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EXISTENCE OF SOLUTION FOR A KIRCHHOFF TYPE SYSTEM WITH WEIGHT AND NONLINEARITY INVOLVING A (p,q)-SUPERLINEAR TERM AND CRITICAL CAFFARELLI–KOHN–NIRENBERG GROWTH

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Abstract. We study a (p,q)-Laplacian system of Kirchhoff type equations with weight and nonlinearity involving a (p,q)-superlinear term, in which p may be different from q, and with critical Caffarelli–Kohn–Nirenberg exponent. Using the Mountain Pass Theorem, we obtain a nontrivial solution to the problem.

1. Introduction

This paper deals with existence of a nontrivial weak solution to the (p,q)-Laplacian system of Kirchhoff type equations

$$\begin{cases} L_p(u) = \lambda|x|^{-c}F_u(x,u,v) + \alpha|x|^{-\beta}|u|^{\alpha-2}u|v|^{\gamma} & \text{in } \Omega, \\ L_q(v) = \lambda|x|^{-c}F_v(x,u,v) + \gamma|x|^{-\beta}|u|^{\alpha}|v|^{\gamma-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where

$$L_p(u) = -\bigg[M_1\bigg(\int_\Omega |x|^{-a_1p}|\nabla u|^p\,dx\bigg)\bigg]\operatorname{div}(|x|^{-a_1p}|\nabla u|^{p-2}\nabla u),$$
$$L_q(v) = -\bigg[M_2\bigg(\int_\Omega |x|^{-a_2q}|\nabla v|^q\,dx\bigg)\bigg]\operatorname{div}(|x|^{-a_2q}|\nabla v|^{q-2}\nabla v);$$

$\Omega\subset\mathbb{R}^N$ is a bounded smooth domain with $N\geq 3$, $1<p<N$, $1<q<N$, $a_1<(N-p)/p$, $a_2<(N-q)/q$, $c\in\mathbb{R}$, $\alpha/p^*+\gamma/q^*=1$, where $p^*=Np/(N-d_1p)$ and $q^*=Nq/(N-d_2q)$ are the critical Caffarelli–Kohn–Nirenberg exponents with $d_i=1+a_i-b_i$, $a_i\leq b_i<a_i+1$, $i=1,2$, and $\beta=b_1p^*=b_2q^*$. Let $F\colon\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a measurable function in Ω, continuously differentiable in $\mathbb{R}\times\mathbb{R}$, where F_w is its partial derivative with respect to w, and $M_i\colon\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+$ be a continuous function, $i=1,2$.

Problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\begin{cases} u_{tt}-M\bigg(\displaystyle\int_\Omega|\nabla u|^2\,dx\bigg)\Delta u=g(x,u) & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), \end{cases}\tag{1.2}$$

where $M(s)=a+bs$, $a,b>0$. It was proposed by Kirchhoff [14] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations.

Due to the presence of terms $M_i(\int_\Omega|x|^{-a_i}|\nabla w|^r\,dx)$, $i=1,2$, the equations in (1.1) are no longer a pointwise identity, therefore it is often called a nonlocal problem. This phenomenon causes some mathematical difficulties, what makes the study of such class of problems particularly interesting.

In the last years many authors have studied the following nonlocal problem:

$$-M\bigg(\int_\Omega|\nabla u|^2\,dx\bigg)\Delta u=f(x,u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega.\tag{1.3}$$

Problems of type (1.3) may be used to model several physical and biological problems, see [1] for more references. Many interesting results for problems of the Kirchhoff type have already been obtained, see for example [1], [5], [11], and the references therein. The study of Kirchhoff type equations has been extended to the case involving the p-Laplacian operator, see [7], [9], and [12]. Systems of Kirchhoff type equations were considered for example in [6] and [8].

To enunciate the main results, we shall pose some hypotheses on the functions M_1, M_2, and F. Hypotheses on the continuous functions $M_i\colon\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+$, $i=1,2$, are the following:

(M1) There exist $m_1>0$ and $m_2>0$ such that $M_i(t)\geq m_i$, for all $t\geq 0$, $i=1,2$.

(M2) The functions M_i, $i = 1, 2$, are increasing.

Let the function $F\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function in Ω, continuously differentiable in $\mathbb{R} \times \mathbb{R}$, and satisfy the following hypotheses:

(F1) $F_u(x, s, t) = -F_u(x, -s, t)$ and $F_v(x, s, t) = -F_v(x, s, -t)$, for all (x, s, t) in $\Omega \times \mathbb{R} \times \mathbb{R}$.

