

CANCELLATIONS FOR CIRCLE-VALUED MORSE FUNCTIONS VIA SPECTRAL SEQUENCES

DAHISY V. DE S. LIMA — OZIRIDE MANZOLI NETO
KETTY A. DE REZENDE — MARIANA R. DA SILVEIRA

ABSTRACT. A spectral sequence analysis of a filtered Novikov complex $(\mathcal{N}_*(f), \Delta)$ over $\mathbb{Z}\langle\langle t \rangle\rangle$ is developed with the goal of obtaining results relating the algebraic and dynamical settings. Specifically, the unfolding of a spectral sequence of $(\mathcal{N}_*(f), \Delta)$ and the cancellation of its modules is associated to a one parameter family of circle-valued Morse functions on a surface and the dynamical cancellations of its critical points. The data of a spectral sequence computed for $(\mathcal{N}_*(f), \Delta)$ is encoded in a family of matrices Δ^r produced by the Spectral Sequence Sweeping Algorithm (SSSA), which has as its initial input the differential Δ . As one “turns the pages” of the spectral sequence, differentials which are isomorphisms produce cancellation of pairs of modules. Corresponding to these cancellations, a family of circle-valued Morse functions f^r is obtained by successively removing the corresponding pairs of critical points of f . We also keep track of all dynamical information on the birth and death of connecting orbits between consecutive critical points, as well as periodic orbits that arise within a family of negative gradient flows associated to f^r .

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1. Introduction

Given $f: M \rightarrow \mathbb{R}$ a Morse function, consider a filtered Morse chain complex associated to it. Once the dynamics is presented at the chain complex level, it is natural to use homology techniques to explore it. In [4], [6], [8], [9] we have chosen to work with a powerful homological tool, namely, a spectral sequence of a filtered Morse chain complex. Spectral sequence analysis has proven to be useful in relating algebraic and dynamical information and has been investigated by other authors, see for example [3], [2]. This relation has been explored for gradient flows generated by Morse functions and their filtered Morse chain complexes.

Let $f: M \rightarrow \mathbb{R}$ be a Morse function on a smooth closed n -manifold. Consider the Morse chain complex of f , $(C_*(f), \partial)$, where $C_*(f)$ is a \mathbb{Z} -module generated by the critical points and graded by their indices, i.e.

$$C_k(f) = \bigoplus_{x \in \text{Crit}_k(f)} \mathbb{Z}\langle x \rangle,$$

where $\text{Crit}_k(f)$ is the set of critical points of f of index k . The differential ∂ is defined on the generators of C_k by

$$\partial_k(x) = \sum_{y \in \text{Crit}_{k-1}(f)} n(x, y)\langle y \rangle,$$

where $n(x, y)$ is the intersection number of x and y . See [13] for more details.

It was proved in [6], [9] that a spectral sequence (E^r, d^r) of a filtered Morse chain complex $(C_*(f), \partial)$ can be retrieved from its differential ∂ via the Spectral Sequence Sweeping Algorithm (SSSA in short). In [8], spectral sequences were considered over \mathbb{Z}_2 and it was proven that changes of generators of the modules E^r determine bifurcation behavior. In [4] and [5], the spectral sequence analysis was realized for an n -dimensional filtered Morse chain complex over \mathbb{Z} , $n = 2$ or $n \geq 5$. This analysis relates the algebraic cancellations caused by non-zero differentials d^r to dynamical cancellations of pairs of critical points. Consequently, as one “turns” the pages of the spectral sequence, dynamical cancellations occur successively. Thus, we obtain a global dynamical cancellation theorem which keeps track of birth and death of connections of critical points.

Motivated by outlook, in this article, we undertake a completely new dynamical setup by considering circle-valued Morse functions $f: M \rightarrow S^1$ on a closed manifold M . Similarly to the Morse complex, the Novikov complex $(\mathcal{N}_*(f), \partial)$ associated to f is a chain complex of free modules generated by the critical points of f , see Section 2. However, instead of the ring of integers, the Novikov complex is defined over $\mathbb{Z}((t))$, the ring of Laurent series in one variable with integral coefficients and finite negative part. The Novikov boundary operator ∂ in this complex is related to the topological-dynamical features of a gradient flow of f . More specifically, ∂ “counts” orbits with signs connecting consecutive critical

points over $\mathbb{Z}((t))$ and it is represented by a matrix denominated as a Novikov matrix Δ .

In this article, we establish the groundwork for the association of a spectral sequence analysis of a filtered two dimensional Novikov chain complex with the dynamical exploration of connections in the negative gradient flow of f . This is done by applying the SSSA to $(\mathcal{N}_*(f), \partial)$. More specifically, as one “turns the pages” of this spectral sequence, i.e. considers progressively modules E^r , one observes that each algebraic cancellation occurring among the E^r , tracked by the SSSA, corresponds to a dynamical cancellation of a pair of critical points of f . Moreover, as r increases, the spectral sequence (E^r, d^r) stabilizes and converges to the Novikov homology $H_*^{\text{Nov}}(M, f)$ of f . Although cancellation of critical points in the Novikov setting has been done, e.g. in [10], the novelty in this article is a global dynamical cancellation theorem for circle-valued Morse functions via spectral sequence analysis.

It is unquestionable that we have developed a phenomenal algebraic tool, SSSA, to compute the modules and differentials d^r of a spectral sequence of a filtered Novikov chain complex. Although this result has considerable algebraic value in itself, this effort is fully compensated by the algebraic-dynamical cancellation results herein. A significant result uses the differentials d^r 's of a spectral sequence, that are responsible for algebraic cancellations, to understand critical point cancellations within a family of circle-valued Morse functions f^r and the respective family of associated negative gradient flows φ^r . The dynamical significance of these algebraic cancellations, ultimately, provides a backstory of death and birth of orbits of φ^r due to cancellations of consecutive critical points of f^r . Also, as the spectral sequences analysis unfolds, it is possible to detect all periodic orbits that arise within the family φ^r .

In Section 3, we recall the definition of a spectral sequence for a filtered chain complex and present the filtrations considered throughout this paper. The main tool behind this correspondence is the Spectral Sequence Sweeping Algorithm (SSSA), which is presented in Section 4. In Section 5, Theorem 5.1 addresses the fact that the SSSA can be applied to a Novikov complex $(\mathcal{N}_*(f), \Delta)$ in dimension 2. Also, within this section, it is shown that the SSSA produces a sequence of matrices, which we call Novikov matrices, that are characterized in Theorems 5.10 and 5.11. In Theorem 5.13, we prove the surprising result that the last matrix produced by the SSSA has polynomial entries ⁽¹⁾ in $\mathbb{Z}((t))$, although the intermediate Novikov matrices may exhibit entries which are infinite series. In Section 6, we prove in Theorems 6.3 and 6.4 that the modules and differentials (E^r, d^r) of the spectral sequence may be retrieved from the sequence of Novikov matrices produced by the SSSA. More specifically, the SSSA provides a system

⁽¹⁾ A polynomial in $\mathbb{Z}((t))$ is a finite sum of powers of t , t^ℓ , with $\ell \in \mathbb{Z}$.

which spans E^r in terms of the original basis of $\mathcal{N}_*(f)$ as well as identifies all differentials $d_p^r: E_p^r \rightarrow E_{p-r}^r$. Finding a system of $(r-1)$ -cycles that span E^r in terms of the original basis of $\mathcal{N}_*(f)$ is a non-trivial matter. In Section 7 we obtain in Theorem 7.2 a global dynamical cancellation theorem for circle-valued Morse functions via spectral sequences analysis.

Hence, the emphasis in this article is on the spectral sequence analysis and its correlation to the dynamics. The importance of this method is that it establishes the foundation for future investigation on the algebraic implications to the dynamics in the Novikov setting in higher dimensions.

2. Characterization of the Novikov complex

The goal of this section is to prove a characterization of the Novikov differential in the case of orientable surfaces, which will be essential for both dynamical and algebraic further results. Initially we present some background material on circle-valued functions and Novikov complex over $\mathbb{Z}((t))$. Further details can be found in [12].

Denote by $\mathbb{Z}[t, t^{-1}]$ the Laurent polynomial ring. Let $\mathbb{Z}((t))$ be the set consisting of all Laurent series

$$\lambda = \sum_{i \in \mathbb{Z}} a_i t^i$$

in one variable with coefficients $a_i \in \mathbb{Z}$, such that the part of λ with negative exponents is finite, i.e. there is $n = n(\lambda)$ such that $a_k = 0$ if $k < n(\lambda)$. In fact, $\mathbb{Z}((t))$ has a natural ring structure such that the inclusion $\mathbb{Z}[t, t^{-1}] \subset \mathbb{Z}((t))$ is a homomorphism. Moreover, $\mathbb{Z}((t))$ is a Euclidean ring.

Let M be a closed connected manifold and $f: M \rightarrow S^1$ be a smooth map from M to the one-dimensional sphere ⁽²⁾. Given a point $x \in M$ and a neighbourhood V of $f(x)$ in S^1 diffeomorphic to an open interval of \mathbb{R} , the map $f|_{f^{-1}(V)}$ is identified to a smooth map from $f^{-1}(V)$ to \mathbb{R} . Therefore, in this context one can define non-degenerate critical points and Morse index as in the classical case of smooth real-valued function. A smooth map $f: M \rightarrow S^1$ is called a *circle-valued Morse function* if its critical points are non-degenerate. The set of critical points of f will be denoted by $\text{Crit}(f)$ and $\text{Crit}_k(f)$ is the set of critical points of f of index k .

Considering the exponential function $\text{Exp}: \mathbb{R} \rightarrow S^1$ given by $t \mapsto e^{2\pi i t}$ and the infinite cyclic covering $E: \overline{M} \rightarrow M$ induced by the map $f: M \rightarrow S^1$ from the universal covering Exp , there exists a map $F: \overline{M} \rightarrow \mathbb{R}$ which makes the

⁽²⁾ S^1 is viewed as the submanifold $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and is endowed with the corresponding smooth structure.

following diagram commutative:

$$\begin{array}{ccc}
 \overline{M} & \xrightarrow{F} & \mathbb{R} \\
 E \downarrow & & \downarrow \text{Exp} \\
 M & \xrightarrow{f} & \mathbb{S}^1.
 \end{array}$$

Moreover, f is a circle-valued Morse function if and only if F is a real-valued Morse function. Observe that if $\text{Crit}(F)$ is non-empty then it has infinite cardinality. If \overline{M} is non-compact, one cannot apply the classical Morse theory to study F , however, one can restrict F to a fundamental cobordism W of \overline{M} , which is compact, and apply the techniques of Morse theory. The *fundamental cobordism* W is defined as $W = F^{-1}([a - 1, a])$, where a is a regular value of F . The cobordism W can be viewed as the manifold M obtained by cutting along the submanifold $V = f^{-1}(\alpha)$, where $\alpha = \text{Exp}(a)$. Hence, W is a cobordism with both boundary components diffeomorphic to V .

Given a circle-valued Morse function f , consider the vector field $v = -\nabla f$. One says that f satisfies the transversality condition if the lift of v to \overline{M} satisfies the classical transversality condition on the unstable and stable manifolds.

From now on, we consider circle-valued Morse functions f such that $v = -\nabla f$ satisfies the transversality condition. Denote by \bar{v} the lift of v to \overline{M} and arbitrarily choose orientations for all unstable manifolds $W^u(p)$ of critical points of f .

Given $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_{k-1}(f)$, the *Novikov incidence coefficient* between p and q is defined as

$$N(p, q; f) = \sum_{\ell \in \mathbb{Z}} n(p, t^\ell q; \bar{v}) t^\ell,$$

where $n(p, t^\ell q; \bar{v})$ is the intersection number between the critical points p and $t^\ell q$ of F , i.e. the number obtained by counting with signs the flow lines of \bar{v} from p to $t^\ell q$, when one considers the orientations on the unstable manifolds $W^u(p)$ and $W^u(t^\ell q)$ according to the previous fixed orientations in $W^u(p)$ and $W^u(q)$. For more details, see [1] and [12].

Let \mathcal{N}_k be the $\mathbb{Z}((t))$ -module freely generated by the critical points of f of index k . Consider the k -th boundary operator $\partial_k: \mathcal{N}_k \rightarrow \mathcal{N}_{k-1}$ which is defined on a generator $p \in \text{Crit}_k(f)$ by

$$\partial_k(p) = \sum_{q \in \text{Crit}_{k-1}(f)} N(p, q; f) q$$

and extended to all chains by linearity. In [12] it is proved that $\partial_k \circ \partial_{k+1} = 0$, hence $(\mathcal{N}_*(f), \partial)$ is a chain complex which is called the *Novikov complex* associated to the pair (f, v) .

One can view the Novikov differential ∂ as a matrix Δ where each column corresponds to generators $p, q \in \text{Crit}(f)$ and the entries are the coefficients $N(p, q; f)$ of the Novikov differential ∂ . Moreover, without loss of generality, one assumes that the columns of Δ are ordered with respect to the Morse indices of the critical points, e.g. in increasing order of Morse index. See Figure 1.

$$\Delta = \begin{matrix} & \mathcal{N}_0 & \cdots & \mathcal{N}_{k-1} & \mathcal{N}_k & \mathcal{N}_{k+1} & \mathcal{N}_{k+2} & \cdots & \mathcal{N}_n \\ \mathcal{N}_0 & \left(\begin{array}{cccccccc} 0 & & & & & & & & \\ & & & & & & & & \\ \vdots & & & & & & & & \\ \mathcal{N}_{k-2} & 0 & & \boxed{\Delta_{k-1}} & & & & & 0 \\ \mathcal{N}_{k-1} & 0 & & 0 & \boxed{\Delta_k} & & & & \\ \mathcal{N}_k & 0 & & & 0 & \boxed{\Delta_{k+1}} & & & \\ \mathcal{N}_{k+1} & 0 & & & & 0 & \boxed{\Delta_{k+2}} & & \\ \vdots & & & & & & & & \\ \mathcal{N}_n & 0 & & & & & & & 0 \end{array} \right) \end{matrix}$$

FIGURE 1. Novikov differential viewed as a matrix, where Δ_k is the matrixial representation of ∂_k .

In the case when M is a surface, the columns of Δ may be partitioned into subsets J_0, J_1, J_2 such that J_s are the columns associated with critical points of index s , i.e. the generators of \mathcal{N}_s . Hence, the non-zero entries of the matrix Δ are in the block $J_0 \times J_1$, which corresponds to connections from saddles to sinks, and in the block $J_1 \times J_2$, which corresponds to connections from sources to saddles. The block $J_0 \times J_1$ (respectively, $J_1 \times J_2$) is referred to as the first block (respectively, second block) of the matrix Δ . Figure 2 illustrates a possible structure for a Novikov differential associated to a circle-valued Morse function on a surface.

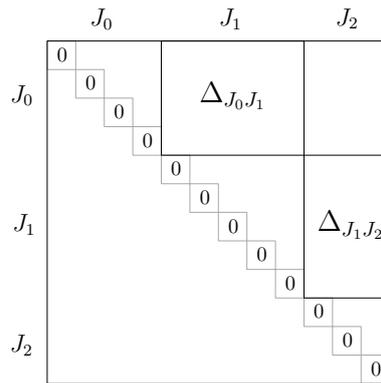


FIGURE 2. Novikov differential, with $J_0 = \{1, 2, 3, 4\}$, $J_1 = \{5, 6, 7, 8, 9\}$ and $J_2 = \{10, 11, 12\}$.

From this point on, we will use the notation Δ to denote interchangeably the Novikov differential ∂ and its matrix representation.

The following theorem presents special characteristics of the Novikov differential Δ by describing the Novikov incidence coefficients. In order to do this, we must define a fundamental domain. A cobordism $W = F^{-1}([a - 1; a + \lambda])$, where a is a regular value of F and $\lambda \in \mathbb{N}$, is said to be a *fundamental domain* for $(M; f)$ if the following property is satisfied: given $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_{k-1}(f)$, W contains a lift of each orbit of the flow v from p to q .

THEOREM 2.1. *Let M be an orientable surface and $(\mathcal{N}_*(f), \Delta)$ be the Novikov complex associated to a circle-valued Morse function $f: M \rightarrow S^1$. The Novikov incidence coefficient $N(p, q; f)$ is either zero, a monomial $\pm t^\ell$ or a binomial $t^{\ell_1} - t^{\ell_2}$.*

PROOF. Given an orientable surface M , let $W_\lambda = F^{-1}([a - 1, a + \lambda])$ be a fundamental domain for (M, f) . Then W_λ is an orientable compact surface with boundary ∂W_λ , possibly empty. If $\partial W_\lambda = \emptyset$, define $\widetilde{W}_\lambda = W_\lambda$. In the case that the boundary ∂W_λ is non-empty, it is the disjoint union of $\partial W_\lambda^- = F^{-1}(a - 1)$ and $\partial W_\lambda^+ = F^{-1}(a + \lambda)$. Let \widetilde{W}_λ be the closed surface obtained from W_λ by gluing a 2-dimensional disk along its boundary to each connected component of ∂W_λ . Moreover, one can assume that each of its disks contains a singularity, more specifically, a source if the disk is glued to ∂W_λ^+ and a sink if the disk is glued to ∂W_λ^- . This procedure extends the Morse function $F: W_\lambda \rightarrow \mathbb{R}$ to a classical Morse function $\widetilde{F}: \widetilde{W}_\lambda \rightarrow \mathbb{R}$ on a closed surface.

Given $p \in \text{Crit}_k(f)$ and $q \in \text{Crit}_{k-1}(f)$, the Novikov incidence coefficient $N(p, q; f)$ counts the number of flow lines from p to q with signs. Since, each of these flow lines has a lift in W_λ , then $N(p, q; f)$ can be obtained by analysing W_λ . On the other hand, the intersection number $n(p, t^\ell q; \bar{v})$ between critical points p and $t^\ell q$ in W_λ is the same as when considered in the surface \widetilde{W}_λ . Since \widetilde{W}_λ is closed, $n(p, t^\ell q; \bar{v})$ is zero, when there are two flow lines from p to $t^\ell q$. It is -1 or $+1$, when there is one flow line from p to $t^\ell q$. See [4]. Therefore, the Novikov incidence coefficient $N(p, q; f)$ is:

- (a) 0, if there are two flow lines from p to q in v which intersect the level set $f^{-1}(a)$ the same number of times;
- (b) $\pm t^\ell$, if there is only one flow line from p to q which intersects ℓ times the level set $f^{-1}(a)$;
- (c) $t^{\ell_1} - t^{\ell_2}$, if there are two flow lines from p to q in v , one intersecting ℓ_1 times the level set $f^{-1}(a)$ and the other one intersecting ℓ_2 times this level set. □

COROLLARY 2.2 (Characterization of the Novikov differential on orientable surfaces). *Let M be an orientable surface and $(\mathcal{N}_*(f), \Delta)$ be the Novikov complex associated to a circle-valued Morse function $f: M \rightarrow S^1$ such that the chosen*

orientation on the unstable manifold of each critical point of index 2 is the same.

Then there are three possibilities for either a column or a row $j \in J_1$ of Δ :

- (a) all entries are null,
- (b) exactly one non-zero entry which is a binomial $t^{\ell_1} - t^{\ell_2}$, for some $\ell_1, \ell_2 \in \mathbb{Z}$,
- (c) exactly two non-zero entries which are monomials t^{ℓ_1} and $-t^{\ell_2}$, for some $\ell_1, \ell_2 \in \mathbb{Z}$,
- (d) exactly one non-zero entry which is a monomial t^ℓ , for some $\ell \in \mathbb{Z}$.

PROOF. If f does not have a critical point of index 1, then the Novikov matrix is null. Suppose that f has at least one critical point of index 1, i.e. a saddle. In this case, it is clear that, given a row (respectively, column) $j \in J_1$ of Δ , there are at most two non-zero entries in this row (respectively, column). In fact, there are exactly two flow lines whose ω -limit (respectively, α -limit) sets are the same saddle. By [4], choosing the same orientation for each unstable manifold of critical points of index 2, the signs on flow lines associated to the stable (respectively, unstable) manifold of a saddle are opposite. Therefore, if there are two flow lines from a source (respectively, saddle) p to a saddle (respectively, sink) q intersecting a regular level set $f^{-1}(a)$ ℓ times, then $N(p, q; f) = t^\ell - t^\ell = 0$. On the other hand, if there is a flow line from a source (respectively, saddle) p to a saddle (respectively, sink) q intersecting ℓ_1 times a regular level set $f^{-1}(a)$ and a flow line from a source (respectively, saddle) p' to the saddle (respectively, sink) q intersecting ℓ_2 times the same regular level set, then $N(p, q; f) = \pm t^{\ell_1}$ and $N(p', q; f) = \mp t^{\ell_2}$. In particular, if $p' = p$ then $N(p, q; f) = \pm t^{\ell_1} \mp t^{\ell_2}$. Case (d) occurs in two situations. First when, given a saddle q , one of the flow lines in $W^s(q)$ has as α -limit a source p and the other flow line in $W^s(q)$ has as α -limit a periodic orbit. Hence $N(p, q; f) = \pm t^\ell$. Secondly, given a saddle p , one of the flow lines in $W^u(p)$ has as ω -limit a sink q and the other flow line in $W^u(p)$ has as ω -limit a periodic orbit. Hence $N(p, q; f) = \pm t^\ell$. \square

3. Spectral sequence for a Novikov complex

In this section, we present the basic results on spectral sequences associated to filtered chain complexes. This section is based on the references [7] and [14].

