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ON THE TOPOLOGICAL PRESSURE OF THE SATURATED SET WITH NON-UNIFORM STRUCTURE

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Abstract. We derive a conditional variational principle of the saturated set for systems with the non-uniform structure. Our result applies to a broad class of systems including β-shifts, S-gap shifts and their subshift factors.

1. Introduction

Most results in multifractal analysis are applied to study the local asymptotic quantities, such as Birkhoff averages, Lyapunov exponents, local entropies, and pointwise dimensions, which reveal information about a single point or trajectory. It is of interest to study the level set for these quantities. A topological dynamical system (X,d,σ) (or (X,σ) for short) consists of a compact metric space (X,d) and a continuous map $\sigma\colon X\to X$. For a continuous function $\psi\colon X\to\mathbb{R}$, we always consider the following set:

$$X(\psi,\alpha)=\bigg\{x\in X:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\psi(\sigma^i x)=\alpha\bigg\}.$$

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The level set $X(\psi,\alpha)$ is the multifractal decomposition set of ergodic averages of ψ. There are fruitful results about the descriptions of the structure (Hausdorff dimension or topological entropy or topological pressure) of these level sets in topological dynamical systems. We refer the reader to [1], [2], [7]–[9], [11], [13], [12], [16] and the references therein. In [11], Pfister and Sullivan consider the saturated sets and obtain a conditional variational principle. Let $C(X)$ be the space of continuous functions from X to $\mathbb{R}$. For $\varphi \in C(X)$ and $n \geq 1$, denote $S_n\varphi(x) := \sum_{i=0}^{n-1} \varphi(\sigma^i x)$. Denote by $M(X), M_\sigma(X)$ and $M_\sigma^e(X)$ the set of Borel probability measures on X, the collection of all σ-invariant Borel probability measures and all σ-ergodic invariant Borel probability measures, respectively. It is well-known that $M(X)$ and $M_\sigma(X)$ equipped with weak* topology are compact metrizable spaces. There exists a countable and separating set of continuous functions $\{f_1, f_2, \ldots\}$ with $0 \leq f_i(x) \leq 1$ on X such that

$$D(\mu,\nu) := \|\mu - \nu\| = \sum_{k\geq 1} \frac{1}{2^k} \left| \int f_k \, d\mu - \int f_k \, d\nu \right|,$$

defines a metric for the weak* topology on $M(X)$. Denote the limit point set of $\{x_n\}_{n\geq 1}$ by $A(x_n)$. Define

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\sigma^i x}.$$

The generic set for $\mu \in M_\sigma(X)$ can be denoted by

$$G_\mu(X,\sigma) := \{x \in X : A(\mathcal{E}_n(x)) = \{\mu\}\}.$$

For any compact connected subset $K \subset M_\sigma(X)$, define the saturated set for K as follows:

$$G_K(X,\sigma) := \{x \in X : A(\mathcal{E}_n(x)) = K\}.$$

We define the multifractal spectrum for $\psi \in C(X)$ to be $\mathcal{L}_\psi := \{\alpha \in \mathbb{R} : X(\psi,\alpha) \neq \emptyset\}$. In [11], Pfister and Sullivan showed the following theorem.

Theorem 1.1. *If (X,σ) satisfies the g-almost product property and the uniform separation property, then for any compact connected non-empty set $K \subset M_\sigma(X)$,*

$$\inf\{h_\mu(\sigma) : \mu \in K\} = h_{\mathrm{top}}(G_K(X,\sigma)),$$

where $h_{\mathrm{top}}(\,\cdot\,)$ denotes the Bowen topological entropy.

We can see that the above theorem needs two conditions: g-almost product property and the uniform separation property. The g-almost product property is a kind of specification property which holds for the case of β-shifts. In this paper, we consider a class of symbolic systems which was studied in [5]. That is, (X,σ) is a symbolic system with non-uniform structure: there exists $\mathcal{G} \subset \mathcal{L}(X)$

which has (W)-specification and $\mathcal{L}(X)$ is edit approachable by $\mathcal{G}$. The details of definitions will be given in the next section. This can be considered as another kind of specification property which holds for both β-shifts, S-gap shifts and their subshift factors.

Our main results are the following.

Theorem 1.2. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$ and $\varphi\colon X \to \mathbb{R}$ be a continuous function. Suppose that $\mathcal{G} \subset \mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any compact connected subset $K \subset M_\sigma(X)$. We have*

$$P_{G_K(X,\sigma)}(\varphi) = \inf_{\mu\in K}\left\{h_\mu(\sigma) + \int \varphi\, d\mu\right\},$$

where $P_\bullet(\varphi)$ denotes the topological pressure.

Accordingly, we investigated the size of the irregular set. For $\psi \in C(X)$, set

$$\widehat{X}(\psi) := \left\{x \in X : \lim_{n\to\infty} \frac{S_n\psi(x)}{n} \text{ does not exist}\right\}.$$

Theorem 1.3. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$ and $\varphi\colon X \to \mathbb{R}$ be a continuous function. Suppose that $\mathcal{G} \subset \mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any $\psi \in C(X)$, either $\widehat{X}(\psi) = \emptyset$, or*

$$P_{\widehat{X}(\psi)}(\varphi) = P_X(\varphi),$$

where $P_\bullet(\varphi)$ denotes topological pressure.

2. Preliminaries

In this paper, we consider the symbolic space. Let $p \geq 2$ be an integer and $\mathcal{A} = \{1, \dots, p\}$. Define

$$\mathcal{A}^{\mathbb{N}} = \{(w_i)_{i=1}^{\infty} : w_i \in \mathcal{A} \text{ for } i \geq 1\},$$

which is compact with the product discrete topology. We can define the metric on $\mathcal{A}^{\mathbb{N}}$, for any $u, v \in \mathcal{A}^{\mathbb{N}}$, define

$$d(u, v) := e^{-|u\wedge v|},$$

where $|u \wedge v|$ denotes the maximal length n such that $u_1 = v_1, \dots, u_n = v_n$ and $d(u, v) = 0$ if $u = v$. We say that (X, σ) is a subshift over $\mathcal{A}$ if X is a compact subset of $\mathcal{A}^{\mathbb{N}}$, and $\sigma(X) \subset X$, where σ is the left shift map on $\mathcal{A}^{\mathbb{N}}$ defined by

$$\sigma((w_i)_{i=1}^{\infty}) = (w_{i+1})_{i=1}^{\infty}, \quad \text{for all } (w_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}.$$

In particular, (X, σ) is called the full shift over $\mathcal{A}$ if $X = \mathcal{A}^{\mathbb{N}}$. The language of X, denoted by $\mathcal{L} = \mathcal{L}(X)$, is the set of finite words that appear in some $x \in X$ that is,

$$\mathcal{L}(X) = \{w \in \mathcal{A}^* : [w] \neq \emptyset\},$$

where $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$ and $[w]$ is the central cylinder for w, which is the set of sequences $x \in X$ that begin with the word w. For any collection $\mathcal{D} \subset \mathcal{L}$, let $\mathcal{D}_n$ denote $\{w \in \mathcal{D} : |w| = n\}$. Thus, $\mathcal{L}_n$ is the set of all words of length n that appear in sequences belonging to X. Given words u, v, we use juxtaposition uv to denote the word obtained by concatenation.