(F2) There exist positive constants C_1, C_2 with $C_1 < C_2$ and $\theta, \delta > 1$ with $\theta/p + \delta/q > 1$ and $\theta/p^* + \delta/q^* < 1$ such that

$$\begin{aligned} C_1\theta|s|^{\theta-1}|t|^{\delta} &\le F_u(x, s, t) \le C_2\theta|s|^{\theta-1}|t|^{\delta}, \\ C_1\delta|s|^{\theta}|t|^{\delta-1} &\le F_v(x, s, t) \le C_2\delta|s|^{\theta}|t|^{\delta-1}, \end{aligned}$$

for all $(x, s, t) \in \Omega \times (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})$.

(F3) There exist $\xi_1 \in (p, p^*)$ and $\xi_2 \in (q, q^*)$ such that

$$F(x, u, v) \le \frac{1}{\xi_1} F_u(x, u, v) \cdot u + \frac{1}{\xi_2} F_v(x, u, v) \cdot v,$$

for all $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

We observe that from (F1) we have

(F1') $F(x, s, t) = F(x, -s, t) = F(x, s, -t) = F(x, -s, -t)$, for all (x, s, t) in $\Omega \times \mathbb{R} \times \mathbb{R}$.

Moreover, from (F1') and (F2) we also have

(F2') $C_1|s|^{\theta}|t|^{\delta} \le F(x, s, t) \le C_2|s|^{\theta}|t|^{\delta}$, for all $(x, s, t) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

In this paper we study a (p, q)-Laplacian system of Kirchhoff type equations with weight and nonlinearity involving a (p, q)-superlinear term, in which p may be different from q, and with critical Caffarelli–Kohn–Nirenberg exponent. Due to the presence of nonlocal terms in system (1.1), it is necessary to make a truncation on the Kirchhoff type functions that appear in the operator, creating an auxiliary problem. Finding solutions of this auxiliary problem, we can find solutions for problem (1.1). The presence of the term with critical growth in the system also causes a difficulty in solving the problem due to the lack of compactness.

We establish two results for problem (1.1). In both results we make use of the Mountain Pass Theorem to find solutions. The first one covers the case when p may be different from q. In this case we find a nontrivial solution for problem (1.1) with $\lambda > \lambda^*$. The "$p \neq q$-problem" is bypassed using a version of the concentration-compactness principle due to Lions (cf. [15, Lemma 2.1]) and by controlling the level of the Palais–Smale sequence obtained with the Mountain Pass Theorem. In second case, working with extremal functions, it is possible to find a nontrivial solution for all $\lambda > 0$, but under the condition $p = q$. To the best of our knowledge, our work is the first in the literature to deal with the (p,q)-superlinear system of Kirchhoff type equations with critical growth, in which p is different from q.

The main results of our paper are the following:

THEOREM 1.1. *Assume* (M1), (M2), (F1)–(F3) *hold, and* $\alpha/p^* + \gamma/q^* = 1$. *Then, there exists* $\lambda^* > 0$ *such that problem* (1.1) *has a nontrivial solution for each* $\lambda \in (\lambda^*, +\infty)$.

THEOREM 1.2. *Suppose* $p = q$. *Assume* (M1), (M2), (F1)–(F3) *hold,* $a_1 = a_2$, *and* $\alpha + \gamma = p^*$. *Then, for all* $\lambda > 0$, *problem* (1.1) *has a nontrivial solution.*

This paper is organized as follows. In Section 2, we provide some preliminary results, the variational framework and a version of the concentration-compactness principle. In Section 3, we construct an auxiliary problem. Section 4 is devoted to the Palais–Smale condition for the Euler–Lagrange functional associated to problem (1.1). In Sections 5 and 6, we prove Theorems 1.1 and 1.2, respectively.

2. Preliminary results and variational framework

Consider $\Omega \subset \mathbb{R}^N$ a bounded smooth domain with $0 \in \Omega$, $N \geq 3$, $1 < l < N$, $a < (N-l)/l$, $a \leq b < a+1$, and $l^* = Nl/(N-dl)$, where $d = 1 + a - b$. From [4], [17] we have

$$\left(\int_\Omega |x|^{-\eta}|w|^r\,dx\right)^{l/r} \leq C\int_\Omega |x|^{-al}|\nabla w|^l\,dx, \quad \text{for all } w \in \mathcal{D}_a^{1,l}, \tag{2.1}$$

where $1 \leq r \leq Nl/(N-l)$, $\eta \leq (a+1)r + N(1-r/l)$, and $\mathcal{D}_a^{1,l}$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|w\| = \left(\int_\Omega |x|^{-al}|\nabla w|^l\,dx\right)^{1/l};$$

i.e. we have the continuous embedding of $\mathcal{D}_a^{1,l}$ in $L^r(\Omega, |x|^{-\eta})$, where $L^r(\Omega, |x|^{-\eta})$ is the weighted $L^r(\Omega)$ space with the norm

