of the function defines a singular foliation on the surface. Morse functions are generic in the set of twice differentiable functions, Morse–Bott functions (Definition 2.2) are a wider class of functions. The singularities can be manifolds but the restriction of the function to a normal plane to the singular manifold is a Morse function. In [13] we classified up to an ambient homeomorphism simple closed curves and eights on orientable closed surfaces. The classification in [13] can be applied to Morse–Bott foliations and Morse–Bott integrable flows allowing us to define complete invariants.

In this paper we extend some results of [13] to non-orientable surfaces. In some sense the non-orientable case contains the orientable case if we do not care of one-sided curves. One can find some results on the classification of simple curves in the Klein bottle in [24] and the topological classification of Morse functions defined on non-orientable surfaces obtained from the Reeb graph associated to the function in [9]. In [6] the classification of simple curves is an important tool used to simplify proofs.

Non-orientable surfaces appear in several areas as manifolds of internal states of a medium (see [15]), as level sets of functions on manifolds and in blow-ups of singularities. They find applications to nematic crystals (see [10]) and barcodes. Level sets on a surface can be considered as cuts of an image or a data set and allow to define some graphs or structures.

After some kind of identification (for instance described in [7]) non-orientable surfaces are obtained.

In Sections 3 and 4 we classify the basic leaves, closed curves and saddles with their separatrices on non-orientable surfaces. These classifications differ from the homotopic or homological classifications, and can be applied to various areas. Theorem 3.6 states that the number of non-equivalent embeddings of S^1 on $\Upsilon(h,0)$ is $h+3$ if $h \geq 3$ and Theorem 4.10 enumerates all eights.

In Section 4 we investigate the existence of invariants for Morse–Bott foliations on non-orientable surfaces using their basic leaves and closed curves. We recall the definition of Reeb graphs.

Definition 1.3. Let f be a continuous function on a compact manifold W to $\mathbb{R}$. A *fiber* is a connected component of a level set of f. If the fiber contains a singular point then it is called a *singular fiber.* The Reeb graph, $R_G(f)$, is the graph that as a set is the space obtained from W by contracting each connected component of the level sets to a point and whose vertices are the singular fibers of the function. See [21].

Our invariant will be a new graph (Definition 5.5) derived from the Reeb graph and two indications on saddle singularities: the way to attach one small neighbourhood to the graph and the *ah* characterization. See subsection 5.2.

2. Definitions and background concepts

Consider a compact manifold W with the Riemannian metric induced by an embedding of W in $\mathbb{R}^m$ and let f be a function from W to $\mathbb{R}$, $f \in C^k$, $k \geq 2$.

Definition 2.1. A point $p \in W$ is called a *singular point* if the rank of $\mathrm{df}(p)$ is less than one. Otherwise p is a *regular point.* A value $b \in \mathbb{R}$ is called a *singular value* if $f^{-1}(b)$ contains a singular point. Otherwise $b \in \mathbb{R}$ is called a *regular value.*

The *singular set* of the function f denoted here by $\mathrm{Sing}(f)$, is the collection of all singular points.

Definition 2.2. A smooth submanifold $S \subset \mathrm{Sing}(f)$ is said to be a *nondegenerate singular submanifold* if the following hold:

- S is compact and connected,
- for all $s \in S$, we have $T_sS = \ker \mathrm{Hess}_s f$.

The function f is called a *Morse–Bott function* ($\mathcal{MB}$ function from now on) if the set $\mathrm{Sing}(f)$ consists of nondegenerate singular submanifolds. See [3], [5], [18].

Let $p \in S \subset \mathrm{Sing}(f)$, then the Morse–Bott Lemma says that there is a local chart of W around p and a local splitting of the normal bundle of S, $N_p(S) = N_p^+(S) \oplus N_p^-(S)$ so that if $p = (s, x, y)$, $s \in S$, $x \in N_p^+(S)$, $y \in N_p^-(S)$:

$$\begin{aligned} T_p(W) &= T_p(S) \oplus N_p^+(S) \oplus N_p^-(S), \\ f(p) &= f(s) + |x|^2 - |y|^2. \end{aligned}$$

The dimension of $N_p^-(S)$ is the index of S.

A foliation defined by the level sets of an $\mathcal{MB}$ function g will be denoted by $\mathcal{F}(g)$. In this paper we will assume that $\mathcal{F}(g)$ is *simple*, in the sense that the singular points of a singular fiber form a unique connected subset.

From now on we will assume that W is a surface.

Definition 2.3. We will say that two foliations on a surface Π are *topologically equivalent* if there exists a homeomorphism on Π that sends the leaves of one foliation to the leaves of the other one.

Definition 2.4. We say that l is a *regular leaf* of a foliation $\mathcal{F}$ if there exists a neighbourhood of l such that $\mathcal{F}$ restricted to this neighbourhood is equivalent to the foliation in $S^1 \times I$ given by (α, x) whose leaves are defined by $x = k$, with $k \in I \subset \mathbb{R}$. A *singular leaf* of a foliation is a leaf that is not a regular leaf.

Remark 2.5. The singularities of f and the singularities of $\mathcal{F}(f)$ are not the same. An oriented critical circle (o) of the function is not a singularity of $\mathcal{F}(f)$.

Let f be an $\mathcal{MB}$ function, the singular fibers of $\mathcal{F}(f)$ are:

- center points (c_i),
- one-sided circles (o_i^-),
- eights $(\mathfrak{B})$.

For the case of Morse functions, see the book [22]. On the other hand, $R_G(f)$ has vertices of degree one, two or three. Vertices of degree one can be associated to center points or critical circumferences that are one-sided curves. These o_i^- are maximum or minimum of the function f. Other vertices of degree two and vertices of degree three are associated to saddle points of the function.

Definition 2.6. An $\mathcal{MB}$ *foliation* is a foliation that has a finite number of singularities and which singularities are topologically equivalent to singularities of $\mathcal{F}(f)$ where f is an $\mathcal{MB}$ function. We will denote such foliation by $\mathcal{F}_{\mathcal{MB}}$.

For instance, a singularity locally defined by $f(x_1, x_2) = x_1^4 + x_2^8$ will be an admissible singularity of an $\mathcal{F}_{\mathcal{MB}}$ foliation.

3. Simple closed curves

In this section we classify up to an ambient homeomorphism simple closed curves on $\Upsilon(q,m)$. The m boundary curves of a surface will be denoted by $(J_1,\dots,J_m)$. First we recall some extension theorems.

Definition 3.1 ([25]). A *concordant orientation* of $(J_1,\dots,J_m)$ consists of an orientation on each $J_1,\dots,J_m$, such that the orientation induced on a surface $\Pi(q,m)$ by the orientation on J_i is independent of $i=1,\dots,m$.

Theorem 3.2 ([14], [25]). *A homeomorphism $h\colon (J_1^1,\dots,J_m^1)\to(J_1^2,\dots,J_m^2)$ can be extended to a homeomorphism between $\Sigma^1(g,m)$ and $\Sigma^2(g,m)$ if and only if h carries a concordant orientation of $(J_1^1,\dots,J_m^1)$ into a concordant orientation of $(J_1^2,\dots,J_m^2)$.*

Theorem 3.3 ([14], [25]). *A homeomorphism $h\colon (J_1^1,\dots,J_m^1)\to(J_1^2,\dots,J_m^2)$ can always be extended to a homeomorphism between $\Upsilon^1(g,m)$ and $\Upsilon^2(g,m)$.*

Definition 3.4. An *embedded circle* on Π is image of an embedding $\phi\colon S^1\to\Pi$. An *oriented embedded circle*, or shortly, an oriented circle, is an embedded circle with one of the two possible orientations.

Definition 3.5. Two embeddings ϕ_i, $i=1,2$, of S^1 into Π are *topologically equivalent* if there is a homeomorphism $h\colon\Pi\to\Pi$ such that

$$h(\phi_1(S^1))=\phi_2(S^1).$$

Two embedded circles are *ah-equivalent* if they can be defined by equivalent embeddings ϕ_i. Two oriented circles are *equivalent* if the homeomorphism that conjugates the embedded circles preserves their orientations.

Given an embedding $\phi\colon S^1\to\Pi$ such that $\phi(\alpha)=\beta$ and given an orientation of S^1 then ϕ induces an orientation on its image; the embedding $\phi^-\colon S^1\to\Pi$ defined by $\phi^-(\alpha)=-\beta$ induces the opposite orientation.

Next, we introduce some notation about surgery on a surface.

If $\phi(S^1)=J$ is the image of an embedding then $\Pi(q,0)\setminus J$ could have one or two connected components. Denote by K the compact surface with holes that is the closure of $\Pi(q,0)\setminus J$ and by K_i, $i=1,2$, the connected components of $\overline{\Pi(q,0)\setminus J}$ if K is not connected. Moreover, we denote by $\mathcal{P}(K)$ (resp. $\mathcal{P}(K_i)$) the patched surface obtained by attaching discs to the holes of K (resp. K_i).

$\Pi(q,0)$ is the connected sum of $\mathcal{P}(K_1)$ and $\mathcal{P}(K_2)$ and K is equipped with a homeomorphism k between two of its boundary components. The above can be generalized in an obvious way to the case of $\Pi(q,0)\setminus\bigcup_i J_i$.

We say that J is *essential* if it is not homotopic to zero. J is said to be *two-sided* if a regular neighbourhood of J is a cylinder and *one-sided* if a regular neighbourhood of J is homeomorphic to a Möbius band.

By l we denote a class of ah-simple closed curves J. We will say that l is the *topological type* of J.

Now, we introduce some more notation. The coherence of this notation is a consequence of Theorem 3.6. To indicate that J is an essential curve we will add the capital letter K as subscript of l. We add a sign to K indicating that K is orientable ($+$) or not ($-$). Otherwise, if J is homologous to zero we write an integer subscript m of l related with the genus of K_i with a double sign. The sign $\pm$ indicates that one of the connected components of $\Upsilon(h,0)\setminus J$ is orientable and the other one is non-orientable, and the sign $=$ means that both of the connected components of $\Upsilon(h,0)\setminus J$ are non-orientable. The double plus sign will be omitted. The superscript of l indicates if J is a one-sided or two-sided curve, l^+ points out that J is a two-sided curve and l^- that it is a one-sided curve. Cutting along a one-sided curve cannot separate a surface. Therefore, we have the following notations:

- $l_{\pm m}$: if $\Upsilon(h,0)\setminus J$ is the disjoint union of two surfaces, K_1 orientable of genus m and K_2 non-orientable of genus $h-2m$,
- $l_{=m}$: if $\Upsilon(h,0)\setminus J$ is the disjoint union of two non-orientable surfaces K_1, K_2 of genus h_1 and h_2 where $m=\min(h_1,h_2)$ and $h=h_1+h_2$,
- l_m: if $\Upsilon(h,0)\setminus J$ is the disjoint union of two orientable surfaces K_1 and K_2 of genus g_1 and g_2, respectively, with $m=\min(g_1,g_2)$,
- l_K^+: if $\Upsilon(h,0)\setminus J$ is an orientable surface and J is two-sided,
- l_K^-: if $\Upsilon(h,0)\setminus J$ is an orientable surface and J is one-sided,
- l_{-K}^-: if $\Upsilon(h,0)\setminus J$ is a non-orientable surface and J is one-sided,
- l_{-K}^+: if $\Upsilon(h,0)\setminus J$ is a non-orientable surface and J is two-sided.

If $l_K^-\subset\Upsilon(h,0)$ (respectively, $l_K^+\subset\Upsilon(h,0)$) then h must be odd (respectively, h must be even).