A spectral sequence E over $\mathbb{Z}((t))$ is a sequence $\{E^r, \partial^r\}$, for $r \geq 0$, such that: E^r is a bigraded module over $\mathbb{Z}((t))$, i.e. an indexed collection of R -modules $E_{p,q}^r$, for all $p, q \in \mathbb{Z}$; d^r is a differential of degree $(-r, r-1)$ on E^r , i.e. an indexed collection of homomorphisms $d^r: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$, for all $p, q \in \mathbb{Z}$, and $(d^r)^2 = 0$. Moreover, there exists an isomorphism $H(E^r) \approx E^{r+1}$, for all $r \geq 0$, where

$$H_{p,q}(E^r) = \frac{\text{Ker } d^r: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r}{\text{Im } d^r: E_{p+r, q-r+1}^r \rightarrow E_{p,q}^r}.$$

Given a chain complex (C_*, ∂) , an *increasing filtration* \mathcal{F} in C_* is a sequence of submodules $\mathcal{F}_p C$ for all integers p such that $\mathcal{F}_p C \subset \mathcal{F}_{p+1} C$. Since C is a graded module, the filtration must be compatible with the gradation, i.e. $\mathcal{F}_p C$ is graded by $\{\mathcal{F}_p C_q\}$.

Let $(\mathcal{N}_*(f), \Delta)$ be a Novikov complex and $\mathcal{F} = \{\mathcal{F}_p \mathcal{N}\}$ be a filtration on this complex defined by

$$(3.1) \quad F_p \mathcal{N} = \mathbb{Z}\langle t \rangle [h_{k_1}^1, h_{k_2}^2, \dots, h_{k_{p+1}}^{p+1}].$$

Note that, for each $p \in \mathbb{Z}$, there is only one singularity in $F_p \mathcal{N} \setminus F_{p-1} \mathcal{N}$, hence the filtration F is called a *finest filtration*. The filtration F is *convergent*, i.e. $\bigcap_p F_p \mathcal{N} = 0$ and $\bigcup_p F_p \mathcal{N} = \mathcal{N}$. In fact, F is *finite*, that is, $F_p \mathcal{N} = 0$ for some p and $F_{p'} \mathcal{N} = \mathcal{N}$ for some p' . Moreover, the filtration F is *bounded below* (i.e. given q , there exists $\varepsilon(q)$ such that $F_{\varepsilon(q)} \mathcal{N}_q = 0$) and hence, the induced filtration on $H_*(\mathcal{N})$ is also bounded below.

Since the filtration F on the chain complex $(\mathcal{N}_*(f), \Delta)$ is bounded below and convergent, then there exists a convergent spectral sequence with

$$\begin{aligned} E_{p,q}^0 &= F_p \mathcal{N}_{p+q} / F_{p-1} \mathcal{N}_{p+q} = G(\mathcal{N})_{p,q}, \\ E_{p,q}^1 &\approx H_{(p+q)}(F_p \mathcal{N}_{p+q} / F_{p-1} \mathcal{N}_{p+q}) \end{aligned}$$

and E^∞ is isomorphic to the module $GH_*(\mathcal{N})$. This result associates a spectral sequence to a filtered chain complex and its proof can be found in [14]. The algebraic formulas for the modules are

$$E_{p,q}^r = Z_{p,q}^r / (Z_{p-1,q+1}^{r-1} + \partial Z_{p+r-1,q-r+2}^{r-1}),$$

where

$$Z_{p,q}^r = \{c \in F_p \mathcal{N}_{p+q} \mid \partial c \in F_{p-r} \mathcal{N}_{p+q-1}\}.$$

Whenever the filtration considered is a finest filtration F , the only q such that $E_{p,q}^r$ is non-zero is $q = k - p$. Hence, we omit reference to q , i.e. E_p^r is in fact $E_{p,k-p}^r$. Note that, E^∞ does not determine $H_*(\mathcal{N})$ completely, but

$$E_{p,q}^\infty \approx GH_*(\mathcal{N})_{p,q} = \frac{F_p H_{p+q}(\mathcal{N})}{F_{p-1} H_{p+q}(\mathcal{N})}.$$

However, it is a well-known fact [7] that whenever $GH_*(\mathcal{N})_{p,q}$ is free and the filtration is bounded,

$$(3.2) \quad \bigoplus_{p+q=k} GH_*(\mathcal{N})_{p,q} \approx H_{p+q}(\mathcal{N}).$$

In the more general setting of a circle-valued function on an n -dimensional manifold, (3.2) need not be true. However, as a consequence of Theorem 5.1, which will be proved in Section 5, in the 2-dimensional setting, $GH_*(\mathcal{N})_{p,q}$ is free for all p and q and thus (3.2) holds. In this case, it follows that the spectral sequence

associated to the filtered Novikov complex (\mathcal{N}, Δ) , defined by the pair (f, M) , converges to the Novikov homology of M :

$$E_{p,q}^\infty \approx GH_*(\mathcal{N})_{p,q} = \frac{F_p H_{p+q}(\mathcal{N})}{F_{p-1} H_{p+q}(\mathcal{N})} \approx H_*^{\text{Nov}}(M, f).$$

4. Spectral Sequence Sweeping Algorithm for a Novikov complex

A square matrix will be called a *Novikov matrix* if it is a strictly upper triangular matrix with square zero and entries in the ring $\mathbb{Z}((t))$. In particular, a Novikov differential is a Novikov matrix.

In this section, we introduce the Spectral Sequence Sweeping Algorithm (SSSA) for a Novikov matrix associated to a Novikov complex $(\mathcal{N}_*(f), \Delta)$ on an orientable surface. The SSSA constructs a family of Novikov matrices $\{\Delta^r, r \geq 0\}$ recursively, where $\Delta^0 = \Delta$, by considering at each stage the r -th diagonal.

Spectral Sequence Sweeping Algorithm (SSSA). For a fixed r -th diagonal ⁽³⁾, the method described below must be applied for all Δ_k for $k = 0, 1, 2$ simultaneously.

A. Initial step.

1. Without loss of generality, we assume that the first diagonal of Δ contains non-zero entries $\Delta_{i,j}$ where $j \in J_k$ and $i \in J_{k-1}$. Whenever the first diagonal contains only zero entries, we define $\Delta^1 = \Delta$ and we repeat this step until we reach a diagonal of Δ which contains non-zero entries.

The non-zero entries $\Delta_{i,j}$ of the first diagonal are called *index k primary pivots*. It follows that the entries $\Delta_{s,j}$ for $s > i$ are all zero.

We end this first step by defining Δ^1 as Δ with the index k primary pivots on the first diagonal marked.

2. Consider the matrix Δ^1 and let $\Delta_{i,j}^1$ be the entries in Δ^1 where the $i \in J_{k-1}$ and $j \in J_k$. Analogously to step one, we assume without loss of generality that the second diagonal contains non-zero entries $\Delta_{i,j}^1$. We now construct a matrix Δ^2 following the procedure:

Given a non-zero entry $\Delta_{i,j}^1$ on the second diagonal of Δ^1 :

- (a) If there are no primary pivots in row i and column j , mark it as an index k primary pivot and the numerical value of the entry remains the same, i.e. $\Delta_{i,j}^2 = \Delta_{i,j}^1$.
- (b) If this is not the case, consider the entries in column j and in a row s with $s > i$ in Δ^1 .
 - (b1) If there is an index k primary pivot in an entry in column j and in a row s , with $s > i$, then the numerical value remains the same and the entry is left unmarked, i.e. $\Delta_{i,j}^2 = \Delta_{i,j}^1$.

⁽³⁾ By r -th diagonal one means the collection of entries Δ_{ij} of Δ such that $j - i = r$.

(b2) If there are no primary pivots in column j below $\Delta_{i,j}^1$ then there is an index k primary pivot in row i , say a column u of Δ^1 , with $u < j$. In this case, we define $\Delta_{i,j}^2 = \Delta_{i,j}^1$ and this entry is marked as a *change-of-basis pivot*.

Note that we have defined a matrix Δ^2 which is actually equal to Δ^1 except that the second diagonal is marked with primary and change-of-basis pivots.

B. Intermediate step.

In this step we consider a matrix Δ^r with the primary and change-of-basis pivots marked on the ξ -th diagonal for all $\xi \leq r$. We now describe how Δ^{r+1} is defined. If there does not exist a change-of-basis pivot on the r -th diagonal we go directly to step B.2, that is, we define $\Delta^{r+1} = \Delta^r$ with diagonal $(r + 1)$ marked with primary and change-of-basis pivots as in B.2.

B.1. Change of basis. Suppose $\Delta_{i,j}^r$ is a change-of-basis pivot on the r -th diagonal. Since we have a change-of-basis pivot in row i , there is a column, namely u -th column, associated to a k -chain such that $\Delta_{i,u}^r$ is a primary pivot. Then, perform a change of basis on Δ^r in order to zero out the entry $\Delta_{i,j}^r$ without introducing non-zero entries in $\Delta_{s,j}^r$ for $s > i$. We will prove in Theorem 5.1 that all the entries in Δ^r which are primary pivots are equal to $\pm t^{l_1} \pm t^{l_2}$ and, since these entries are invertible in $\mathbb{Z}((t))$, it is always possible choosing a particular change of basis using only columns j and u of Δ^r .

Once this is done, we obtain a k -chain associated to column j of Δ^{r+1} . It is a linear combination over $\mathbb{Z}((t))$ of columns u and j of Δ^r such that $\Delta_{i,j}^{r+1} = 0$. It is also a particular linear combination of the columns of Δ in J_k on and to the left of column j .

Let k_1 be the column of Δ^r which is associated to a k -chain. We denote by $\sigma_k^{j,r}$ the Morse index k -chain corresponding to column j of Δ^r . We have

$$\sigma_k^{j,r} = \sum_{\ell=k_1}^j c_{\ell}^{j,r} h_{\ell}^k,$$

and column j of Δ^{r+1} corresponds to

$$(4.1) \quad \sigma_k^{j,r+1} = \sigma_k^{j,r} - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \sigma_k^{u,r} = c_{k_1}^{j,r+1} h_{k_1}^k + \dots + c_{j-1}^{j,r+1} h_{j-1}^{j-1} + c_j^{j,r+1} h_j^j$$

where $c_{\ell}^{j,r+1} \in \mathbb{Z}((t))$ and $c_j^{j,r+1} = 1$.

Therefore the matrix Δ^{r+1} has entries determined by a change of basis over $\mathbb{Z}((t))$ of Δ^r . In particular, all the change-of-basis pivots on the r -th diagonal of Δ^r are zero in Δ^{r+1} .

Once the above procedure is done for all change-of-basis pivots of the r -th diagonal of Δ^r , we can define a change-of-basis matrix T^r such that $\Delta^{r+1} = (T^r)^{-1} \Delta^r T^r$.

B.2. Marking the diagonal $(r+1)$ of Δ^{r+1} . Consider the matrix Δ^{r+1} defined in the previous step. We mark diagonal $(r+1)$ with primary and change-of-basis pivots as follows:

Given a non-zero entry $\Delta_{i,j}^{r+1}$:

- (a) If there are no primary pivots in row i and column j , mark it as an index k primary pivot.
- (b) If this is not the case, consider the entries in column j and in a row $s > i$ in Δ^{r+1} .
 - (b1) If there is an index k primary pivot in the entries in column j below $\Delta_{i,j}^{r+1}$ then leave the entry unmarked.
 - (b2) If there are no primary pivots in column j below $\Delta_{i,j}^{r+1}$ then there is an index k primary pivot in row i , say in the column u of Δ^{r+1} , with $u < j$. In this case, mark it as a change-of-basis pivot.

C. Final step.

We repeat the above procedure until all diagonals have been considered.

Note that in the SSSA the columns of the matrix Δ are not necessarily ordered with respect to k , or equivalently, that the singularities h_k are not ordered with respect to the filtration. In this work, without loss of generality, we consider the singularities to be ordered with respect to the Morse index for the sole reason of simplifying notation.

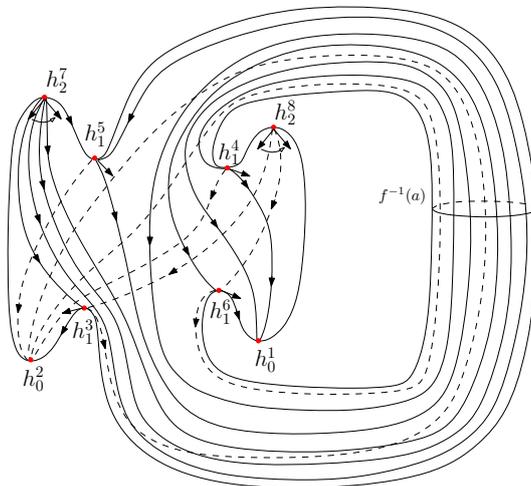


FIGURE 3. Circle-valued Morse function on the torus.

In order to perform the particular change of basis (4.1) in step B.1 of the SSSA, the primary pivots must be invertible polynomials in the ring $\mathbb{Z}((t))$. Otherwise, the change of basis in (4.1) is not well defined. The example below

$$\begin{array}{c}
 h_0^1 \\
 h_0^2 \\
 h_1^3 \\
 h_1^4 \\
 h_1^5 \\
 h_1^6 \\
 h_2^7 \\
 h_2^8
 \end{array}
 \begin{pmatrix}
 h_0^1 & h_0^2 & h_1^3 & h_1^4 & h_1^5 & h_1^6 & h_2^7 & h_2^8 \\
 \boxed{0} & & \boxed{0} & \boxed{1} & \boxed{t} & \boxed{1} & & \\
 & \boxed{0} & \boxed{t-1} & \boxed{-1} & \boxed{-1} & \boxed{-t} & & \\
 & & \boxed{0} & & & & \boxed{1} & \boxed{-1} \\
 & & & \boxed{0} & & & \boxed{-t^2} & \boxed{1} \\
 & & & & \boxed{0} & & \boxed{t-1} & \boxed{0} \\
 & & & & & \boxed{0} & \boxed{t} & \boxed{-1} \\
 & & & & & & \boxed{0} & \\
 & & & & & & & \boxed{0}
 \end{pmatrix}$$

FIGURE 4. Novikov matrix.

$$\begin{array}{c}
 h_0^1 \\
 h_0^2 \\
 h_1^3 \\
 h_1^4 \\
 h_1^5 \\
 h_1^6 \\
 h_2^7 \\
 h_2^8
 \end{array}
 \begin{pmatrix}
 h_0^1 & h_0^2 & h_1^3 & h_1^4 & h_1^5 & h_1^6 & h_2^7 & h_2^8 \\
 \boxed{0} & & \boxed{0} & \boxed{1} & \boxed{t} & \boxed{1} & & \\
 & \boxed{0} & \boxed{t-1} & \boxed{-1} & \boxed{-1} & \boxed{-t} & & \\
 & & \boxed{0} & & & & \boxed{1} & \boxed{-1} \\
 & & & \boxed{0} & & & \boxed{-t^2} & \boxed{1} \\
 & & & & \boxed{0} & & \boxed{t-1} & \boxed{0} \\
 & & & & & \boxed{0} & \boxed{t} & \boxed{-1} \\
 & & & & & & \boxed{0} & \\
 & & & & & & & \boxed{0}
 \end{pmatrix}$$

$$\sigma_0^{j,1} = h_0^j \text{ for } j \in J_0; \sigma_1^{j,1} = h_1^j \text{ for } j \in J_1; \sigma_2^{j,1} = h_2^j \text{ for } j \in J_2.$$

FIGURE 5. Δ^1 , marking primary pivots.

$$\begin{array}{c}
 h_0^1 \\
 h_0^2 \\
 h_1^3 \\
 h_1^4 \\
 h_1^5 \\
 h_1^6 \\
 h_2^7 \\
 h_2^8
 \end{array}
 \begin{pmatrix}
 h_0^1 & h_0^2 & h_1^3 & h_1^4 & h_1^5 & h_1^6 & h_2^7 & h_2^8 \\
 \boxed{0} & & \boxed{0} & \boxed{1} & \boxed{t} & \boxed{1} & & \\
 & \boxed{0} & \boxed{t-1} & \boxed{-1} & \boxed{-1} & \boxed{-t} & & \\
 & & \boxed{0} & & & & \boxed{1} & \boxed{-1} \\
 & & & \boxed{0} & & & \boxed{-t^2} & \boxed{1} \\
 & & & & \boxed{0} & & \boxed{t-1} & \boxed{0} \\
 & & & & & \boxed{0} & \boxed{t} & \boxed{-1} \\
 & & & & & & \boxed{0} & \\
 & & & & & & & \boxed{0}
 \end{pmatrix}$$

$$\sigma_0^{j,2} = h_0^j \text{ for } j \in J_0; \sigma_1^{j,2} = h_1^j \text{ for } j \in J_1; \sigma_2^{j,2} = h_2^j \text{ for } j \in J_2.$$

FIGURE 6. Δ^2 , marking change-of-basis pivot.

shows a Novikov differential for which all changes of bases are well defined and hence the SSSA is correct.

EXAMPLE 4.1. Figure 3 illustrates a flow on the torus T^2 associated to a circle-valued Morse function f defined on T^2 , where a is a regular value

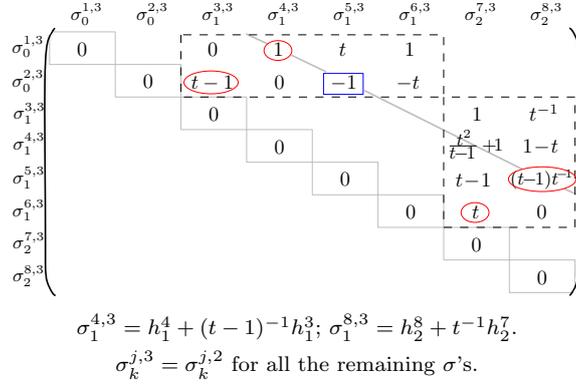


FIGURE 7. Δ^3 , marking pivots.

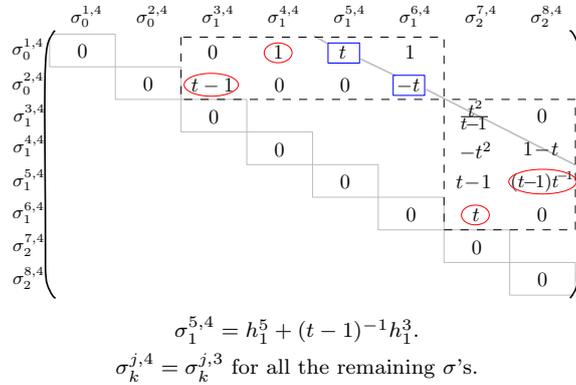


FIGURE 8. Δ^4 , marking change-of-basis pivot.

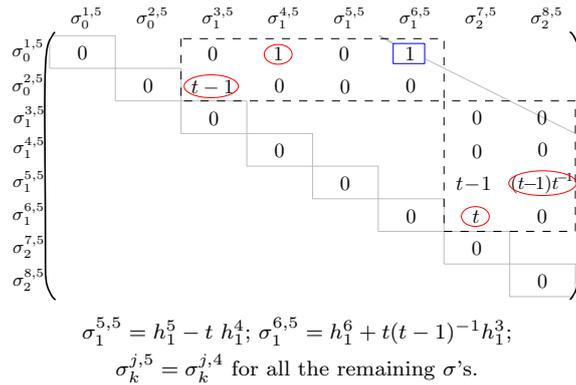


FIGURE 9. Δ^5 , marking change-of-basis pivot.

of f . The Novikov chain groups are generated by the critical points as follows: $\mathcal{N}_0 = \{h_0^1, h_0^2\}$, $\mathcal{N}_1 = \{h_1^3, h_1^4, h_1^5, h_1^6\}$ and $\mathcal{N}_2 = \{h_2^7, h_2^8\}$. Choosing orientations

for the unstable manifolds of the critical points of f as indicated in Figure 3, the Novikov matrix associated to ∂ is presented in Figure 4.

Applying the SSSA to the Novikov matrix Δ in Figure 4, one obtains the sequence of Novikov matrices $\Delta^1, \dots, \Delta^6$ presented in Figures 5–10, respectively. In these figures, the markup process at the r -th iteration is done as follows: primary pivots are encircled and change-of-basis pivots are encased in boxes.

Note that, in this example, each marked primary pivot in Δ^r is invertible in the ring $\mathbb{Z}((t))$, making it possible to apply the SSSA and obtain the next Novikov matrix Δ^{r+1} .

$$\begin{array}{c}
 \begin{array}{cccccccc}
 & \sigma_0^{1,6} & \sigma_0^{2,6} & \sigma_1^{3,6} & \sigma_1^{4,6} & \sigma_1^{5,6} & \sigma_1^{6,6} & \sigma_2^{7,6} & \sigma_2^{8,6} \\
 \sigma_0^{1,6} & \boxed{0} & & & & & & & \\
 \sigma_0^{2,6} & & \boxed{0} & & & & & & \\
 \sigma_1^{3,6} & & & \boxed{t-1} & & & & & \\
 \sigma_1^{4,6} & & & & \boxed{1} & & & & \\
 \sigma_1^{5,6} & & & & & \boxed{0} & & & \\
 \sigma_1^{6,6} & & & & & & \boxed{0} & & \\
 \sigma_2^{7,6} & & & & & & & \boxed{t-1} & \boxed{(t-1)t^2} \\
 \sigma_2^{8,6} & & & & & & & & \boxed{0}
 \end{array} \\
 \\
 \sigma_1^{6,6} = h_1^6 - h_1^4, \\
 \sigma_k^{j,6} = \sigma_k^{j,5} \text{ for all the remaining } \sigma\text{'s.}
 \end{array}$$

FIGURE 10. Δ^6 for Example 4.1.