We give the definition of topological pressure for non-compact set in symbolic systems.

Definition 2.1. Let X be a subshift space on a finite alphabet and $Z \subset X$ be an arbitrary Borel set. For $N \in \mathbb{N}$ and $t \in \mathbb{R}$, we define the following quantities:

$$M(Z, t, \varphi, N) = \inf\bigg\{\sum_{[w_0\ldots w_m]\in\mathcal{S}} \exp\bigg(-t(m+1) + \sup_{w\in[w_0\ldots w_m]} \sum_{k=0}^{m} \varphi(\sigma^k x)\bigg)\bigg\},$$

where the infimum is taken over all finite or countable collections $\mathcal{S}$ of cylinder sets $[w_0 \ldots w_m]$ with $m \geq N$ which cover Z. Define

$$M(Z, t, \varphi) = \lim_{N\to\infty} M(Z, t, \varphi, N).$$

The existence of the limit is guaranteed since the function $M(Z, t, \varphi, N)$ does not decrease with N. By standard techniques, we can show the existence of

$$P_Z(\varphi) = \inf\{t : M(Z, t, \varphi) = 0\},$$

and then, we define the topological pressure of Z by $P_Z(\varphi)$. If $\varphi = 0$, then $P_Z(0) = h_{\mathrm{top}}(Z)$, where $h_{\mathrm{top}}(Z)$ denotes the Bowen topological entropy.

Definition 2.2 ([5]). Given a shift space X and its language $\mathcal{L}$, consider a subset $\mathcal{G} \subset \mathcal{L}$. Given $\tau \in \mathbb{N}$, we say that $\mathcal{G}$ has (W)-specification with gap length τ if for every $v, w \in \mathcal{G}$ there is $u \in \mathcal{L}$ such that $v'uw' \in \mathcal{G}$ and $|u| \leq \tau$, whenever v' is suffix of v, w' is prefix of w.

Definition 2.3 ([5]). Define an edit of a word $w = w_1 \ldots w_n \in \mathcal{L}$ to be a transformation of w by one of the following actions, where $u^j \in \mathcal{L}$ are arbitrary words and $a, a' \in \mathcal{A}$ are arbitrary symbols.

(a) Substitution: $w = u^1 a u^2 \mapsto w' = u^1 a' u^2$.
(b) Insertion: $w = u^1 u^2 \mapsto w' = u^1 a' u^2$.
(c) Deletion: $w = u^1 a u^2 \mapsto w' = u^1 u^2$.

Given $v, w \in \mathcal{L}$, define the edit distance between v and w to be the minimum number of edits required to transform the word v into the word w: we will denote this by $\widehat{d}(v, w)$. The following lemma about describes the size of balls in the edit metric.

Proposition 2.4 ([5]). *There is $C > 0$ such that given $n \in \mathbb{N}$, $w \in \mathcal{L}_n$, and $\delta > 0$, we have*

$$\#\{v \in \mathcal{L} : \widehat{d}(v, w) \leq \delta n\} \leq Cn^C (e^{C\delta} e^{-\delta \log \delta})^n.$$

Next we introduce the key definition, which requires that any word in $\mathcal{L}$ can be transformed into a word in $\mathcal{G}$ with a relatively small number of edits.

DEFINITION 2.5 ([5]). Say that a non-decreasing function $g\colon \mathbb{N} \to \mathbb{N}$ is a mistake function if $g(n)/n$ converges to 0. We say that $\mathcal{L}$ is edit approachable by $\mathcal{G}$, where $\mathcal{G} \subset \mathcal{L}$, if there is a mistake function g such that for every $w \in \mathcal{L}$, there exists $v \in \mathcal{G}$ with $\widehat{d}(v, w) \le g(|w|)$.

LEMMA 2.6 ([5]). *For any continuous function $\varphi \in C(X)$ and any mistake function $g(n)\colon \mathbb{N} \to \mathbb{N}$, there is a sequence of positive numbers $\delta_n \to 0$ such that if $x, y \in X$ and $m, n \in \mathbb{N}$ are such that $\widehat{d}(x_1 \dots x_n, y_1 \dots y_m) \le g(n)$, then*

$$\left| \frac{1}{n}\, S_n\varphi(x) - \frac{1}{m}\, S_m\varphi(y) \right| \le \delta_n.$$

Similarly to the above lemma, we can give another lemma for measures.

LEMMA 2.7. *For any mistake function $g(n)$, there is a sequence of positive numbers $\delta_n \to 0$ such that if $x, y \in X$ and $m, n \in \mathbb{N}$ are such that $\widehat{d}(x_1 \dots x_n, y_1 \dots y_m) \le g(n)$, then $D(\mathcal{E}_n(x), \mathcal{E}_m(y)) \le \delta_n$.*

PROOF. Let $\{f_1, f_2, \dots\}$ with $0 \le f_i(x) \le 1$ be a countable and separating set of continuous functions on X and

$$D(\mu, \nu) := \|\mu - \nu\| = \sum_{k \ge 1} \frac{1}{2^k} \left| \int f_k \, d\mu - \int f_k \, d\nu \right|$$

for any $\mu, \nu \in M(X)$. From Lemma 2.6, we can choose $\delta_n^i \to 0$ as $n \to \infty$ for each f_i with $i \ge 1$. Hence, if $x, y \in X$ and $m, n \in \mathbb{N}$ are such that $\widehat{d}(x_1 \dots x_n, y_1 \dots y_m) \le g(n)$, then

$$\begin{aligned} D(\mathcal{E}_n(x), \mathcal{E}_m(y)) &= \sum_{k \ge 1} \frac{1}{2^k} \left| \int f_k \, d\mathcal{E}_n(x) - \int f_k \, d\mathcal{E}_m(y) \right| \\ &= \sum_{k \ge 1} \frac{1}{2^k} \left| \frac{S_n f_k(x)}{n} - \frac{S_m f_k(y)}{m} \right| 2^k \le \sum_{k \ge 1} \frac{\delta_n^k}{2^k}. \end{aligned}$$

We set $\delta_n := \sum_{k \ge 1} \delta_n^k / 2^k > 0$ and we have $D(\mathcal{E}_n(x), \mathcal{E}_m(y)) \le \delta_n$. Then

$$\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \sum_{k \ge 1} \frac{\delta_n^k}{2^k} = \sum_{k \ge 1} \lim_{n \to \infty} \frac{\delta_n^k}{2^k} = 0.$$

So we are done. □

We can get the following lemma, by applying [5, Proposition 4.2 and Lemma 4.3].

PROPOSITION 2.8 ([5]). *If $\mathcal{G}$ has (W)-specification, then there exists $\mathcal{F} \subset \mathcal{L}$, which has free concatenation property (for all $u, w \in \mathcal{F}$, we have $uw \in \mathcal{F}$) such that $\mathcal{L}$ is edit approachable by $\mathcal{F}$.*

3. Proof of Theorem 1.2

The upper bound is easy to get by Theorem 3.1 in [9]. To prove the lower bound, we begin with a proposition about measures.