The classification of embedded circles on orientable surfaces is given in [12]. The classification of the embedded circles in the projective plane $\Upsilon(1,0)$ and in the Klein bottle $\Upsilon(2,0)$ can be found in [16], [24].

In the next theorem we classify all embeddings of S^1 on $\Upsilon(h,0)$. As usual, $\lceil a\rceil$ and $\lfloor a\rfloor$ will represent the ceiling and floor values of a.

Theorem 3.6. *Let $\Upsilon(h,0)$ be a non-orientable surface of genus $h\geq 1$. The number of non-equivalent embeddings of S^1 on $\Upsilon(h,0)$ is $h+3$ if $h\geq 3$ with representative classes:*

- $l_{\pm 0}, l_{\pm 1},\dots,l_{\pm m},\dots,l_{\pm\lfloor (h-1)/2\rfloor}$;
- $l_{=1}, l_{=2},\dots,l_{=m},\dots,l_{=\lfloor h/2\rfloor}$;
- l_K^- *with h odd*;
- l_{-K}^-;
- l_K^+ *with h even*;
- l_{-K}^+.

For $\Upsilon(1,0)$ *we have two classes of embeddings represented by* $l_{\pm 0}$ *and* l_K^- *(see Figure* 1*) and for* $\Upsilon(2,0)$ *there are four classes of embeddings represented by* $l_{\pm 0}$, $l_{=1}$, l_K^+, l_{-K}^- *(see Figure* 2*).*

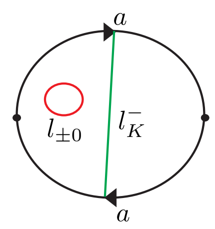


FIGURE 1. Classes of embeddings of S^1 into $\Upsilon(1,0)$.

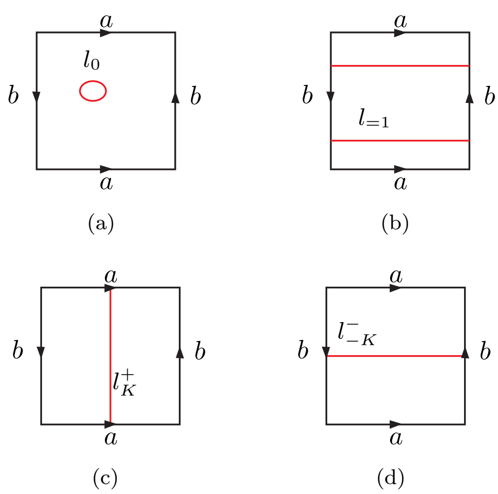


FIGURE 2. Classes of embeddings of S^1 into $\Upsilon(2,0)$.

PROOF. The proof proceeds in two steps. First we divide the set of all embeddings in classes such that two embeddings J_1 and J_2 belong to the same class if and only if $\Upsilon(h,0)\setminus J_1 \cong \Upsilon(h,0)\setminus J_2$. Then in Step 2 we will show that each class has only one element.

Step 1. (a) If $\overline{\Upsilon(h,0)\setminus J}$ is connected, then J is a not a null homologous curve. Moreover, $\chi(\Upsilon(h,0)) = \chi(\overline{\Upsilon(h,0)\setminus J})$, so

$$2 - h = \chi(\overline{\Upsilon(h,0)\setminus J}). \tag{3.1}$$

Let J be one-side and $h>1$. If $\Upsilon(h,0)\setminus J$ is non-orientable then it is homeomorphic to $\Upsilon(h-1,1)$ by (3.1). Otherwise, it is homeomorphic to $\Sigma((h-1)/2,1)$. In this second case the genus h of $\Upsilon(h,0)$ must be odd. Now, if $h=1$ we have only

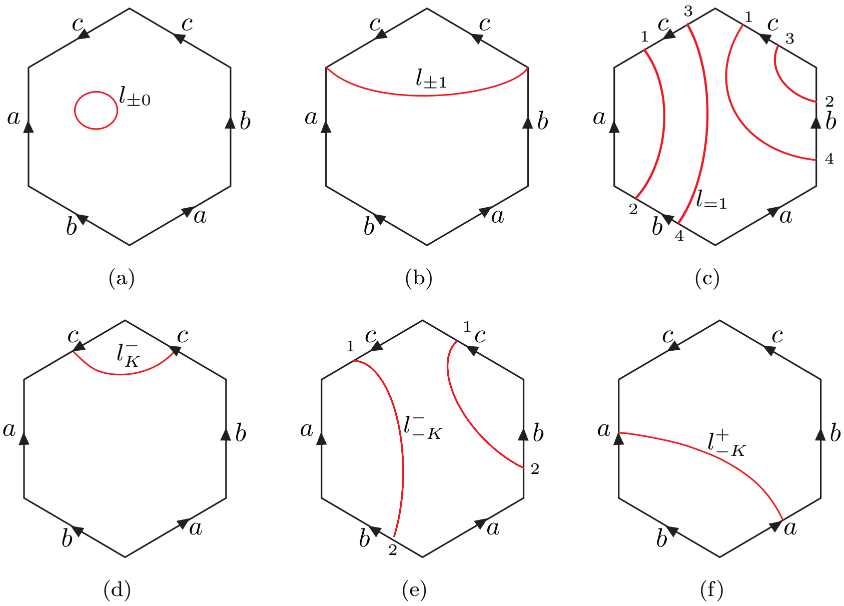


Figure 3. Non-equivalent embeddings of S^1 on $\Upsilon(3,0)$.

one curve and this is the unique not homologous to zero curve in the projective plane.

Similarly, if J is a two-sided curve $\Upsilon(h,0)\setminus J$ has two holes and it can be orientable or not. In the first case, $\Upsilon(h,0)\setminus J\cong\Sigma(g,2)$ and by (3.1), $g=h/2-1$ and h must be even. In the second case, $\Upsilon(h,0)\setminus J\cong\Upsilon(h-2,2)$, $h\geq 3$. In fact, $\Upsilon(h,0)$ can be reconstructed gluing a handle. If $h=2$, $\Upsilon(2,0)\setminus J$ will be a cylinder.

(b) If $\overline{\Upsilon(h,0)\setminus J}$ has two connected components K_i, $i=1,2$, then J on $\Upsilon(h,0)$ is homologous to zero and two-sided.

Each K_i has one hole on the boundary. Moreover, $\Upsilon(h,0)$ is the connected sum $\mathcal{P}(K_1)\#\mathcal{P}(K_2)$ and

$$\chi(\Upsilon(h,0))=\chi(K_1)+\chi(K_2). \tag{3.2}$$

The connected components K_i, $i=1,2$, cannot be both orientable surfaces because in this case the connected sum $\Upsilon(h,0)=\mathcal{P}(K_1)\#\mathcal{P}(K_2)$ will be an orientable surface.

Let us suppose that $K_1\simeq\Sigma(g_1,1)$ is an orientable surface and $K_2\simeq\Upsilon(h_2,1)$ is a non-orientable surface then from (3.2) we have $2g_1+h_2=h$. Fixing $h\geq 1$ and giving values for $g_1\in\{0,\ldots,-1+\lceil h/2\rceil\}$ we obtain $\lceil h/2\rceil$ cases that satisfy the equation $2g_1+h_2=h$. Moreover, if K_1 and K_2 are non-orientable then $h_1+h_2=h$; and fixing $h\geq 1$ we have $\lfloor h/2\rfloor$ cases.

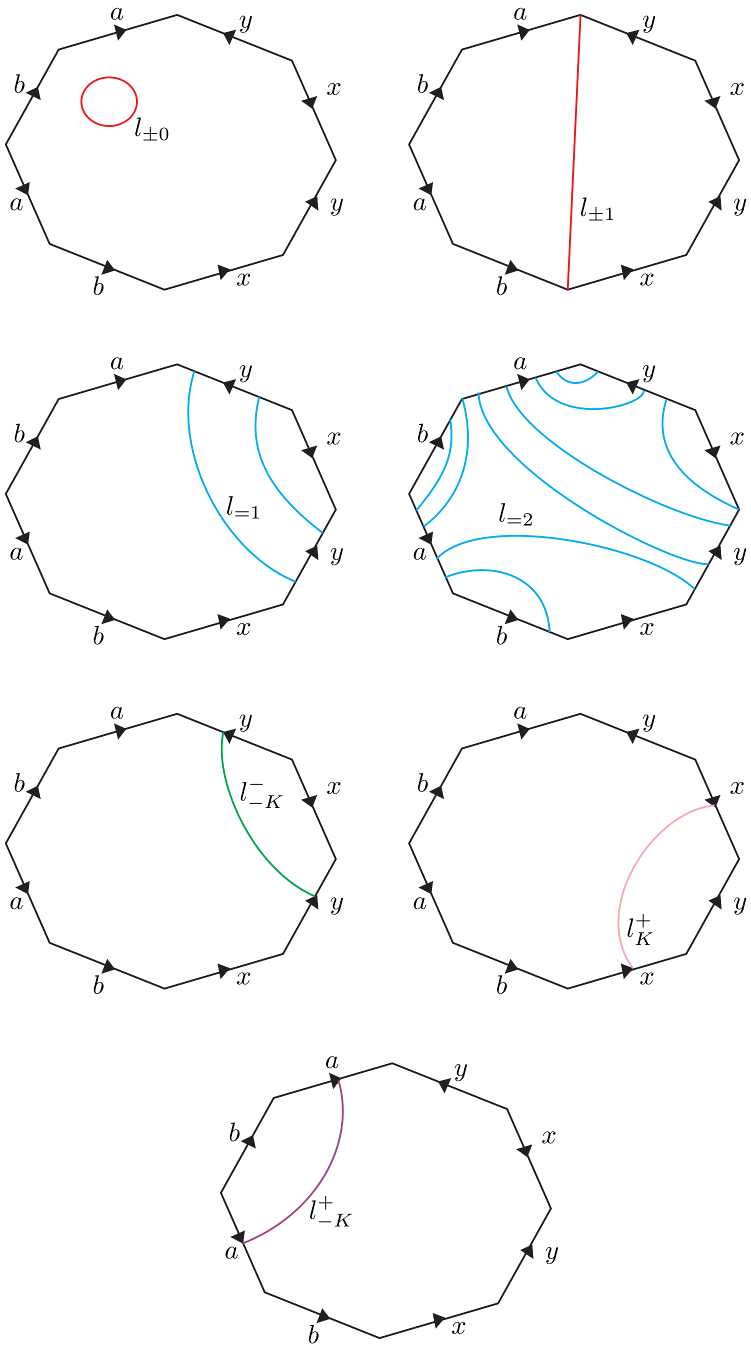


Figure 4. Non-equivalent embeddings of S^1 on $\Upsilon(4,0)$.

Step 2. Let $J_1 = \phi_1(S^1)$ and $J_2 = \phi_2(S^1)$ be two embeddings of S^1 into $\Upsilon(h,0)$ that belong to the same class. Consider the map $H\colon J_1 \to J_2$ given by $h = \phi_1^{-1} \circ \phi_2$. We must prove that there is an extension of H to $\Upsilon(h,0)$.

If c, $i = 1,2$, has two connected components K^i_j, $j = 1,2$, then each of these components has one closed boundary curve where H is defined. The extension of H to $\Upsilon(h,0)$ is a direct consequence of Theorems 3.2, 3.3 and the Pasting Lemma.

In the case that $\Upsilon(h,0) \setminus J_i$, $i = 1,2$, is connected, say K^i, $i = 1,2$, then we need to consider some cases. If K^i, $i = 1,2$, is a non-orientable surface or has a unique hole, the extension of H to $\Upsilon(h,0)$ again follows from Theorems 3.2 and 3.3.

