Topological **M**ethods in **N**onlinear **A**nalysis
Volume 49, No. 1, 2017, 377–380
DOI: 10.12775/TMNA.2016.092

MICHAEL'S SELECTION THEOREM FOR A MAPPING DEFINABLE IN AN O-MINIMAL STRUCTURE DEFINED ON A SET OF DIMENSION 1

Małgorzata Czapla — Wiesław Pawłucki

Abstract. Let R be a real closed field and let some o-minimal structure extending R be given. Let $F\colon X \rightrightarrows R^m$ be a definable multivalued lower semicontinuous mapping with nonempty definably connected values defined on a definable subset X of R^n of dimension 1 (X can be identified with a finite graph immersed in R^n). Then F admits a definable continuous selection.

1. Introduction

Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [3] or [1].)

Let $F\colon X \rightrightarrows R^m$ be a multivalued mapping defined on a subset X of R^n; i.e. a mapping which assigns to each point $x \in X$ a nonempty subset $F(x)$ of R^m. F can be identified with its graph; i.e. a subset of $R^n \times R^m$. If this subset is definable we will call F *definable*. F is called *lower semicontinuous* if for each $x \in X$ and each $u \in F(x)$ and any neighbourhood U of u, there exists

2010 *Mathematics Subject Classification.* Primary: 14P10; Secondary: 54C60, 54C65, 32B20, 49J53.

Key words and phrases. Michael's selection theorem; o-minimal structure; finite graph.

a neighbourhood V of x such that $U \cap F(y) \neq \emptyset$, for each $y \in V$. A mapping $\varphi\colon A \to R^m$, where $A \subset X$, is called a *selection* of F on A if $\varphi(x) \in F(x)$, for each $x \in A$.

The aim of the present article is the following version of Michael's Selection Theorem.

Theorem 1.1. (Main Theorem) *Let $F\colon X \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping with nonempty definably connected* (1) *values defined on a definable subset X of R^n of dimension* 1 *(X can be identified with a finite graph in R^n). Let $\varphi\colon A \to R^m$ be any continuous definable selection of F on a definable closed subset A of X. Then there exists a continuous definable selection $f\colon X \to R^m$ of F on X such that $f|A = \varphi$.*

Let us notice that our Main Theorem is independent of classical Michael's Selection Theorem (cf. [4, Theorem 1.2]). To see this, consider as an example the following semialgebraic multivalued mapping $F\colon R \rightrightarrows R^2$ defined by the formula

$$F(x) := \begin{cases} \{(y,z) \in R^2 : y^2 - zx^2 = 0\}, & \text{when } x \neq 0, \\ \{(y,z) \in R^2 : y = 0,\ z \geq 0\}, & \text{when } x = 0. \end{cases}$$

(The graph of F is the famous *Whitney umbrella*.) By our theorem, for any semialgebraic closed subset $A \subset R$ and any semialgebraic continuous selection $\varphi\colon A \to R^2$ of F on A there exists a semialgebraic continuous selection of F on R extending φ. However, the family $\{F(x) : x \in R\}$ is obviously not equi-LC^0 in the sense of Michael [4] and if we consider the following (non-semialgebraic) continuous selection $\varphi\colon A \to R^2$ on $A = \{1/n : n = 1, 2, \dots\} \cup \{0\}$ defined by:

$$\varphi(x) := \begin{cases} \left(\dfrac{1}{n}, 1\right) & \text{when } x = \dfrac{1}{n},\ n \text{ is even}, \\ \left(-\dfrac{1}{n}, 1\right) & \text{when } x = \dfrac{1}{n},\ n \text{ is odd}, \\ (0,1) & \text{when } x = 0, \end{cases}$$

then it is easy to see that there is no extension of φ to a continuous selection of F on a neighbourhood of 0.

As an application of Main Theorem we can see that in the counterexample from [2] the dimension 2 of the domain is the smallest possible.

2. Proof of Main Theorem

The proof is based on the following three fundamental tools of the o-minimal geometry: Curve Selection Lemma (see [3, Chapter 6, (1.5)] or [1, Theorem 3.2]),

---

(1) In fact any definably connected subset is definably arcwise connected; i.e. arcwise connected by definable arcs. Besides, if R is the field of real numbers $\mathbb{R}$, then definable connectedness coincides with usual connectedness.

Trivialization Theorem (see [4, Chapter 9, (1.2)] or [1; Theorem 5.22]) and Triangulation Theorem (see [3, Chapter 8, (2.9)] or [1, Theorem 4.4]). Replacing F by the mapping G defined by the formula

$$G(x) := \begin{cases} F(x) & \text{when } x \in X \setminus A, \\ \{\varphi(x)\} & \text{when } x \in A, \end{cases}$$

we reduce the general case to that with $A = \emptyset$, so in what follows we assume that $A = \emptyset$.