PROPOSITION 3.1. *Let $\alpha_i, \beta_i \geq 0$ with $\sum\limits_{i=1}^{k} \alpha_i = 1$ and $\sum\limits_{i=1}^{k} \beta_i = 1$, for $\mu_i, m_i \in M(X)$, then we have*

$$D\bigg(\sum_{i=1}^{k} \alpha_i \mu_i, \sum_{i=1}^{k} \beta_i m_i\bigg) \leq \sum_{i=1}^{k} \alpha_i D(\mu_i, m_i) + \sum_{i=1}^{k} |\alpha_i - \beta_i| \|m_i\|,$$

where

$$\|m\| := \sup_{0<\|f\|\leq 1} \frac{1}{\|f\|} \bigg| \int f \, dm \bigg|.$$

PROOF.

$$\begin{aligned} D\bigg(\sum_{i=1}^{k} \alpha_i \mu_i, \sum_{i=1}^{k} \beta_i m_i\bigg) &\leq D\bigg(\sum_{i=1}^{k} \alpha_i \mu_i, \sum_{i=1}^{k} \alpha_i m_i\bigg) + D\bigg(\sum_{i=1}^{k} \alpha_i m_i, \sum_{i=1}^{k} \beta_i m_i\bigg) \\ &\leq \sum_{i=1}^{k} \alpha_i D(\mu_i, m_i) + \sum_{i=1}^{k} |\alpha_i - \beta_i| \|m_i\|. \qquad \square \end{aligned}$$

To estimate the lower bound, we need the following *Horseshoe theorem* given in [5].

THEOREM 3.2 ([5]). *Let X be a shift space. Suppose that $\mathcal{G} \subset \mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$. Then there exists an increasing sequence $\{X_n\}$ of compact σ-invariant subsets of X with the following properties*:

(a) *Each X_n is a topological transitive sofic shift.*
(b) *There is $T \in \mathbb{N}$ such that for every n and every $w \in \mathcal{L}(X_n)$, there are $u, v \in \mathcal{L}$ with $|u|, |v| \leq n + T$ such that $uwv \in \mathcal{G}$.*
(c) *Every invariant measure on X is entropy approachable by ergodic measures on X_n: for any $\eta > 0$, any $\mu \in \mathcal{M}_\sigma(X)$, and any neighborhood U of μ in $\mathcal{M}_\sigma(X)$, there exists $n \geq 1$ and $\mu' \in \mathcal{M}_\sigma^e(X_n) \cap U$ such that $h_{\mu'}(\sigma) > h_\mu(\sigma) - \eta$ holds.*

LEMMA 3.3 ([11]). *Let $K \subset M_\sigma(X)$ be a compact connected non-empty set. Then there exists a sequence $\{\alpha_1, \alpha_2, \ldots\}$ of K, such that*

$$\overline{\{\alpha_j : j \in \mathbb{N}, j > n\}} = K, \quad \textit{for all } n \in \mathbb{N}, \quad \textit{and} \quad \lim_{j\to\infty} d(\alpha_j, \alpha_{j+1}) = 0.$$

DEFINITION 3.4. Given $\mu \in M_\sigma(X)$ and $\varepsilon > 0$, let

$$\mathcal{L}_n^{\mu,\varepsilon} := \{w \in \mathcal{L}_n : D(\mathcal{E}_n(x), \mu) < \varepsilon \text{ for all } x \in [w]\}.$$

By Proposition 1.1 in [11], we have the following result.

LEMMA 3.5. *For any ergodic measure $\mu \in M_\sigma^e(X)$ and $\varepsilon, \delta > 0$, there exists $N(\mu, \varepsilon, \delta) \in \mathbb{N}$ such that for $n \geq N(\mu, \varepsilon, \delta)$, we have*

$$\# \mathcal{L}_n^{\mu,\varepsilon} \geq e^{n(h_\mu(\sigma)-\delta)},$$

where # denotes the cardinality of a set.

Now we begin to show the lower bound of Theorem 1.2.

3.1. Choose a sequence $\{n_j\}_{j\geq 1}$. Let $\eta > 0$ and

$$h^* = \inf \left\{ h_\mu(\sigma) + \int \varphi \, d\mu : \mu \in K \right\} - \eta.$$

We only need to prove that $P_{G_K(X,\sigma)}(\varphi) \geq h^*$. We choose ε_0 small enough, such that for any $\mu, \alpha \in M(X)$ with $D(\alpha, \mu) < \varepsilon_0$, we have

$$\left| \int \varphi \, d\mu - \int \varphi \, d\alpha \right| < \frac{\eta}{2}. \tag{3.1}$$

By Horseshoe Theorem 3.2, we choose a sequence of measures $\{\alpha_j\}_{j\geq 1}$ in K satisfying Lemma 3.3. Then, for any j, there exist $X_j \subset X$, $\mu_j \in M_\sigma^e(X_j)$ and $\varepsilon_j \to 0$ with $\varepsilon_j < \varepsilon_0$ such that

$$D(\mu_j, \alpha_j) < \frac{\varepsilon_j}{2} \quad \text{and} \quad h_{\mu_j}(\sigma) > h_{\alpha_j}(\sigma) - \frac{\eta}{2}.$$

It follows from Lemma 3.5 that there exists $\widehat{N}_j$ such that $n > \widehat{N}_j$ satisfies

$$\# \mathcal{L}_n^{\mu_j,\varepsilon_j/2} \geq e^{n(h_{\mu_j}(\sigma)-\eta/2)}.$$

Thus, for any $n > \widehat{N}_j$,

$$\# \mathcal{L}_n^{\alpha_j,\varepsilon_j} \geq \# \mathcal{L}_n^{\mu_j,\varepsilon_j/2} \geq e^{n(h_{\alpha_j}(\sigma)-\eta)}.$$

By the assumptions and Proposition 2.8, $\mathcal{L}$ is edit approachable by some $\mathcal{F}$ which has free concatenation property. We can define a map $\phi_{\mathcal{F}} \colon \mathcal{L} \to \mathcal{F}$ such that $\widehat{d}(w, \phi_{\mathcal{F}}(w)) \leq g(|w|)$, and then we can define a map $\Phi \colon \mathcal{L}^* \to \mathcal{F}$ by editing then gluing. That is

$$(w^1, \ldots, w^n) \mapsto \phi_{\mathcal{F}}(w^1)\phi_{\mathcal{F}}(w^2) \ldots \phi_{\mathcal{F}}(w^n),$$

where $\mathcal{L}^* := \{(w^1, \ldots, w^n) : w^j \in \mathcal{L}, 1 \leq j \leq n, n \in \mathbb{N}\}$. Let $\chi \ll \eta$ be small enough with $C\chi - \chi \log \chi < \eta$; one can pick $M > 0$ such that

$$Cn^C (e^{C\chi} e^{-\chi \log \chi})^n < e^{(n-g(n))\eta} \tag{3.2}$$

for all $n \geq M$. From Lemma 2.6, (3.1) and the fact $g(n)/n \to 0$, we can choose $n_j \to \infty$, such that $n_j > \max\{\widehat{N}_j, M\}$,