5. Characterization of the Novikov matrices

The primordial aim in this section is to show that the SSSA is proved to be correct for all Novikov differentials of a 2-dimensional Novikov complex. This is done by showing that, given a Novikov differential Δ , all primary pivots determined by the SSSA are invertible polynomials in the ring $\mathbb{Z}((t))$. In fact, they are monomials with coefficient ± 1 or binomials with coefficients ± 1 . Throughout this section, the term monomial (respectively, binomial) will be used to refer to polynomials in $\mathbb{Z}((t))$ of the form $\pm t^\ell$ (respectively, $t^{\ell_1} - t^{\ell_2}$), where $\ell \in \mathbb{Z}$ (respectively, $\ell_1, \ell_2 \in \mathbb{Z}$).

THEOREM 5.1. *Given a Novikov differential Δ , the primary pivots and the change-of-basis pivots in the SSSA are polynomials of the form t^ℓ or $t^{\ell_1} - t^{\ell_2}$, where $\ell, \ell_1, \ell_2 \in \mathbb{Z}$.*

The proof of Theorem 5.1 is an immediate consequence of Theorem 5.10, Lemma 5.3 and Theorem 5.11. Theorem 5.10 gives a characterization of the columns of the first block $J_0 \times J_1$ of Δ and Theorem 5.11 gives a characterization of the rows of the second block $J_0 \times J_1$ of Δ .

Lemma 5.2 asserts that we cannot have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in row i , then there is no primary pivot in column i .

LEMMA 5.2. *Let Δ be a Novikov differential for which the SSSA is proved correct up to step R . Let $\Delta^1, \dots, \Delta^R$ be the family of Novikov matrices produced by the SSSA until step R . Given two primary pivots, the ij -th entry $\Delta_{i,j}^r$ and the ml -th entry $\Delta_{m,l}^r$, then $\{i, j\} \cap \{m, l\} = \emptyset$.*

We omit the proof of Lemma 5.2, since it is similar in nature to proof of Proposition 3.2 in [6], where the SSSA was designed for a Morse chain complex over \mathbb{Z} . The next lemma implies that, in order to know the pivots which will appear during the execution of the SSSA, one can apply this algorithm separately in block $J_0 \times J_1$ and $J_1 \times J_2$.

LEMMA 5.3. *Let Δ be a Novikov differential for which the SSSA is proved correct up to step R . Then, the change of basis caused by change-of-basis pivots in block $J_0 \times J_1$ do not affect the pivots in block $J_1 \times J_2$. In other words, multiplication by $(T^r)^{-1}$ does not change the primary and change-of-basis pivots in block $J_1 \times J_2$.*

PROOF. Without loss of generality, suppose that there is only one change-of-basis pivot $\Delta_{i,j}^r$ in Δ^r with $j \in J_1$, and let $\Delta_{i,u}^r$ be the primary pivot in row i . The change-of-basis matrix T^r has unit diagonal and the only non-zero entry off the diagonal is $T_{u,j}^r = -\Delta_{i,j}^r(\Delta_{i,u}^r)^{-1}$. Hence, $(T^r)^{-1}$ has unit diagonal and the only non-zero entry off the diagonal is $(T^r)^{-1}_{u,j} = -T_{u,j}^r = \Delta_{i,j}^r(\Delta_{i,u}^r)^{-1}$. Therefore, multiplication by $(T^r)^{-1}$ will only affect row u of Δ^r . By Lemma 5.2, there are no primary pivots in row u and hence there are no change-of-basis pivots in row u as well. \square

Before proving Theorem 5.1, we introduce the notation and terminology that will be used in the proof. From now on, we consider the SSSA without realising the pre-multiplication by $(T^r)^{-1}$, unless mentioned otherwise.

Let $\Delta_{i,j}^r$ be a change-of-basis pivot caused by a primary pivot $\Delta_{i,u}^r$. Suppose a change of basis determined by $\Delta_{i,j}^r$ is performed by the SSSA in the matrix Δ^r , i.e. in the step $(r + 1)$, one has

$$\sigma_k^{j,r+1} = \sigma_k^{j,r} - \Delta_{i,j}^r(\Delta_{i,u}^r)^{-1}\sigma_k^{u,r}.$$

Hence, whenever this change of basis occurs, only column j of the matrix Δ^r is modified, in fact, for each $s = 0, \dots, i$, the entry $\Delta_{s,j}^r$ is added to a multiple of the entry $\Delta_{s,u}^r$, where u is the column of the primary pivot in row i . In other words, $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{i,j}^r(\Delta_{i,u}^r)^{-1}\Delta_{s,u}^r$. See the matrix in Figure 11 (this figure shows part of the block associated with index k , as the r -th diagonal is swept).

$$\begin{array}{c}
 \begin{array}{cc}
 & u & & j \\
 & \vdots & & \vdots \\
 s & \cdots & \Delta_{s,u}^r & \cdots & \Delta_{s,j}^r & \cdots \\
 & \vdots & & \ddots & \vdots & \\
 i & \cdots & \Delta_{i,u}^r & \cdots & \Delta_{i,j}^r & \cdots \\
 & \vdots & & & \vdots &
 \end{array}
 \end{array}$$

 FIGURE 11. $\Delta_{J_{k-1}J_k}^r$; marking change-of-basis pivot.

DEFINITION 5.4. In the situation described above and represented in Figure 11, we assert that the entry $\Delta_{s,u}^r$ in column u generates⁽⁴⁾ the entry $\Delta_{s,j}^{r+1}$ in column j in Δ^{r+1} , whenever $\Delta_{s,u}^r \neq 0$.

Note that if an entry in a column j generates another entry in a column t then $t > j$, i.e. $\Delta_{s,j}^r$ generates an entry in a column on the right of the column j .

EXAMPLE 5.5. It is helpful to keep in mind some configurations that allow an entry $\Delta_{s,u}^r \neq 0$ to generate another entry $\Delta_{s,j}^{r+1}$. Consider for instance that $\Delta_{i,u}^r = t^\ell$ and $\Delta_{i,j}^r = -t^{\tilde{\ell}}$. We list some of the possibilities for the entries in positions (s, u) and (s, j) of Δ^r :

- (a) $\Delta_{s,u}^r = t^\ell$ and $\Delta_{s,j}^r = 0$. In this case, $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \Delta_{s,u}^r = t^{\tilde{\ell}} t^{-\ell} t^\ell$, see Figure 12.
- (b) $\Delta_{s,u}^r = -t^\ell$ and $\Delta_{s,j}^r = t^{\tilde{\ell}}$. In this case, $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \Delta_{s,u}^r = t^{\tilde{\ell}} - t^{\tilde{\ell}} t^{-\ell} t^\ell$, see Figure 13.
- (c) $\Delta_{s,u}^r = t^\ell - t^{\tilde{\ell}}$ and $\Delta_{s,j}^r = 0$. In this case, $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \Delta_{s,u}^r = t^{\tilde{\ell}} t^{-\ell} (t^\ell - t^{\tilde{\ell}})$, see Figure 14.

$$\begin{array}{c}
 \begin{array}{cc}
 & u & & j \\
 & \vdots & & \vdots \\
 s & \cdots & t^\ell & \cdots & 0 & \cdots \\
 & \vdots & & \ddots & \vdots & \\
 i & \cdots & t^\ell & \cdots & -t^{\tilde{\ell}} & \cdots \\
 & \vdots & & & \vdots &
 \end{array}
 \xrightarrow{\text{SSSA}}
 \begin{array}{c}
 \begin{array}{cc}
 & u & & j \\
 & \vdots & & \vdots \\
 s & \cdots & t^\ell & \cdots & t^{\tilde{\ell}-\ell+\ell} & \cdots \\
 & \vdots & & \ddots & \vdots & \\
 i & \cdots & t^\ell & \cdots & 0 & \cdots \\
 & \vdots & & & \vdots &
 \end{array}
 \end{array}
 \end{array}$$

 FIGURE 12. $\Delta_{J_{k-1}J_k}^r$ and $\Delta_{J_{k-1}J_k}^{r+1}$, respectively.

Figures 12–14 show part of the block associated with index k , as the r -th diagonal is swept.

⁽⁴⁾ If an entry $\Delta_{s,u}^\xi$ with $\xi < r$ does not change until step r , i.e. $\Delta_{s,u}^\xi = \Delta_{s,u}^{\xi+1} = \dots = \Delta_{s,u}^r$, and $\Delta_{s,u}^r$ generates $\Delta_{s,j}^{r+1}$, we say that $\Delta_{s,u}^\xi$ generates $\Delta_{s,j}^{r+1}$.

is marked in a row during the SSSA until step R , then every lineage is formed by a unique generation sequence.

PROOF. By hypothesis, one has that in each row i at most one change-of-basis pivot is marked through out the algorithm and if so the mark up is done in step $2 \leq \xi \leq m - 1$, where m is the order of Δ . Then an entry $\Delta_{s,j_0}^{\xi_0}$ in row s generates at most one entry $\Delta_{s,j_1}^{\xi_1}$ through out the algorithm, and this entry will necessarily be in a column $j_1 > j_0$ and diagonal ξ_1 with $\xi_0 < \xi_1 \leq m - 1$. In fact, if $\Delta_{s,j_0}^{\xi_0}$ generates two entries, then either there would be two change-of-basis pivots in row i , which contradicts our initial hypothesis, or two primary pivots in column j_0 , which cannot occur by the definition of primary pivots.

Now, if $\xi_1 \leq m - 1$, then $\Delta_{s,j_1}^{\xi_1}$ can generate at most a unique entry $\Delta_{s,j_2}^{\xi_2}$ where $\xi_1 < \xi_2 \leq m - 1$ and $j_2 > j_1$ and this can be done successively. More specifically, the entry $\Delta_{s,u}^{\xi}$ is responsible for generating a unique immediate successor and this successor can in turn generate a unique immediate successor and thus it determines a full lineage of entries represented in one finite sequence $\{\Delta_{s,j_0}^{\xi_0}, \Delta_{s,j_1}^{\xi_1}, \dots\}$, where $2 \leq \xi_0 < \xi_1 < \dots \leq m - 1$, and $j_0 < j_1 < \dots \leq m$. \square

COROLLARY 5.9. Let Δ be a matrix for which the SSSA is proved correct up to step R . Suppose that the second block of Δ has the property that at most one change-of-basis pivot is marked in each row from the beginning until the end of the SSSA. Therefore, each $\Delta_{s,u}^{\xi}$ -lineage with $s \in J_1$ is formed by a unique generation sequence.

PROOF. The SSSA applied to the first block $J_0 \times J_1$ of Δ does not interfere in the number of change-of-basis pivots identified in block $J_1 \times J_2$. \square

$$\Delta_{J_a J_{a+1}}^{\xi} = \begin{matrix} & & u_f & & j_f & & \\ & & \vdots & & \vdots & & \\ s & \left(\begin{array}{cccccc} \cdots & \Delta_{s,u_f}^{\xi} & \cdots & \Delta_{s,j_f}^{\xi} & \cdots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \cdots & \Delta_{i,u_f}^{\xi} & \cdots & \Delta_{i,j_f}^{\xi} & \cdots & \\ \vdots & \vdots & & \vdots & \vdots & \end{array} \right) \\ i & & & & & & \end{matrix}$$

FIGURE 15. Generating a binomial from two monomial lineages.

Consider a $\Delta_{s,u}^{\xi}$ -lineage which is formed by a unique generation sequence. If this generation sequence contains only monomials (binomials, resp.) then one says that $\Delta_{s,u}^{\xi}$ -lineage is a *monomial (binomial, resp.)* lineage. However, if $\Delta_{s,u}^{\xi}$ is a monomial, then the $\Delta_{s,u}^{\xi}$ -lineage could eventually contain binomials. One way that this can occur is when row s is of type 3 in Δ , $\Delta_{s,u}^{\xi}$ and

$\Delta_{s,j}^\zeta$ are monomials and the lineage determined by these entries merge giving rise to a binomial. More specifically, suppose that the first binomial in row s appears in $\Delta^{\zeta+1}$ then one has two monomial lineages $\{\Delta_{s,u}^\xi, \Delta_{s,u_1}^{\xi_1}, \dots, \Delta_{s,u_f}^{\xi_f}\}$ and $\{\Delta_{s,j}^\zeta, \Delta_{s,j_1}^{\zeta_1}, \dots, \Delta_{s,j_f}^{\zeta_f}\}$, where $\xi_f, \zeta_f \leq \zeta$; observe that $\Delta_{s,u_f}^{\xi_f} = \Delta_{s,u_f}^\zeta$ and $\Delta_{s,j_f}^{\zeta_f} = \Delta_{s,j_f}^\zeta$. The binomial will appear in $\Delta^{\zeta+1}$ as a consequence of a change of basis caused by a change-of-basis pivot Δ_{i,j_f}^ζ and a primary pivot Δ_{i,u_f}^ζ in a row $i > s$, as in Figure 15. In this case, Δ_{s,u_f}^ζ is the generator of the binomial $\Delta_{s,j_f}^{\zeta+1}$. Hence, we say that the $\Delta_{s,j}^\zeta$ -lineage *ceases*, i.e, this lineage remains the same until Δ^{r+1} and the $\Delta_{s,u}^\zeta$ -lineage is an *eventual binomial lineage*. From this point on, this lineage contains only binomials.

Once an element of a lineage is marked as a pivot, this lineage ceases, since pivots do not generate entries, by Lemma 5.6.

The next theorem provides a characterization of columns in the first block $J_0 \times J_1$ of the Novikov matrices Δ^r as the diagonals are swept.

THEOREM 5.10 (First block characterization). *Let Δ be a Novikov differential for which the SSSA is proved correct up to step R . Then we have the following possibilities for a column j of the matrix Δ^r with $j \in J_1$ and $r \leq R$:*

- (1) *all the entries in column j are equal to zero, i.e. $\Delta_{\bullet,j}^r = 0$;*
- (2) *there is only one non-zero entry in column j and it is a binomial $t^\ell - t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$;*
- (3) *there are exactly two non-zero entries in column j and they are monomials t^ℓ and $-t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$;*
- (4) *there is only one non-zero entry in column j and it is a monomial t^ℓ , where $\ell \in \mathbb{Z}$.*

PROOF. The proof is done by induction. Note that the result is trivial for Δ^1 and Δ^2 . In fact, the first change-of-basis pivot can only be detected from the second diagonal of Δ , which implies that the entries of Δ may change from $r = 3$ onwards. Because of that the base of the induction is $r = 3$.

Step $r = 3$. To prove that the rows of Δ^3 are of type (1)–(4), we will analyze the effect a change-of-basis pivot marked in Δ^2 has on Δ^3 . Suppose, without loss of generality, that there is a change-of-basis pivot $\Delta_{i,i+2}^2$ on the second diagonal. Consequently, $\Delta_{i,i+1}^2$ is a primary pivot marked in Δ^1 . Recall that columns in Δ^2 are characterized by Corollary 2.2. Hence, one has the following possibilities:

- 1. Column $i + 1$ is of type (2). Column $i + 2$ can be of type (2), (3) or (4). In each case, $\Delta_{i,i+2}^3 = 0$ and all the other entries in column $i + 2$ remain the same, i.e. $\Delta_{s,i+2}^3 = \Delta_{s,i+2}^2$ for all $s \neq i$. If column $i + 2$ is of type (2) or (4) (respectively, (3)) it turns into a column of type (1) (respectively, (4)).
- 2. Column $i + 1$ is of type 3. Then $\Delta_{i,i+1}^2$ is a monomial and there is $s < i$ such that $\Delta_{s,i+1}^2$ is also a monomial.

- (a) If the column $i + 2$ is of type (2) (respectively, type (4)), then $\Delta_{i,i+2}^3 = 0$ and $\Delta_{s,i+2}^3 = -\Delta_{s,i+1}^2(\Delta_{i,i+1}^2)^{-1}\Delta_{i,i+2}^2$, which is a binomial (respectively, monomial). Hence, the column $i + 2$ remains of type (2) (respectively, type (4)).
- (b) If the column $i + 2$ is of type (3), then change-of-basis pivot $\Delta_{i,i+2}^2$ is a monomial and there is $\bar{s} < i$ such that $\Delta_{\bar{s},i+2}^2$ is also a monomial. If $s = \bar{s}$, then $\Delta_{i,i+2}^3 = 0$ and $\Delta_{s,i+2}^3 = \Delta_{s,i+2}^2 - \Delta_{s,i+1}^2(\Delta_{i,i+1}^2)^{-1}\Delta_{i,i+2}^2$, which is either a binomial or zero. Hence, column $i + 2$ turns into a column of type (2) or (1), respectively. On the other hand, if $s \neq \bar{s}$, then $\Delta_{i,i+2}^3 = 0$, $\Delta_{s,i+2}^3 = -\Delta_{s,i+1}^2(\Delta_{i,i+1}^2)^{-1}\Delta_{i,i+2}^2$, which is a monomial, and the other entries of column $i + 2$ remain the same. Hence, column $i + 2$ remains of type (3).

3. Column $i + 1$ is of type (4). It is analogous to case 1.

Induction hypothesis. Suppose that the conclusion of the theorem holds for $3 \leq r < R$. We will show, that it also holds for $r + 1$.

Suppose that $\Delta_{i,j}^r$ is a change-of-basis pivot in the r -th diagonal. Then there is a primary pivot $\Delta_{i,u}^r$ in a column $u < j$. We have one of the possibilities:

- 1. Column u is of type (2). Column j can be of type (2), (3) or (4). In all cases, $\Delta_{i,j}^{r+1} = 0$ and all the other entries in column $r + 1$ remain the same. If column j is of type (2) or (4) (respectively, type (3)) it turns into a column of type (1) (respectively, type (4)).
- 2. Column u is of type (3). Then $\Delta_{i,u}^r$ is a monomial and there is $s < i$ such that $\Delta_{s,u}^r$ is also a monomial.

- (a) If the column j is of type (2) (respectively, type (4)), then the change-of-basis pivot $\Delta_{i,j}^r$ is a binomial (respectively, monomial). In this case, $\Delta_{i,j}^{r+1} = 0$ and $\Delta_{s,j}^{r+1} = -\Delta_{s,u}^r(\Delta_{i,u}^r)^{-1}\Delta_{i,j}^r$, which is a binomial (respectively, monomial). Hence, the column j remains of type (2), (respectively, type (4)).
- (b) If the column j is of type 3, then the change-of-basis pivot $\Delta_{i,j}^r$ is a monomial and there exists $\bar{s} < i$ such that $\Delta_{\bar{s},j}^r$ is also a monomial. If $s = \bar{s}$, then $\Delta_{i,j}^{r+1} = 0$ and $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{s,u}^r(\Delta_{i,u}^r)^{-1}\Delta_{i,j}^r$, which is either a binomial or zero. Hence, column j turns into a column of type 2 or 1, respectively. On the other hand, if $s \neq \bar{s}$, then $\Delta_{i,j}^{r+1} = 0$, $\Delta_{s,j}^{r+1} = -\Delta_{s,u}^r(\Delta_{i,u}^r)^{-1}\Delta_{i,j}^r$, which is a monomial, and the other entries of column j remain the same. Hence, column j remains of type (3).

3. Column u is of type (4). The only non-zero entry in column u is the primary pivot $\Delta_{i,u}^r$ which is a monomial. Hence, in column j all the entries remain the same besides the change-of-basis pivot $\Delta_{i,j}^{r+1} = 0$. If j is a column of type (2) or (4) it turns into a column of type (1); if j is of type (3) it turns into a column of type (4). □

THEOREM 5.11 (Second block characterization). *Let Δ be a Novikov differential for which the SSSA is proved correct up to step R . Then we have the following possibilities for a non-zero row $s \in J_1$ of the matrix Δ^r produced by the SSSA in step $r \leq R$ without realizing the pre-multiplication by $(T^r)^{-1}$:*

- (A) all non-null entries are binomials of the form $t^\ell - t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$;
- (B) all non-null entries are monomials of the form t^ℓ , where $\ell \in \mathbb{Z}$;
- (C) all non-null entries are either monomials t^ℓ or binomials $t^\ell - t^{\tilde{\ell}}$. Moreover, if a column $j \in J_2$ contains a binomial, then there are no monomials in columns $j' \in J_2$ with $j' > j$.