$$\|w\|_{r,\eta} = \left(\int_\Omega |x|^{-\eta}|w|^r\,dx\right)^{1/r}.$$

Moreover, this embedding is compact if $1 \leq r < Nl/(N-l)$ and $\eta < (a+1)r + N(1-r/l)$. The best constant of the weighted Caffarelli–Kohn–Nirenberg type (see [4]) inequality will be denoted by $C_{a,l}^*$, which is characterized by

$$C_{a,l}^* = \inf_{w \in \mathcal{D}_a^{1,l}\setminus\{0\}} \left\{ \frac{\displaystyle\int_\Omega |x|^{-al}|\nabla w|^l\,dx}{\left(\displaystyle\int_\Omega |x|^{-bl^*}|w|^{l^*}\,dx\right)^{l/l^*}} \right\}.$$

We will denote the Sobolev space by $E = A \times B$, where $A = \mathcal{D}^{1,p}_{a_1}$ and $B = \mathcal{D}^{1,q}_{a_2}$, and endow it with the norm

$$\|(u,v)\| = \|u\|_A + \|v\|_B = \left(\int_\Omega |x|^{-a_1 p}|\nabla u|^p\,dx\right)^{1/p} + \left(\int_\Omega |x|^{-a_2 q}|\nabla v|^q\,dx\right)^{1/q}.$$

We will look for solutions of problem (1.1) by finding critical points of the Euler–Lagrange functional $I\colon E \to \mathbb{R}$, given by

$$I(u,v) = \frac{1}{p}\widehat{M_1}(\|u\|_A^p) + \frac{1}{q}\widehat{M_2}(\|v\|_B^q) - \lambda\int_\Omega |x|^{-c}F(x,u,v)\,dx - \int_\Omega |x|^{-\beta}|u|^\alpha|v|^\gamma dx,$$

for all $(u,v) \in E$, where $\widehat{M_i}(t) := \int_0^t M_i(s)\,ds$, $i = 1,2$. Note that $I \in C^1(E,\mathbb{R})$ and, for all $(\varphi,\psi) \in E$,

$$\begin{aligned} I'(u,v)(\varphi,\psi) = {}& M_1(\|u\|_A^p)\int_\Omega |x|^{-a_1 p}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx \\ &+ M_2(\|v\|_B^q)\int_\Omega |x|^{-a_2 q}|\nabla v|^{q-2}\nabla v\nabla\psi\,dx \\ &- \lambda\int_\Omega |x|^{-c}F_u(x,u,v)\varphi\,dx - \lambda\int_\Omega |x|^{-c}F_v(x,u,v)\psi\,dx \\ &- \alpha\int_\Omega |x|^{-\beta}|u|^{\alpha-2}u|v|^\gamma\varphi\,dx - \gamma\int_\Omega |x|^{-\beta}|u|^\alpha|v|^{\gamma-2}v\psi\,dx. \end{aligned}$$

The next proposition is a version of the concentration-compactness principle due to Lions (cf. [15, Lemma 2.1]), it will be useful in showing that the functional I satisfies a local Palais–Smale condition. This version is a more general version of the theorem given by Silva and Xavier [16], adapted to our problem.

Let $Q \in C^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ be a nonnegative function satisfying $Q(x,0,0) = 0$, for every $x \in \Omega$ and

(Q_0) there is $C > 0$ such that, for every $(x,u,v) \in \Omega\times\mathbb{R}\times\mathbb{R}$,

$$\begin{aligned} |Q_u(x,u,v)| &\le C(|u|^{p^*-1} + |v|^{q^*(p^*-1)/p^*} + 1), \\ |Q_v(x,u,v)| &\le C(|u|^{p^*(q^*-1)/q^*} + |v|^{q^*-1} + 1). \end{aligned}$$

PROPOSITION 2.1. *Let $1 \le p < N$ and $1 \le q < N$. Let $Q \in C^1(\overline{\Omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ be a nonnegative function satisfying (Q_0) and $Q(x,0,0) = 0$, for every $x \in \Omega$. Let $\{(u_n,v_n)\} \subset E$ be such that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in E. Suppose that*

$$|x|^{-a_1 p}|\nabla u_n|^p\,dx \rightharpoonup \mu, \quad |x|^{-a_2 q}|\nabla v_n|^q\,dx \rightharpoonup \sigma, \quad |x|^{-\beta}Q(x,u_n,v_n)dx \rightharpoonup \nu$$

weakly in the sense of measures, where μ,σ, and ν are nonnegative and bounded measures on $\overline{\Omega}$. Then there are an at most countable index set Λ, families $(\mu_j)_{j\in\Lambda}, (\sigma_j)_{j\in\Lambda}$, and $(\nu_j)_{j\in\Lambda}$ of positive numbers, and a family $(x_j)_{j\in\Lambda}$ of points on $\overline{\Omega}$ such that

$$\nu = |x|^{-\beta}Q(x,u,v)\,dx + \sum_{j\in\Lambda}\nu_j\delta_{x_j}, \quad \mu \ge |x|^{-a_1 p}|\nabla u|^p\,dx + \sum_{j\in\Lambda}\mu_j\delta_{x_j},$$

and

$$\sigma \geq |x|^{-a_2 q}|\nabla v|^q dx + \sum_{j\in\Lambda} \sigma_j \delta_{x_j}.$$

Moreover, there exists a constant $C > 0$ such that $\mu_j^{p^*/p} + \sigma_j^{q^*/q} \geq C\nu_j$.