Using the semialgebraic homeomorphism

$$R^n \ni (x_1, \dots, x_n) \mapsto \left(\frac{x_1}{1+|x_1|}, \dots, \frac{x_n}{1+|x_n|} \right) \in (-1,1)^n$$

we can assume without any loss of generality that X is bounded. By the Triangulation Theorem, we can assume that there is a finite subset $\{x_0, \dots, x_r\} \subset R^n$ (with $x_{i_1} \neq x_{i_2}$, when $i_1 \neq i_2$) such that

$$X \setminus \{x_0, \dots, x_r\} = \bigcup_{j=1}^{s} (y_j, z_j),$$

where $s \in \mathbb{Z}$, $s > 0$, $(y_j, z_j) = \{ty_j + (1-t)z_j : t \in (0,1)\}$, $y_j, z_j \in \{x_0, \dots, x_r\}$, $y_j \neq z_j$, for each j, and $\{y_{j_1}, z_{j_1}\} \neq \{y_{j_2}, z_{j_2}\}$, when $j_1 \neq j_2$. Moreover, by the Trivialization Theorem, applied to the natural projection $\pi\colon F \to X$ of the graph of F onto its domain ([2]), we can assume that π is definably trivial over every (y_j, z_j); i.e. there exists a definable subset L_j of R^m and a definable homeomorphism $h_j\colon F|(y_j, z_j) \to (y_j, z_j) \times L_j$ such that the following diagram is commutative:

$$\begin{tikzcd} F|(y_j, z_j) \arrow[r, "h_j"] \arrow[d, "\pi"'] & (y_j, z_j) \times L_j \arrow[d, "p_j"] \\ (y_j, z_j) \arrow[r, equal] & (y_j, z_j), \end{tikzcd}$$

where $p_j\colon (y_j, z_j) \times L_j \to (y_j, z_j)$ denotes the natural projection. Let $\omega_j(t) := ty_j + (1-t)z_j$, for each $t \in [0,1]$.

For each x_i select arbitrarily a point $u_i \in F(x_i)$. Now we will extend this selection to a selection f to every (y_j, z_j). There are four possibilities:

(I) $y_j \notin X$ and $z_j \notin X$. Then fix any $w_j \in L_j$ and put $f(\xi) := h_j^{-1}(\xi, w_j)$, for each $\xi \in (y_j, z_j)$.

(II) $y_j \in X$ and $z_j \notin X$. Then $y_j = x_i$, for some i. By the assumption of lower semicontinuity and by the Curve Selection Lemma there is a continuous map $f\colon \omega_j([0,\varepsilon]) \to R^m$ such that $\varepsilon \in (0,1)$, $f(\omega_j(t)) \in F(\omega_j(t))$, for each $t \in [0,\varepsilon]$, and $\varphi(\omega_j(0)) = f(y_j) = f(x_i) = u_i$. Now we extend φ to the whole

---

([2]) We identify a mapping with its graph and denote them by the same letter.

(y_j, z_j) by putting $f(\omega_j(t)) := h_j^{-1}(\omega_j(t), q_j(f(\varepsilon)))$, for each $t \in [\varepsilon, 1)$, where $q_j\colon (y_j, z_j) \times L_j \to L_j$ is the natural projection.

(III) $y_j \notin X$ and $z_j \in X$. The definition of f is symmetrical to case (II).

(IV) $y_j, z_j \in X$. Then $y_j = x_{i_1}$ and $z_j = x_{i_2}$, for some i_1, i_2. By the argument from case (II), there exists $\varepsilon \in (0, 1/2)$ and a continuous selection $f\colon \omega_j([0,\varepsilon] \cup [1-\varepsilon, 1]) \to R^m$ of F on $\omega_j([0,\varepsilon] \cup [1-\varepsilon, 1])$ such that $f(\omega_j(0)) = u_{i_1}$ and $f(\omega_j(1)) = u_{i_2}$. Since L_j is definably arcwise connected (cf. [1, Corollary 3.10]) there is a definable continuous arc $\lambda\colon [\varepsilon, 1-\varepsilon] \to L_j$ such that $\lambda(\varepsilon) = q_j(f(\omega_j(\varepsilon))$ and $\lambda(1-\varepsilon) = q_j(f(\omega_j(1-\varepsilon))$. Put now $f(\omega(t)) := h_j^{-1}(\omega_j(t), \lambda(t))$, for each $t \in [\varepsilon, 1-\varepsilon]$, in order to get a continuous selection on the whole (y_j, z_j).

Since $f\colon X \to R^m$ is continuous on the closure of every (y_j, z_j) in X it is continuous.

Acknowledgements. The authors thank the referees for valuable comments.

References

[1] M. Coste, *An Introduction to O-minimal Geometry*, Dottorato di Ricerca in Matematica, Dipartimento di Matematica, Università di Pisa, Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[2] M. Czapla and W. Pawłucki, *Michael's theorem for Lipschitz cells in o-minimal structures*, Ann. Polon. Math. **117** (2016), no. 2, 101–107.

[3] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge University Press, 1998.

[4] E. Michael, *Continuous selections* II, Ann. of Math. (2) **64** (1956), 562–580.

Manuscript received March 14, 2016

accepted June 14, 2016

Małgorzata Czapla and Wiesław Pawłucki
Instytut Matematyki
Uniwersytet Jagielloński
ul. Prof. S. Łojasiewicza 6
30-348 Kraków, POLAND

E-mail address: Malgorzata.Czapla@im.uj.edu.pl, Wieslaw.Pawlucki@im.uj.edu.pl