Finally, if J_i is a two-sided curve and K^i are orientable surfaces, then each surface K^i, $i = 1,2$, has two closed curves on their boundaries equivalent to J_i. If $h = 2$, $\overline{\Upsilon(2,0) \setminus J}$ is a cylinder with the structure of an interval not trivially fibred by circles, Ψ. The extension for $h = 2$ is straightforward. If $h > 2$, $\overline{\Upsilon(h,0) \setminus J} \cong \Sigma(h/2 - 1, 2)$. We extend H to $\Sigma(h/2 - 1, 2)$ as in Theorem 18 of [12]. Consider Ψ, whose boundary consists of two copies of J_1 and an extension of f to Ψ. As $\Upsilon(h,0)$ can be obtained gluing $\Sigma(h/2 - 1, 2)$ and Ψ along their boundaries, applying the Pasting Lemma, we get the required extension H. $\square$

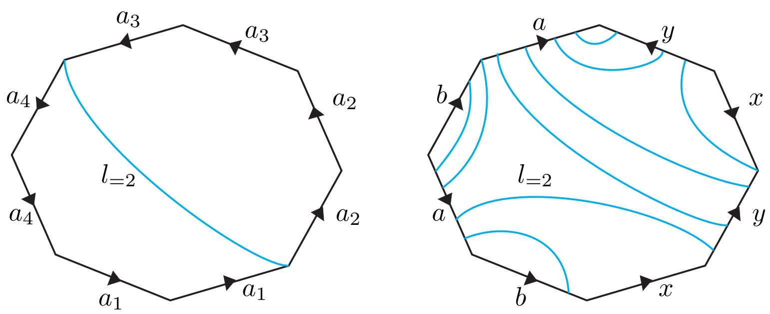


Figure 5. The curve $l_{=2}$ on $\Upsilon(4,0)$. On the left: $\Upsilon(4,0)$ is represented as the connected sum of four projective planes. On the right: $\Upsilon(4,0)$ as the connected sum of a torus and the Klein bottle.

By Theorem 3.6 we have six non-equivalent embeddings of S^1 on $\Upsilon(3,0)$ represented in Figure 3 and there are seven non-equivalent embeddings of S^1 on $\Upsilon(4,0)$, see Figure 4. In these figures we display the embeddings using the same polygonal representation of the surfaces. Some embeddings can have a simpler representative in one polygonal representation of a surface than in another one. For instance choosing two different polygonal representations of $\Upsilon(4,0)$ we get two different ways to represent the class $l_{=2}$ (see Figure 5).

4. Figure eight curves

With the notation of Definition 1.2, let $\mathfrak{B}$, $\mathfrak{B}'$ be eights on the surface $\Pi(q,0)$, and h a homeomorphism $h\colon \mathfrak{B} \to \mathfrak{B}'$. Then h is composed of two homeomorphisms $h_i\colon s_i \to s'_i$ such that $h_1(p) = h_2(p) = p'$. We assume that the definitions of *ah*-equivalence and topological type of simple circles are extended in an obvious way to eights.

Lemma 4.1. *Let d be the dimension of the subgroup of the first group of homology of $\Pi(q,0)$ spanned by the components of $\mathfrak{B}$, s_1 and s_2. Then $\mathfrak{B}$ splits $\Pi(q,0)$ in $3-d$ connected regions.*

Proof. Consider $\mathfrak{B}$ as a graph whose vertices are the vertices of two triangles glued by a common vertex and the edges are the edges of the two triangles. Thus by [8, p. 181], we should get

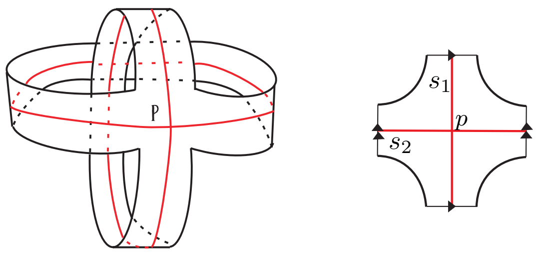
$$\alpha_0(\mathfrak{B}) - \alpha_1(\mathfrak{B}) + r = k + 1 - d, \tag{4.1}$$

where d is the dimension of the image of $i_*\colon H_1(\mathfrak{B},2) \to H_1(\Pi(g,0),2)$, r is the number of connected regions of $\Pi(g,0) \setminus \mathfrak{B}$, k is the number of components of $\mathfrak{B}$ and $\alpha_p = |p\text{-simplexes}|$ of $\mathfrak{B}$. Obviously $\alpha_0(\mathfrak{B}) = 5$, $\alpha_1(\mathfrak{B}) = 6$ and as $\mathfrak{B}$ is connected we have $k = 1$. From (4.1) we obtain $r = 3 - d$. □

Definition 4.2. We will say that an eight $\mathfrak{B}$ is a *non-separating* eight if $r = 1$. Otherwise, we will say that $\mathfrak{B}$ is a *separating* eight.

Denote by $\mathfrak{N}\mathfrak{B}$ a closed regular neighbourhood of $\mathfrak{B} = s_1 \underset{p}{\cup} s_2$. The next result classifies these $\mathfrak{N}\mathfrak{B}$.

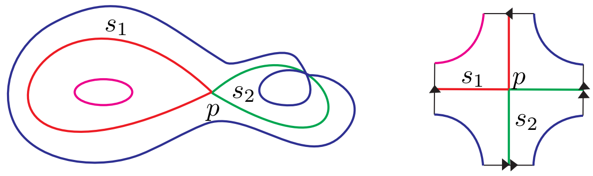
(a) $\mathfrak{N}\mathfrak{B} \cong \Sigma(0,3)$



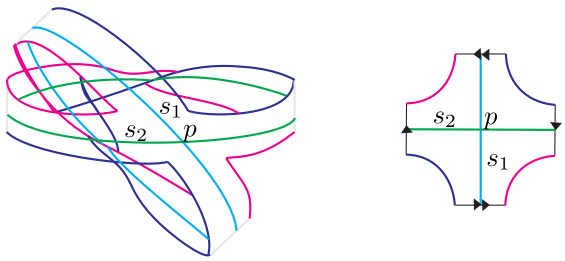
(b) $\mathfrak{N}\mathfrak{B} \cong \Sigma(1,1)$

Figure 6. Orientable closed regular neighbourhood of $\mathfrak{B}$.

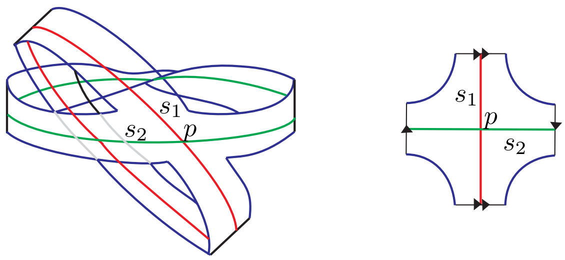
Lemma 4.3. *Let $\mathfrak{N}\mathfrak{B}$ be a closed regular neighbourhood of an eight $\mathfrak{B}$ on $\Upsilon(h,0)$. If $\mathfrak{N}\mathfrak{B}$ is orientable then $\mathfrak{N}\mathfrak{B}$ is homeomorphic either to $\Sigma(0,3)$ or to $\Sigma(1,1)$. If $\mathfrak{N}\mathfrak{B}$ is non-orientable then $\mathfrak{N}\mathfrak{B}$ is homeomorphic either to $\Upsilon(1,2)$ or*



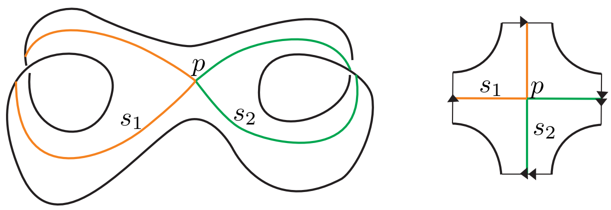
(a) $\mathfrak{NB} \cong \Upsilon(1,2)$, s_1 is a two-sided curve and s_2 is a one-sided curve.



(b) $\mathfrak{NB} \cong \Upsilon(1,2)$, s_1 and s_2 are one-sided curves.



(c) $\mathfrak{NB} \cong \Upsilon(2,1)$, s_1 is a two-sided curve and s_2 is a one-sided curve.



(d) $\mathfrak{NB} \cong \Upsilon(2,1)$, s_1 and s_2 are one-sided curves.

Figure 7. Non orientable closed regular neighbourhood of $\mathfrak{B}$.

to $\Upsilon(2,1)$ *and the components* s_1 *and* s_2 *of* $\mathfrak{B}$ *can be both one-sided curves or one component is a one-sided curve and the other one is a two-sided curve.*

Proof. Consider on the surface $\mathfrak{NB}$ a circle centered at the point p and small enough so that $\mathfrak{B}$ cuts the circle in exactly four points. Taking into account the component that corresponds to each point we have two cyclic orderings:

s_1, s_1, s_2, s_2 and s_1, s_2, s_1, s_2. Also taking into account if the components $s_i^{'}$ of $\mathfrak{B}$ are one-sided or not, we get the result. See Figures 6 and 7. □

DEFINITION 4.4. Let $\mathfrak{B}$ be an eight in $\Upsilon(h, 0)$ with $h \geq 1$. We will say that $\mathfrak{B}$ is

(a) A *planar* eight if $\mathfrak{NB} \cong \Sigma(0, 3)$ (see Figure 6 (a)).
(b) A *toroidal* eight if $\mathfrak{NB} \cong \Sigma(1, 1)$ (see Figure 6 (b)).
(c) A *projective* eight *of the type* $\mathfrak{NB}^-$ if $\mathfrak{NB} \cong \Upsilon(1, 2)$ and one component of $\mathfrak{B}$ is a one-sided curve and the other one is a two-sided curve (see Figure 7 (a)).
(d) A *projective* eight *of the type* $\mathfrak{NB}^+$ if $\mathfrak{NB} \cong \Upsilon(1, 2)$ and both components of the $\mathfrak{B}$ are one-sided curves (see Figure 7 (b)).
(e) A *Klein* eight *of the type* $\mathfrak{NB}^-$ if $\mathfrak{NB} \cong \Upsilon(2, 1)$ and one component of $\mathfrak{B}$ is a one-sided curve and the other one is a two-sided curve (see Figure 7 (c)).
(f) A *Klein* eight *of the type* $\mathfrak{NB}^+$ if $\mathfrak{NB} \cong \Upsilon(2, 1)$ and both components of $\mathfrak{B}$ are one-sided curves (see Figure 7 (d)).

Lemma 4.5 classifies the set $\overline{\Upsilon(h, 0) \setminus \mathfrak{NB}}$ according to the type of $\mathfrak{NB}$ and its connected components.