The proof of Proposition 2.1 is an adaptation of [16, Proposition 2.1].

The next lemma will be also useful for us, it was proved by Ghoussoub and Yuan in [10, Lemma 4.1].

Lemma 2.2 (S_+ condition). *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$, and $u_n \in \mathcal{D}_a^{1,p}$ is such that*

$$\begin{cases} u_n \rightharpoonup u \quad \text{as } n \to +\infty, \\ \limsup\limits_{n\to\infty} \displaystyle\int_\Omega |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\,dx \leq 0, \end{cases}$$

then there exists a subsequence strongly convergent in $\mathcal{D}_a^{1,p}$.

3. Auxiliary problem

In order to prove Theorems 1.1 and 1.2 , we will make use of a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2], but since we are working with critical growth and a nonlocal operator without information about the behavior of functions M_1 and M_2 at infinity, we need to make a truncation on these functions. So we will prove that the Euler–Lagrange functional associated to problem (1.1) has the Mountain Pass geometry.

Define $m_0 = \min\{m_1, m_2\}$. It follows from (M2) that there exist $t_1, t_2 > 0$ such that $m_0 \leq M_1(0) < M_1(t_1) < \xi_1 m_0/p$ and $m_0 \leq M_2(0) < M_2(t_2) < \xi_2 m_0/q$. We set

$$M_{t_1}(t) := \begin{cases} M_1(t) & \text{if } 0 \leq t \leq t_1, \\ M_1(t_1) & \text{if } t \geq t_1, \end{cases}$$

and

$$M_{t_2}(t) := \begin{cases} M_2(t) & \text{if } 0 \leq t \leq t_2, \\ M_2(t_2) & \text{if } t \geq t_2. \end{cases}$$

From (M2) we get

$$m_0 \leq M_{t_1}(t) < \frac{\xi_1}{p} m_0 \quad \text{and} \quad m_0 \leq M_{t_2}(t) < \frac{\xi_2}{q} m_0, \quad \text{for all } t \geq 0. \tag{3.1}$$

The proofs of Theorems 1.2 and 1.1 are based on a careful study of solutions of the following auxiliary problem:

$$\begin{cases} L_p^1(u) = \lambda|x|^{-c}F_u(x,u,v) + \alpha|x|^{-\beta}|u|^{\alpha-2}u|v|^\gamma & \text{in } \Omega, \\ L_q^2(v) = \lambda|x|^{-c}F_v(x,u,v) + \gamma|x|^{-\beta}|u|^{\alpha}|v|^{\gamma-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where

$$L_p^1(u) := -\left[M_{t_1}\left(\int_\Omega |x|^{-a_1p}|\nabla u|^p\,dx\right)\right]\operatorname{div}(|x|^{-a_1p}|\nabla u|^{p-2}\nabla w),$$
$$L_q^2(v) := -\left[M_{t_2}\left(\int_\Omega |x|^{-a_2q}|\nabla v|^q\,dx\right)\right]\operatorname{div}(|x|^{-a_2q}|\nabla v|^{q-2}\nabla v).$$

The Euler–Lagrange functional, $J\colon E\to\mathbb{R}$, associated to problem (3.2), is given by

$$J(u,v) = \frac{1}{p}\widehat{M}_{t_1}(\|u\|_A^p) + \frac{1}{q}\widehat{M}_{t_2}(\|v\|_B^q) - \lambda\int_\Omega |x|^{-c}F(x,u,v)\,dx - \int_\Omega |x|^{-\beta}|u|^\alpha|v|^\gamma\,dx,$$

for all $(u,v)\in E$, where $\widehat{M}_{t_i}(t) := \int_0^t M_{t_i}(s)\,ds$, $i=1,2$. Note that $J\in C^1(E,\mathbb{R})$.

4. The Palais–Smale condition

In this section we verify that, under hypotheses (M1), (M2), (F1) and (F2), the functional J satisfies the Palais–Smale condition below a given level.