PROOF. We will prove this theorem by induction in $r \leq R$. In the course of the proof, we will also prove the following set of statements:

- (i) If an entry t^ℓ is a primary pivot in row i then at most one entry will be marked as a change-of-basis pivot in row i .
- (ii) An entry $t^\ell - t^{\tilde{\ell}}$ is never marked as a change-of-basis pivot, i.e. all change-of-basis pivots are monomials t^ℓ , for some $\ell \in \mathbb{Z}$.
- (iii) If $\Delta_{s,j}^r = t^\ell - t^{\tilde{\ell}}$, $\Delta_{i,j}^r$ is a change-of-basis pivot in row $i > s$ and $\Delta_{i,u}^r$ is the primary pivot in row i with $u < j$, then $\Delta_{s,u}^r$ is zero.
- (iv) A primary pivot $\Delta_{i,u}^r$ and a change-of-basis pivot $\Delta_{i,j}^r$ in row i are always monomials with opposite signs, i.e. $\Delta_{i,j}^r = \pm t^\ell$ and $\Delta_{i,u}^r = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.
- (v) A monomial $\Delta_{s,u}^r$ above a primary pivot $\Delta_{i,u}^r$ and a monomial $\Delta_{s,j}^r$ above a change-of-basis pivot $\Delta_{i,j}^r$ always have opposite signs, i.e. $\Delta_{s,j}^r = \pm t^\ell$ and $\Delta_{s,u}^r = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.

Observe that the matrices Δ^1 and Δ^2 differ from the initial matrix Δ only in the mark-ups of primary and change-of-basis pivots, since the entries can only change as of the 3-rd step of the SSSA.

Base case $r = 3$. In order to prove that the rows of Δ^3 satisfy conditions (A), (B) and (C) of the theorem, we must analyze the effect on a row of Δ^3 caused by a change-of-basis pivot marked in the second step $r = 2$ of the SSSA.

$$\Delta_{J_1 J_2}^2 = \begin{matrix} & & i+1 & i+2 & & \\ & & \vdots & \vdots & & \\ s & \left(\begin{array}{cccc} \cdots & \Delta_{s,i+1}^2 & \Delta_{s,i+2}^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \Delta_{i,i+1}^2 & \Delta_{i,i+2}^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \right) & & & \\ i & & & & & \end{matrix}$$

FIGURE 16. Primary and change-of-basis pivots in the first and second diagonals of Δ^2 , respectively.

Suppose, without loss of generality, that there is a change-of-basis pivot $\Delta_{i,i+2}^2$ on the second diagonal. Consequently, $\Delta_{i,i+1}^2$ is a primary pivot marked in the first step of the SSSA, see Figure 16. Recall that the rows in Δ^2 are characterized by Corollary 2.2. A change-of-basis pivot only occurs in row i and column $i + 2$ if this row is of type (3). In what follows, we analyze the effect of this change of basis on a row s with $s < i$:

1. If row s is null (i.e. of type (1)) then only row i is altered and becomes a row of type (B).
2. Suppose that row s is of type (2). If the only non-zero entry in row s is in a column different from $i + 1$, then row s remains unaltered and row i turns into a row of type (B). On the other hand, if this non-zero entry is in the column $i + 1$, which is the same column as that of the primary pivot, then row i turns into a row of type (B) and row s turns into a row of type (A), as one can see in Figure 17.

$$\begin{array}{c}
 \begin{array}{cc} & i+1 & i+2 \\ & \vdots & \vdots \\ s & \cdots & t^l - t^{\bar{l}} & 0 & \cdots \\ & \vdots & \vdots \\ i & \cdots & \pm t^{\bar{\ell}} & \mp t^{\ell} & \cdots \\ & \vdots & \vdots \end{array} & \xrightarrow{\text{SSSA}} & \begin{array}{cc} & i+1 & i+2 \\ & \vdots & \vdots \\ s & \cdots & t^l - t^{\bar{l}} & t^{-\bar{\ell}l} - t^{-\bar{\ell}\bar{l}} & \cdots \\ & \vdots & \vdots \\ i & \cdots & \pm t^{\bar{\ell}} & 0 & \cdots \\ & \vdots & \vdots \end{array}
 \end{array}$$

FIGURE 17. $\Delta_{J_1 J_2}^2$ and $\Delta_{J_1 J_2}^3$, respectively.

3. Suppose that row s is of type (3). If $\Delta_{s,i+1}$ is zero, then row s remains unaltered. If $\Delta_{s,i+1}^2 = t^l$, one has two possibilities for $\Delta_{s,i+2}^2$, namely, 0 or $t^{\bar{l}}$. In the first case, after performing the change of basis, row s turns into a row of type (B) (see Figure 18), and in the second case it turns into a row of type (C) (see Figure 19).

$$\begin{array}{c}
 \begin{array}{cc} & i+1 & i+2 \\ & \vdots & \vdots \\ s & \cdots & \pm t^l & 0 & \cdots \\ & \vdots & \vdots \\ i & \cdots & \pm t^{\bar{\ell}} & \mp t^{\ell} & \cdots \\ & \vdots & \vdots \end{array} & \xrightarrow{\text{SSSA}} & \begin{array}{cc} & i+1 & i+2 \\ & \vdots & \vdots \\ s & \cdots & \pm t^l & \pm t^{\bar{\ell}-\ell+l} & \cdots \\ & \vdots & \vdots \\ i & \cdots & \pm t^{\bar{\ell}} & 0 & \cdots \\ & \vdots & \vdots \end{array}
 \end{array}$$

FIGURE 18. $\Delta_{J_1 J_2}^2$ and $\Delta_{J_1 J_2}^3$, respectively.

In the base case, it is ease to see that (i) through (v) hold.

In order to prove (i) note that, the only case that needs to be analyzed is when $\Delta_{i,i+1}^2$ is a primary pivot and $\Delta_{i,i+2}^2$ is a change-of-basis pivot. Observe

$$\begin{array}{c}
 \begin{array}{c} i+1 \quad i+2 \\ \vdots \quad \vdots \\ \cdots \quad \pm t^l \quad \mp t^{\bar{l}} \quad \cdots \\ \vdots \quad \vdots \\ \cdots \quad \boxed{\pm t^{\ell}} \quad \boxed{\mp t^{\bar{\ell}}} \quad \cdots \\ \vdots \quad \vdots \end{array} \\
 \begin{array}{c} s \\ i \end{array}
 \end{array}
 \xrightarrow{\text{SSSA}}
 \begin{array}{c}
 \begin{array}{c} i+1 \quad i+2 \\ \vdots \quad \vdots \\ \cdots \quad \pm t^l \quad \mp t^{\bar{l}} \pm t^{\bar{l}-\ell+l} \quad \cdots \\ \vdots \quad \vdots \\ \cdots \quad \boxed{\pm t^{\ell}} \quad 0 \quad \cdots \\ \vdots \quad \vdots \end{array} \\
 \begin{array}{c} s \\ i \end{array}
 \end{array}$$

FIGURE 19. $\Delta_{J_1 J_2}^2$ and $\Delta_{J_1 J_2}^3$, respectively.

that $\Delta_{i,i+3}^2 = 0$, since rows of Δ^2 have at most two non-zero entries. As pivots do not generate entries, by Lemma 5.6, the entry $\Delta_{i,i+3}^2$ is not altered by change-of-basis pivots marked in step 2. Hence, $\Delta_{i,i+3}^3 = \Delta_{i,i+3}^2 = 0$ and it is not a change-of-basis pivot.

In order to prove (ii), we must consider each row s where the entry $\Delta_{s,s+3}^3 = t^l - t^{\bar{l}}$ was generated in Δ^3 , otherwise this entry would be in a row of type (2) and hence, could not be a change-of-basis pivot. There are exactly two ways that this entry can be generated: when row s was of type (2) and of type (3) in Δ^2 . Suppose by contradiction that $\Delta_{s,s+3}^3$ is a change-of-basis pivot. Since primary pivots do not generate entries and there exists at most one change-of-basis pivots in a row then row s could not be of type (2) in Δ^2 , see Figure 13. If row s was of type (3) in Δ^2 then the primary pivot would generate $\Delta_{s,s+3}^3$ which also contradicts Lemma 5.6, see Figure 14. Consequently, $\Delta_{s,s+3}^3$ is not a change-of-basis pivot.

In order to prove (iii), suppose by contradiction that $\Delta_{s,u}^3 \neq 0$. Since rows of the initial matrix Δ do not admit two non-zero entries with one of them being a binomial, see Corollary 2.2, then $\Delta_{s,u}^2 = \Delta_{s,u}^3$ generates $\Delta_{s,i+3}^3$ in Δ^3 . By Definition 5.4, there must exist a change-of-basis pivot in the second diagonal in a row $\tilde{i} > i$ and a primary pivot in row \tilde{i} and column u , which contradicts the fact that each column has at most one primary pivot.

Items (iv) and (v) are trivially true.

Induction hypothesis. Suppose that Theorem 5.11 and item (i) through (v) hold for $\xi \leq r < R$. We will show that they also hold for $r + 1$. First, note that by the induction hypothesis, at most one entry in a fixed row in J_1 is marked as change-of-basis pivot up to step r . By Corollary 5.9, given an entry $\Delta_{s,u}$, the $\Delta_{s,u}$ -lineage is formed by a unique generation sequence until Δ^{r+1} , which is either a binomial lineage, if $\Delta_{s,u}$ is a binomial; or a monomial lineage or an eventual binomial lineage, if $\Delta_{s,u}$ is a monomial ⁽⁵⁾. More specifically, if row s is of type (2) in Δ , where $\Delta_{s,u} \neq 0$ is a binomial, then the $\Delta_{s,u}$ -lineage is a binomial lineage. If row s is of type (3), where $\Delta_{s,u}$ and $\Delta_{s,j}$ are monomials, then each

⁽⁵⁾ Note that in this proof, whenever we assume the induction hypothesis for r , the index r is shifted by one, i.e. $r + 1$, when referring to the lineages.

monomial determines a lineage, which are either both monomial lineages or one monomial lineage which ceases and merges with the other to create an eventual binomial lineage. If row s is of type (4) in Δ , where $\Delta_{s,u}$ is a monomial, then $\Delta_{s,u}$ -lineage is monomial lineage. It is important to keep in mind that, if there is at most one change of basis per row up to step r , then the $\Delta_{s,u}$ -lineage is formed by a unique generation sequence until Δ^{r+1} . This follows since entries in Δ^{r+1} can only be generated by change of basis determined in step r .

By the induction hypothesis, it follows that if $\Delta_{i,j}^r$ is a change-of-basis pivot on the r -th diagonal and $\Delta_{i,u}^r$ is the primary pivot of row i , then these entries must be monomials. Moreover, $\Delta_{i,j}^r = \pm t^\ell$ and $\Delta_{i,u}^r = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.

Now we will prove that the statement of the Theorem 5.11 and item (i) through (v) hold for $r + 1$.

- We will first show that the non-zero rows in Δ^{r+1} are of type (A), (B) or (C). In order to prove this, we will perform all the possible changes of basis that could occur due to a change-of-basis pivot in the r -th diagonal. Let the monomial $\Delta_{i,u}^r$ be the primary pivot in row i and the monomial $\Delta_{i,j}^r$ be a change-of-basis pivot in diagonal r . Observe that after the change of basis, row i will remain of the same type. Each row $s < i$ in Δ^r is of type (A), (B) or (C), and thus one has the following cases to analyze the effect a change of basis causes in row s (bear in mind the configuration of the matrix in Figure 11):

1. If $\Delta_{s,u}^r = 0$, then row s remains unaltered after performing the change of basis.
2. If $\Delta_{s,u}^r \neq 0$ and $\Delta_{s,j}^r = 0$, then $\Delta_{s,j}^{r+1} = -\Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \Delta_{s,u}^r$, which is a monomial if $\Delta_{s,u}^r$ is a monomial, or a binomial if $\Delta_{s,u}^r$ is a binomial. Hence, row s remains of the same type.
3. If $\Delta_{s,u}^r \neq 0$ and $\Delta_{s,j}^r \neq 0$, by the induction hypothesis the only possible case is $\Delta_{s,u}^r = \pm t^\ell$ and $\Delta_{s,j}^r = \mp t^{\tilde{\ell}}$. Then $\Delta_{s,j}^{r+1} = \Delta_{s,j}^r - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \Delta_{s,u}^r$, which ensures that $\Delta_{s,j}^{r+1}$ is a zero entry or a binomial with coefficients equal to ± 1 . Hence, row s turns into a row of type (B) or (C), respectively.

Hence, every row $s \in J_1$ of Δ^{r+1} is also of type (A), (B) or (C).

- We will now show that item (i) holds for Δ^{r+1} . Let $\Delta_{i,j}^{r+1}$ be a change-of-basis pivot marked in diagonal $r+1$. Suppose by contradiction that an entry $\Delta_{i,t}^\xi$, where $t < j$, was marked as a change-of-basis pivot in an earlier step $\xi < r + 1$. Consequently, the primary pivot in row i , which is in a column $u < t$, was marked in a previous step $< \xi$. By item (ii) of the induction hypothesis, this primary pivot is a monomial and it cannot generate entries, which implies that $\Delta_{i,t}^\xi$ and $\Delta_{i,j}^{r+1}$ are not in the lineage of this primary pivot. Therefore, $\Delta_{i,t}^\xi$ and $\Delta_{i,j}^{r+1}$ must be in the same lineage, since these entries were generated up to step $r + 1$ by the change-of-basis pivots up to step r , and by the induction hypothesis, there

is only one change-of-basis pivot per row up to step r . This is a contradiction, since by Lemma 5.6 the change-of-basis pivot $\Delta_{i,t}^\xi$ cannot generate entries.

- We will prove item (ii) for $r + 1$. Let $\Delta_{i,j}^{r+1}$ be a binomial in diagonal $r + 1$. Suppose by contradiction that this entry is a change-of-basis pivot marked in step $r + 1$. Let u be the column of the primary pivot in row i , hence $u < j$.

If row i was originally of type 2 in Δ^2 , then there is only one lineage in row i until Δ^{r+1} , which is a binomial lineage. Hence, $\Delta_{i,j}^{r+1}$ is a successor of the primary pivot in row i , which must have generated an entry, contradicting Lemma 5.6.

If row i was of type (3) in Δ^2 , then originally there were two lineages that merged in order to create a binomial. Note that the primary pivot in row i cannot be a monomial, i.e. it cannot be marked before the two sequences have merged, since pivots do not generate entries. Hence the primary pivot must be a binomial. As we have seen in the previous paragraph, this contradicts Lemma 5.6.

Note that row i could not be originally of type (4) in Δ^2 , since this row would not contain binomials.

- We will prove item (iii) for $r + 1$. Let $\Delta_{i,j}^{r+1}$ be a change-of-basis pivot marked in diagonal $r + 1$, $\Delta_{s,j}^{r+1}$ be a binomial and let $\Delta_{i,u}^{r+1}$ be the primary pivot in row i . Suppose by contradiction that the entry $\Delta_{s,u}^{r+1}$ is non-zero.

If row s is of type (2) in Δ^2 , then $\Delta_{s,u}^{r+1}$ and $\Delta_{s,j}^{r+1}$ are in the same lineage, i.e. $\Delta_{s,j}^{r+1}$ must be a successor of $\Delta_{s,u}^{r+1}$, which is a contradiction. In fact, $\Delta_{s,j}^{r+1}$ is not an immediate successor of $\Delta_{s,u}^{r+1}$, since in this case it would imply the existence of two primary pivots in column u . Moreover, $\Delta_{s,j}^{r+1}$ is not an eventual successor of $\Delta_{s,u}^{r+1}$, since it would imply the existence of two change-of-basis pivots in row i marked up to step r .

If row s is of type (3) in Δ^2 and if $\Delta_{s,u}^{r+1}$ is a binomial then the argument is the same as the one above. However, if $\Delta_{s,u}^{r+1}$ is a monomial, one has two cases to consider: $\Delta_{s,u}^{r+1}$ and $\Delta_{s,j}^{r+1}$ are in the same lineage or in different lineages. In both cases there exists a $\xi < r + 1$ such that the entry $\Delta_{s,u}^\xi$ generated an entry in $\Delta^{\xi+1}$. Indeed, if they are in the same lineage, which is an eventual binomial lineage, then the (s, j) -entry is a successor of the (s, u) -entry. Now, if they are in different lineages, the lineage containing the (s, u) -entry ceases due to the appearing of the first binomial in a step $\xi^* < r + 1$, which appears in the entry corresponding to the last element of this lineage. Hence, the last element in this lineage cannot be $\Delta_{s,u}^{\xi^*-1}$ since the entry $\Delta_{s,u}^{r+1}$ is still a monomial. Therefore there exists a $\xi < \xi^* < r + 1$ such that $\Delta_{s,u}^\xi$ generates an entry.

Note that row s could not be originally of type (4) in Δ^2 , since this row would not contain binomials.

- We will prove item for (iv) for $r + 1$, i.e. that a primary pivot $\Delta_{i,u}^{r+1}$ and a change-of-basis pivot $\Delta_{i,j}^{r+1}$ in row i are always monomials with opposite signs.

As a consequence of the induction hypothesis (iv) and (v) for r , all elements in a monomial lineage have the same sign, up to step $\xi \leq r + 1$. Moreover, if row i was of type (3) in Δ , then the two monomial lineages of this row have opposite signs. Now, suppose that there is a change-of-basis pivot $\Delta_{i,j}^{r+1}$ in diagonal $r + 1$ and let $\Delta_{i,u}^{r+1}$ be the primary pivot in row i with $u < j$. Since pivots do not generate entries by Lemma 5.6, the entries $\Delta_{i,j}^{r+1}$ and $\Delta_{i,u}^{r+1}$ are clearly in different lineages, therefore they have opposite signs.

- We will prove item (v) for $r + 1$, i.e. that a monomial $\Delta_{s,u}^{r+1}$ above a primary pivot $\Delta_{i,u}^{r+1}$ and a monomial $\Delta_{s,j}^{r+1}$ above a change-of-basis pivot $\Delta_{i,j}^{r+1}$ always have opposite signs.

As a consequence of the induction hypothesis (iv) and (v), until Δ^{r+1} , all elements in a monomial lineage have the same coefficient, which is either $+1$ or -1 . Moreover, if row i was of type (2) in Δ , then the two monomial lineages of this row have opposite signs. Let $\Delta_{s,u}^{r+1}$ be a monomial above a primary pivot $\Delta_{i,u}^{r+1}$ and $\Delta_{s,j}^{r+1}$ be a monomial above a change-of-basis pivot $\Delta_{i,j}^{r+1}$. Observe that, by the induction hypothesis, $\Delta_{s,u}^{r+1}$ and $\Delta_{s,j}^{r+1}$ are not in the same lineage. Hence, they have opposite signs. □

We now proceed with the proof Theorem 5.1.

PROOF OF THEOREM 5.1. Let Δ be a Novikov differential. By the characterization of the initial matrix, see Corollary 2.2, the entries of Δ are invertible in $\mathbb{Z}((t))$; hence, one can apply the SSSA in Δ . Since the first change of basis can only occur from step 2 to step 3, then the SSSA is correct until step 2, and the entries of Δ^1 are equal to the entries of Δ^2 . Now, using Lemma 5.3, one can apply the SSSA to each block of Δ . Theorems 5.10 and 5.11 imply that the pivots in Δ^3 are invertible. Hence, the SSSA is also correct for Δ^3 . By an induction argument, one can suppose that the SSSA is proved correct until step r . Theorems 5.10 and 5.11 also imply that the SSSA is proved correct for Δ^{r+1} . Therefore, Theorem 5.1 follows. □

Observe that, if Δ^L is the last matrix produced by the SSSA, then the non-null columns of Δ^L are the columns containing primary pivots. The primary pivots are non-zero and are unalterable after being identified. Moreover, $\Delta^L \circ \Delta^L = 0$.

The next example shows that during the application of the SSSA infinite series can appear as entries of an intermediate matrix due to multiplication by $(T^r)^{-1}$. However, the last matrix produced by the SSSA does not contain entries which are infinite series.

EXAMPLE 5.12. Applying the SSSA to the Novikov matrix Δ in Figure 4, one obtains the sequence of Novikov matrices $\Delta^1, \dots, \Delta^6$ presented in Figures 5–10, respectively. Observe that the entry $\Delta_{3,7}^3$ is an infinite Laurent series.

The next two results imply that the last matrix Δ^L produced by the SSSA never contains an entry which is an infinite series, rather all entries are Laurent polynomials in $\mathbb{Z}((t))$. The proof of Theorem 5.13 follows the same steps of the proof of its analogous version in [9], where the SSSA is done over a field \mathbb{F} .