LEMMA 4.5. *Let $\mathfrak{B}$ be an eight on $\Upsilon(h, 0)$ with $h \geq 1$.*

(a) *If $\mathfrak{B}$ is a planar eight then $\overline{\Upsilon(h, 0) \setminus \mathfrak{NB}}$ is homeomorphic to one of the following sets* (*where $\sqcup$ denotes the disjoint union*):
  (a$_1$) $\Sigma(g_1, 1) \sqcup \Sigma(g_2, 1) \sqcup \Upsilon(h_3, 1)$, $2g_1 + 2g_2 + h_3 = h$;
  (a$_2$) $\Sigma(g_1, 1) \sqcup \Upsilon(h_2, 1) \sqcup \Upsilon(h_3, 1)$, $2g_1 + h_2 + h_3 = h$;
  (a$_3$) $\Upsilon(h_1, 1) \sqcup \Upsilon(h_2, 1) \sqcup \Upsilon(h_3, 1)$, $h_1 + h_2 + h_3 = h$;
  (a$_4$) $\Sigma(g_1, 2) \sqcup \Sigma(g_2, 1)$, *if* $2g_1 + 2g_2 = h - 2$;
  (a$_5$) $\Sigma(g_1, 2) \sqcup \Upsilon(h_2, 1)$, *if* $2g_1 + h_2 = h - 2$;
  (a$_6$) $\Sigma(g_1, 1) \sqcup \Upsilon(h_2, 2)$, *if* $2g_1 + h_2 = h - 2$;
  (a$_7$) $\Upsilon(h_1, 1) \sqcup \Upsilon(h_2, 2)$, *if* $h_1 + h_2 = h - 1$;
  (a$_8$) $\Upsilon(h_1, 3)$, $h_1 = h - 4$, $h \geq 5$ *or* $\Sigma((h - 4)/2, 3)$ *with* $h \geq 4$ *even.*
(b) *If* $\mathfrak{NB} \cong \Sigma(1, 1)$ *then* $\overline{\Upsilon(h, 0) \setminus \mathfrak{NB}} \cong \Upsilon(h - 2, 1)$.
(c) *If* $\mathfrak{NB} \cong \Upsilon(1, 2)$ *then* $\overline{\Upsilon(h, 0) \setminus \mathfrak{NB}}$ *is homeomorphic to one of the following sets*:
  (c$_1$) $\Sigma(g_1, 1) \sqcup \Sigma(g_2, 1)$, $2g_1 + 2g_2 = h - 1$;
  (c$_2$) $\Sigma(g_1, 1) \sqcup \Upsilon(h_2, 1)$, $2g_1 + h_2 = h - 1$;
  (c$_3$) $\Upsilon(h_1, 1) \sqcup \Upsilon(h_2, 1)$, $h_1 + h_2 = h - 1$;
  (c$_4$) $\Sigma(g_1, 2)$, $2g_1 = h -; 3$
  (c$_5$) $\Upsilon(h_1, 2)$, $h_1 = h - 3$.
(d) *If* $\mathfrak{NB} = \Upsilon(2, 1)$ *then* $\overline{\Upsilon(h, 0) \setminus \mathfrak{NB}}$ *is homeomorphic to* $\Sigma((h - 2)/2, 1)$ *or* $\Upsilon(h - 2, 1)$.

Proof. Let $\chi(\mathfrak{N}\mathfrak{B}) = k_1$ and $\chi(\overline{\Upsilon(h,0)\setminus\mathfrak{N}\mathfrak{B}}) = k_2$. Then $\chi(\Upsilon(h,0)) = k_1+k_2$. So, using Lemma 4.1 and depending on the components of $\mathfrak{B}$, s_1 and s_2, we calculate the number of the components of the set $\Upsilon(h,0)\setminus\mathfrak{N}\mathfrak{B}$. The rest of the statements follows from a direct computation of the Euler characteristic. □

The boundary curves J of $\mathfrak{N}\mathfrak{B}$ that are contractible to s_1 or to s_2 will be distinguish by a subindex s: J_s. The other boundary curves will be denoted by $J_{\mathfrak{B}}$. An orientation on an s_i induces an orientation in J_s that can be contracted to s_i.

Lemma 4.6. *Let $\mathfrak{B}$ and $\mathfrak{B}'$ be eights on Π and $h\colon \mathfrak{B}\to\mathfrak{B}'$ a homeomorphism. Suppose that the two h_i, $i=1,2$, are simultaneously preserving or reversing orientation homeomorphisms. Then the map h can be extended to the regular neighbourhood $\mathfrak{N}\mathfrak{B}$.*

Proof. Consider $\Upsilon(h,0)\setminus\mathfrak{B}$ and the notations of Section 3. If the boundary of K_i comes from a component s_i of $\mathfrak{B}$ we have the homeomorphism $h_i\colon \partial K_i\to\partial K_i'$. If ∂K_i comes from $\mathfrak{B}$, then there exists a homeomorphism between ∂K_i and $\partial K_i'$ if and only if the two h_i, $i=1,2$, are simultaneously preserving or reversing orientation homeomorphisms. Let us assume that this is the case. Then by Theorems 3.2 and 3.3 it is possible to extend the homeomorphism on the boundary to all K_i and in particular to each component of $K\cap\mathfrak{N}\mathfrak{B}$. Pasting together these homeomorphisms we get the extension required in the lemma. □

s_1 \ s_2	$l_{\pm0}$	l_K^-
$l_{\pm0}$	$\Sigma(0,1)$ $\Sigma(0,1)$ $\Upsilon(1,1)$	
l_K^-	$\Sigma(0,1)$ $\Sigma(0,1)$	$\Sigma(0,1)$ $\Sigma(0,1)$

Table 1. Combinations of representative curves and connected components of the closure of $\Upsilon(1,0)\setminus\mathfrak{N}\mathfrak{B}$.

In the next two propositions we introduce the simplest examples of types of $\mathfrak{N}\mathfrak{B}$.

Proposition 4.7. *In $\Upsilon(1,0)$ there are three non ah-equivalent eights, $\mathfrak{B}$, which are listed in Table 1.*

Proof. Two *ah*-equivalent eights must contain *ah*-equivalent s_i and homeomorphic K_i. These conditions are also sufficient.

If $h\colon s_i \to s_i'$ is a homeomorphism, then h^- is also a homeomorphism, therefore one can always apply Lemma 4.6 and extend the homeomorphism between $\mathfrak{B}$ and $\mathfrak{B}'$ to all $\Upsilon(1,0)$.

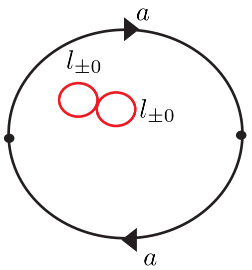


Figure 8. The eight $\mathfrak{B}(l_{\pm 0}, l_{\pm 0})$ on $\Upsilon(1,0)$.

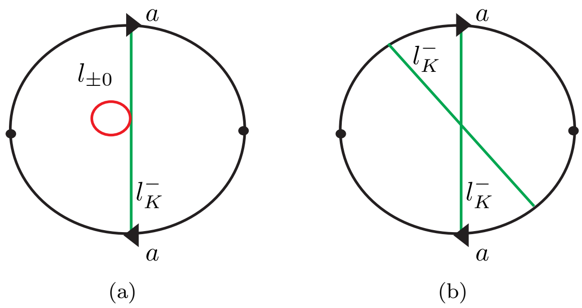


Figure 9. On the left: the eight $\mathfrak{B}(l_{\pm 0}, l_K^-)$; on the right: the eight $\mathfrak{B}(l_K^-, l_K^-)$.

By Theorem 3.6 we have two non-equivalent embeddings of S^1 into $\Upsilon(1,0)$ which representative curves are $l_{\pm 0}$ and l_K^-. Then in order to obtain the number of non-equivalent eights on $\Upsilon(1,0)$, we need to make combinations of $l_{\pm 0}$ with l_K^- and then to analyze the connected components of the set $\Upsilon(1,0) \setminus \mathfrak{N}\mathfrak{B}$. If $\mathfrak{N}\mathfrak{B} \cong \Sigma(0,3)$, by Lemma 4.5, $\overline{\Upsilon(h,0) \setminus \mathfrak{N}\mathfrak{B}} \cong \Sigma(0,1) \sqcup \Sigma(0,1) \sqcup \Upsilon(1,1)$. On the other hand, if $\mathfrak{N}\mathfrak{B} \cong \Sigma(0,3)$, then the components of $\mathfrak{B}$ must be two-sided curves and consequently with representative curves $l_{\pm 0}$ (see Figure 8).

The cases (b) and (d) of Lemma 4.5 cannot happen if $h = 1$. Finally, if $\mathfrak{N}\mathfrak{B} \cong \Upsilon(1,2)$, by Lemma 4.5, $\overline{\Upsilon(1,0) \setminus \mathfrak{N}\mathfrak{B}} \cong \Sigma(0,1) \sqcup \Sigma(0,1)$ since the cases from (c2) to (c5) cannot happen when $h = 1$. In fact, for this possibility we have two non-equivalent eights, see Figure 9. $\square$

Proposition 4.8. *There are* 9 *of non-equivalent eights on* $\Upsilon(2,0)$. *They are described in Table* 2.

s_1 / s_2	$l_{\pm 0}$	$l_{=1}$	l_K^+	l_{-K}^-
$l_{\pm 0}$	$\Sigma(0,1)$ $\Sigma(0,1)$ $\Upsilon(2,1)$			
$l_{=1}$	$\Sigma(0,1)$ $\Upsilon(1,1)$ $\Upsilon(1,1)$	$\Sigma(0,1)$ $\Upsilon(1,1)$		
l_K^+	$\Sigma(0,2)$ $\Sigma(0,1)$	No	$\Sigma(0,2)$ $\Sigma(0,1)$	
l_{-K}^-	$\Sigma(0,1)$ $\Upsilon(1,1)$	$\Sigma(0,1)$ $\Upsilon(1,1)$	$\Sigma(0,1)$	$\Upsilon(1,1)$ $\Sigma(0,1)$

Table 2. Combinations of representative curves and connected components of the closure of $\Upsilon(2,0) \setminus \mathfrak{NB}$.

Proof. The arguments are similar to those used in Proposition 4.7. Let $\mathfrak{B}$ be an eight on $\Upsilon(2,0)$. If $\mathfrak{NB} \cong \Sigma(0,3)$, by Lemma 4.5 we have that the closure of $\Upsilon(2,0) \setminus \mathfrak{NB}$ must be:

(a1) $\Sigma(0,1) \sqcup \Sigma(0,1) \sqcup \Upsilon(2,1)$. This is the eight $\mathfrak{B}(l_{\pm 0}, l_{\pm 0})$. See Figure 10.

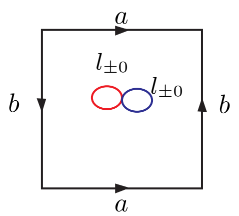


Figure 10. $\mathfrak{B}(l_{\pm 0}, l_{\pm 0})$.

(a2) $\Sigma(0,1) \sqcup \Upsilon(1,1) \sqcup \Upsilon(1,1)$. For this case we have two non-equivalent eights on $\Upsilon(2,0)$. See Figure 11.

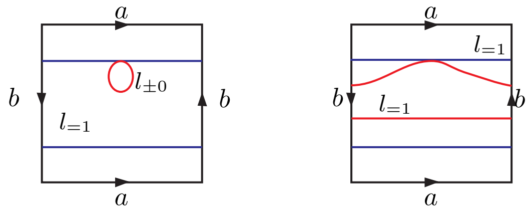
(a4) $\Sigma(0,2) \sqcup \Sigma(0,1)$. There are only two non-equivalents eights and they are shown in the Figure 12.

In $\Upsilon(2,0)$, $\mathfrak{NB}$ cannot be homeomorphic to $\Sigma(1,1)$.

When $\mathfrak{NB} \cong \Upsilon(1,2)$, $\overline{\Upsilon(2,0) \setminus \mathfrak{NB}}$ is the disjoint union of $\Sigma(0,1)$ and $\Upsilon(1,1)$. These eights are represented in Figure 13.