Lemma 4.1. *Let $\{(u_n,v_n)\}$ be a bounded sequence in E such that*

$$J(u_n,v_n)\to c_\lambda \quad \text{and} \quad J'(u_n,v_n)\to 0 \quad \text{in } E^{-1} \text{ (dual of } E\text{), as } n\to\infty.$$

Suppose that (M1), (M2), (F1) *and* (F2) *hold and*

$$c_\lambda < \left(\frac{\alpha}{\xi_1}+\frac{\gamma}{\xi_2}-1\right)\frac{m_0}{\alpha+\gamma}K_{p,q},$$

where

$$K_{p,q} = \min\left\{\left(\frac{m_0}{2(\alpha+\gamma)C}\right)^{p/(p^*-p)}, \left(\frac{m_0}{2(\alpha+\gamma)C}\right)^{q/(q^*-q)}\right\},$$

then there exists a subsequence strongly convergent in E.

Proof. Since $\{(u_n,v_n)\}$ is bounded in E, passing to a subsequence, if necessary, we have

$$\begin{aligned}
&(u_n,v_n) \rightharpoonup (u,v) \quad \text{in } E,\\
&(u_n,v_n) \to (u,v) \quad \text{in } L^r(\Omega,|x|^{-a})\times L^s(\Omega,|x|^{-b}),\\
&u_n(x)\to u(x) \quad \text{a.e. in } \Omega \quad \text{and} \quad v_n(x)\to v(x) \quad \text{a.e. in } \Omega,\\
&\|u_n\|_A\to t_0\ge 0 \qquad \text{and} \quad \|v_n\|_B\to s_0\ge 0,
\end{aligned}$$

as $n\to\infty$, where $1\le r<p^*$, $1\le s<q^*$, $a<(a_1+1)r+N(1-r/p)$, and $b<(a_2+1)s+N(1-s/q)$. Moreover, since $Q(x,u,v)=|u|^\alpha|v|^\gamma$ satisfies (Q_0), we can apply Proposition 2.1 to obtain an at most countable index set Λ and sequences $\{x_j\}\subset\mathbb{R}^N$, $\{\mu_j\},\{\sigma_j\},\{\nu_j\}\subset(0,+\infty)$ such that

$$|x|^{-a_1p}|\nabla u_n|^p\,dx \rightharpoonup \mu, \quad |x|^{-a_2q}|\nabla v_n|^q\,dx \rightharpoonup \sigma, \quad |x|^{-\beta}|u_n|^\alpha|v_n|^\gamma\,dx \rightharpoonup \nu, \tag{4.1}$$

as $n \to +\infty$, in weak*-sense of measures, where

$$\nu = |x|^{-\beta}|u|^{\alpha}|v|^{\gamma}x + \sum_{j\in\Lambda}\nu_j\delta_{x_j}, \ \mu \geq |x|^{-a_1p}|\nabla u|^p dx + \sum_{j\in\Lambda}\mu_j\delta_{x_j},$$
$$\sigma \geq |x|^{-a_2q}|\nabla v|^q dx + \sum_{j\in\Lambda}\sigma_j\delta_{x_j},$$

for all $j \in \Lambda$, where δ_{x_j} is the Dirac mass at $x_j \in \Omega$, and there exists a constant $C > 0$ such that

$$\mu_j^{p^*/p} + \sigma_j^{q^*/q} \geq C\nu_j, \quad \text{for all } j \in \Lambda. \tag{4.2}$$

Now let $k \in \mathbb{N}$. Without loss of generality we can suppose $B_2(0) \subset \Omega$, then for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_k)/\varrho)$, where $\psi \in C_0^\infty(\Omega, [0,1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$, and $|\nabla\psi| \leq 1$. Observe that $(\psi_\varrho u_n, \psi_\varrho v_n)$ is bounded in E. So we have $J'_\lambda(u_n, v_n)(\psi_\varrho u_n, \psi_\varrho v_n) \to 0$, as $n \to +\infty$, that is,

$$\begin{aligned}
&M_{t_1}(\|u_n\|_A^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{a_1p}}\,dx \\
&\qquad + M_{t_2}(\|v_n\|_B^q)\int_\Omega \frac{v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho}{|x|^{a_2q}}\,dx \\
&\qquad \leq -m_0\int_\Omega |x|^{-a_1p}|\nabla u_n|^p\psi_\varrho\,dx - m_0\int_\Omega |x|^{-a_2q}|\nabla v_n|^q\psi_\varrho\,dx \\
&\qquad + \lambda\int_\Omega |x|^{-c}F_u(x,u_n,v_n)\psi_\varrho u_n\,dx + \lambda\int_\Omega |x|^{-c}F_v(x,u_n,v_n)\psi_\varrho v_n\,dx \\
&\qquad + (\alpha+\gamma)\int_\Omega |x|^{-\beta}|u_n|^\alpha|v_n|^\gamma\psi_\varrho\,dx + o_n(1).
\end{aligned}$$