$$\frac{g(n_j)}{n_j} \ll \chi \ll \eta, \tag{3.3}$$

$$\left| \frac{S_{|\phi_{\mathcal{F}}(w^j)|}\varphi(y)}{|\phi_{\mathcal{F}}(w^j)|} - \frac{S_{n_j}\varphi(x)}{n_j} \right| \leq \frac{\eta}{2}, \qquad \left| \frac{S_{n_j}\varphi(x)}{n_j} - \int \varphi \, d\alpha_j \right| \leq \frac{\eta}{2}, \tag{3.4}$$

and

$$\frac{g(n_j)h_{\mathrm{top}}(\sigma)}{n_j - g(n_j)} \leq \eta, \tag{3.5}$$

for any $w^j \in \mathcal{L}_{n_j}^{\alpha_j,\varepsilon_j}$, $x \in [w^j]$, $y \in [\phi_{\mathcal{F}}(w^j)]$. Moreover, by Lemma 2.7, we obtain

$$D(\mathcal{E}_{n_j}(x), \mathcal{E}_{|\phi_{\mathcal{F}}(w^j)|}(y)) \to 0, \tag{3.6}$$

for each $w^j \in \mathcal{L}_{n_j}^{\alpha_j,\varepsilon_j}$ and any $x \in [w^j]$, $y \in [\phi_{\mathcal{F}}(w^j)]$. By Proposition 2.4, (3.2) and (3.3),

$$\begin{aligned}\#\{w \in \mathcal{L}_{n_j}^{\alpha_j,\varepsilon_j} : \phi_{\mathcal{F}}(w) = v\} &\leq C n_j^C (e^{C\chi} e^{-\chi \log \chi})^{n_j} \\ &< e^{(n_j - g(n_j))\eta} \leq e^{|v|\eta}.\end{aligned} \tag{3.7}$$

3.2. Construction of the Moran set H. For brevity of notation, we write $\mathcal{D}_j = \mathcal{L}_{n_j}^{\alpha_j,\varepsilon_j}$. Moreover, we pick a strictly increasing sequence $N_k \to \infty$ with $N_k \in \mathbb{N}$,

$$\lim_{k\to\infty} \frac{n_{k+1} + g(n_{k+1})}{\sum_{j=1}^{k}(n_j - g(n_j))N_j} = 0, \qquad \lim_{k\to\infty} \frac{\sum_{j=1}^{k}(n_j + g(n_j))N_j}{\sum_{j=1}^{k+1}(n_j - g(n_j))N_j} = 0. \tag{3.8}$$

We now define new sequences $\{n'_j\}, \{\alpha'_j\}$ and $\{\mathcal{D}'_j\}$ by setting for $j = N_1 + \ldots + N_{k-1} + q$ with $1 \leq q \leq N_k$,

$$\varepsilon'_j := \varepsilon_k, n'_j := n_k, \qquad \alpha'_j := \alpha_k, \mathcal{D}'_j := \mathcal{D}_k.$$

Consider the map $\Phi\colon \prod_{j=1}^{\infty} \mathcal{D}'_j \to X$, defined by editing then gluing. More precisely, given $\mathbf{w} = \{w^j\}_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \mathcal{D}'_j$, let $v^j = \phi_{\mathcal{F}}(w^j) \in \mathcal{F}$ and $\Phi(\mathbf{w}) = v^1 v^2 \ldots$

Put $H := \Phi\Big(\prod_{j=1}^{\infty} \mathcal{D}'_j\Big)$. Next we will prove the fact that $H \subset G_K(X, \sigma)$. For any $\mathbf{w} = \{w^j\}_{j=1}^{\infty} \in \prod_{j=1}^{\infty} \mathcal{D}'_j$, we define $l_j = l_j(w^j) = |\phi_{\mathcal{F}}(w^j)|$ for the length of the words associated to the index j. Clearly,

$$|l_j - n'_j| \leq g(n'_j). \tag{3.9}$$

From the construction of α'_j, we have $A(\alpha'_j) = A(\alpha_j) = K$. For any $j = N_1 + \ldots + N_{k-1} + q$, $1 \leq q \leq N_k$, we define $t_j = \sum_{i=1}^{j} l_i$. By (3.8) and (3.9), we obtain

$$\frac{l_j}{t_j} \leq \frac{n'_j + g(n'_j)}{\sum_{i=1}^{j} l_i} \to 0.$$

Hence, $A(\mathcal{E}_{t_j}(\Phi(\mathbf{w}))) = A(\mathcal{E}_n(\Phi(\mathbf{w})))$. Then it is sufficient to show that

$$\lim_{j\to\infty} D(\mathcal{E}_{t_j}(\Phi(\mathbf{w})), \alpha'_j) = 0.$$

Assume that $j = N_1 + \ldots + N_{k-1} + q$, $1 \le q \le N_k$, we have $\alpha'_j = \alpha_k$. Define $c_k := N_1 + \ldots + N_k$. Then we can make the following estimate:

$$\begin{aligned}
D(\mathcal{E}_{t_j}(\Phi(\mathbf{w})), \alpha'_j) \le{}& \frac{t_{c_{k-2}}}{t_j} D(\mathcal{E}_{t_{c_{k-2}}}(\Phi(\mathbf{w})), \alpha'_j) \\
&+ \frac{t_{c_{k-1}} - t_{c_{k-2}}}{t_j} D(\mathcal{E}_{t_{c_{k-1}} - t_{c_{k-2}}}(\sigma^{t_{c_{k-2}}}\Phi(\mathbf{w})), \alpha'_j) \\
&+ \frac{t_j - t_{c_{k-1}}}{t_j} D(\mathcal{E}_{t_j - t_{c_{k-1}}}(\sigma^{t_{c_{k-1}}}\Phi(\mathbf{w})), \alpha'_j) \\
\le{}& \frac{t_{c_{k-2}}}{t_j} D(\mathcal{E}_{t_{c_{k-2}}}(\Phi(\mathbf{w})), \alpha'_j) \\
&+ \frac{t_{c_{k-1}} - t_{c_{k-2}}}{t_j} D(\mathcal{E}_{t_{c_{k-1}} - t_{c_{k-2}}}(\sigma^{t_{c_{k-2}}}\Phi(\mathbf{w})), \alpha_{k-1}) \\
&+ \frac{t_{c_{k-1}} - t_{c_{k-2}}}{t_j} D(\alpha_{k-1}, \alpha_k) \\
&+ \frac{t_j - t_{c_{k-1}}}{t_j} D(\mathcal{E}_{t_j - t_{c_{k-1}}}(\sigma^{t_{c_{k-1}}}\Phi(\mathbf{w})), \alpha_k).
\end{aligned}$$

From (3.8), (3.9) and Lemma 3.3, we obtain

$$\begin{aligned}
(3.10)\quad & \frac{t_{c_{k-2}}}{t_j} \\
\le{}& \frac{N_1 n_1 + \ldots + N_{k-2} n_{k-2} + N_1 g(n_1) + \ldots + N_{k-2} g(n_{k-2})}{N_1 n_1 + \ldots + N_{k-1} n_{k-1} + q n_k - N_1 g(n_1) - \ldots - N_{k-1} g(n_{k-1}) - q g(n_k)} \to 0, \\
& D(\alpha_{k-1}, \alpha_k) \to 0.
\end{aligned}$$