THEOREM 5.13. *Given a Novikov complex $(\mathcal{N}_*(f), \Delta)$ on surfaces, let Δ^L be the last matrix produced by the SSSA over $\mathbb{Z}((t))$. If column j of Δ^L is non-null then row j is null.*

PROOF. The statement of the lemma is equivalent to say that $\Delta_{j\bullet}^L \Delta_{\bullet j}^L = 0$ for all j . If $\Delta_{\bullet j}^L = 0$ then it is trivial that $\Delta_{j\bullet}^L \Delta_{\bullet j}^L = 0$. Suppose that $\Delta_{\bullet j}^L \neq 0$. Let s be an integer such that $j \in J_s$. Labelling the primary pivots in block J_s such that, if $\Delta_{i_1, j_1}^L, \dots, \Delta_{i_a, j_a}^L$ are the primary pivots in block J_s , then $i_1 < \dots < i_a$, one has that j_1, \dots, j_a are the non-null columns of J_s . Moreover, Δ_{i_a, j_a}^L is the unique non-zero entry in row i_a . Row i_{a-1} has non-zero entry in column j_{a-1} and may have another one non-zero entry in column j_a , and so on. Since $\Delta^L \circ \Delta^L = 0$, one has

$$0 = \Delta_{i_a \bullet}^L \Delta_{\bullet j'}^L = \Delta_{i_a j_a}^L \Delta_{j_a j'}^L,$$

for all j' . Since $\Delta_{i_a j_a}^L$ is a primary pivot, hence non-null, then $\Delta_{j_a j'}^L = 0$ for all j' , i.e. $\Delta_{j_a \bullet}^L = 0$. Analogously, one has

$$0 = \Delta_{i_{a-1} \bullet}^L \Delta_{\bullet j'}^L = \Delta_{i_{a-1} j_{a-1}}^L \Delta_{j_{a-1} j'}^L + \Delta_{i_{a-1} j_a}^L \Delta_{j_a j'}^L,$$

for all j' . Since $\Delta_{i_{a-1} j_{a-1}}^L \neq 0$ and $\Delta_{j_a j'}^L = 0$, it follows that $\Delta_{j_{a-1} j'}^L = 0$ for all j' , i.e. $\Delta_{j_{a-1} \bullet}^L = 0$. Proceeding in this way, one can show the nullity of rows j_{a-2}, \dots, j_1 . □

COROLLARY 5.14. *Given a Novikov complex $(\mathcal{N}_*(f), \Delta)$ on a surface, let Δ^L be the last matrix produced by the SSSA over $\mathbb{Z}((t))$. Then, the entries of Δ^L are monomials t^ℓ or binomials $t^\ell - t^{\tilde{\ell}}$.*

PROOF. By Theorems 5.10 and 5.11, without performing the pre-multiplication by $(T^r)^{-1}$, the entries of Δ^r are monomials t^ℓ or binomials $t^{\ell_1} - t^{\ell_2}$. Moreover, the pre-multiplication by $(T^r)^{-1}$ only affects row j if column j contains a primary pivot. However, by Lemma 5.13, these rows will be zeroed out by the time SSSA reaches the last matrix Δ^L . □

6. Spectral Sequence (E^r, d^r) associated to SSSA

Let M be a smooth closed orientable 2-dimensional manifold, $f: M \rightarrow S^1$ be a circle-valued Morse function. Let $(\mathcal{N}_*(f), \Delta)$ be a Novikov chain complex

with the finest filtration F and (E^r, d^r) be the associated spectral sequence, as in Section 3. In this section, we show that the SSSA provides a mechanism to recover the modules E^r and differentials d^r of the spectral sequence. More specifically, the SSSA provides a system to detect the generators of E^r in terms of the original basis of $\mathcal{N}_*(f)$ and to identify the differentials d^r with the primary pivots in the r -th diagonal.

For instance, consider the Novikov complex $(\mathcal{N}_*(f), \Delta)$ presented in Example 4.1. The matrices produced by the SSSA are illustrated in Figures 5–10. The Novikov homology of the complex $(\mathcal{N}_*(f), \Delta)$ is given by

$$H_0^{\text{Nov}}(M, f) = 0, \quad H_1^{\text{Nov}}(M, f) = 0, \quad H_2^{\text{Nov}}(M, f) = 0.$$

Consider the filtration F on $(\mathcal{N}_*(f), \Delta)$ defined by

$$F_p \mathcal{N}_k = \bigoplus_{h_k^\ell, \ell \leq p+1} \mathbb{Z}((t)) \langle h_k^\ell \rangle.$$

The spectral sequence associated to $(\mathcal{N}_*(f), \Delta)$ endowed with this filtration F is presented in Figure 20. Note that:

- (1) Each $\mathbb{Z}((t))$ -module E_p^r is generated by a k -chain $\sigma_k^{p+1, r}$ determined in the r -th step of the SSSA.
- (2) The differentials d_2^1, d_6^1, d_3^3 and d_7^3 are isomorphisms which could be interpreted as the multiplication by the invertible polynomials $\Delta_{2,3}^1, \Delta_{6,7}^1, \Delta_{1,4}^3$ and $\Delta_{5,8}^3$, respectively.
- (3) The spectral sequence (E^r, d^r) converges to the Novikov homology of $(\mathcal{N}_*(f), \Delta)$.

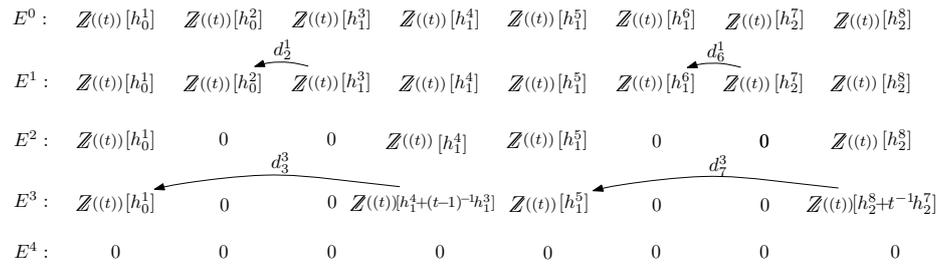


FIGURE 20. A spectral sequence associated to the Novikov complex (\mathcal{N}_*, Δ) presented in Example 4.1.

In the 2-dimensional setting, we will prove that the remarks in (1) and (2) hold for any Novikov chain complex endowed with a finest filtration. The first step in this direction is the next proposition, which establishes a formula for the modules $Z_{p, k-p}^r$ via the chains $\sigma_k^{i, j}$'s determined by the SSSA applied to Δ .

PROPOSITION 6.1. *Let κ be the first column in Δ associated to a k -chain and consider $\mu^{j,\zeta} = 0$ whenever the primary pivot of column j is below row $(p-r+1)$ and $\mu^{j,\zeta} = 1$ otherwise. Then*

$$Z_p^r = \mathbb{Z}((t))[\mu^{p+1,r} \sigma_k^{p+1,r}, \mu^{p,r-1} \sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa} \sigma_k^{\kappa,r-p-1+\kappa}].$$

PROOF. By definition, $\sigma_k^{p+1-\xi,r-\xi}$ is associated to column $(p+1-\xi)$ of the matrix $\Delta^{r-\xi}$, for $\xi \in \{0, \dots, p+1-\kappa\}$, and $\mu^{p+1-\xi,r-\xi} = 1$ if and only if the primary pivot on column $(p+1-\xi)$ is in or above row $(p+1-\xi)-(r-\xi) = p-r+1$ or if this column does not have a primary pivot. If $\sigma_k^{p+1-\xi,r-\xi}$ is such that $\mu^{p+1-\xi,r-\xi} = 1$, we show that $\sigma_k^{p+1-\xi,r-\xi}$ is a k -chain which corresponds to a generator of Z_p^r . In fact, $\sigma_k^{p+1-\xi,r-\xi}$ is in $F_p \mathcal{N}_k$ for $\xi \geq 0$. Furthermore, all nonzero entries of column $(p+1-\xi)$ of $\Delta^{r-\xi}$ are in or above row $(p-r+1)$, since step $(r-\xi)$ of SSSA has zeroed out all entries below diagonal $(r-\xi)$. Hence, the boundary of $\sigma_k^{p+1-\xi,r-\xi}$ is in $F_{p-r} \mathcal{N}_{k-1}$.

On the other hand, any element in Z_p^r is a linear combination over $\mathbb{Z}((t))$ of $\mu^{p+1-\xi,r-\xi} \sigma_k^{p+1-\xi,r-\xi}$ for $\xi = 0, \dots, p+1-\kappa$, as we prove below by multiple induction in p and r .

Base case. 1. Denote by κ the first column of Δ associated to a k -chain and let ξ be such that the boundary of h_k^κ is in $F_{\kappa-1-\xi} \mathcal{N}_k$ but it is not in $F_{\kappa-1-\xi-1} \mathcal{N}_k$. We will show that $Z_{\kappa-1}^r = \mathbb{Z}((t))[\mu^{\kappa,r} \sigma_k^{\kappa,r}]$. Since $Z_{\kappa-1}^r$ is generated by k -chain in $F_{\kappa-1} \mathcal{N}_k$ with boundaries in $F_{\kappa-1-r} \mathcal{N}_{k-1}$ and there is only one chain h_k^κ in $F_{\kappa-1} \mathcal{N}_k$ then:

- (a) If $\xi < r$ then $\partial h_k^\kappa \notin F_{\kappa-1-r} \mathcal{N}_{k-1}$. Thus, $Z_{\kappa-1}^r = 0$.
- (b) If $\xi \geq r$ then $\partial h_k^\kappa \in F_{\kappa-1-r} \mathcal{N}_{k-1}$. Thus, $Z_{\kappa-1}^r = \mathbb{Z}((t))[h_k^\kappa]$.

On the other hand, since there is no change of basis caused by the SSSA that affects the first column of Δ_k , $\sigma_k^{\kappa,r} = h_k^\kappa$, where $\sigma_k^{\kappa,r}$ is a k -chain associated to the column κ of Δ^r . Furthermore, $\mu^{\kappa,r} = 1$ if and only if the boundary of $h_k^\kappa = \sigma_k^{\kappa,r}$ is in or above the r -th diagonal. Hence

- (a) If $\xi < r$ then $\mu^{\kappa,r} = 0$. Thus $\mathbb{Z}((t))[\mu^{\kappa,r} \sigma_k^{\kappa,r}] = 0$.
- (b) If $\xi \geq r$ then $\mu^{\kappa,r} = 1$. Thus $\mathbb{Z}((t))[\mu^{\kappa,r} \sigma_k^{\kappa,r}] = \mathbb{Z}((t))[\sigma_k^{\kappa,r}] = \mathbb{Z}((t))[h_k^\kappa]$.

It follows that in both cases $Z_{\kappa-1}^r = \mathbb{Z}((t))[\mu^{\kappa,r} \sigma_k^{\kappa,r}]$.

2. Denote by ξ_1 the first diagonal in Δ which contains a non-zero entry in Δ_k . Note that the non-zero entries of the columns of Δ corresponding to the chains $h_k^\kappa, \dots, h_k^{p+1}$ are in or above the row $(p-\xi_1+1)$. We will show that $Z_p^{\xi_1} = \mathbb{Z}((t))[\mu^{p+1,\xi_1} \sigma_k^{p+1,r}, \dots, \mu^{\kappa,\kappa-p+1+\xi_1} \sigma_k^{\kappa,\kappa-p+1+\xi_1}]$. By the definition of $Z_p^{\xi_1}$ and the remark above we have that $Z_p^{\xi_1} = \mathbb{Z}((t))[h_k^{p+1}, \dots, h_k^\kappa]$. On the other hand, it is easy to see that $\sigma_k^{j,\xi_1} = h_k^j$, $j = \kappa, \dots, p+1$ and $\mu^{j,\xi_1} = 1$, $j = \kappa, \dots, p+1$. Hence,

$$\mathbb{Z}((t))[\mu^{p+1,\xi_1} \sigma_k^{p+1,r}, \dots, \mu^{\kappa,\kappa-p+1+\xi_1} \sigma_k^{\kappa,\kappa-p+1+\xi_1}] = \mathbb{Z}((t))[h_k^{p+1}, \dots, h_k^\kappa].$$

Induction hypothesis. Suppose

$$Z_{p-1}^{r-1} = \mathbb{Z}((t))[\mu^{p,r-1}\sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}].$$

Note that if $\mu^{p+1,r} = 0$ then $Z_p^r = Z_{p-1}^{r-1}$ and result in this case follows by the induction hypothesis. Suppose that $\mu^{p+1,r} = 1$ and let $\mathfrak{h}_k = b^{p+1}h_k^{p+1} + \dots + b^\kappa h_k^\kappa \in Z_{p,k-p}^r$. If $b^{p+1} = 0$ then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows. If $b^{p+1} \neq 0$, then

$$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + (b^p - b^{p+1}c_p^{p+1,r})h_k^p + \dots + (b^\kappa - b^{p+1}c_\kappa^{p+1,r})h_k^\kappa$$

and hence $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} \in F_{p-1}\mathcal{N}_k$. Since \mathfrak{h}_k and $\sigma_k^{p+1,r}$ have their boundaries on and above row $p-r+1$ then the boundary of $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r}$ is on and above row $p-r+1$. Hence $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} \in Z_{p-1}^{r-1}$. By the induction hypotheses, we have that $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} = \alpha_p\mu^{p,r-1}\sigma_k^{p,r-1} + \dots + \alpha_\kappa\mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}$, i.e.

$$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + \alpha_p\mu^{p,r-1}\sigma_k^{p,r-1} + \dots + \alpha_\kappa\mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}. \quad \square$$

LEMMA 6.2. *If $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$, then $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1} = Z_p^r$.*

PROOF. Since $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$ then $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ is a submodule of

$$Z_p^r = \mathbb{Z}((t))[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}],$$

where κ is the first column associated to a k -chain, but it is not a submodule of

$$Z_{p-1}^{r-1} = \mathbb{Z}((t))[\mu^{p,r-1}\sigma_k^{p,r-1}, \mu^{p-1,r-2}\sigma_k^{p-1,r-2}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}].$$

Then $\mu^{(p+1),r} = 1$ and $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ contains a multiple of $\sigma_k^{p+1,r}$ over $\mathbb{Z}((t))$. We will show that $\sigma_k^{p+1,r} \in Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$. Note that

$$(6.1) \quad \partial Z_{p+r-1}^{r-1} = \mathbb{Z}((t))[\mu^{p+r,r-1}\partial\sigma_{k+1}^{p+r,r-1}, \mu^{p+r-1,r-2}\partial\sigma_{k+1}^{p+r-1,r-2}, \dots, \mu^{\bar{\kappa},\bar{\kappa}-p-1}\partial\sigma_{k+1}^{\bar{\kappa},\bar{\kappa}-p-1}],$$

where $\bar{\kappa}$ is the first column associated to a $(k+1)$ -chain. For $\xi = 0, \dots, p+r-\bar{\kappa}$ with $\mu^{p+r-\xi,r-1-\xi} = 1$ we have $\Delta_{i,p+r-\xi}^{r-1-\xi} = 0$ for all $i > p+1$ and hence

$$\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi} = \Delta_{p+1,p+r-\xi}^{r-1-\xi}\sigma_k^{p+1,r-1-\xi} + \dots + \Delta_{\kappa,p+r-\xi}^{r-1-\xi}\sigma_k^{\kappa,r-1-\xi}.$$

In fact, the boundaries $\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}$ with $\Delta_{i,p+r-\xi}^{r-1-\xi} \neq 0$ for some $i > p+1$ correspond exactly to the columns which have the primary pivots below the $(p+1)$ -st row and therefore $\mu^{p+r-\xi,r-1-\xi} = 0$. Hence, for $\xi = 0, \dots, p+r-\bar{\kappa}$, when $\mu^{p+r-\xi,r-1-\xi} = 1$ we have

$$(6.2) \quad Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}] = Z_{p-1}^{r-1} + [\Delta_{p+1,p+r-\xi}^{r-1-\xi}\sigma_k^{p+1,r-1-\xi} + \dots + \Delta_{\kappa,p+r-\xi}^{r-1-\xi}\sigma_k^{\kappa,r-1-\xi}].$$

On the other hand, since $Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi, r-1-\xi}] \subset Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$, then

$$(6.3) \quad Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi, r-1-\xi}] \\ = [\ell_\xi \sigma_k^{p+1, r}, \mu^{p, r-1} \sigma_k^{p, r-1}, \mu^{p-1, r-2} \sigma_k^{p-1, r-2}, \dots, \mu^{\kappa, r-p-1+\kappa} \sigma_k^{\kappa, r-p-1+\kappa}].$$

The coefficient of h_k^{p+1} on the set of generators of the $\mathbb{Z}((t))$ -module in (6.2) is $\Delta_{p+1, p+r-\xi}^{r-1-\xi}$. On the other hand, the coefficient of h_k^{p+1} on the set of the generators of the $\mathbb{Z}((t))$ -module in (6.3) is ℓ_ξ . Since for each $\xi = 0, \dots, p+r-\bar{\kappa}$, $\Delta_{i, p+r-\xi}^{r-1-\xi} = 0$ for all $i > p+1$, then $\Delta_{p+1, p+r-\xi}^{r-1-\xi}$ is either a pivot or a zero entry. Note that the entries $\Delta_{p+1, p+r-\xi}^{r-1-\xi}$ cannot be all zeros, since it would contradict the hypothesis of $\partial Z_{p+r-1}^{r-1} \not\subset Z_{p-1}^{r-1}$. It follows from Lemma 5.1 that, if $\Delta_{p+1, p+r-\xi}^{r-1-\xi}$ is nonzero, then it is invertible in $\mathbb{Z}((t))$. Then $\sigma_k^{p+1, r} \in Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ and hence $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1} = Z_p^r$. \square

THEOREM 6.3. *The matrix Δ^r obtained from the SSSA applied to Δ determines*

$$E_p^r = \frac{Z_p^r}{Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}}.$$

More specifically, E_p^r is either zero or a finitely generated $\mathbb{Z}((t))$ -module whose generator corresponds to a k -chain associated to column $(p+1)$ of Δ^r .

PROOF. The entry $\Delta_{p-r+1, p+1}^r$ on the r -th diagonal plays a crucial role in determining the generators of E_p^r . We will analyze each possibility of $\Delta_{p-r+1, p+1}^r$ and prove how to determine E_p^r for each case.

1. The entry $\Delta_{p-r+1, p+1}^r$ is identified by the SSSA as a primary pivot, a change-of-basis pivot or a zero entry with column of zeros below it.

In these cases $\Delta_{s, p+1}^r = 0$ for all $s > p-r+1$ and hence the generator $\sigma_k^{p+1, r}$ corresponding to the k -chain associated to column $(p+1)$ in Δ^r is a generator of Z_p^r . Thus we must analyze row $(p+1)$. We have the following possibilities:

(a) $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$, i.e, all the boundaries of the elements in Z_{p+r-1}^{r-1} are above row p .

In this case, as before, by Proposition 6.1, $E_p^r = \mathbb{Z}((t))[\sigma_k^{p+1, r}]$.

(b) $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$, i.e, there exist elements in Z_{p+r-1}^{r-1} whose boundary has a non-zero entry in row $(p+1)$. By Proposition 6.1 and Lemma 6.2, $E_p^r = 0$.

Note that if $\Delta_{p-r+1, p+1}^r$ has been identified by the SSSA as a primary pivot then

$\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$. For more details see [6].

In fact, by Lemma 5.2, row $(p+1)$ cannot contain a primary pivot since we have assumed that column $(p+1)$ has a primary pivot. Therefore, the entries of these h_{k+1} columns in row $(p+1)$ must be zeroes. It follows that ∂Z_{p+r-1}^{r-1} does not contain in its set of generators the generator $\sigma_k^{p+1, r}$.

2. The entry $\Delta_{p-r+1,p+1}^r$ is an entry above a primary pivot, i.e. there exists $s > p-r+1$ such that $\Delta_{s,p+1}^r$ is a primary pivot. In this case, the generator $\sigma_k^{p+1,r}$ corresponding to the k -chain associated to column $(p+1)$ is not a generator of Z_p^r and hence $Z_{p-1}^{r-1} = Z_p^r$. It follows that $E_p^r = 0$.

3. The entry $\Delta_{p-r+1,p+1}^r$ is not in Δ_k^r . This includes the case $p-r+1 < 0$, i.e. $\Delta_{p-r+1,p+1}^r$ is not on the matrix Δ^r .

The analysis of E_p^r is very similar to the previous one, i.e. we have two possibilities:

- (a) There is a primary pivot in column $(p+1)$ in a diagonal $\bar{r} < r$. In this case the generator corresponding to the k -chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ is not a generator of Z_p^r . Hence $Z_{p-1}^{r-1} = Z_p^r$ and $E_p^r = 0$.
- (b) All the entries in Δ^r in column $(p+1)$ in diagonals lower than r are zero, i.e. the generator corresponding to the k -chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ in Δ^r is a generator of Z_p^r . Then we have to analyze row $(p+1)$.
 - (i) If $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$ then, by Proposition 6.1, $E_p^r = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$.
 - (ii) If $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$ then $E_p^r = 0$ by Proposition 6.1 and Lemma 6.2. □

We will describe how the SSSA applied to Δ induces the differentials $d_p^r: E_p^r \rightarrow E_{p-r}^r$ of spectral sequence.

THEOREM 6.4. *If E_p^r and E_{p-r}^r are both non-zero, then the map $d_p^r: E_p^r \rightarrow E_{p-r}^r$ is induced by the multiplication by the entry $\Delta_{p-r+1,p+1}^r$.*

PROOF. Suppose that E_p^r and E_{p-r}^r are both non-zero. Let

$$\delta_p^r: \mathbb{Z}((t))[\sigma_k^{p+1,r}] \rightarrow \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$$

be the multiplication by the entry $\Delta_{p-r+1,p+1}^r$ and $\tilde{\delta}_p^r$ the induced map in E_p^r . We must show that

$$\frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r} \cong E_p^{r+1}.$$

Since we are considering E_p^r non-zero, it follows from Theorem 6.3, that we must consider three cases for the entry $\Delta_{p-r+1,p+1}^r$: primary pivot, change-of-basis pivot and zero with a column of zeroes below it. However, if $\Delta_{p-r+1,p+1}^r$ is a change-of-basis pivot then there exists a primary pivot in row $(p-r+1)$ on a diagonal below the r -th diagonal and hence $E_{p-r}^r = 0$. Hence, whenever E_p^r and E_{p-r}^r are both non-zero, the entry $\Delta_{p-r+1,p+1}^r$ in Δ^r is either a primary pivot or a zero with a column of zero entries below it.