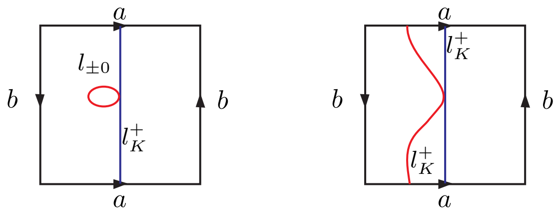
Finally, if $\mathfrak{NB} \cong \Upsilon(2,1)$ then $\overline{\Upsilon(2,0) \setminus \mathfrak{NB}} \cong \Sigma(0,1)$. See Figure 14. $\square$

In Theorem 4.10 we classify eights for any $\Upsilon(h,0)$, but in Proposition 4.9 we describe all eights in $\Upsilon(3,0)$ in order to give an example of the general case in



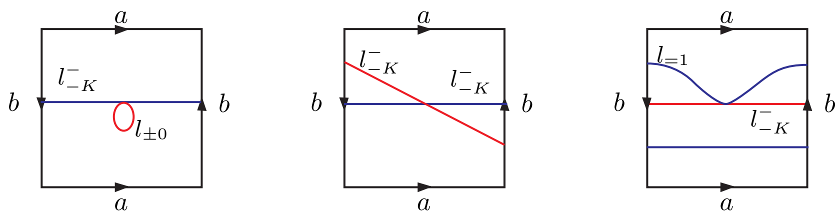
(a) (b)

Figure 11. On the left: the eight $\mathfrak{B}(l_{\pm 0}, l_{=1})$; on the right the eight $\mathfrak{B}(l_{=1}, l_{=1})$.



(a) (b)

Figure 12. On the left: the eight $\mathcal{B}(l_{\pm 0}, l^+_K)$; on the right: the eight $\mathcal{B}(l^+_K, l^+_K)$.



(a) (b) (c)

Figure 13. (a) $\mathcal{B}(l_{\pm 0}, l^-_{-K})$; (b) $\mathcal{B}(l^-_{-K}, l^-_{-K})$; (c) $\mathcal{B}(l_{=1}, l^-_{-K})$.

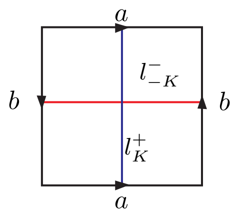


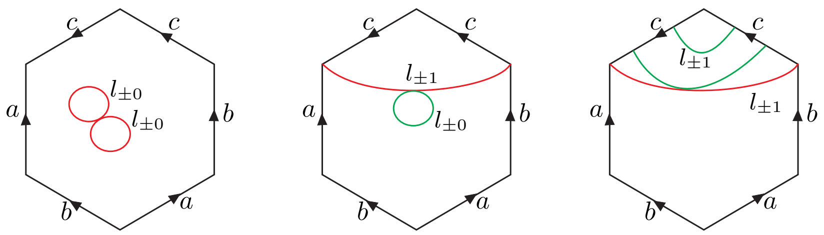
Figure 14. The eight $\mathcal{B}(l^+_K, l^-_{-K})$.

a non-orientable surface, when this surface is the connected sum of an n-torus and $\Upsilon(1,0)$ or $\Upsilon(2,0)$.

Proposition 4.9. *The number of non-equivalent eights on* $\Upsilon(3,0)$ *is* 23. *They are detailed in Table* 3.

s_2 \ s_1	$l_{\pm 0}$	$l_{\pm 1}$	$l_{=1}$		l_K^-	l_{-K}^-		l_{-K}^+	
$l_{\pm 0}$	$\Sigma(0,1)$ $\Sigma(0,1)$ $\Upsilon(3,1)$								
$l_{\pm 1}$	$\Sigma(0,1)$ $\Sigma(1,1)$ $\Upsilon(1,1)$	$\Sigma(0,1)$ $\Sigma(1,1)$ $\Upsilon(1,1)$							
$l_{=1}$	$\Sigma(0,1)$ $\Upsilon(1,1)$ $\Upsilon(2,1)$	No	$\Sigma(0,1)$ $\Upsilon(1,1)$ $\Upsilon(2,1)$	$\Upsilon(1,1)$ $\Upsilon(1,1)$ $\Upsilon(1,1)$					
l_K^-	$\Sigma(0,1)$ $\Sigma(1,1)$	$\Sigma(0,1)$ $\Sigma(1,1)$	No		$\Sigma(0,1)$ $\Sigma(1,1)$				
l_{-K}^-	$\Sigma(0,1)$ $\Upsilon(2,1)$	No	$\Upsilon(1,1)$ $\Upsilon(1,1)$	$\Sigma(0,1)$ $\Upsilon(2,1)$	$\Sigma(0,2)$	$\Sigma(0,1)$ $\Upsilon(2,1)$	$\Upsilon(1,1)$		
l_{-K}^+	$\Sigma(0,1)$ $\Upsilon(1,2)$	$\Sigma(0,2)$ $\Upsilon(1,1)$	$\Sigma(0,2)$	$\Upsilon(1,1)$	$\Sigma(0,2)$	$\Sigma(0,2)$	$\Upsilon(1,1)$	$\Sigma(0,1)$ $\Upsilon(1,2)$	$\Upsilon(1,1)$

Table 3. Combinations of curves and connected components of $\Upsilon(3,0) \setminus \mathfrak{N}\mathfrak{B}$.



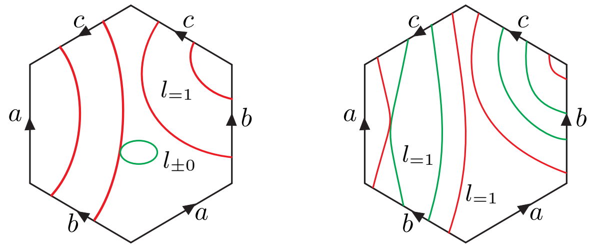
(a) (b) (c)

Figure 15. (a) the eight $\mathcal{B}(l_{\pm 0}, l_{\pm 0})$; (b) the eight $\mathcal{B}(l_{\pm 0}, l_{\pm 1})$; (c) the eight $\mathcal{B}(l_{\pm 1}, l_{\pm 1})$.

Proof. The number of non-equivalent embeddings of S^1 into $\Upsilon(3,0)$ is six and their representative curves are: $l_{\pm 0}$, $l_{\pm 1}$, $l_{=1}$, l_K^-, l_{-K}^- and l_{-K}^+. Using Lemma 4.5 we describe the connected components of $\Upsilon(3,0) \setminus \mathfrak{N}\mathfrak{B}$, for distinct closed regular neighbourhoods of $\mathfrak{B}$ and describe all eights as in the previous propositions. $\square$

From now on, we denote by $\mathcal{B}(\tau, \upsilon, q, n)$ the class of eights on a non-orientable surface $\Upsilon(h,0)$ where τ, υ denote the representative curves of the components s_1 and s_2 of $\mathcal{B}$ respectively, q is the minimum value of the genus of the connected components of the closure of $\Upsilon(h,0) \setminus \mathfrak{N}\mathfrak{B}$ and n indicates the number of non-orientable components of $\overline{\Upsilon(h,0) \setminus \mathfrak{N}\mathfrak{B}}$. If there is no ambiguity, we use the simplified notation $\mathcal{B}(\tau, \upsilon)$.

Next theorem gives us the ah-classification of eights on non-orientable surfaces.



(a) (b)

Figure 16. On the left: the eight $\mathcal{B}(l_{\pm 0}, l_{=1})$; on the right: the eight $\mathcal{B}(l_{=1}, l_{=1})$.

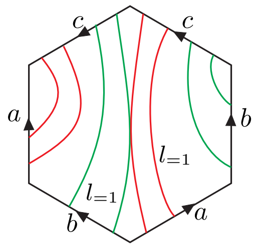
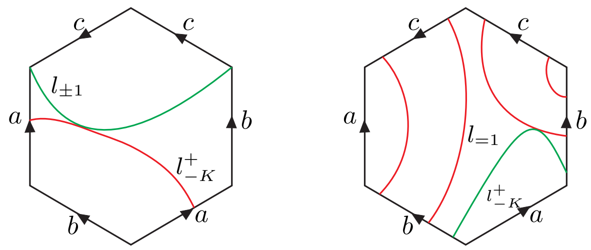
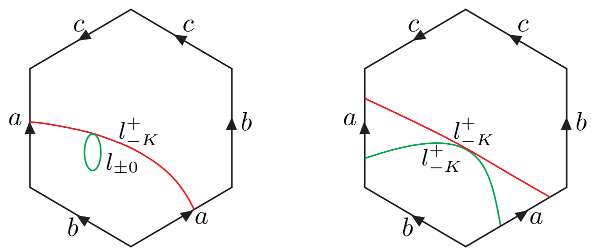


Figure 17. $\mathcal{B}(l_{=1}, l_{=1})$.



(a) (b)

Figure 18. On the left: the eight $\mathcal{B}(l_{\pm 1}, l^+_{-K})$; on the right: the eight $\mathcal{B}(l_{=1}, l^+_{-K})$.



(a) (b)

Figure 19. On the left: the eight $\mathcal{B}(l_{\pm 0}, l^+_{-K})$; on the right: the eight $\mathcal{B}(l^+_{-K}, l^+_{-K})$.

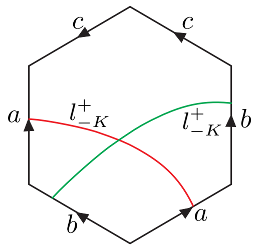
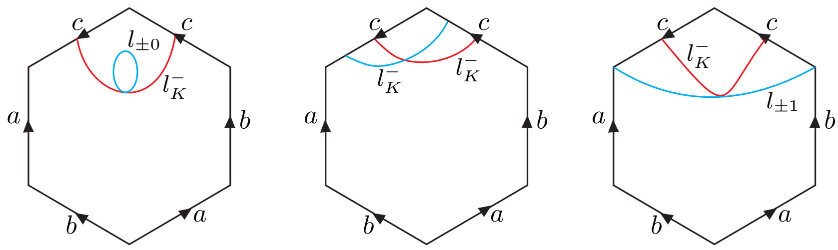
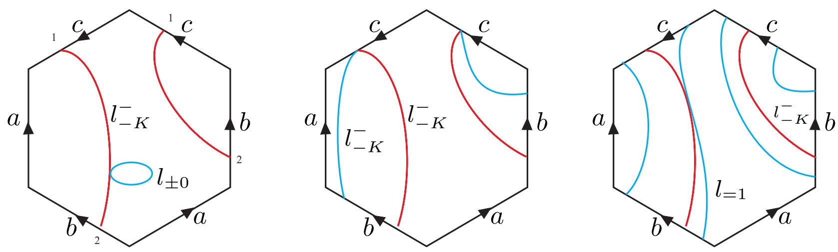


Figure 20. $\mathcal{B}(l^+_{-K}, l^+_{-K})$.



(a) (b) (c)

Figure 21. (a) the eight $\mathcal{B}(l_{\pm 0}, l^-_K)$; (b) the eight $\mathcal{B}(l^-_K, l^-_K)$; (c) the eight $\mathcal{B}(l^-_K, l_{\pm 1})$.



(a) (b) (c)

Figure 22. (a) the eight $\mathcal{B}(l_{\pm 0}, l^-_{-K})$; (b) the eight $\mathcal{B}(l^-_{-K}, l^-_{-K})$; (c) the eight $\mathcal{B}(l_{=1}, l^-_{-K})$.

Theorem 4.10. *Let $\Upsilon(h,0)$ be a non-orientable surface with genus $h \geq 1$. Then the number of non-equivalent eights on $\Upsilon(h,0)$ is*

(a) 3, *if* $h = 1$;
(b) 9, *if* $h = 2$;
(c) $(h+4)^2/2$, *if h is even and* $h \geq 3$;
(d) $(h+4)^2/2 - 3/2$, *if h is odd and* $h \geq 3$.

Proof. First we distinguish the eights according to the ah-class of s_i and the connected components of $\overline{\Upsilon(h,0) \setminus \mathfrak{NB}}$.