For $1 \le i \le N_{k-1}$, taking any $x_i \in [w^{c_{k-2}+i}]$, we can make the following estimate:

$$\begin{aligned}
& D(\mathcal{E}_{t_{c_{k-1}} - t_{c_{k-2}}}(\sigma^{t_{c_{k-2}}}\Phi(\mathbf{w})), \alpha_{k-1}) \\
&\le D\Bigg(\sum_{i=1}^{N_{k-1}} \frac{l_{c_{k-2}+i}}{t_{c_{k-1}} - t_{c_{k-2}}} \mathcal{E}_{l_{c_{k-2}+i}}(\sigma^{t_{c_{k-2}+i-1}}\Phi(\mathbf{w})), \sum_{i=1}^{N_{k-1}} \frac{n_{k-1}}{N_{k-1} n_{k-1}} \mathcal{E}_{n_{k-1}}(x_i) \Bigg) \\
&\quad + D\Bigg(\sum_{i=1}^{N_{k-1}} \frac{n_{k-1}}{N_{k-1} n_{k-1}} \mathcal{E}_{n_{k-1}}(x_i), \alpha_{k-1} \Bigg) \\
&\le \sum_{i=1}^{N_{k-1}} \frac{n_{k-1}}{N_{k-1} n_{k-1}} D(\mathcal{E}_{l_{c_{k-2}+i}}(\sigma^{t_{c_{k-2}+i-1}}\Phi(\mathbf{w})), \mathcal{E}_{n_{k-1}}(x_i))
\end{aligned}$$

$$+\sum_{i=1}^{N_{k-1}}\left|\frac{l_{c_{k-2}+i}}{t_{c_{k-1}}-t_{c_{k-2}}}-\frac{n_{k-1}}{N_{k-1}n_{k-1}}\right|\|\mathcal{E}_{l_{c_{k-2}+i}}(\sigma^{t_{c_{k-2}+i-1}}\Phi(\mathbf{w}))\|$$
$$+\sum_{i=1}^{N_{k-1}}\frac{1}{N_{k-1}}D(\mathcal{E}_{n_{k-1}}(x_i),\alpha_{k-1}),$$

where the above inequality follows from Proposition 3.1. By (3.6), we have

$$\sum_{i=1}^{N_{k-1}}\frac{n_{k-1}}{N_{k-1}n_{k-1}}D(\mathcal{E}_{l_{c_{k-2}+i}}(\sigma^{t_{c_{k-2}+i-1}}\Phi(\mathbf{w})),\mathcal{E}_{n_{k-1}}(x_i))\to 0\quad (j\to\infty);$$

by the definition of $\mathcal{D}'_j$, we have

$$\sum_{i=1}^{N_{k-1}}\frac{1}{N_{k-1}}D(\mathcal{E}_{n_{k-1}}(x_i),\alpha_{k-1})\to 0.$$

We estimate the following

$$\begin{aligned}
&\sum_{i=1}^{N_{k-1}}\left|\frac{l_{c_{k-2}+i}}{t_{c_{k-1}}-t_{c_{k-2}}}-\frac{n_{k-1}}{N_{k-1}n_{k-1}}\right|\|\mathcal{E}_{l_{c_{k-2}+i}}(\sigma^{t_{c_{k-2}+i-1}}\Phi(\mathbf{w}))\|\\
&\quad\le\sum_{i=1}^{N_{k-1}}\left|\frac{l_{c_{k-2}+i}}{t_{c_{k-1}}-t_{c_{k-2}}}-\frac{n_{k-1}}{N_{k-1}n_{k-1}}\right|=\sum_{i=1}^{N_{k-1}}\left|\frac{l_{c_{k-2}+i}}{t_{c_{k-1}}-t_{c_{k-2}}}-\frac{1}{N_{k-1}}\right|\\
&\quad=\sum_{i=1}^{N_{k-1}}\left|\frac{l_{c_{k-2}+i}N_{k-1}-(t_{c_{k-1}}-t_{c_{k-2}})}{(t_{c_{k-1}}-t_{c_{k-2}})N_{k-1}}\right|\\
&\quad=\sum_{i=1}^{N_{k-1}}\left|\frac{(l_{c_{k-2}+i}-n_{k-1})N_{k-1}+n_{k-1}N_{k-1}-(t_{c_{k-1}}-t_{c_{k-2}})}{(t_{c_{k-1}}-t_{c_{k-2}})N_{k-1}}\right|\\
&\quad\le\sum_{i=1}^{N_{k-1}}\left|\frac{2N_{k-1}g(n_{k-1})}{(t_{c_{k-1}}-t_{c_{k-2}})N_{k-1}}\right|\le\sum_{i=1}^{N_{k-1}}\left|\frac{2N_{k-1}g(n_{k-1})}{N_{k-1}^2(n_{k-1}-g(n_{k-1}))}\right|\\
&\quad=\frac{2}{N_{k-1}}\sum_{i=1}^{N_{k-1}}\left|\frac{g(n_{k-1})}{n_{k-1}-g(n_{k-1})}\right|\to 0
\end{aligned}$$

as $k\to\infty$, since

$$\lim_{k\to 0}\frac{g(n_{k-1})}{n_{k-1}-g(n_{k-1})}=\lim_{k\to 0}\frac{g(n_{k-1})}{n_{k-1}}\to 0.$$

So we have

$$D(\mathcal{E}_{t_{c_{k-1}}-t_{c_{k-2}}}(\sigma^{t_{c_{k-2}}}\Phi(\mathbf{w})),\alpha_{k-1})\to 0 \tag{3.11}$$

as $k\to\infty$.

Likewise, choose any $y_i \in [w^{c_{k-1}+i+1}]$, for $0 \le i \le q-1$, it follows that

$$\begin{aligned}
&D(\mathcal{E}_{t_j - t_{c_{k-1}}}(\sigma^{t_{c_{k-1}}}\Phi(\mathbf{w})), \alpha_k) \\
&= D\biggl(\sum_{i=0}^{q-1} \frac{l_{c_{k-1}+i+1}}{t_j - t_{c_{k-1}}}\, \mathcal{E}_{l_{c_{k-1}+i+1}}(\sigma^{t_{c_{k-1}+i}}\Phi(\mathbf{w})), \alpha_k\biggr) \\
&\le D\biggl(\sum_{i=0}^{q-1} \frac{l_{c_{k-1}+i+1}}{t_j - t_{c_{k-1}}}\, \mathcal{E}_{l_{c_{k-1}+i+1}}(\sigma^{t_{c_{k-1}+i}}\Phi(\mathbf{w})), \sum_{i=0}^{q-1} \frac{n_k}{qn_k}\, \mathcal{E}_{n_k}(y_i)\biggr) \\
&\quad + D\biggl(\sum_{i=0}^{q-1} \frac{n_k}{qn_k}\, \mathcal{E}_{n_k}(y_i), \alpha_k\biggr) \\
&\le \sum_{i=0}^{q-1} \frac{1}{q}\, D(\mathcal{E}_{l_{c_{k-1}+i+1}}(\sigma^{t_{c_{k-1}+i}}\Phi(\mathbf{w})), \mathcal{E}_{n_k}(y_i)) \\
&\quad + \sum_{i=0}^{q-1} \left| \frac{l_{c_{k-1}+i+1}}{t_j - t_{c_{k-1}}} - \frac{1}{q} \right| \|\mathcal{E}_{n_k}(y_i)\| + \sum_{i=0}^{q-1} \frac{1}{q}\, D(\mathcal{E}_{n_k}(y_i), \alpha_k),
\end{aligned}$$

where the above inequality follows from Proposition 3.1. By (3.6), we have

$$\sum_{i=0}^{q-1} \frac{1}{q}\, D(\mathcal{E}_{l_{c_{k-1}+i+1}}(\sigma^{t_{c_{k-1}+i}}\Phi(\mathbf{w})), \mathcal{E}_{n_k}(y_i)) \quad (k \to \infty).$$