In this case $E_p^r = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$ and $E_{p-r}^r = \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$.

1. Suppose $\Delta_{p-r+1,p+1}^r$ is a primary pivot. Since $\delta_p^r: \mathbb{Z}((t))[\sigma_k^{p+1,r}] \rightarrow \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$ is multiplication by $\Delta_{p-r+1,p+1}^r$, which is invertible in $\mathbb{Z}((t))$,

then $\text{Ker } \delta_p^r = 0$. Since $\tilde{\delta}_p^r = \delta_p^r$, then $\text{Ker } \tilde{\delta}_p^r / \text{Im } \tilde{\delta}_{p+r}^r = 0$. On the other hand, since $\Delta_{p-r+1, p+1}^r$ is a primary pivot then $\sigma_k^{p+1, r+1} = \sigma_k^{p+1, r} \notin Z_p^{r+1}$. Thus $Z_p^{r+1} = Z_{p-1}^r$ and $E_p^{r+1} = 0$.

2. Suppose $\Delta_{p-r+1, p+1}^r = 0$ with a column of zeroes below it. In this case $\text{Ker } \delta_p^r \cong E_p^r = \text{Ker } \tilde{\delta}_p^r$ and $\sigma_k^{p+1, r} = \sigma_k^{p+1, r+1}$. There are three cases to consider:

- (a) If $\Delta_{p+1, p+r+1}^r$ is an entry above a primary pivot then we have $E_{p+r}^r = 0$ and hence $\text{Im } \tilde{\delta}_{p+r}^r = 0$. Thus,

$$\frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r} = E_p^r.$$

On the other hand, since $\mu^{p+r+1, r} = 0$, then $E_p^{r+1} = E_p^r$.

- (b) If $\Delta_{p+1, p+r+1}^r$ is a primary pivot then $E_{p+r}^r = \mathbb{Z}((t))[\sigma_k^{p+r+1, r}]$. Therefore $\tilde{\delta}_{p+r}^r$ is an isomorphism and hence

$$\frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r} \cong \frac{\mathbb{Z}((t))[\sigma_k^{p+1, r}]}{\mathbb{Z}((t))[\sigma_k^{p+1, r}]} = 0.$$

On the other hand, since $\Delta_{p-r+1, p+1}^r$ is zero with a column of zero entries below it then $\sigma_k^{p+1, r+1} \in Z_{p, k-p}^{r+1}$ and hence $Z_{p-1}^r \subsetneq Z_{p, k-p}^{r+1}$. Moreover, since $E_p^r = \mathbb{Z}((t))[\sigma_k^{p+1, r}]$ then $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$. But the difference between ∂Z_{p+r-1}^{r-1} and ∂Z_{p+r}^r is that the last one includes the boundary of column $(p+r+1)$. The element in column $(p+r+1)$ and row $(p+1)$ is $\Delta_{p+1, p+r+1}^r$. Since $\Delta_{p+1, p+r+1}^r$ is a primary pivot then $\partial Z_{p+r}^r \not\subseteq Z_{p-1}^r$ and $E_p^{r+1} = 0$.

- (c) If $\Delta_{p+1, p+r+1}^r = 0$ with a column of zero entries below it then $\text{Im } \tilde{\delta}_{p+r}^r = 0$ and

$$\frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r} = E_p^r.$$

Analogously to the previous case, $\sigma_k^{p+1, r+1} \in Z_{p, k-p}^{r+1}$ and hence $Z_{p-1}^r \subsetneq Z_p^{r+1}$. Moreover, $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$ and the only difference between ∂Z_{p+r-1}^{r-1} and ∂Z_{p+r}^r is that the last one includes the boundary of column $(p+r+1)$. Since the element in column $(p+r+1)$ and row $(p+1)$ is $\Delta_{p+1, p+r+1}^r = 0$ then $\partial Z_{p+r}^r \subseteq Z_{p-1}^r$ and $E_p^{r+1} = \mathbb{Z}((t))[\sigma_k^{p+1, r}]$.

Note that the case where $\Delta_{p+1, p+r+1}^r$ is a change-of-basis pivot does not have to be considered, since in this case $E_p^r = 0$. Therefore, in all cases

$$\frac{\text{Ker } d_p^r}{\text{Im } d_{p+r}^r} = E_p^{r+1} = \frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r}. \quad \square$$

COROLLARY 6.5. *Each non-zero differential d_p^r of the spectral sequence (E^r, d^r) is an isomorphism.*

PROOF. In fact, by the proof of Theorem 6.4, non-zero differentials of (E^r, d^r) are induced by primary pivots. Theorem 5.1 states that each primary pivot produced by the SSSA is an invertible polynomial, hence each induced non-zero differential is an isomorphism. \square

The next corollary states that the spectral sequence (E^r, d^r) converges to the Novikov homology of $(\mathcal{N}_*(f), \Delta)$.

COROLLARY 6.6. *The modules $E_{p,q}^\infty$ of the spectral sequence are free for all p and q . Moreover, the spectral sequence converges to the Novikov homology.*

PROOF. By Corollary 6.5, the non-zero differentials $d^r: E_p^r \rightarrow E_{p-r}^r$ of the spectral sequence are isomorphisms. Since, $E_p^{r+1} \cong \text{Ker } d_p^r / \text{Im } d_{p-r}^r$, it follows that the modules E_p^r 's are free for all $r \geq 0$ and $p \geq 0$. Therefore,

$$E_{p,q}^\infty \approx GH_*(\mathcal{N})_{p,q} = \frac{F_p H_{p+q}(\mathcal{N})}{F_{p-1} H_{p+q}(\mathcal{N})} \approx H_*^{\text{Nov}}(M, f)$$

by equation (3.2). \square

7. Cancellation of critical points

The results obtained for a spectral sequence of a filtered two dimensional Novikov chain complex $(\mathcal{N}_*(f), \Delta)$ associated to a circle-valued Morse function $f: M \rightarrow S^1$ lay the groundwork for dynamical exploration of connections in a negative gradient flow associated to f . In Sections 5 and 6 we proved that the SSSA produces from $(\mathcal{N}_*(f), \Delta)$ a sequence of Novikov matrices from which the modules and differentials of the spectral sequence (E^r, d^r) may be retrieved. By Corollary 6.5, if $d^r: E_p^r \rightarrow E_{p-r}^r$ is a non-zero differential, then the modules E_p^{r+1} and E_{p-r}^{r+1} are zero. We refer to this situation as an *algebraic cancellation*. Hence, as one “turns the pages” of the spectral sequence, i.e. considers progressively modules E^r , one observes algebraic cancellations occurring within the E^r 's.

We will now show how the dynamics follows the algebraic unfolding of the spectral sequence, that is, how these algebraic cancellations can be associated to dynamical cancellations of singularities of the negative gradient flow on M . Let f be a circle-valued Morse function and $(\mathcal{N}_*(f), \Delta)$ the Novikov complex associated to f . Let the Morse function $F: \overline{M} \rightarrow \mathbb{R}$ be the lift of f to the infinite cyclic covering space \overline{M} , as defined in Section 2.

Consider a finest filtration $\mathcal{F} = \{\mathcal{F}_p \mathcal{N}\}_{p \in P}$ on $(\mathcal{N}_*(f), \Delta)$, where

$$F_p \mathcal{N} = \mathbb{Z}((t)) [h_{k_1}^1, h_{k_2}^2, \dots, h_{k_{p+1}}^{p+1}]$$

and $P = \{0, 1, \dots, m\}$ is an indexing set for the filtration \mathcal{F} , where $\# \text{ Crit}(f) = m + 1$. From the filtration \mathcal{F} one can induce an infinite filtration $\{\mathcal{F}_{\lambda,p}\}_{\lambda \in \mathbb{Z}, p \in P}$ in the covering space \overline{M} which corresponds to

$$\mathcal{F}_{\lambda,p} \mathcal{N} = \mathbb{Z} \langle t^\delta h_k^j \mid \delta = \lambda \text{ and } j \leq p + 1 \text{ or } \delta > \lambda \text{ and } j \in \{1, \dots, m\} \rangle,$$

where $t^\lambda h_k^{p+1}$ represents the lift of the point h_k^{p+1} at level λ , for $\lambda \in \mathbb{Z}$. Note that, for $p \in \{0, 1, \dots, m\}$ and $\lambda \in \mathbb{Z}$, we have

$$\mathcal{F}_{\lambda,p-1}\mathcal{N} \subset \mathcal{F}_{\lambda,p}\mathcal{N} \quad \text{and} \quad \mathcal{F}_{\lambda,m}\mathcal{N} \subset \mathcal{F}_{\lambda-1,0}\mathcal{N},$$

and also

$$\mathcal{F}_{\lambda,p}\mathcal{N} \setminus \mathcal{F}_{\lambda,p-1}\mathcal{N} = t^\lambda h_k^{p+1} \quad \text{and} \quad \mathcal{F}_{\lambda-1,0}\mathcal{N} \setminus \mathcal{F}_{\lambda,m}\mathcal{N} = t^{\lambda-1} h_k^1.$$

Without loss of generality, one can assume that f has one critical point per critical level set. Let c_p be a critical value of f such that $f(h_k^p) = c_p$. Hence, F also has one critical point per critical level set and $c_{\lambda,p} := c_p - \lambda$ is a critical value of F such that $F(t^\lambda h_k^p) = c_{\lambda,p}$, for all $\lambda \in \mathbb{Z}$.

EXAMPLE 7.1. Consider the negative gradient flow on a torus as illustrated in Figure 21 which is associated to a circle-valued Morse function f .

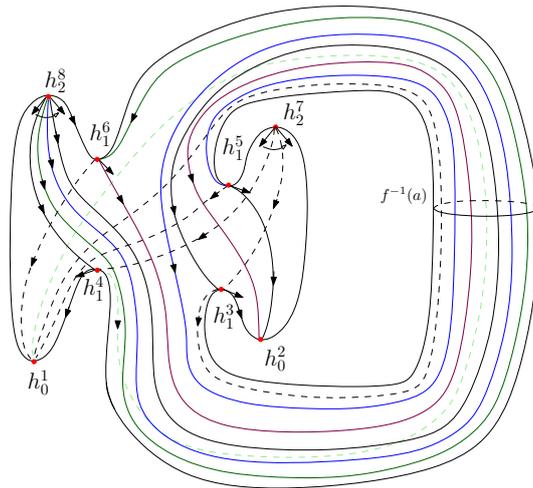


FIGURE 21. Circle-valued Morse function on the torus.

Let $(\mathcal{N}_*(f), \Delta)$ be the Novikov complex associated to f and define a filtration $\mathcal{F} = \{\mathcal{F}_p\mathcal{N}\}$ by:

$$\begin{aligned} F_0\mathcal{N} &= \mathbb{Z}((t))[h_0^1], & F_4\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3, h_1^4, h_1^5], \\ F_1\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2], & F_5\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3, h_1^4, h_1^5, h_1^6], \\ F_2\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3], & F_6\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3, h_1^4, h_1^5, h_1^6, h_2^7], \\ F_3\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3, h_1^4], & F_7\mathcal{N} &= \mathbb{Z}((t))[h_0^1, h_0^2, h_1^3, h_1^4, h_1^5, h_1^6, h_2^7, h_2^8]. \end{aligned}$$

The Novikov differential Δ is presented in Figure 22 and we illustrated the associated covering space enriched with the finest filtration $\{\mathcal{F}_{\lambda,p}\}$ in Figure 23.

$$\begin{array}{c}
 h_0^1 \quad h_0^2 \quad h_1^3 \quad h_1^4 \quad h_1^5 \quad h_1^6 \quad h_2^7 \quad h_2^8 \\
 \left(\begin{array}{cccccccc}
 0 & & -t & t-1 & -1 & -1 & & \\
 & 0 & 1 & 0 & 1 & t & & \\
 & & 0 & & & & -1 & t \\
 & & & 0 & & & -1 & 1 \\
 & & & & 0 & & 1 & -t^2 \\
 & & & & & 0 & 0 & t-1 \\
 & & & & & & 0 & \\
 & & & & & & & 0
 \end{array} \right)
 \end{array}$$

FIGURE 22. Novikov differential Δ .

The next result will provide an algorithmic approach to the dynamical cancellation of critical points of a circle-valued Morse function $f: M \rightarrow S^1$ by computing the spectral sequence of the associated Novikov chain complex of f .

THEOREM 7.2 (Dynamical cancellation theorem via spectral sequences in the Novikov setting). *Let $f: M \rightarrow S^1$ be a circle-valued Morse function on a closed n -manifold M and $(\mathcal{N}_*(f), \Delta)$ be the Novikov chain complex associated to f . Let \mathcal{F} be a finest filtration in $(\mathcal{N}_*(f), \Delta)$ and let (E^r, d^r) be the associated spectral sequence for this filtered chain complex. There exists a one-to-one correspondence between the algebraic cancellations of the modules E^r with dynamical cancellations of critical points of f , which produces a family of circle-valued Morse functions f^r , where f^{r+1} is obtained from f^r by removing the pairs of critical points corresponding to the cancelled modules of the r -th page of the spectral sequence. Moreover, for each $r \geq 0$, the $(r + 1)$ -th page of the spectral sequence completely determines all Novikov incidence coefficients, hence all new flow lines, between consecutive critical points of f^{r+1} .*

The following example illustrates how the dynamics follows the algebraic unfolding of the spectral sequence.

EXAMPLE 7.3. Consider the circle-valued Morse function f presented in Example 7.1. Applying the SSSA to the Novikov differential Δ in Figure 22, one obtains the sequence of Novikov matrices $\Delta^1, \dots, \Delta^6$ presented in Figures 24–29, respectively. The primary pivots detected by the SSSA induce the differentials of the spectral sequence, as we have shown in Theorem 6.4. By Theorem 5.1, these pivots are invertible, hence responsible for algebraic cancellations of the modules of the spectral sequence. More specifically:

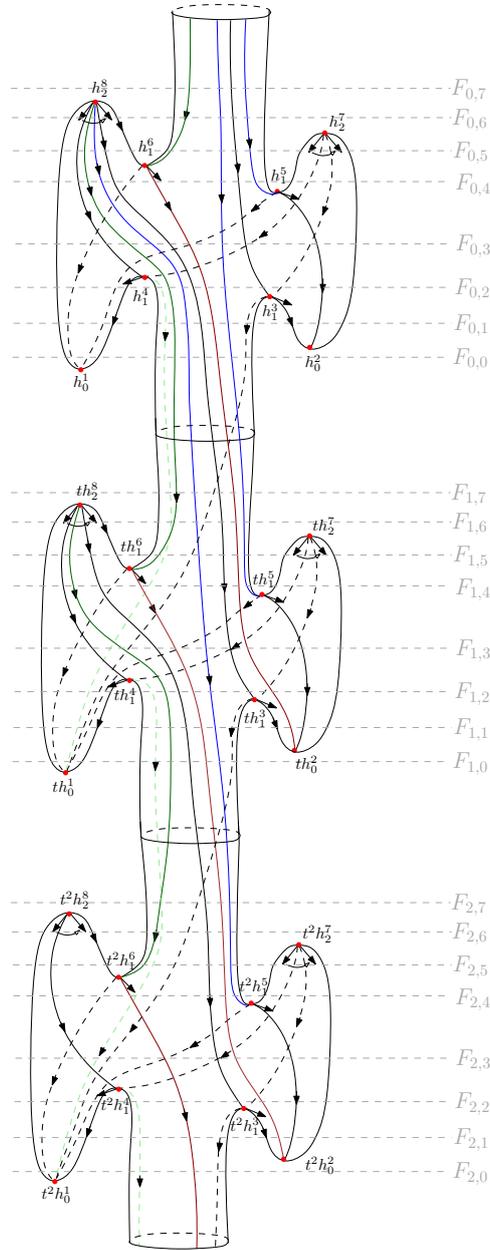
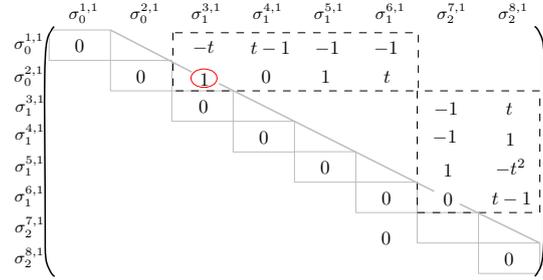


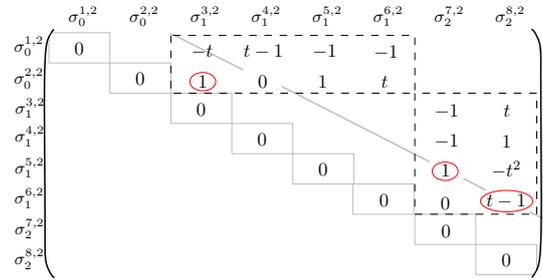
FIGURE 23. Filtration in the infinite cyclic covering.

- the primary pivot $\Delta_{2,3}^1$ induces the differential $d_2^1: E_2^1 \rightarrow E_1^1$ which in turn causes the algebraic cancellation $E_2^2 = E_1^2 = 0$;
- the primary pivot $\Delta_{5,7}^2$ induces the differential $d_6^2: E_6^2 \rightarrow E_4^2$ which in turn causes the algebraic cancellation $E_6^3 = E_4^3 = 0$;



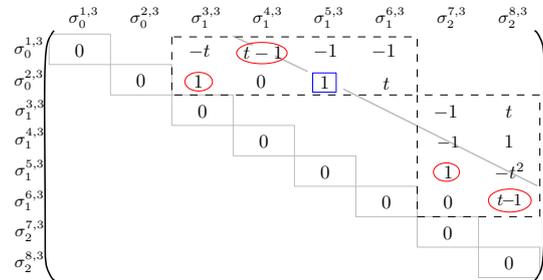
$$\sigma_0^{j,1} = h_0^j \text{ for } j \in J_0; \sigma_1^{j,1} = h_1^j \text{ for } j \in J_1; \sigma_2^{j,1} = h_2^j \text{ for } j \in J_2.$$

FIGURE 24. Δ^1 , marking primary pivots.



$$\sigma_0^{j,2} = h_0^j \text{ for } j \in J_0; \sigma_1^{j,2} = h_1^j \text{ for } j \in J_1; \sigma_2^{j,2} = h_2^j \text{ for } j \in J_2.$$

FIGURE 25. Δ^2 , marking change-of-basis pivot.



$$\sigma_0^{j,3} = h_0^j \text{ for } j \in J_0; \sigma_1^{j,3} = h_1^j \text{ for } j \in J_1; \sigma_2^{j,3} = h_2^j \text{ for } j \in J_2.$$

FIGURE 26. Δ^3 , marking pivots.

- the primary pivot $\Delta_{6,8}^2$ induces the differential $d_7^2: E_7^2 \rightarrow E_5^2$ which in turn causes the algebraic cancellation $E_7^3 = E_5^3 = 0$;
- the primary pivot $\Delta_{1,4}^3$ induces the differential $d_3^3: E_3^3 \rightarrow E_0^3$ which in turn causes the algebraic cancellation $E_3^4 = E_0^4 = 0$.

Figures 30–33 show the dynamical cancellations of pairs of critical points (h_0^2, h_1^3) , (h_1^5, h_2^7) , (h_1^6, h_2^8) and (h_0^1, h_1^4) corresponding to the above algebraic cancellations, as explained in what follows.

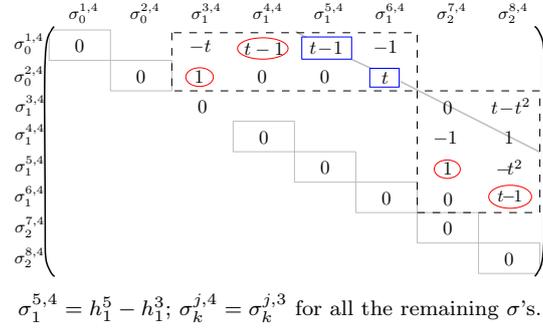


FIGURE 27. Δ^4 , marking change-of-basis pivot.

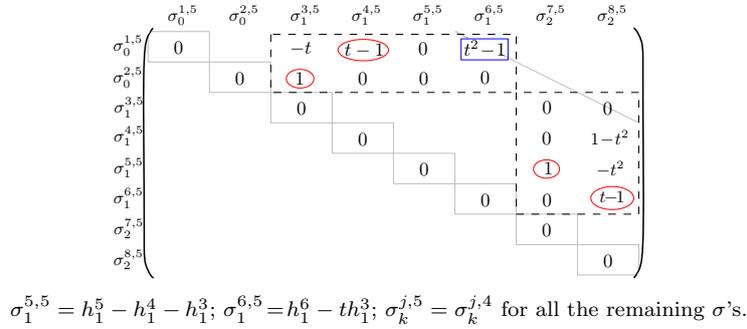


FIGURE 28. Δ^5 , marking change-of-basis pivot.