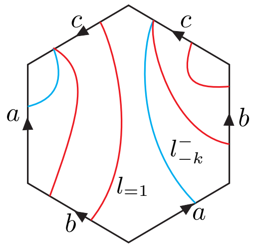
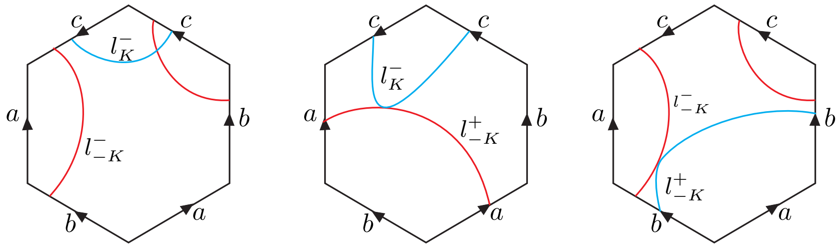
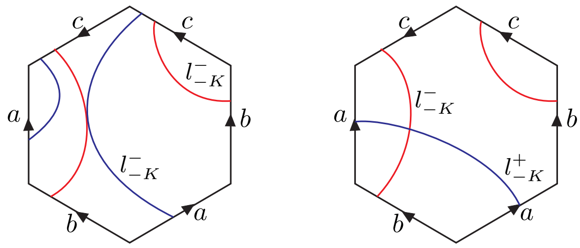


Figure 23. The eight $\mathcal{B}(l_{=1}, l^-_{-k})$.



(a) (b) (c)

Figure 24. (a) the eight $\mathcal{B}(l^-_K, l^-_{-K})$; (b) the eight $\mathcal{B}(l^+_{-K}, l^-_K)$; (c) the eight $\mathcal{B}(l^-_{-K}, l^+_{-K})$.



(a) (b)

Figure 25. (a) $\mathcal{B}(l^-_{-K}, l^-_{-K})$, (b) $\mathcal{B}(l^-_{-K}, l^+_{-K})$.

Theorem 3.6 classifies the embeddings of S^1 into $\Upsilon(h, 0)$. These curves are the possible components of an eight on $\Upsilon(h, 0)$. We eliminate the pairs of curves that do not generate an eight because they intercept in two or more points (see [1]). When they define an eight we analyze the type of their regular neighbourhood. Lemma 4.5 allows us to obtain the connected components of the closure of $\Upsilon(h, 0) \setminus \mathcal{B}$.

We arrange the embeddings of S^1 into $\Upsilon(h, 0)$ in three groups. The first group is composed by $l_{\pm p}$ curves, the second group consists of $l_{=m}$ curves and the third one is formed by $l^{\pm}_{\pm K}$ curves. We begin looking for the eights $\mathcal{B}(l_{\pm m}, l)$ where l is a curve of arbitrary type.

s_2 \ s_1	$l_{\pm 0}$	$l_{\pm 1}$	$\cdots$	$l_{\pm m}$	$\cdots$	$l_{\pm\lfloor\frac{h-1}{2}\rfloor}$
$l_{\pm 0}$	$\Sigma(0,1)$ $\Sigma(0,1)$ $\Upsilon(h,1)$					
$l_{\pm 1}$	$\Sigma(0,1)$ $\Sigma(1,1)$ $\Upsilon(h-2,1)$	$\Sigma(0,1)$ $\Sigma(1,1)$ $\Upsilon(h-2,1)$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$l_{\pm m}$	$\Sigma(0,1)$ $\Sigma(m,1)$ $\Upsilon(h-2m,1)$	$\Sigma(1,1)$ $\Sigma(m-1,1)$ $\Upsilon(h-2m,1)$	$\cdots$	$\Sigma(0,1)$ $\Sigma(m,1)$ $\Upsilon(h-2m,1)$, $h>2m+1$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$l_{\pm\lfloor\frac{h-1}{2}\rfloor}$	$\Sigma(0,1)$ $\Sigma\left(\lceil\frac{h}{2}\rceil,1\right)$ $\Upsilon(n,1)$ $n=1,2$	$\Sigma(1,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor-1,1\right)$ $\Upsilon\left(h-2\lfloor\frac{h-1}{2}\rfloor,1\right)$	$\cdots$	$\Sigma(m,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor-m,1\right)$ $\Upsilon(j,1)$ $j=1,2$	$\cdots$	$\Sigma(0,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,1\right)$ $\Upsilon(j,1)$, $j=1,2$
$l_{=1}$	$\Sigma(0,1)$ $\Upsilon(1,1)$ $\Upsilon(h-1,1)$	$\Sigma(1,1)$ $\Upsilon(h-3,1)$ $\Upsilon(1,1)$, $h>3$	$\cdots$	$\Sigma(m,1)$ $\Upsilon(1,1)$ $\Upsilon(h-1-2m,1)$, if $h>2m+1$	$\cdots$	$\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,1\right)$ $\Upsilon(1,1)$ $\Upsilon(1,1)$, if h is even
$l_{=2}$	$\Sigma(0,1)$ $\Upsilon(2,1)$ $\Upsilon(h-2,1)$	$\Sigma(1,1)$ $\Upsilon(2,1)$, $h>5$ $\Upsilon(h-4,1)$	$\cdots$	$\Sigma(m,1)$ $\Upsilon(2,1)$ $\Upsilon(h-2-2m,1)$, $h>2m+3$	$\cdots$	No
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$l_{=p}$	$\Sigma(0,1)$ $\Upsilon(p,1)$ $\Upsilon(h-p,1)$	$\Sigma(1,1)$ $\Upsilon(p,1)$ $\Upsilon(h-2-p,1)$ $h-2-p\geq 1$	$\cdots$	$\Sigma(m,1)$ $\Upsilon(p,1)$ $\Upsilon(h-2m-p,1)$ $h\geq 2(m+p)$ even $h\geq 1+2(m+p)$ odd	$\cdots$	No
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$l_{=\lfloor\frac{h}{2}\rfloor}$	$\Sigma(0,1)$ $\Upsilon\left(\lfloor\frac{h}{2}\rfloor,1\right)$ $\Upsilon\left(\lfloor\frac{h}{2}\rfloor,1\right)$	No	$\cdots$	No, if h is even and $p\leq h/2-m$. No, if h is odd and $p\leq\lfloor\frac{h-1}{2}\rfloor-m$.	$\cdots$	No
l_K^-	$\Sigma(0,1)$ $\Sigma\left(\lfloor\frac{h}{2}\rfloor,1\right)$ h odd	$\Sigma(1,1)$ $\Sigma\left(\frac{h-3}{2},1\right)$ $h>2$ odd	$\cdots$	$\Sigma(m,1)$ $\Sigma\left(\frac{h-1-2m}{2},1\right)$ $h\geq 1+2m$ odd	$\cdots$	$\Sigma(0,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,1\right)$ if h is odd
l_{-K}^-	$\Sigma(0,1)$ $\Upsilon(h-1,1)$	$\Sigma(1,1)$ $\Upsilon(h-3,1)$ $h>3$	$\cdots$	$\Sigma(m,1)$ $\Upsilon(h-2m,1)$ $h\geq 2m+1$	$\cdots$	$\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,1\right)$ $\Upsilon(1,1)$ if h is even
l_K^+	$\Sigma(0,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,2\right)$ h even	$\Sigma(1,1)$ $\Sigma(h-2,2)$ $h>3$ even	$\cdots$	$\Sigma(m,1)$ $\Sigma(h/2-m-1,2)$ $h\geq 2(1+m)$ even	$\cdots$	$\Sigma(0,2)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor,1\right)$ if h is even
l_{-K}^+	$\Sigma(0,1)$ $\Upsilon(h-2,2)$	$\Sigma(0,2)$ $\Upsilon(h-2,1)$ $\Sigma(1,1)$ $\Upsilon(h-4,2)$	$\cdots$	$\Sigma(m-1,2)$ $\Upsilon(h-2m,1)$ $\Sigma(m,1)$ $\Upsilon(h-2m-2,2)$	$\cdots$	$\Sigma\left(\lfloor\frac{h-1}{2}\rfloor-1,2\right)$ $\Upsilon(1,1)$ $\Sigma\left(\lfloor\frac{h-1}{2}\rfloor-1,2\right)$ $\Upsilon(2,1)$
Total	$h+3$	$h+2$	$\cdots$	$h+4-2m$	$\cdots$	5, if h is even 3, if h odd

Table 4. Combinations with l of type $l_\pm$.

Assume $l = l_{\pm p}$. As these curves are two-sided curves and homologous to zero then the eight formed by them has a closed regular neighbourhood homeomorphic to $\Sigma(0,3)$. According to Lemma 4.5 the only possibility here is (a1). Let us suppose that $m \le p$. Then the closed connected components of $\Upsilon \setminus \mathcal{B}$ are homeomorphic to $\Sigma(m,1)$, $\Sigma(p-m,1)$ and $\Upsilon(h-2p,1)$.

If $l = l_{=p}$ by Lemma 4.5 the only possibility is (a2). So $\overline{\Upsilon(h,0) \setminus \mathfrak{NB}}$ is the disjoint union of $\Sigma(m,1)$, $\Upsilon(p,1)$ and $\Upsilon(h-2m-p,1)$. But we need to exclude the cases where the curves $l_{\pm m}$ and $l_{\pm p}$ have more than one interception point. Then $h \ge 2(m+p)$ if h is even and $h \ge 1+2(m+p)$ if h is odd.

When $l = l^-_K$ or $l = l^-_{-K}$ the curves $l_{\pm m}$ and l cannot have more than one interception point so the eight $\mathcal{B}(l_{\pm m}, l)$ has a closed regular neighbourhood homeomorphic to $\Upsilon(1,2)$ that corresponds to cases (c1) and (c2) of Lemma 4.5. The connected components of the closure of $\Upsilon(h,0) \setminus \Upsilon(1,2)$ in case (c1) are $\Sigma(m,1)$ and $\Sigma((h-1-2m)/2,1)$, with $(h-1-2m)/2 \ge 0$, h even. In case (c2) the closure of $\Upsilon(h,0) \setminus \Upsilon(1,2)$ is the union of $\Sigma(m,1)$, $\Upsilon(h-2m,1)$, where $h-2m \ge 1$ and h is odd.

When $l = l^+_{-K}$ or $l = l^+_K$, as in the previous case we have an eight which closed regular neighbourhood is homeomorphic to $\Sigma(0,3)$. Therefore we are in cases (a5) or (a6) of Lemma 4.5. The connected components of the set $\overline{\Upsilon(h,0) \setminus \Sigma(0,3)}$ in case (a5) are $\Sigma(m,1)$ and $\Upsilon(h-2m-2,2)$ or $\Sigma(m-1,2)$ and $\Upsilon(h-2m,1)$. In case (a6) connected components of the set $\overline{\Upsilon(h,0) \setminus \Sigma(0,3)}$ are $\Sigma(m,1)$ and $\Sigma(h/2-m-1,2)$, for h even.

Now we compute the number of non-equivalent eights of type $\mathcal{B}(l_{\pm m}, l)$. If $m \ge 1$ it follows from the previous considerations that there are $h+4-2m$ non-equivalent eights. If $m = 0$ there are $h+3$ eights (note that $h+3$ is the number of non-equivalents embeddings of S^1 on $\Upsilon(h,0)$). If $m = \lfloor (h-1)/2 \rfloor$ we have five cases for h even and three for h odd. Then the total is $(h^2+10h-3)/4$, if $h \ge 5$ and h is odd and, $(h^2+10h-8)/4$, if $h \ge 4$ and h is even. See Table 4.