It follows from the definition of $\mathcal{D}_j^{'}$ that,

$$\sum_{i=0}^{q-1} \frac{1}{q}\, D(\mathcal{E}_{n_k}(y_i), \alpha_k) \to 0 \quad (k \to \infty).$$

Next, we estimate the following:

$$\begin{aligned}
&\sum_{i=0}^{q-1} \left| \frac{l_{c_{k-1}+i+1}}{t_j - t_{c_{k-1}}} - \frac{1}{q} \right| \|\mathcal{E}_{n_k}(y_i)\| \le \sum_{i=0}^{q-1} \left| \frac{l_{c_{k-1}+i+1}}{t_j - t_{c_{k-1}}} - \frac{1}{q} \right| \\
&= \sum_{i=0}^{q-1} \left| \frac{q l_{c_{k-1}+i+1} - (t_j - t_{c_{k-1}})}{q(t_j - t_{c_{k-1}})} \right| \\
&= \sum_{i=0}^{q-1} \left| \frac{q(l_{c_{k-1}+i+1} - n_k) + qn_k - (t_j - t_{c_{k-1}})}{q(t_j - t_{c_{k-1}})} \right| \\
&\le \sum_{i=0}^{q-1} \left| \frac{2qg(n_k)}{q^2(n_k - g(n_k))} \right| = \frac{2}{q} \sum_{i=0}^{q-1} \left| \frac{g(n_k)}{(n_k - g(n_k))} \right| \to 0
\end{aligned}$$

as $k \to \infty$, since

$$\lim_{k \to 0} \frac{g(n_k)}{(n_k - g(n_k))} = \lim_{k \to 0} \frac{g(n_{k-1})}{n_{k-1}} \to 0.$$

Hence,

$$D(\mathcal{E}_{t_j - t_{c_{k-1}}}(\sigma^{t_{c_{k-1}}}\Phi(\mathbf{w})), \alpha_k) \to 0 \tag{3.12}$$

as $k \to \infty$. By (3.10)–(3.12), one has

$$\lim_{j\to\infty} D(\mathcal{E}_{t_j}(\Phi(\mathbf{w})), \alpha_j') = 0.$$

So we proved $H \subset G_K(X,\sigma)$.

3.3. To estimate the lower bound. Now we prove $P_{G_K(X,\sigma)}(\varphi) \geq h^*$. From the choice of N_j and the fact that $|l_j - n_j'| \leq g(n_j')$, one can readily verify that $\lim_{j\to\infty} t_j/t_{j+1} = 1$. For any $j \in \mathbb{N}$, we have

$$\#\mathcal{D}_j' = \#\,\mathcal{L}_{n_j'}^{\alpha_j',\varepsilon_j'} \geq e^{n_j'(h_{\alpha_j'}(\sigma)-\eta)}. \tag{3.13}$$

Let $\mathbf{w} = (w^1, w^2, \ldots) \in \prod_{i=1}^{\infty} \mathcal{D}_j'$. Then, for any w^j, by (3.4), we have

$$\begin{aligned} &\bigg|\int \varphi\, d\mathcal{E}_{l_j}(\sigma^{t_{j-1}}\Phi(\mathbf{w})) - \int \varphi\, d\alpha_j'\bigg| \\ &\qquad = \bigg|\frac{S_{l_j}\varphi(\sigma^{t_{j-1}}\Phi(\mathbf{w}))}{l_j} - \frac{S_{n_j'}\varphi(x)}{n_j'}\bigg| + \bigg|\frac{S_{n_j'}\varphi(x)}{n_j'} - \int \varphi d\alpha_j'\bigg| \leq \eta, \end{aligned} \tag{3.14}$$

where $j \geq 1$, $x \in [w^j]$.

Clearly, H is a compact set. We can just consider finite cover $\mathcal{C}$ of H by cylinder sets with the property that if $[w] \in \mathcal{C}$, then $[w] \cap H \neq \emptyset$. For each $[w] \in \mathcal{C}$, $|w| > N$, we define the cover $\mathcal{C}'$, in which each cylinder $[w] \in \mathcal{C}$ is replaced by its prefix $[w|_{t_j}]$ where $t_j \leq |w| < t_{j+1}$. Then, for any $\widehat{h} < h^* - 4\eta$,

$$\begin{aligned} &M(H, \widehat{h}, \varphi, N) \\ &\qquad = \inf\bigg\{\sum_{[w_0\ldots w_m]\in\mathcal{C}} \exp\bigg(-\widehat{h}(m+1) + \sup_{x\in[w_0\ldots w_m]} \sum_{k=0}^{m} \varphi(\sigma^k x)\bigg)\bigg\} \\ &\qquad \geq \inf\bigg\{\sum_{[w_0\ldots w_{t_j-1}]\in\mathcal{C}'} \exp\bigg(-\widehat{h}t_{j+1} + \sup_{x\in[w_0\ldots w_m]} \sum_{k=0}^{m} \varphi(\sigma^k x)\bigg)\bigg\}. \end{aligned}$$

Consider a specific $\mathcal{C}'$ and let s be the largest value of j such that there exists $[w|_{t_j}] \in \mathcal{C}^{'}$. In the following, we set

$$\mathcal{W}_k := \prod_{i=1}^{k} \Phi(\mathcal{D}_j'), \qquad \overline{\mathcal{W}}_s := \bigcup_{k=1}^{s} \mathcal{W}_k.$$

For any $w^j \in \mathcal{D}_j'$ and $l_j = l_j(w^j)$, we can use (3.5) and (3.13) to get

$$\#\,\mathcal{D}_j' \geq e^{n_j'(h_{\alpha_j'}(\sigma)-\eta)} \geq e^{l_j(h_{\alpha_j'}(\sigma)-\eta)-g(n_j')(h_{\alpha_j'}(\sigma)-\eta)} \geq e^{l_j(h_{\alpha_j'}(\sigma)-\eta)-l_j\eta}. \tag{3.15}$$