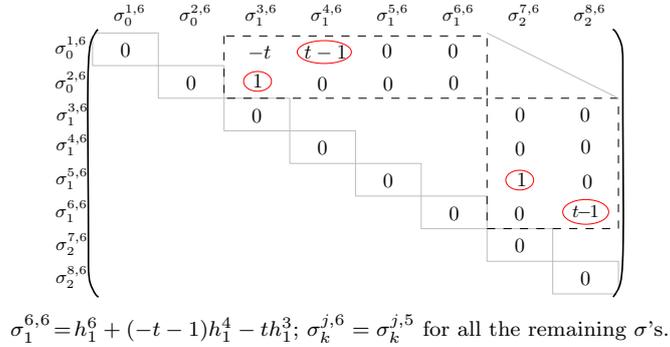


FIGURE 29. Δ^6 for Example 4.1.

On the first page of the spectral sequence, the differential d_2^1 determines the algebraic cancellation $E_2^2 = E_1^2 = 0$. Corresponding to this algebraic cancellation, we have the dynamical cancellation of the pair (h_0^2, h_1^3) , which determines a circle-valued Morse function f^2 such that $\text{Crit}(f^2) = \text{Crit}(f) \setminus \{h_0^2, h_1^3\}$ and is equal (up to isotopy) to f outside a neighbourhood of the connection of this critical points. In the corresponding negative gradient flow φ^2 (see Figure 30),

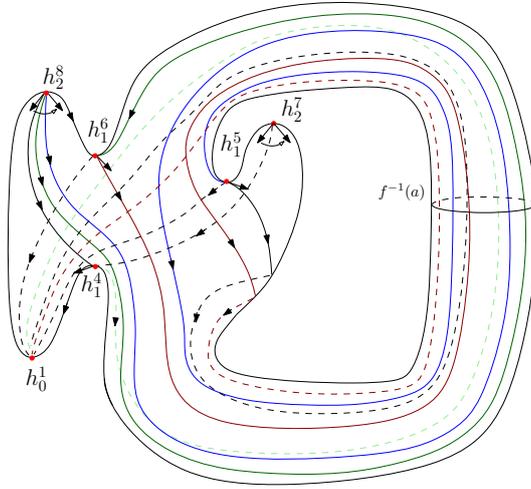


FIGURE 30. Cancellation of the pair (h_0^2, h_1^3) .

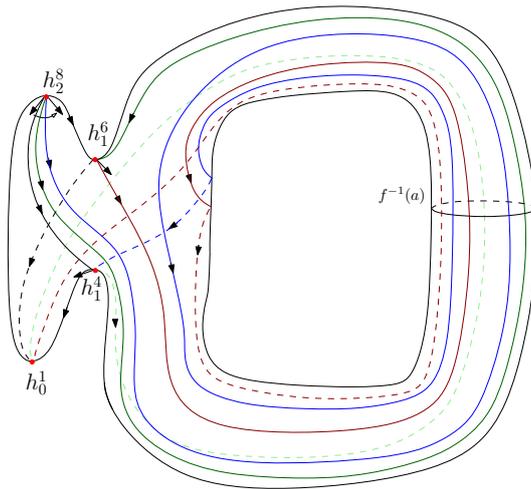


FIGURE 31. Cancellation of the pair (h_1^5, h_2^7) .

we note birth of flow lines from h_1^5 to h_0^1 and from h_1^6 to h_0^1 . Moreover, we have the following relation between the Novikov incidence coefficients:

$$\begin{aligned}
 N(h_1^5, h_0^1; f^2) &= N(h_1^5, h_0^2; f^1)t + N(h_1^5, h_0^1; f^1) = 0, \\
 N(h_1^6, h_0^1; f^2) &= N(h_1^6, h_0^2; f^1)t + N(h_1^6, h_0^1; f^1) = t^2 - 1.
 \end{aligned}$$

The other flow lines between consecutive critical points remain the same.

On the second page of the spectral sequence, there are two non-zero differentials, namely d_6^2 and d_7^2 , which determine the algebraic cancellation $E_6^3 = E_4^3 = 0$ and $E_7^3 = E_5^3 = 0$. The corresponding pairs of critical points which will cancel

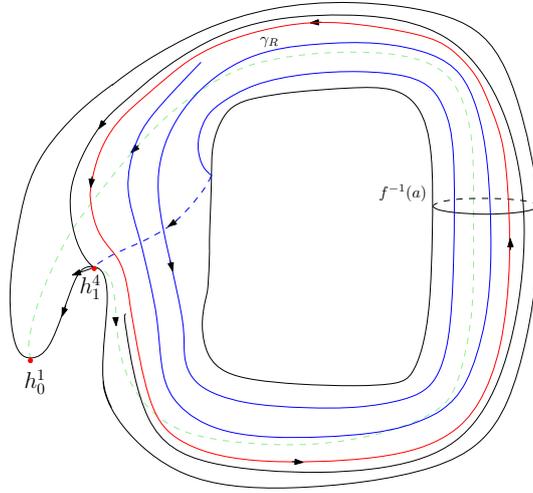


FIGURE 32. Cancellation of the pair (h_1^6, h_2^8) .

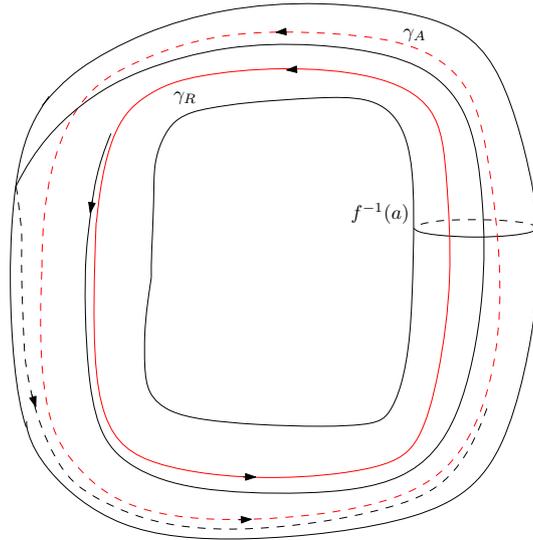


FIGURE 33. Cancellation of the pair (h_0^1, h_1^4) .

are (h_1^5, h_2^7) and (h_1^6, h_2^8) , respectively. In order to obtain the circle-valued Morse function f^3 , we cancel the pair (h_1^5, h_2^7) , see Figure 31, and then the pair (h_1^6, h_2^8) , see Figure 32. In the corresponding negative flow φ^3 (see Figure 32), the cancellation of the pair of critical points (h_1^6, h_2^8) gives rise to the birth of a repeller periodic orbit γ_R and two flow lines whose α -limit set is γ_R and whose ω -limit set is the saddle h_1^4 . Note that the birth of the periodic orbit γ_R is due to a primary pivot $\Delta_{6,8}^2$ which is a binomial and the period of γ_R is equal to the difference

between the exponents of the binomial. Moreover, we have the following relation between the Novikov incidence coefficient: $N(h_1^4, h_0^1; f^3) = N(h_1^4, h_0^1; f^2) = t - 1$.

On the third page of the spectral sequence, the differential d_3^3 determines the algebraic cancellation $E_4^4 = E_0^4 = 0$, which corresponds to the dynamical cancellation of the pair (h_0^1, h_1^4) , creating the circle-valued Morse function f^4 . In the corresponding negative gradient flow φ^4 , see Figure 33, an attractor periodic orbit γ_A is born due to the primary pivot $\Delta_{1,4}^3 = t - 1$. Therefore, the periodic orbits that will appear after the cancellations are codified in the Novikov matrices.

In order to prove this relation between dynamical and algebraic cancellations, we need to use an adaptation of the SSSA which is more conducive to dynamical interpretations. The dynamical inspiration for this adaptation of the SSSA is as follows. Note that the changes of basis caused by pivots in row $j - r$ reflect all the changes in connecting orbits caused by the cancellation of h_k^j and h_{k-1}^{j-r} . However, when the pair of critical points h_k^j and h_{k-1}^{j-r} is removed, all the connecting orbits between index k critical points and h_{k-1}^{j-r} and also all the ones between h_k^j and index $k - 1$ critical points are immediately removed and new ones take their place. Hence, in order to interpret dynamically the SSSA, we have to perform the changes of basis that occur therein in a different order to reflect the death and birth of connections. More specifically, if $\Delta_{j-r,j}^r$ is a primary pivot marked in step r of the SSSA, all changes of basis caused by $\Delta_{j-r,j}^r$ will be performed in step $r+1$. This modification in the SSSA is called *Row Cancellation Algorithm* (RCA).

Row Cancellation Algorithm

Initialization step:

$$\left[\begin{array}{l} r = 0 \\ \tilde{\Delta}^r = \Delta \\ \tilde{T}^r = I \text{ (} m \times m \text{ identity matrix)} \end{array} \right.$$

Iterative step: Repeat until all diagonals parallel and to the right of the main diagonal have been swept

$$\left[\begin{array}{l} \text{Matrix } \Delta \text{ update} \\ r \leftarrow r + 1 \\ \tilde{\Delta}^r = (\tilde{T}^{r-1})^{-1} \tilde{\Delta}^{r-1} \tilde{T}^{r-1} \end{array} \right.$$

$$\left[\begin{array}{l} \text{Markup} \\ \text{Sweep entries of } \tilde{\Delta}^r \text{ in the } r\text{-th diagonal:} \\ \text{If } \tilde{\Delta}_{j-r,j}^r \neq 0 \text{ and } \tilde{\Delta}_{\bullet,j}^r \text{ does not contain a primary pivot} \\ \text{Then permanently mark } \tilde{\Delta}_{j-r,j}^r \text{ as a primary pivot} \end{array} \right.$$

Matrix \tilde{T}^r construction
 $\tilde{T}^r \leftarrow I$
For each primary pivot $\tilde{\Delta}_{p-r,p}^r$ such that $j < m$,
change the p -th row of \tilde{T}^r as follows
 $\tilde{T}_{p,\ell}^r \leftarrow -(1/\tilde{\Delta}_{p-r,p}^r)\tilde{\Delta}_{p-r,\ell}^r$, for $\ell = p + 1, \dots, m$

Final step:

Matrix Δ update
 $r \leftarrow r + 1$
 $\tilde{\Delta}^r = (T^{r-1})^{-1}\tilde{\Delta}^{r-1}T^{r-1}$

In [5], it was proved that, in the case of matrices with entries over \mathbb{Z} , the primary pivots on the r -th diagonal of $\tilde{\Delta}^r$ marked in the r -th step of the RCA coincide in position and value with the ones on the r -th diagonal of Δ^r marked in the r -th step of the SSSA. More details on this algorithm are presented in [5]. The analogous result for the case over $\mathbb{Z}((t))$ is stated in the next lemma and its proof is completely analogous to the one obtained for coefficients over \mathbb{Z} .

LEMMA 7.4. *Let $f: M \rightarrow S^1$ be a circle-valued Morse function on a closed n -manifold M and $(\mathcal{N}_*(f), \Delta)$ be the Novikov chain complex associated to f . Let \mathcal{F} be a finest filtration in $(\mathcal{N}_*(f), \Delta)$ and let (E^r, d^r) be the associated spectral sequence for this filtered chain complex. The primary pivots of the matrices $\tilde{\Delta}^r$ of the RCA coincide in position and value with the ones of the matrices Δ^r of the SSSA.*

EXAMPLE 7.5. The following set of matrices (see Figures 34–37) illustrate the depletion of rows and columns in the RCA when applied to Example 6.1.

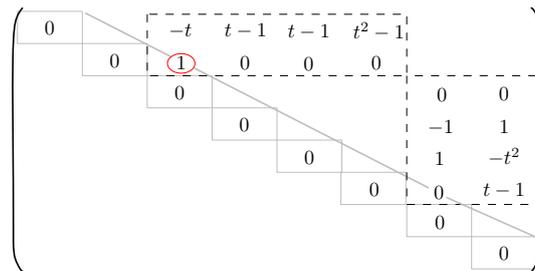


FIGURE 34. $\tilde{\Delta}^1$ in RCA.

PROOF OF THEOREM 7.2. By Theorem 6.4, the non-zero differentials of the spectral sequence are induced by the pivots. Theorem 5.1 states that the primary pivots are always invertible polynomials, hence the differentials $d_p^r: E_p^r \rightarrow E_{p-r}^r$ associated to primary pivots are isomorphisms and the ones associated to change-of-basis pivots always correspond to zero maps. Consequently, the non-zero

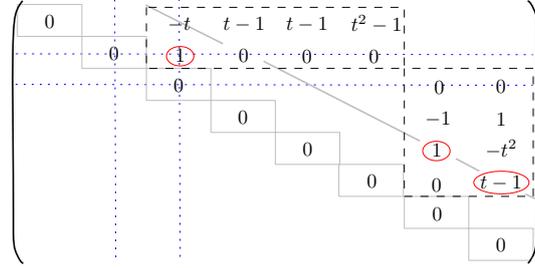


FIGURE 35. $\tilde{\Delta}^2$ in RCA.

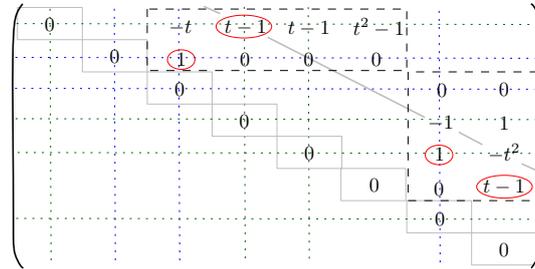


FIGURE 36. $\tilde{\Delta}^3$ in RCA.

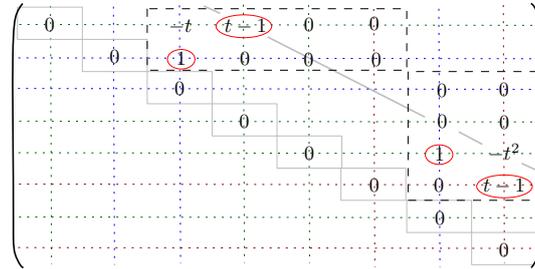


FIGURE 37. $\tilde{\Delta}^4$ in RCA.

differentials are isomorphisms and this implies that at the next stage of the spectral sequence they produce algebraic cancellations, i.e. $E_p^{r+1} = E_{p-r}^{r+1} = 0$.

Note that the algebraic cancellation $E_p^{r+1} = E_{p-r}^{r+1} = 0$ is associated to the primary pivot $\Delta_{p-r+1,p+1}^r$ on the r -th diagonal of Δ^r in the SSSA. The row $p-r+1$ is associated to $h_{k-1}^{p-r+1} \in \mathcal{F}_{p-r}C_{k-1} \setminus \mathcal{F}_{p-r-1}C_{k-1}$ and the column $p+1$ is associated to $h_k^{p+1} \in \mathcal{F}_pC_k \setminus \mathcal{F}_{p-1}C_k$. Moreover, the primary pivot $\Delta_{p-r+1,p+1}^r$ of the SSSA is equal to the primary pivot $\tilde{\Delta}_{p-r+1,p+1}^r$ of the RCA.

We will now prove the theorem by associating dynamical cancellations to the algebraic cancellations of the SSSA via the RCA. Given φ , the negative gradient flow associated to f , we recursively construct a family of negative gradient flows $\{\varphi^r\}$ of circle-valued Morse functions f^r , $r = 0, \dots, \omega$, where $f^0 = f$ and f^{r+1}

is obtained from f^r removing the pairs of critical points corresponding to the cancelled modules of the r -th page of the spectral sequence, i.e. the pairs associated to the primary pivots on the r -th diagonal of Δ^r . In order to do this, we have to prove that whenever a primary pivot $\tilde{\Delta}_{p-r+1,p+1}^r$ on the r -th diagonal of $\tilde{\Delta}^r$ is marked, it is actually a Novikov incidence coefficient $N(h_k^{p+1}, h_{k-1}^{p-r+1}; f^r)$ between two singularities h_k^{p+1} and h_{k-1}^{p-r+1} of consecutive indices of φ^r .

Without loss of generality, the Novikov differential Δ associated to f is considered to be the one where all orientations on the unstable manifolds of critical points of index 2 are the same. In this case, we have the characterization of the Novikov differential Δ given by Corollary 2.2.

The proof follows by induction in r . The base case $r = 1$ is trivial, since the differentials d^0 are null and hence no pivots are marked in this step. Thus there are no algebraic cancellations, $f^0 = f^1$, $\varphi^0 = \varphi$ and $\tilde{\Delta}^1 = \Delta(M, \varphi^1) = \Delta(M, \varphi)$, since no columns and rows are removed. Furthermore, $N(h_k^j, h_{k-1}^i; f^1) = \tilde{\Delta}_{ij}^1$ for all $h_k^j, h_{k-1}^i \in \text{Crit}(f^1)$.

Induction hypothesis. Suppose that the Morse function f^r is well defined and let φ^r be the negative gradient associated to f^r , where the matrix of the Novikov operator $\Delta(M, \varphi^r)$ associated to f^r is the submatrix of $\tilde{\Delta}^r$ obtained when we remove all columns and rows corresponding to all primary pivots marked until diagonal $r - 1$ of $\tilde{\Delta}^r$, i.e. $N(h_k^j, h_{k-1}^i; f^r) = \tilde{\Delta}_{ij}^r$ for all $h_k^j, h_{k-1}^i \in \text{Crit}(f^r)$.

We now prove the case $r + 1$. This proof requires three steps. The first step performs dynamical cancellations of the lift of certain critical points of f^r in the covering space. The second step analyzes the effect that the dynamical cancellations in the covering space has on the relation between the Novikov incidence of f^r and f^{r+1} . The third and final step links the algebra of the RCA with the dynamics of f^{r+1} , i.e. establishes that $N(h_k^j, h_{k-1}^i; f^{r+1}) = \tilde{\Delta}_{ij}^{r+1}$ for all $h_k^j, h_{k-1}^i \in \text{Crit}(f^{r+1})$.

Step 1. Let $\tilde{\Delta}_{p-r+1,p+1}^r$ be a primary pivot on the r -diagonal of $\tilde{\Delta}^r$. By the induction hypothesis, this primary pivot represents the Novikov incidence coefficient between two singularities of the flow φ^r , namely h_k^{p+1} and h_{k-1}^{p-r+1} . By Theorem 5.1, the primary pivot is either $\pm t^\ell$ or $\mp t^{\tilde{\ell}} \pm t^\ell$, where $\ell, \tilde{\ell} \in \mathbb{Z}$, and we assume $\ell < \tilde{\ell}$.

Now, consider the pair of critical points $t^\lambda h_k^{p+1}$ and $t^{\lambda+\tilde{\ell}} h_{k-1}^{p-r+1}$ in the covering space \overline{M} . Denote by F^r the Morse function which makes the following diagram commutative:

$$(7.1) \quad \begin{array}{ccc} \overline{M} & \xrightarrow{F^r} & \mathbb{R} \\ E \downarrow & & \downarrow \text{Exp} \\ M & \xrightarrow{f^r} & \mathbb{S}^1 \end{array}$$

Since F^r is a Morse–Smale function, it can be perturbed if necessary so that there are no critical points in $(F^r)^{-1}[(c_{\lambda+\ell, p-r+1}, c_{\lambda, p+1})]$. For a nice proof of this fact see Lemma 3.2 in [13]. Using the Smale’s Cancellation Theorem, we can equivariantly cancel the pairs of critical points $t^\lambda h_k^{p+1}$ and $t^{\lambda+\ell} h_{k-1}^{p-r+1}$, for all $\lambda \in \mathbb{Z}$, obtaining a new Morse–Smale function $F^{r+1}: \overline{M} \rightarrow \mathbb{R}$ such that

$$\text{Crit}(F^{r+1}) = \text{Crit}(F^r) \setminus \{t^\lambda h_k^{p+1}, t^{\lambda+\ell} h_{k-1}^{p-r+1} \mid \lambda \in \mathbb{Z}\}.$$

Moreover, we have the following correspondence between the intersection numbers with respect to the function F^r and F^{r+1} , denoted by $n(\cdot, \cdot; F^r)$ and $n(\cdot, \cdot; F^{r+1})$, respectively:

1. For $k = 1$:

1.1. If the saddle $t^\lambda h_1^{p+1}$ connects with exactly two sinks, namely $t^{\lambda+\ell} h_0^{p-r+1}$ and $t^{\lambda+\tilde{\ell}} h_0^{i_0}$, then given a saddle $t^\delta h_1^j \in \text{Crit}(F^{r+1})$, we have that

$$(7.2) \quad n(t^\delta h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^{r+1}) = n(t^\delta h_1^j, t^{\lambda+\ell} h_0^{p-r+1}; F^r) + n(t^\delta h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^r),$$

$$(7.3) \quad n(t^\delta h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^{r+1}) = n(t^\delta h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^r), \quad i \neq i_0.$$

In the particular case that $i_0 = p - r + 1$, since the cancellation is done equivariantly, an eventual flow line in φ^r from $t^\delta h_1^j$ to $t^{\lambda+\tilde{\ell}} h_0^{i_0}$ gives place to a flow line starting at $t^\delta h_1^j$ with empty ω -limit set.

1.2. If the saddle $t^\lambda h_1^{p+1}$ connects with exactly one sink, namely $t^{\lambda+\ell} h_0^{p-r+1}$, then the intersection numbers between critical points of F^{r+1} remain the same.