We shall study now the eights $\mathcal{B}(l_{=m}, l)$, where $l = l_{=p}$ and $l = l^{\pm}_{\pm K}$, with $m \le p$ and $m,\, p \ne \{1,\, \lfloor h/2 \rfloor\}$. When $l = l_{=p}$ the regular neighbourhood of the eight is homeomorphic to $\Sigma(0,3)$. Then according to item (a3) of Lemma 4.5 we obtain two different decompositions for the closure of $\Upsilon(h,0) \setminus \Sigma(0,3)$: $\Upsilon(m,1)$, $\Upsilon(h-p,1)$, $\Upsilon(p-m,1)$ or $\Upsilon(m,1)$, $\Upsilon(p,1)$, $\Upsilon(h-p-m,1)$.

The representative curves $l_{=m}$ and $l = l^-_K$ intercept in two or more points, so they do not form an eight. The same happens when $l = l^+_K$ because l^+_K is by definition an exceptional curve, so they intercept $l_{=m}$ even number of times (see [1]). The eights $\mathcal{B}(l_{=m}, l^-_{-K})$ has a closed regular neighbourhood homeomorphic to $\Upsilon(1,2)$ and the closure of the components of $\Upsilon(h,0) \setminus \Upsilon(1,2)$ can be obtained by the items (c2) and (c3) of Lemma 4.5. Whence we have two possible decompositions: $\Upsilon(m,1)$, $\Upsilon(h-1-m,1)$ and $\Upsilon(n,1)$, $\Sigma((h-1-n)/2,1)$, where if m is even then $n = m$ and if m is odd then $n = h-m$.

Note that when the components s_i of the eight are of types $l_{=m}$ and l^+_{-K}, with $1 < m < \lfloor h/2 \rfloor$, then the decomposition of the closure of $\Upsilon(h,0) \setminus \Sigma(0,3)$ is given

s_1 / s_2	$l_{=1}$	$l_{=2}$	$\cdots$	$l_{=m}$	$\cdots$	$l_{=\lfloor \frac{h}{2} \rfloor}$
$l_{=1}$	$\Sigma(0,1)$ $\Upsilon(1,1)$ $\Upsilon(h-1,1)$ $\Upsilon(1,1)$ $\Upsilon(1,1)$ $\Upsilon(h-2,1)$		$\cdots$		$\cdots$	
$l_{=2}$	$\Upsilon(1,1)$ $\Upsilon(1,1)$ $\Upsilon(h-2,1)$ $\Upsilon(1,1)$ $\Upsilon(2,1)$ $\Upsilon(h-3,1)$	$\Sigma(0,1)$ $\Upsilon(2,1)$ $\Upsilon(h-2,1)$ $\Upsilon(2,1)$ $\Upsilon(2,1)$ $\Upsilon(h-4,1)$	$\cdots$		$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$l_{=p}$	$\Upsilon(1,1)$ $\Upsilon(p-1,1)$ $\Upsilon(h-p,1)$ $\Upsilon(1,1)$ $\Upsilon(p,1)$ $\Upsilon(h-p-1,1)$	$\Upsilon(2,1)$ $\Upsilon(p-2,1)$ $\Upsilon(h-p,1)$ $\Upsilon(2,1)$ $\Upsilon(p,1)$ $\Upsilon(h-p-2,1)$	$\cdots$	$\Upsilon(m,1)$ $\Upsilon(p-m,1)$ $\Upsilon(h-p,1)$ $\Upsilon(m,1)$ $\Upsilon(p,1)$ $\Upsilon(h-p-m,1)$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$l_{=\lfloor \frac{h}{2} \rfloor}$	$\Upsilon(1,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor-1,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon(1,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$	$\Upsilon(2,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor-1,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon(2,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ if h is odd $\Upsilon(2,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor-1,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor,1\right)$ if h is even	$\cdots$	$\Upsilon(m,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor-m,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon(m,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor-m,1\right)$	$\cdots$	$\Sigma(0,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ if h is even $\Sigma(0,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon(1,1)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ if h is odd
l_K^-	No	No	$\cdots$	No	$\cdots$	No
l_{-K}^-	$\Sigma(0,1)$ $\Upsilon(h-1,1)$ $\Upsilon(1,1)$ $\Upsilon(h-2,1)$	$\Sigma\left(\frac{h-3}{2},1\right)$ $\Upsilon(2,1)$ $\Upsilon(2,1)$ $\Upsilon(h-3,1)$	$\cdots$	$\Sigma\left(\frac{h-1-m}{2},1\right)$ $\Upsilon(m,1)$ $\Upsilon(m,1)$ $\Upsilon(h-1-m,1)$ if m is even and h odd $\Sigma\left(\frac{h-1-m}{2},1\right)$ $\Upsilon(m,1)$ $\Upsilon(h-m,1)$ $\Sigma\left(\frac{m-1}{2},1\right)$ if m is odd and h even	$\cdots$	$\Sigma\left(\frac{h-1-\lfloor \frac{h}{2} \rfloor}{2},1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(h-1-\lfloor \frac{h}{2} \rfloor,1\right)$ if $\lfloor \frac{h}{2} \rfloor$ is even and h odd $\Sigma\left(\frac{h-1-\lfloor \frac{h}{2} \rfloor}{2},1\right)$ $\Upsilon\left(\lfloor \frac{h}{2} \rfloor,1\right)$ $\Upsilon\left(h-\lfloor \frac{h}{2} \rfloor,1\right)$ $\Sigma\left(\frac{\lfloor \frac{h}{2} \rfloor-1}{2},1\right)$ if $\lfloor \frac{h}{2} \rfloor$ is odd and h even
l_K^+	No	No	$\cdots$	No	$\cdots$	No
l_{-K}^+	$\Sigma(0,2)$ $\Upsilon(h-2,1)$	$\Upsilon(2,1)$ $\Upsilon(h-4,2)$	$\cdots$	$\Upsilon(m,1)$ $\Upsilon(h-m-2,2)$	$\cdots$	$\Sigma(0,2)$ $\Upsilon(h-2,1)$
Total	$h+2$	$h+2$	$\cdots$	$h+2$	$\cdots$	3, if h is even 5, if h is odd

Table 5. Combinations with l of type $l_{=}$.

by: $\Upsilon(m,1)$, $\Upsilon(h-m-2,2)$. When $m=1$ or $m=\lfloor h/2 \rfloor$, the decomposition is: $\Sigma(0,2)$, $\Upsilon(h-2,1)$. Therefore, the number of all eights with a component of type $l_{=m}$, $1 \le m \le \lfloor h/2 \rfloor$, is $(h+7)(h-1)/4$, if h is odd and $h(h+6)/4$, if h is even. See Table 5 for details.

Finally, we consider the eights $\mathcal{B}(l^{\pm}_{\pm K}, l^{\pm}_{\pm K})$. In this case we have ten eights if h is even and nine if h is odd. See Tables 6 and 7 for details.

s_2 \ s_1	l^-_K	l^-_{-K}	l^+_K	l^+_{-K}
l^-_K	No			
l^-_{-K}	No	$\Sigma(0,1)$ $\Upsilon(h-1,1)$		
l^+_K	No	$\Sigma(\frac{h-2}{2},1)$	$\Sigma(0,1)$ $\Sigma(\frac{h-2}{2},2)$ $\Sigma(0,1)$ $\Upsilon(h-2,2)$	
l^+_{-K}	No	$\Upsilon(h-3,2)$ $\Upsilon(h-2,1)$	$\Sigma(\frac{h-4}{2},3)$	$\Sigma(0,1)$ $\Upsilon(h-2,2)$ $\Upsilon(h-2,1)$ $\Upsilon(h-4,3)$

Table 6. Combinations of curves and connected components of $\Upsilon(h,0) \setminus \mathfrak{NB}$ for h even, l essential.

s_2 \ s_1	l^-_K	l^-_{-K}	l^+_K	l^+_{-K}
l^-_K	$\Sigma(0,1)$ $\Sigma(\frac{h-1}{2},1)$			
l^-_{-K}	$\Sigma(\frac{h-3}{2},2)$	$\Upsilon(h-2,1)$		
l^+_K	No	No	No	
l^+_{-K}	$\Sigma(\frac{h-3}{2},2)$	$\Sigma(\frac{h-3}{2},2)$ $\Upsilon(h-2,1)$	No	$\Sigma(0,1)$ $\Upsilon(h-2,2)$ $\Upsilon(h-2,1)$ $\Upsilon(h-4,3)$

Table 7. Combinations of curves and connected components of $\Upsilon(h,0) \setminus \mathfrak{NB}$ for each possible combination, h odd, l essential.

On the other hand, classes of eights distinguished by the ah-class of s_i and the connected components of $\Upsilon(h,0) \setminus \mathfrak{NB}$ contain only one ah-type of eight due to similar arguments as in the previous propositions. Therefore, the number of non-equivalent eights on a non-orientable surface $\Upsilon(h,0)$ where $h \geq 3$ is $(h+4)^2/2$ if h is even and $((h+4)^2-3)/2$ if h is odd. It follows from Propositions 4.7 and 4.8 that there are 9 non-equivalent eights on $\Upsilon(2,0)$ and 3 on $\Upsilon(1,0)$. □

Note that some eights have the same components, but distinct decompositions for the complementary set $\Upsilon(h,0) \setminus \mathfrak{NB}$, see for instance the eights $\mathcal{B}(l_{=1}, l_{=1})$. From the proof of Theorem 4.10, the cases where the full notation $\mathcal{B}(\tau, \upsilon, g, n)$ is necessary to avoid ambiguity are the following ones:

$$\mathcal{B}(l_{=p}, l_{=p}, g, n), \qquad \mathcal{B}(l_{=p}, l^-_{-K}, g, n), \qquad \mathcal{B}(l^+_{-K}, l^+_{-K}, g, n),$$
$$\mathcal{B}(l^-_{-K}, l^-_{-K}, g, n), \qquad \mathcal{B}(l^+_{-K}, l^-_{-K}, g, n), \qquad \mathcal{B}(l^+_K, l^-_{-K}, g, n).$$

5. Invariant for $\mathcal{F}_{\mathcal{MB}}$ on non-orientable surfaces

In [12] we present a complete topological invariant for $\mathcal{MB}$ systems on orientable surfaces. The case of Morse functions on orientable or non-orientable surfaces is considered among others in [23] and for functions with not necessarily Morse singularities in [11]. In this section we define a complete invariant for $\mathcal{MB}$ foliations on non-orientable surfaces. First we introduce some basic properties of $\mathcal{MB}$ foliations.

5.1. Morse–Bott foliations.

Proposition 5.1. *On a non-orientable surface, a toroidal eight, a projective eight of type $\mathfrak{NB}^-$ and a Klein eight are not admissible in $\mathcal{F}_{\mathcal{MB}}$.*

Proof. Let $\mathfrak{B}$ be a toroidal eight or a projective eight of type $\mathfrak{NB}^+$ or a Klein eight of an $\mathcal{MB}$ foliation $\mathcal{F}(g)$ on Υ and $g(\mathfrak{B}) = 0$. Assuming that p is a saddle singular point we can find two sectors in a neighbourhood of p where g is positive. In the complementary sectors g is negative. But, the interior of $\mathfrak{NB}$ is filled by regular closed curves contractible to $\mathfrak{B}$. As these curves connect positive and negative sectors, they cannot be level curves of a map g and consequently cannot be leaves of $\mathcal{F}_{\mathcal{MB}}$. $\square$

Proposition 5.2. *On a non-orientable surface $\mathcal{F}_{\mathcal{MB}}$ have only one topological type of projective eight.*

Proof. Since a saddle singularity is not a maximum nor a minimum, $\Upsilon(h,0)\setminus \mathfrak{NB}$ must have two holes. By the cases stated in Theorem 4.10, there exists only one topological type. $\square$

From now on we assume that all eights are planar or projective eight of type $\mathfrak{NB}^+$. The next proposition shows that the structure of an $\mathcal{MB}$ foliation differs from the structure of a Morse foliation.