Furthermore, (3.7), (3.15) and (3.14) yield

(3.16)
$$\begin{aligned}\#\,\Phi(\mathcal{D}'_j) &\geq e^{l_j(h_{\alpha'_j}(\sigma)-\eta)-2\widehat{l}_j\eta} \\ &\geq \exp\left(l_j\left(h_{\alpha'_j}(\sigma)+\int\varphi\,d\alpha'_j-\eta\right)-S_{l_j}\varphi(\sigma^{t_{j-1}}\Phi(\mathbf{w}))-3\widehat{l}_j\eta\right) \\ &\geq \exp\left(l_jh^*-S_{l_j}\varphi(\sigma^{t_{j-1}}\Phi(\mathbf{w}))-3\widehat{l}_j\eta\right),\end{aligned}$$

for any $\mathbf{w}=(w^1,w^2,\dots)\in\prod_{i=1}^{\infty}\mathcal{D}'_j$ and $l_j=l_j(w^j)$, $\widehat{l}_j:=\max_{w^j\in\mathcal{D}'_j}l_j$. By (3.16), we obtain

$$|\mathcal{W}_k|\geq\exp\left(t_kh^*-3\widehat{t}_k\eta-S_{t_k}\varphi(\Phi(\mathbf{w}))\right), \tag{3.17}$$

for any $\mathbf{w}=(w^1,w^2,\dots)\in\prod_{i=1}^{\infty}\mathcal{D}'_j$ and $\widehat{t}_k:=\sum_{i=1}^{k}\widehat{l}_i$. For $1\leq j\leq k$, we say the word $v_1\dots v_j\in\mathcal{W}_j$ is a prefix of $w=w_1\dots w_k\in\mathcal{W}_k$ if $v_i=w_i$, $i=1,\dots,j$. Note that each $w\in\mathcal{W}_k$ is the prefix of exactly $|\mathcal{W}_s|/|\mathcal{W}_k|$ words of $\mathcal{W}_s$. If $\mathcal{W}\subset\overline{\mathcal{W}}_s$ contains a prefix of each word of $\mathcal{W}_s$, then

$$\sum_{k=1}^{s}\frac{|\mathcal{W}\cap\mathcal{W}_k||\mathcal{W}_s|}{|\mathcal{W}_k|}\geq|\mathcal{W}_s|.$$

If $\mathcal{W}$ contains a prefix of each word of $\mathcal{W}_s$, we have

$$\sum_{k=1}^{s}\frac{|\mathcal{W}\cap\mathcal{W}_k|}{|\mathcal{W}_k|}\geq 1.$$

It follows from (3.17) that

$$\sum_{\mathcal{C}'}\exp\left(-t_jh^*+S_{t_j}\varphi(\Phi(\mathbf{w}))+3\widehat{t}_j\eta\right)\geq 1.$$

By (3.8), we have $t_j/t_{j+1}\to 1$ and $\widehat{t}_j/t_{j+1}\to 1$, moreover, we *claim* that for any $\mathbf{w}=(w^1,w^2,\dots)\in\prod_{i=1}^{\infty}\mathcal{D}'_j$,

$$(t_jh^*-S_{t_j}\varphi(\Phi(\mathbf{w}))-3\widehat{t}_j\eta)-\left(\widehat{h}t_{j+1}-\sup_{x\in[\Phi(\mathbf{w})|_{(m+1)}]}\sum_{k=0}^{m}\varphi(x)\right)>0.$$

Then, for N large enough,

$$\begin{aligned}M(H,\widehat{h},\varphi,N)&\geq\inf\left\{\sum_{[w_0\dots w_{t_j-1}]\in\mathcal{C}'}\exp\left(-\widehat{h}t_{j+1}+\sup_{x\in[w_0\dots w_m]}\sum_{k=0}^{m}\varphi(\sigma^kx)\right)\right\} \\ &\geq\sum_{\mathcal{C}'}\exp\left(-t_jh^*+S_{t_j}\varphi(\Phi(\mathbf{w}))+3\widehat{t}_j\eta\right)\geq 1.\end{aligned}$$

Moreover, $M(H,\widehat{h},\varphi)\geq 1$, so we have $P_H(\varphi)\geq\widehat{h}$. Hence, $P_{G_K(X,\sigma)}(\varphi)\geq P_H(\varphi)\geq\widehat{h}$. Together with $\widehat{h}<h^*-4\eta$ and η is arbitrary small, this completes the proof. $\square$

Corollary 3.6. *Let X be a shift space with $\mathcal{L}=\mathcal{L}(X)$. Suppose that $\mathcal{G}\subset\mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any $\mu\in M_\sigma(X)$, we have $G_\mu(X,\sigma)\neq\emptyset$.*

Proposition 3.7. *Let X be a shift space with $\mathcal{L}=\mathcal{L}(X)$. Suppose that $\mathcal{G}\subset\mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then $\mathcal{L}_\psi$ is a non-empty bounded interval. Furthermore, $\mathcal{L}_\psi=\{\int\psi\,d\mu:\mu\in M_\sigma(X)\}$.*

Proof. Let $\mathcal{I}_\psi:=\{\int\psi\,d\mu:\mu\in M_\sigma(X)\}$. Since $M_\sigma(X)$ is compact, then $\mathcal{I}_\psi$ is bounded. We will show $\mathcal{I}_\psi=\mathcal{L}_\psi$. For any $\alpha\in\mathcal{I}_\psi$, we can choose $\mu\in M_\sigma(X)$ such that $\alpha=\int\psi\,d\mu$. By Corollary 3.6, there exists $x\in G_\mu(X,\sigma)$, so $\alpha\in\mathcal{L}_\psi$, this shows $\mathcal{I}_\psi\subset\mathcal{L}_\psi$. On the other hand, for any $\alpha\in\mathcal{L}_\psi$, we can choose $x\in X(\psi,\alpha)$. Let μ be any weak* limit point of the sequence $\mathcal{E}_n(x)$. It is a standard result that $\mu\in M_\sigma(X)$, and it is easy to show that $\int\psi\,d\mu=\alpha$. So we have $\mathcal{I}_\psi=\mathcal{L}_\psi$. Secondly, we show $\mathcal{I}_\psi$ is an interval using the convexity of $M_\sigma(X)$. Assume $\mathcal{I}_\psi$ is not a single point. Let $\alpha_1,\alpha_2\in\mathcal{I}_\psi$. Let $\beta\in(\alpha_1,\alpha_2)$, and μ_i satisfying $\int\psi\,d\mu_i=\alpha_i$ for $i=1,2$. Let $t\in(0,1)$ satisfy $\beta=t\alpha_1+(1-t)\alpha_2$. If $m:=t\mu_1+(1-t)\mu_2$, then $\int\psi\,d\mu=\beta$. □

We give the following conditional variational principle.