Using (4.1) and Lesbegue's Dominated Convergence Theorem, we obtain

$$\begin{aligned}
&\limsup_{n\to+\infty}\bigg[M_{t_1}(\|u_n\|_A^p)\int_\Omega \frac{u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho}{|x|^{a_1p}}\,dx \\
&\qquad + M_{t_2}(\|v_n\|_B^q)\int_\Omega \frac{v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho}{|x|^{a_2q}}\,dx\bigg] \\
&\qquad \leq -m_0\int_\Omega |x|^{-a_1p}|\nabla u|^p\psi_\varrho\,dx - m_0\sum_{j\in\Lambda}\mu_j\delta_j(\psi_\varrho) \\
&\qquad - m_0\int_\Omega |x|^{-a_2q}|\nabla v|^q\psi_\varrho\,dx - m_0\sum_{j\in\Lambda}\sigma_j\delta_j(\psi_\varrho) \\
&\qquad + \lambda\int_\Omega |x|^{-c}F_u(x,u,v)\psi_\varrho u\,dx + \lambda\int_\Omega |x|^{-c}F_v(x,u,v)\psi_\varrho v\,dx \\
&\qquad + (\alpha+\gamma)\int_\Omega |x|^{-\beta}|u|^\alpha|v|^\gamma\psi_\varrho\,dx + (\alpha+\gamma)\sum_{j\in\Lambda}\nu_j\delta_j(\psi_\varrho).
\end{aligned}$$

Using Lesbegue's Dominated Convergence Theorem, again, we have

$$\int_\Omega |x|^{-a_1 p}|\nabla u|^p \psi_\varrho\, dx = o_\varrho(1), \qquad \int_\Omega |x|^{-a_2 q}|\nabla v|^q \psi_\varrho\, dx = o_\varrho(1),$$
$$\int_\Omega |x|^{-\delta} F_u(x,u,v)\psi_\varrho u\, dx = o_\varrho(1), \qquad \int_\Omega |x|^{-\delta} F_v(x,u,v)\psi_\varrho v\, dx = o_\varrho(1),$$
$$\int_\Omega |x|^{-\beta}|u|^\alpha |v|^\gamma \psi_\varrho\, dx = o_\varrho(1),$$

where $\lim\limits_{\varrho\to 0^+} o_\varrho(1) = 0$. Thus, we get

$$\begin{aligned}
(4.3)\qquad &\lim_{\rho\to 0^+}\Bigg\{\limsup_{n\to+\infty}\bigg[M_{t_1}(\|u_n\|_A^p)\int_\Omega |x|^{-a_1 p} u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho\, dx\\
&\qquad + M_{t_2}(\|v_n\|_B^q)\int_\Omega |x|^{-a_2 q} v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho\, dx\bigg]\Bigg\}\\
&\le \lim_{\rho\to 0^+}\bigg[-m_0\sum_{j\in\Lambda}\mu_j\delta_j(\psi_\varrho)\\
&\qquad - m_0\sum_{j\in\Lambda}\sigma_j\delta_j(\psi_\varrho) + (\alpha+\gamma)\sum_{j\in\Lambda}\nu_j\delta_j(\psi_\varrho)\bigg].
\end{aligned}$$

Now we will show that

$$\begin{aligned}
(4.4)\qquad &\lim_{\rho\to 0^+}\Bigg\{\limsup_{n\to+\infty}\bigg[M_{t_1}(\|u_n\|_A^p)\int_\Omega |x|^{-a_1 p} u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\psi_\varrho\, dx\\
&\qquad + M_{t_2}(\|v_n\|_B^q)\int_\Omega |x|^{-a_2 q} v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho\, dx\bigg]\Bigg\} = 0.
\end{aligned}$$

First, observe that, by Hölder's inequality,

$$\left|\int_\Omega |x|^{-a_1 p} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho\, dx\right| \le \|u_n\|_A^{p-1}\left(\int_\Omega |x|^{-a_1 p}|u_n\nabla\psi_\varrho|^p\, dx\right)^{1/p}.$$

Since $\{u_n\}$ is bounded in $\mathcal{D}_a^{1,p}$, M_{t_1} and M_{t_2} are continuous, and $\operatorname{supp}(\psi_\varrho) \subset B(x_k; 2\varrho)$, there exists $L_1 > 0$ such that

$$\begin{aligned}
M_{t_1}(\|u_n\|^p)&\int_\Omega |x|^{-a_1 p} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho\, dx\\
&\le L_1\left(\int_{B(x_k;2\varrho)}\frac{|u_n\nabla\psi_\varrho|^p}{|x|^{a_1 p}}\, dx\right)^{1/p}.
\end{aligned}$$