Proposition 5.3 ([12]). *In a Morse foliation two components of an eight $\mathfrak{B}$ of $\mathcal{F}(g)$ cannot be connected by a family of closed curves. Moreover, two regular cylinders connecting two eights contain only J_s circles.*

As an example, the foliation in Figure 27 cannot be a Morse foliation.

5.2. Reduced graphs and complete invariants.

Remark 5.4. Let f be an $\mathcal{MB}$ function, the vertices of $R_G(f)$ of degree 2 and vertices of degree 3 are associated to saddle points of the function. In this case, one vertex of degree two will be associated to a projective eight and vertices of degree 3 to a planar eight.

DEFINITION 5.5 (Graph for $\mathcal{MB}$ foliations on surfaces). Let $\mathcal{F} = \Theta(\mathcal{F})$ be an $\mathcal{MB}$ foliation on a non-orientable surface. The *graph* $\Theta(\mathcal{F})$ of $\mathcal{F}$ is:

(a) A circle, in the case of a regular foliation by circles on the Klein bottle.
(b) The graph obtained from the Reeb graph of f by replacing each vertex of degree two v associated to critical two-sided circumference of f and their two incidents edges for a new edge.

The graph $\Theta(\mathcal{F})$ carries the information about the surface Π. But this graph is not a complete invariant as shown the next example (see also [12]).

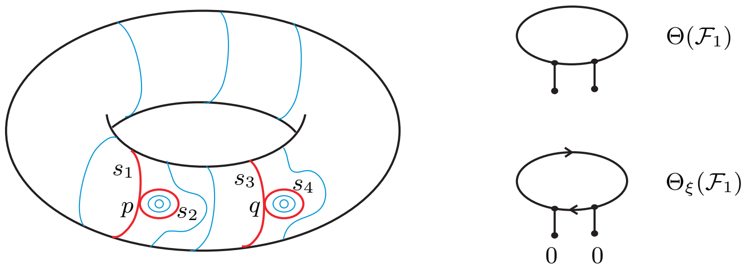


FIGURE 26.

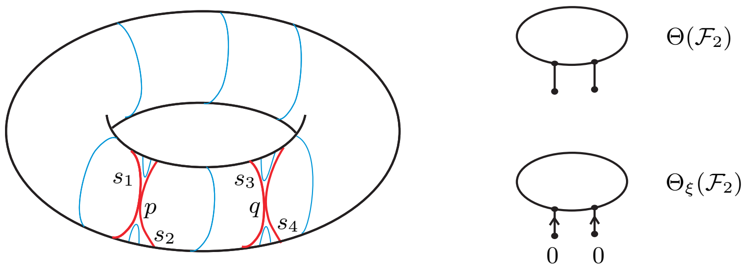


FIGURE 27.

EXAMPLE 5.6. In the Figures 26 and 27 we show two non-topologically equivalent $\mathcal{MB}$ foliations on the torus, $\mathcal{F}_1$ and $\mathcal{F}_2$, whose graphs and spaces of leaves are isomorphic. In fact, would there exist a homeomorphism h which sends the leaves of the first foliation to leaves of the second one then the topological type of the leaves would be the same. But this does not happen for all the components of the saddle points, so such homeomorphism does not exist.

The topological type of an eight must be included in any complete invariant associated with $\mathcal{MB}$ foliations on surfaces. Nevertheless, the topological type associated to a vertex v can be sometimes obtained considering the components of $\Theta(\mathcal{F}) \setminus v$ or with some additional information. Then, an explicit reference to the topological type is not needed in the invariant, it is in some way hidden.

Let ξ be a function on $\Theta(\mathcal{F})$ that associates to each saddle vertex of degree three the edge that contains $J_{\mathfrak{B}}$ circles and τ, the function that associates to each univalent vertex the number zero if it is a center point, the number $+1$ if it is a circle of type l_K^- and -1 if it is of type l_{-K}^-. The triple $\Theta(\mathcal{F})$, ξ, τ, will be noted by $\Theta_{\xi,\tau}(\mathcal{F})$.

Each edge on $\Theta(\mathcal{F})$ can have $0 \leq n \leq 2$ distinctions according to the number of times that ξ distinguishes it. We denote the edge by an n-edge. In [12], a 0-edge is unmarked, a 1-edge is an arrow directed towards the vertex that distinguishes and a 2-edge is represented by a left-right arrow. The sense of the arrow indicates the sense of branching of the graph.

The foliation associated to $\Theta_{\xi,\tau}$ will be denoted by $\mathcal{F}ol(\Theta_{\xi,\tau})$.

We assume here that $\Theta_{\xi,\tau}(\mathcal{F}_1)$ and $\Theta_{\xi,\tau}(\mathcal{F}_2)$ are isomorphic if there exists an isomorphism from $\Theta(\mathcal{F}_1)$ onto $\Theta(\mathcal{F}_2)$ that preserves the assignments of the functions ξ and τ.

5.3. Complete invariant.

Theorem 5.7. $\Theta_{\xi,\tau}(\mathcal{F})$ *is a complete topological invariant for $\mathcal{MB}$ foliations on closed surfaces.*

Proof. *Necessity.* $\Theta(\mathcal{F})$ can be considered as a quotient space by fiber equivalence. Then equivalent foliations must share the same graph. The topological type of the singularities is by definition another necessary invariant. Since it is determined by $\Theta(\mathcal{F})$, ξ and τ, the type of J_s of J_K curve and the assignments of τ must be the same for equivalent foliations.

Sufficiency. Let $\mathcal{F}_1(f)$ and $\mathcal{F}_2(g)$ be two $\mathcal{MB}$ foliations on Π. Assume that there exists an isomorphism $\theta\colon \Theta_{\xi,\tau}(\mathcal{F}_1) \to \Theta_{\xi,\tau}(\mathcal{F}_2)$. Denote by $S(\mathcal{F}_1)_i = f^{-1}(a_i)$ and $S(\mathcal{F}_2)_i = g^{-1}(b_i)$, $b_i = \theta(a_i)$ two related singular levels of the foliations; then there exist homeomorphisms $\theta_i^s\colon S(\mathcal{F}_1)_i \to S(\mathcal{F}_2)_i$. We can assume that all these θ_i^s are orientation preserving. We are going to prove that it is possible to extend θ_i^s to a homeomorphism $h\colon \Pi \to \Pi$ sending leaves of $\mathcal{F}_1$ to leaves of $\mathcal{F}_2$.

We assume that there exist singularities $\overline{c}_i$ that are not centers in order to avoid trivial cases and also that the extension of θ_i^s to $\mathfrak{NB}_i$, denoted here by θ_i^r and defined in Lemma 4.6 and in Step 2 of Theorem 3.6, sends level curves to level curves. The restrictions of θ_i^r to the components of $\partial\mathfrak{NB}\overline{c}_i$ are orientation preserving and concordant by construction. There exist extensions of θ_i^r to each component that we denote by $h_{\mathfrak{NB}_i}$.

Consider two connected singularities $\overline{c}_i$, $\overline{c}_j$ and let $\mathcal{C}_{ij}^1$ be one connecting cylinder. We have a homeomorphism between each components of the border of $\mathcal{C}_{ij}^1$. Then we can construct an extension, $\theta_{\mathcal{C}_{ij}^1}$, of the homeomorphism on the border to the entire cylinder. The homeomorphism obtained pasting together

θ_i^r, θ_j^r, and all $\theta_{C_{ij}^1}$ can be extended to a homeomorphism $h_{\mathfrak{NB}\overline{c}_i,\mathfrak{NB}\overline{c}_j}$ on the entire Π as in the case of one singularity. We iterate the extension to new singularities that are not center points until all of them have been involved, obtaining a homeomorphism $h_{\mathfrak{NB}}: \Pi \to \Pi$.

Finally, consider saturated neighbourhoods $\mathfrak{N}c_k$ of the centers. They are disk trivially foliated by circles and $h_{\mathfrak{NB}}$ defines a homeomorphism between $\partial\mathfrak{N}c_k$ and $\partial\mathfrak{N}\theta(c_k)$. Since we can extend a homeomorphism between the borders of two disks fixing one point in the interior of one disk and its image on the other, we get the desired extension h. □

EXAMPLE 5.8. Let f be the height function on the projective plane. Then the singularities of f are a maximum point, a minimum point and a saddle point. The foliation generated by f is an $\mathcal{MB}$ foliation. In this case, the singular leaves of the foliation are two points and a projective eight of type $\mathfrak{NB}^+$. The foliation and its invariant are shown in Figure 28. The vertex that corresponds to the eight is labeled in the invariant, but this is not necessary because in each case the vertices associated to planar eights have degree two.

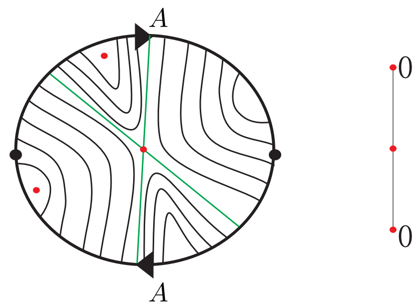


FIGURE 28. On the left: the invariant $\Theta_{\xi,\tau}(\mathcal{F})$; on the right: an $\mathcal{MB}$ foliation on $\Upsilon(1,0)$.

EXAMPLE 5.9. The 3-projective $\Upsilon(3,0)$ can be seen as the connected sum of $\Sigma(1,0)$ and $\Upsilon(1,0)$. An $\mathcal{MB}$ foliation is shown in Figure 29. Suppose that this function has a critical circumference, in Figure 29. It is denoted by the red line drawn in the Möbius band of $\Upsilon(1,0)$. This critical circumference is a one-sided curve so it is a codimension zero singular leaf of the foliation. The remaining singular leaves are two center points and two planar eights.

EXAMPLE 5.10. Recall that the 3-projective $\Upsilon(3,0)$ can be also presented as the result of gluing a hexagon, as shown in the Figure 30. Other example of $\mathcal{MB}$ foliation in $\Upsilon(3,0)$ is shown in Figure 30. This foliation has four singular leaves: one-sided curve, two center points and an eight.

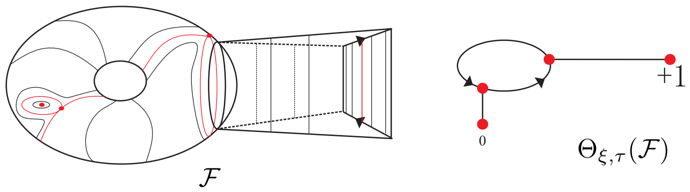


Figure 29. On the left: the invariant $\Theta_{\xi,\tau}(\mathcal{F})$; on the right: an $\mathcal{MB}$ foliation in $\Upsilon(3,0)$.

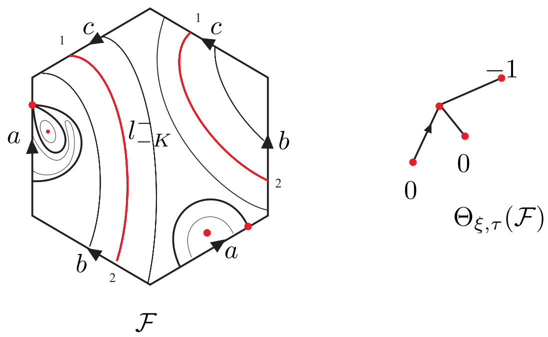


Figure 30. On the left: the invariant $\Theta_{\xi,\tau}(\mathcal{F})$; on the right, an $\mathcal{MB}$ foliation in $\Upsilon(3,0)$.

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