2. For $k = 2$:

2.1. If the saddle $t^{\lambda+\ell} h_1^{p-r+1}$ connects with exactly two sources, namely $t^\lambda h_2^{p+1}$ and $t^{\lambda+\tilde{\ell}} h_2^{j_0}$, then given a saddle $t^\delta h_1^i \in \text{Crit}(F^{r+1})$, we have that

$$(7.4) \quad n(t^{\lambda+\tilde{\ell}} h_2^{j_0}, t^\delta h_1^i; F^{r+1}) = n(t^\lambda h_2^{p+1}, t^\delta h_1^i; F^r) + n(t^{\lambda+\tilde{\ell}} h_2^{j_0}, t^\delta h_1^i; F^r),$$

$$(7.5) \quad n(t^{\lambda+\tilde{\ell}} h_2^{j_0}, t^\delta h_1^i; F^{r+1}) = n(t^{\lambda+\tilde{\ell}} h_2^{j_0}, t^\delta h_1^i; F^r), \quad j \neq j_0.$$

In the particular case that $j_0 = p + 1$, since the cancellation is done equivariantly, an eventual flow line in φ^r from $t^{\lambda+\tilde{\ell}} h_2^{j_0}$ to $t^\delta h_1^i$ gives place to a flow line ending at $t^\delta h_1^i$ with empty α -limit set.

2.2. If the saddle $t^{\lambda+\ell} h_1^{p-r+1}$ connects with exactly one source, namely $t^\lambda h_2^{p+1}$, then the intersection numbers between critical points of F^{r+1} remain the same.

Formulas (7.2)–(7.5) are a consequence of the characterization of the Morse boundary operator, when we consider the orientations on the unstable manifolds of index 2 critical points of F^r being the same. This follows easily since the characteristic signs of the two flow lines of the unstable (stable) manifold of a saddle are opposite as proved in [4].

Step 2. Let $f^{r+1}: M \rightarrow S^1$ be the smooth map such that $f^{r+1}(x) = \text{Exp} \circ F^{r+1}(y)$, where $y \in E^{-1}(x)$ for all $x \in M$. Since the cancellation was done

equivariantly then f^{r+1} is well defined and it is a circle-valued Morse function such that $\text{Crit}(f^{r+1}) = \text{Crit}(f^r) \setminus \{h_k^{p+1}, h_{k-1}^{p-r+1}\}$ which coincides up to isotopy with f^r outside a neighbourhood of the flow line joining these critical points.

As consequence of the formulas in (7.2)–(7.5), we have the following relation between the Novikov incidence coefficients when considering the functions f^r and f^{r+1} :

3. If $k = 1$:

3.1. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell \mp t^{\tilde{\ell}}$ where $\ell < \tilde{\ell}$, then there is a double connection between h_1^{p+1} and h_0^{p-r+1} in the flow φ^r . This corresponds to case 1.1 in the covering space when $i_0 = p-r+1$. After the equivariant cancellation, new connections are not created between consecutive critical points of F^{r+1} , hence

$$N(h_1^j, h_0^i; f^{r+1}) = N(h_1^j, h_0^i; f^r), \quad \text{for all } h_1^j, h_0^i \in \text{Crit}(f^{r+1}).$$

3.2. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell$. Given $h_0^i \in \text{Crit}(f^{r+1})$ with $i < p-r+1$, the entry $\tilde{\Delta}_{i,p+1}^r$ belongs to the submatrix $\Delta(M, \varphi^r)$, consequently we can use Corollary 2.2, which implies that we have two possible cases:

- $\tilde{\Delta}_{i,p+1}^r = 0$ for all $i < p-r+1$ such that $h_0^i \in \text{Crit}(f^{r+1})$. This corresponds to case 1.2 above. After the equivariant cancellation, there are no changes in the connections between consecutive critical points of F^{r+1} , hence $N(h_1^j, h_0^i; f^{r+1}) = N(h_1^j, h_0^i; f^r)$, for $h_1^j \in \text{Crit}(f^{r+1})$.
- There exists $i_0 < p-r+1$ such that $h_0^{i_0} \in \text{Crit}(f^{r+1})$, $\tilde{\Delta}_{i_0,p+1}^r = \mp t^{\tilde{\ell}}$ and $\tilde{\Delta}_{i,p+1}^r = 0$ for all $i < p-r+1$ with $i \neq i_0$ and $h_0^i \in \text{Crit}(f^{r+1})$. In this case, h_1^{p+1} connects with exactly two sinks, namely h_0^{p-r+1} and $h_0^{i_0}$ in the flow φ^r . Then, by case 1.1 above, we have that for all $h_1^j \in \text{Crit}(f^{r+1})$,

$$\begin{aligned} (7.6) \quad N(h_1^j, h_0^{i_0}; f^{r+1}) &= \sum_{\lambda \in \mathbb{Z}} n(h_1^j, t^\lambda h_0^{i_0}; F^{r+1}) t^\lambda \\ &= \sum_{\lambda \in \mathbb{Z}} n(t^{\tilde{\ell}} h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^{r+1}) t^\lambda \\ &= \sum_{\lambda \in \mathbb{Z}} [n(t^{\tilde{\ell}} h_1^j, t^{\lambda+\tilde{\ell}} h_0^{p-r+1}; F^r) + n(t^{\tilde{\ell}} h_1^j, t^{\lambda+\tilde{\ell}} h_0^{i_0}; F^r)] t^\lambda, \\ &= \sum_{\lambda \in \mathbb{Z}} n(h_1^j, t^{\lambda+\tilde{\ell}} h_0^{p-r+1}; F^r) t^\lambda + \sum_{\lambda \in \mathbb{Z}} n(h_1^j, t^\lambda h_0^{i_0}; F^r) t^\lambda \\ &= \sum_{\lambda - (\tilde{\ell} - \tilde{\ell}) \in \mathbb{Z}} n(h_1^j, t^\lambda h_0^{p-r+1}; F^r) t^{\lambda - (\tilde{\ell} - \tilde{\ell})} + N(h_1^j, h_0^{i_0}; f^r) \\ &= N(h_1^j, h_0^{p-r+1}; f^r) t^{\tilde{\ell} - \tilde{\ell}} + N(h_1^j, h_0^{i_0}; f^r), \end{aligned}$$

where the third equality follows from (7.2). Moreover,

$$N(h_1^j, h_0^i; f^{r+1}) = N(h_1^j, h_0^i; f^r) \quad \text{for } i \neq i_0.$$

4. If $k = 2$:

4.1. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell \mp t^{\tilde{\ell}}$ where $\ell < \tilde{\ell}$, then there is a double connection between h_2^{p+1} and h_1^{p-r+1} in the flow φ^r . This corresponds to the case 2.1 in the covering space when $j_0 = p+1$. After the equivariant cancellation new connections are not created between consecutive critical points of F^{r+1} , hence $N(h_2^j, h_1^i; f^{r+1}) = N(h_2^j, h_1^i; f^r)$, for all $h_2^j, h_1^i \in \text{Crit}(f^{r+1})$.

4.2. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell$. Given $h_2^j \in \text{Crit}(f^{r+1})$ with $j > p+1$, the entry $\tilde{\Delta}_{p-r+1,j}^r$ belongs to the submatrix $\Delta(M, \varphi^r)$, consequently we can use Corollary 2.2, which implies that we have two possible cases:

- $\tilde{\Delta}_{p-r+1,j}^r = 0$ for all $j > p+1$ such that $h_2^j \in \text{Crit}(f^{r+1})$. This corresponds to case 2.2 above. After the equivariant cancellation, there are no changes in the connections between consecutive critical points of F^{r+1} , hence $N(h_2^j, h_1^i; f^{r+1}) = N(h_2^j, h_1^i; f^r)$, for $h_1^i \in \text{Crit}(f^{r+1})$.
- There exists $j_0 > p+1$ such that $h_2^{j_0} \in \text{Crit}(f^{r+1})$, $\tilde{\Delta}_{p-r+1,j_0}^r = \mp t^{\tilde{\ell}}$ and $\tilde{\Delta}_{p-r+1,j}^r = 0$ for all $j > p+1$ with $j \neq j_0$ and $h_2^j \in \text{Crit}(f^{r+1})$. In this case, h_1^{p-r+1} connects with exactly two sources, namely h_2^{p+1} and $h_2^{j_0}$ in the flow φ^r . Then, by case 2.1 above, we have that for all $h_1^i \in \text{Crit}(f^{r+1})$,

$$(7.7) \quad N(h_2^{j_0}, h_1^i; f^{r+1}) = N(h_2^{p+1}, h_1^i; f^r)t^{\tilde{\ell}-\ell} + N(h_2^{j_0}, h_1^i; f^r).$$

Moreover, $N(h_2^j, h_1^i; f^{r+1}) = N(h_2^j, h_1^i; f^r)$ for $j \neq j_0$.

Step 3. Now, we link the dynamics with the algebra of the RCA. More specifically, we will show that the matrix of the Novikov operator $\Delta(M, \varphi^{r+1})$ associated to f^{r+1} is the submatrix of $\tilde{\Delta}^{r+1}$ obtained when we remove all columns and rows corresponding to all primary pivots marked up to the r -th diagonal of $\tilde{\Delta}^r$.

Let $\tilde{\Delta}_{p-r+1,p+1}^r$ be a primary pivot and consider an entry $\tilde{\Delta}_{i,j}^{r+1}$ with $i < p-r+1$ and $j > p+1$. If there is a primary pivot in column/row i or column/row j , then the $\tilde{\Delta}_{i,j}^{r+1}$ is not in $\Delta(M, \varphi^{r+1})$. Otherwise, the entry $\tilde{\Delta}_{i,j}^{r+1}$ is given by

$$(7.8) \quad \tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r - \frac{\tilde{\Delta}_{p-r+1,j}^r}{\tilde{\Delta}_{p-r+1,p+1}^r} \tilde{\Delta}_{i,p+1}^r.$$

From now on consider $\tilde{\Delta}_{i,j}^r$ such that $h_k^j, h_{k-1}^i \in \text{Crit}(f^{r+1})$. In this case, there are no primary pivots in column/row i nor in column/row j , since $\tilde{\Delta}_{i,j}^r$ is an entry of the Novikov matrix $\Delta(M, \varphi^r)$ of f^r .

5. If $k = 1$:

5.1. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell \mp t^{\tilde{\ell}}$. By Corollary 2.2, $\tilde{\Delta}_{i,p+1}^r = 0$ for $i < p-r+1$ and $h_0^i \in \text{Crit}(f^{r+1})$. Hence, $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r$. By the induction

hypothesis and by item 3.1 above $\tilde{\Delta}_{i,j}^{r+1} = N(h_1^j, h_0^i; f^{r+1})$, for all j such that $h_1^j \in \text{Crit}(f^{r+1})$.

5.2. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell$. We have two cases to consider.

- $\tilde{\Delta}_{i,p+1}^r = 0$ for all $i < p - r + 1$ such that $h_0^i \in \text{Crit}(f^{r+1})$. Then $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r$. By the induction hypothesis and by case 3.2 above $\tilde{\Delta}_{i,j}^{r+1} = N(h_1^j, h_0^i; f^{r+1})$.
- There exists $i_0 < p - r + 1$ such that $h_0^{i_0} \in \text{Crit}(f^{r+1})$, $\tilde{\Delta}_{i_0,p+1}^r = \mp t^{\tilde{\ell}}$ and $\tilde{\Delta}_{i,p+1}^r = 0$ for all $i < p - r + 1$ with $i \neq i_0$ and $h_0^i \in \text{Crit}(f^{r+1})$. In this case, (7.8) is equivalent to $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r + \tilde{\Delta}_{p-r+1,j}^r t^{\tilde{\ell}-\ell}$. Since by the induction hypothesis $\tilde{\Delta}_{i,j}^r = N(h_1^j, h_0^i; f^r)$ and $\tilde{\Delta}_{p-r+1,j}^r = N(h_1^j, h_0^{i_0}; f^r)$, then by item 3.2 (equation (7.6)) we have $\tilde{\Delta}_{i,j}^{r+1} = N(h_1^j, h_0^i; f^{r+1})$.

6. If $k = 2$:

6.1. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell \mp t^{\tilde{\ell}}$. Given an entry $\tilde{\Delta}_{i,j}^r$ with $j > p + 1$ and $h_2^j, h_1^i \in \text{Crit}(f^{r+1})$, then $\tilde{\Delta}_{p-r+1,j}^r = 0$, since the submatrix $\Delta(M, \varphi^r)$ of $\tilde{\Delta}^r$ satisfies Corollary 2.2. Hence, $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r$ and, by the induction hypothesis, $\tilde{\Delta}_{i,j}^{r+1} = N(h_2^j, h_1^i; f^{r+1})$.

6.2. Suppose that $\tilde{\Delta}_{p-r+1,p+1}^r = \pm t^\ell$. Given an entry $\tilde{\Delta}_{i,j}^r$ with $j > p + 1$ and $h_2^j, h_1^i \in \text{Crit}(f^{r+1})$, then $\tilde{\Delta}_{p-r+1,j}^r = 0$ or $\tilde{\Delta}_{p-r+1,j}^r = \mp t^{\tilde{\ell}}$, since the submatrix $\Delta(M, \varphi^r)$ of $\tilde{\Delta}^r$ satisfies Corollary 2.2. In the case that $\tilde{\Delta}_{p-r+1,j}^r = 0$, then $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r$ and $\tilde{\Delta}_{i,j}^{r+1} = N(h_2^j, h_1^i; f^{r+1})$. On the other hand, if $\tilde{\Delta}_{p-r+1,j}^r = \mp t^{\tilde{\ell}}$, (7.8) is equivalent to $\tilde{\Delta}_{i,j}^{r+1} = \tilde{\Delta}_{i,j}^r + \tilde{\Delta}_{i,p+1}^r t^{\tilde{\ell}-\ell}$. By the induction hypothesis $\tilde{\Delta}_{i,j}^r = N(h_2^j, h_1^i; f^r)$ and $\tilde{\Delta}_{i,p+1}^r = N(h_2^{p+1}, h_1^i; f^r)$. Then by item 4.2 above (equation (7.7)) we have $\tilde{\Delta}_{i,j}^{r+1} = N(h_2^j, h_1^i; f^{r+1})$.

If there are more primary pivots $\{\tilde{\Delta}_{p_\ell-r+1,p_\ell+1}^r\}$ on the r -th diagonal of $\tilde{\Delta}^r$, we repeat the above construction successively by determining neighbourhoods U_ℓ of the closures of the flows lines joining $h_k^{p_\ell+1}$ and $h_{k-1}^{p_\ell-r+1}$. Cancelling the pairs of critical points $\{(h_k^{p_\ell+1}, h_{k-1}^{p_\ell-r+1})\}$ in U_ℓ , we obtain a family of flows $\{\varphi_\ell^r\}$ such that φ_ℓ^r has exactly two less singularities than $\varphi_{\ell-1}^r$. Each one of the cancellations is done as above with the dynamics corresponding to the algebra of the RCA. In this case, the flow φ^{r+1} is obtained from φ^r after cancelling all pairs of critical points corresponding to primary pivots on the r -th diagonal. \square

As a consequence of the proof of Theorem 7.2, the appearance of periodic orbits in the family f^r due cancellations of critical points can be detected by the spectral sequence.

THEOREM 7.6 (Detecting periodic orbits). *There exists a family of circle-valued Morse functions satisfying Theorem 7.2 such that, for each cancelled pair of modules E_p^r and E_{p-r}^r corresponding to a binomial primary pivot $\Delta_{p-r+1,p+1}^r =$*

$\mp t^\ell \pm t^{\tilde{\ell}}$, where $\ell < \tilde{\ell}$, the associated dynamical cancellation of critical points h_k^{p+1} and h_{k-1}^{p-r+1} gives rise to a periodic orbit that crosses the regular value $f^{-1}(a)$ $\tilde{\ell} - \ell$ times. This periodic orbit is either an attractor if $k = 0$ or a repeller if $k = 1$.

PROOF. In order to perform the cancellation of h_k^{p+1} and h_{k-1}^{p-r+1} , we consider the critical points $t^\lambda h_k^{p+1}$ and $t^{\lambda+\ell} h_{k-1}^{p-r+1}$ in the covering space \overline{M} and cancel them equivariantly for all $\lambda \in \mathbb{Z}$ using the First Cancellation Theorem in [11]. Note that, for a fixed λ , when the pair $t^\lambda h_k^{p+1}$ and $t^{\lambda+\ell} h_{k-1}^{p-r+1}$ of critical points of F^r is cancelled, the function F^{r+1} coincides with F^r (up to isotopy) outside a neighbourhood U of the connecting orbit between them and a flow line γ from $t^{\lambda-\ell} h_k^{p+1}$ to $t^{\lambda+\tilde{\ell}} h_{k-1}^{p-r+1}$ arises in the negative gradient flow of F^{r+1} . One can choose the diffeomorphism given by the Assertion 4 in Chapter 5 of [11] such that γ coincides outside of U with the flow lines in the negative gradient flow of F^r going from $t^{\lambda-\ell} h_k^{p+1}$ to $t^{\lambda+\ell} h_{k-1}^{p-r+1}$ and from $t^\lambda h_k^{p+1}$ to $t^{\lambda+\tilde{\ell}} h_{k-1}^{p-r+1}$. Cancelling equivariantly in \overline{M} , one obtains a periodic orbit of the negative gradient flow of f^{r+1} on M . \square

EXAMPLE 7.7. We could have considered an alternative finest filtration in Example 7.3, such as $\mathcal{G} = \{G_p \mathcal{N}\}$, where

$$\begin{aligned} G_0 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1], & G_4 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3, h_1^4, h_1^6], \\ G_1 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2], & G_5 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3, h_1^4, h_1^6, h_1^5], \\ G_2 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3], & G_6 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3, h_1^4, h_1^6, h_1^5, h_2^8], \\ G_3 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3, h_1^4], & G_7 \mathcal{N} &= \mathbb{Z}((t)) [h_0^1, h_0^2, h_1^3, h_1^4, h_1^6, h_1^5, h_2^8, h_2^7]. \end{aligned}$$

In this case, the dynamical cancellations of pairs of critical points associated to algebraic cancellations of the modules of the spectral sequence through the SSSA will be the following pairs of critical points: (h_0^2, h_1^3) , (h_1^5, h_2^8) , (h_0^1, h_1^4) , (h_1^6, h_2^7) . For instance, in the first matrix Δ^1 produced by the SSSA, see Figure 38, the entry $\Delta_{6,7}^1 = -t^2$ is a primary pivot which induces the differential $d_6^1: E_6^1 \rightarrow E_5^1$ of the spectral sequence. Note that by the choice of the finest filtration \mathcal{G} , E_6^1 is generated by the critical point h_2^8 and E_5^1 is generated by the critical point h_1^5 . Therefore, the algebraic cancellation of the modules E_6^1 and E_5^1 is associated to the dynamical cancellation of the pair (h_1^5, h_2^8) . Also note that, despite the fact that h_2^8 and h_1^5 are close with respect to the filtration \mathcal{G} , i.e. the gap between them is 1, the flow line from h_2^8 to h_1^5 is a “long” orbit both in M and \overline{M} represented by $-t^2$ on the Novikov differential. This orbit in the torus M wraps twice around it. Hence, the corresponding cancelled pairs $(t^{\lambda+2} h_1^5, t^\lambda h_2^8)$ in the covering space are also not close compared with the gap in the filtration $\mathcal{G}_{\lambda,p}$, since the gap between them is 19. The algebraic cancellation corresponding to $-t^2$ also has its dynamical counterpart.

$$\begin{array}{c}
 h_0^1 \\
 h_0^2 \\
 h_1^3 \\
 h_1^4 \\
 h_1^5 \\
 h_1^6 \\
 h_2^7 \\
 h_2^8
 \end{array}
 \left(
 \begin{array}{cccccccc}
 h_0^1 & h_0^2 & h_1^3 & h_1^4 & h_1^5 & h_1^6 & h_2^7 & h_2^8 \\
 0 & & & & & & & \\
 & 0 & & & & & & \\
 & & 1 & & & & & \\
 & & 0 & & & & & \\
 & & & 0 & & & & \\
 & & & & 0 & & & \\
 & & & & & 0 & & \\
 & & & & & & 1 & \\
 & & & & & & & 0
 \end{array}
 \right)$$

FIGURE 38. Δ^1 for an alternative filtration \mathcal{G} .

The choice of a different filtration for the Novikov chain complex from the one in Example 6.2, implies in different cancellations of pairs of critical points. The family of negative gradient flows φ^r determined by Theorem 6.1 for this example exhibit new orbits that wrap around the torus many times possibly with different orientations in some of the turns. This dynamical behavior is determined by the primary pivots in the Novikov matrices Δ^r . Once all of the cancellations are performed, the flow associated to the last Novikov matrix is minimal and in this case we are left with no critical points and one attracting and one repelling periodic orbit.

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DAHISY V. DE S. LIMA AND KETTY A. DE REZENDE
 IMECC
 Universidade Estadual de Campinas
 Campinas, SP, BRAZIL

E-mail address: dahisylima@ime.unicamp.br, ketty@ime.unicamp.br

OZIRIDE MANZOLI NETO
 ICMC
 Universidade de São Paulo
 São Carlos, SP, BRAZIL

E-mail address: ozimneto@icmc.usp.br

MARIANA R. DA SILVEIRA
 CMCC
 Universidade Federal do ABC
 Santo André, SP, BRAZIL

E-mail address: mariana.silveira@ufabc.edu.br