Proposition 3.8. *Let X be a shift space with $\mathcal{L}=\mathcal{L}(X)$ and $\varphi\colon X\to\mathbb{R}$ be a continuous function. Suppose that $\mathcal{G}\subset\mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any $\psi\in C(X)$, $\alpha\in\mathbb{R}$*

$$P_{X(\psi,\alpha)}(\varphi)=\sup\bigg\{h_\mu(\sigma)+\int\varphi\,d\mu\;:\;\int\psi\,d\mu=\alpha\bigg\}.$$

Proof. Let $F(\alpha):=\{\mu\in M_\sigma(X):\int\psi\,d\mu=\alpha\}$. $F(\alpha)$ is a closed set. The statement $\lim\limits_{n\to\infty}S_n\psi(x)/n=\alpha$ is equivalent to the statement $\mathcal{E}_n(x)$ has all its limit-points in $F(\alpha)$. Let

$$G^{F(\alpha)}:=\{x\in X:A(\mathcal{E}_n(x))\subset F(\alpha)\}.$$

For any $\mu\in F(\alpha)$, we have $G_\mu(X,\sigma)\subset G^{F(\alpha)}$. So

$$h_\mu(\sigma)+\int\varphi\,d\mu=P_{G_\mu(X,\sigma)}(\varphi)\le P_{G^{F(\alpha)}}(\varphi),$$

and thus $\sup\{h_\mu(\sigma)+\int\varphi\,d\mu:\mu\in F(\alpha)\}\le P_{G^{F(\alpha)}}(\varphi)$. On the other hand, the upper bound can be verified by Theorem 3.1 in [9]. □

Finally, we can show the result about irregular set.

Proof of Theorem 1.3. Since the entropy map is upper semi-continuous, there exists ergodic measure $\mu\in M_\sigma^e(X)$ such that $P_X(\varphi)=h_\mu(\sigma)+\int\varphi\,d\mu$. By Lemma 2.1 in [14], $\widehat{X}(\psi)\neq\emptyset$ implies that there exists another ergodic measure

$\nu \in M_\sigma^e(X)$ and $\int \psi\, d\nu \neq \int \psi\, d\mu$. Let $p_n \in (0,1)$ and $p_n \to 0$, we define $\nu_n := p_n\nu + (1-p_n)\mu$. Clearly, $\nu_n \to \mu$ and

$$h_{\nu_n}(\sigma) + \int \varphi\, d\nu_n \to P_X(\varphi).$$

For any $\eta > 0$, choose $N \geq 1$, such that for any $n \geq N$,

$$h_{\nu_n}(\sigma) + \int \varphi\, d\nu_n \geq P_X(\varphi) - \eta.$$

Furthermore, define the compact connected subset $K_n := \{t\nu_n + (1-t)\mu : t \in [0,1]\} \subset M_\sigma(X)$. For any $n \geq 1$, we have

$$\widehat{X}(\psi) \supset \{x \in X : A(\mathcal{E}_j(x)) = K_n\}.$$

It turns out that

$$\begin{aligned} P_{\widehat{X}(\psi)}(\varphi) &\geq \inf_{m\in K_n}\left\{h_m(\sigma) + \int \varphi\, dm\right\} \\ &= \inf_{t\in[0,1]}\left\{t h_{\nu_n} + (1-t)h_\mu(\sigma) + t\int \varphi\, d\nu_n + (1-t)\int \varphi\, d\mu\right\} \\ &\geq \inf_{t\in[0,1]}\{t(P_X(\varphi)-\eta) + (1-t)P_X(\varphi)\} = P_X(\varphi) - \eta. \end{aligned}$$

Since η can be arbitrary small, the proof is complete. □

4. Applications

4.1. Hausdorff dimension. In this section, we use the Hausdorff dimension to describe the level set. We use a metric defined by Gatzouras and Peres in [6]. Given a strictly positive continuous function $\varphi\colon X \to \mathbb{R}$, we define a metric d_φ on X by

$$d_\varphi(x,y) = \begin{cases} e^{-\sup\limits_{z\in[x\wedge y]} S_{|x\wedge y|}\varphi(z)} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Observe that d_φ induces the product topology on X, since the strict positivity of φ implies that $S_n\varphi(x) \to \infty$ for all $x \in X$. Furthermore, we can use the metric d_φ to define Hausdorff dimension denoted by $\dim_\varphi(\,\cdot\,)$. Together with the definition of topological pressure, we readily get that for any set $Z \subset X$, the Hausdorff dimension of Z is a unique solution of Bowen equation $P_Z(-s\varphi) = 0$, i.e. $s = \dim_\varphi(Z)$. Moreover, by Theorem 1.3 and Proposition 3.8, we have the following conditional variational principles.

Proposition 4.1. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$. Suppose that $\mathcal{G} \subset \mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any $\psi \in C(X)$,*

$$\dim_\varphi(X(\psi,\alpha)) = \sup\left\{\frac{h_\mu(\sigma)}{\int \varphi\, d\mu} : \int \psi\, d\mu = \alpha\right\}.$$

Proposition 4.2. *Let X be a shift space with $\mathcal{L} = \mathcal{L}(X)$ and $\varphi\colon X \to \mathbb{R}$ be a strictly positive continuous function. Suppose that $\mathcal{G} \subset \mathcal{L}$ has (W)-specification and $\mathcal{L}$ is edit approachable by $\mathcal{G}$, then for any $\psi \in C(X)$, either $\widehat{X}(\psi) = \emptyset$, or*

$$\dim_\varphi(\widehat{X}(\psi)) = \dim_\varphi(X).$$

4.2. Subshifts. In this section, we study the level set of two main examples (S-gap shifts and β-shifts).

S-gap shifts. For a subshift of $\{0,1\}^{\mathbb{N}}$, fixed $S \subset \{0,1,\ldots\}$, the number of 0 between consecutive 1 is an integer in S. That is, the language

$$\big\{0^n 10^{n_1} 10^{n_2} 10^{n_3} 1 \ldots 10^{n_k} 10^m : 1 \le i \le k \text{ and } n_i \in S,\, n, m \in \mathbb{N}\big\},$$

together with $\{0^n : n \in \mathbb{N}\}$, and this subshift is denoted by Σ_S.

β-shifts. Fix $\beta > 1$, write $b = \lceil \beta \rceil$ and let $w^\beta \in \{0,1,\ldots,b-1\}^{\mathbb{N}}$ be the greedy β-expansion of 1. Then w^β satisfies $\sum\limits_{j=1}^{\infty} w_j^\beta \beta^{-j} = 1$, and has the property that $\sigma^j(w^\beta) \prec w^\beta$ for all $j \ge 1$, where $\prec$ denotes the lexicographic ordering. The β-shift is defined by

$$\Sigma_\beta = \big\{x \in \{0,1,\ldots,b-1\}^{\mathbb{N}} : \sigma^j(x) \prec w^\beta \text{ for all } j \ge 1\big\}.$$

In fact, in [5], Climenhaga, Thompson and Yamamoto showed that all the subshift factors of S-gap shifts and β-shifts satisfy the the conditions of Theorem 1.2 (i.e. these subshifts have non-uniform structure). Hence, for $X = \Sigma_S$ or Σ_β, and $\varphi, \psi \in C(X)$, $\varphi > 0$, $\alpha \in \mathbb{R}$, we have

$$\dim_\varphi(X(\psi,\alpha)) = \sup\left\{\frac{h_\mu(\sigma)}{\int \varphi\, d\mu} : \int \psi\, d\mu = \alpha\right\},$$

and either $\widehat{X}(\psi) = \emptyset$ or $\dim_\varphi(\widehat{X}(\psi)) = \dim_\varphi(X)$. Accordingly, we also can use the topological pressure to describe these level set just like Theorems 1.2 and 1.3.

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