Analogously, there exists $L_2 > 0$ such that

$$M_{t_2}(\|v_n\|^q)\int_\Omega |x|^{-a_2 q} v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho\, dx \le L_2\left(\int_{B(x_k;2\varrho)}\frac{|v_n\nabla\psi_\varrho|^q}{|x|^{a_2 q}}\, dx\right)^{1/q}.$$

Therefore, using Hölder's inequality, we obtain

$$\begin{aligned}
\limsup_{n\to+\infty}\bigg[&M_{t_1}(\|u_n\|_A^p)\int_\Omega |x|^{-a_1p}u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\varrho\,dx\\
&+M_{t_2}(\|v_n\|_B^q)\int_\Omega |x|^{-a_2q}v_n|\nabla v_n|^{q-2}\nabla v_n\nabla\psi_\varrho\,dx\bigg]\\
&\leq L_1|B(x_k;2\varrho)|^{1/N}\bigg(\int_\Omega \chi_{B(x_k;2\varrho)}(|x|^{-a_1p}|u|^p)^{N/(N-p)}\,dx\bigg)^{(N-p)/(Np)}\\
&+L_2|B(x_k;2\varrho)|^{1/N}\bigg(\int_\Omega \chi_{B(x_k;2\varrho)}(|x|^{-a_2q}|v|^q)^{N/(N-q)}\,dx\bigg)^{(N-q)/(Nq)}.
\end{aligned}$$

Letting $\varrho\to 0^+$ in the above expression, it follows from the Dominated Convergence Theorem that (4.4) occurs. Thus, we conclude from (4.3) that

$$0\leq \lim_{\rho\to 0^+}\bigg[-m_0\sum_{j\in\Lambda}\mu_j\delta_j(\psi_\varrho)-m_0\sum_{j\in\Lambda}\sigma_j\delta_j(\psi_\varrho)+(\alpha+\gamma)\sum_{j\in\Lambda}\nu_j\delta_j(\psi_\varrho)\bigg].$$

That is, $0\leq -m_0(\mu_k+\sigma_k)+(\alpha+\gamma)\nu_k$. So, from (4.2) we obtain

$$m_0(\mu_k+\sigma_k)\leq(\alpha+\gamma)\nu_k\leq(\alpha+\gamma)C\big(\mu_k^{p^*/p}+\sigma_k^{q^*/q}\big). \tag{4.5}$$

Setting $\tau=\mu_k+\sigma_k$, we have $0<m_0\tau\leq(\alpha+\gamma)C(\tau^{p^*/p}+\tau^{q^*/q})$, which implies $m_0/[(\alpha+\gamma)C]\leq\tau^{p^*/p-1}+\tau^{q^*/q-1}$. We define $r_1=p^*/p-1$ and $r_2=q^*/q-1$. Therefore, if $\tau<1$, we have $\tau^{r_1}+\tau^{r_2}\leq 2\tau^{\min\{r_1,r_2\}}$. If $\tau\geq 1$, we have $\tau^{r_1}+\tau^{r_2}\leq 2\tau^{\max\{r_1,r_2\}}$. Therefore,

$$\tau\geq\min\bigg\{\bigg(\frac{m_0}{2(\alpha+\gamma)C}\bigg)^{1/r_1},\bigg(\frac{m_0}{2(\alpha+\gamma)C}\bigg)^{1/r_2}\bigg\}=K_{p,q}. \tag{4.6}$$

Thus from (4.5) and (4.6) we obtain

$$\nu_k\geq\frac{m_0}{\alpha+\gamma}\,\tau\geq\frac{m_0}{\alpha+\gamma}\,K_{p,q}. \tag{4.7}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that (4.7) holds for some $k\in\Lambda$. Thus, once that $m_0\leq M_{t_1}(t)\leq\xi_1m_0/p$ and $m_0\leq M_{t_2}(t)\leq\xi_2m_0/q$, for all $t\in\mathbb{R}$, from (F3) we obtain

$$\begin{aligned}
c_\lambda&=J(u_n,v_n)-J'(u_n,v_n)\cdot\bigg(\frac{u_n}{\xi_1},\frac{v_n}{\xi_2}\bigg)+o_n(1)\\
&\geq\bigg(\frac{\alpha}{\xi_1}+\frac{\gamma}{\xi_2}-1\bigg)\int_\Omega|x|^{-\beta}|u_n|^\alpha|v_n|^\gamma\psi_\varrho\,dx+o_n(1).
\end{aligned}$$

Letting $n\to+\infty$, we get

$$c_\lambda\geq\bigg(\frac{\alpha}{\xi_1}+\frac{\gamma}{\xi_2}-1\bigg)\nu_k\geq\bigg(\frac{\alpha}{\xi_1}+\frac{\gamma}{\xi_2}-1\bigg)\frac{m_0}{\alpha+\gamma}K_{p,q}.$$

But this is a contradiction. Thus Λ is empty and it follows that

$$\int_\Omega |x|^{-\beta}|u_n|^\alpha|v_n|^\gamma\,dx \to \int_\Omega |x|^{-\beta}|u|^\alpha|v|^\gamma\,dx. \tag{4.8}$$

Now we will prove that $(u_n,v_n)\to(u,v)$ in E. Since $(u_n,v_n)\to(u,v)$ in $L^\theta(\Omega,|x|^{-c})\times L^\delta(\Omega,|x|^{-c})$, it follows from the Lesbegue Dominated Convergence Theorem that

$$-\lambda\int_\Omega |x|^{-c}F_u(x,u,v)(u_n-u)\,dx\to 0,$$

as $n\to+\infty$. Also, from (4.8), the Lesbegue Dominated Convergence Theorem and Brezis–Lieb Lemma [3] we have

$$\int_\Omega |x|^{-\beta}|u_n|^{\alpha-2}u_n|v_n|^\gamma(u_n-u)\,dx\to 0,$$

as $n\to+\infty$. Therefore, as $\{(u_n,v_n)\}$ is bounded in E, $J'(u_n,v_n)(u_n-u,0)\to 0$ in $\mathbb{R}$, as $n\to+\infty$. Thus, as $\|u_n\|_A\to t_0\geq 0$, as $n\to+\infty$, and as M_{t_1} is continuous and positive, we obtain

$$\lim_{n\to\infty}\int_\Omega |x|^{-a_1p}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,dx=0.$$

It follows from Lemma 2.2 that $u_n\to u$ in $\mathcal{D}^{1,p}_{a_1}$ as $n\to+\infty$. Using the same arguments, we obtain $v_n\to v$ in $\mathcal{D}^{1,q}_{a_2}$, as $n\to+\infty$. Thus we conclude that $(u_n,v_n)\to(u,v)$ in E as $n\to+\infty$. $\square$

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Here p may be different from q, so that the price to pay is that we cannot get a result for all $\lambda>0$.

The next two lemmas show that the functional J has the Mountain Pass geometry. Before we prove them, note that since $\theta/p+\delta/q>1$ and $\theta/p^*+\delta/q^*<1$, there exist $p_0\in(p,p^*)$ and $q_0\in(q,q^*)$ such that $\theta/p_0+\delta/q_0=1$. Thus, from Young's inequality, we have

$$|u|^\theta|v|^\delta\leq\frac{\theta}{p_0}|u|^{p_0}+\frac{\delta}{q_0}|v|^{q_0}\quad\text{and}\quad |u|^\alpha|v|^\gamma\leq\frac{\alpha}{p^*}|u|^{p^*}+\frac{\gamma}{q^*}|v|^{q^*}. \tag{5.1}$$

LEMMA 5.1. *Assume that conditions* (M1), (M2), (F1) *and* (F2) *hold. Then there exist positive numbers ρ and ζ such that*

$$J(u,v)\geq\zeta>0,\quad \textit{for all } (u,v)\in E \textit{ with } \|(u,v)\|=\rho.$$

PROOF. Let $(u,v)\in E$ be such that $\|(u,v)\|\leq 1$. From (M1), (F1), (F2), (5.1) and the Caffarelli–Kohn–Nirenberg inequality, we obtain

$$J(u,v)\geq\bigg(\frac{m_0}{p}\|u\|_A^p-\bigg(\lambda\widetilde{C_2}+\frac{\alpha}{p^*}\bigg)\|u\|_A^{p_0}\bigg)+\bigg(\frac{m_0}{q}\|v\|_B^q-\bigg(\lambda\widehat{C_2}+\frac{\gamma}{q^*}\bigg)\|v\|_B^{q_0}\bigg).$$

Since $p<p_0$ and $q<q_0$, taking $\rho\in(0,1)$ small enough, there exists $\zeta>0$ such that $J(u,v)\geq\zeta>0$, for all $(u,v)\in E$ with $\|(u,v)\|=\rho$. $\square$

Lemma 5.2. *Assume that conditions* (M1), (M2) *and* (F2) *hold. Then, for all* $\lambda > 0$, *there exists* $e \in E$ *with* $J(e) < 0$ *and* $\|e\| > \rho$.

Proof. Fix $(u_0, v_0) \in E$ with $u_0, v_0 > 0$ in Ω and $\|(u_0, v_0)\| = 1$. Using (3.1) and (F2), we obtain

$$J(t^{1/p}u_0, t^{1/q}v_0) \leq \frac{\xi_1}{p} m_0 t \|u_0\|_A^p + \frac{\xi_2}{q} m_0 t \|v_0\|_B^q - \lambda C_1 t^{\theta/p+\delta/q} \int_\Omega |x|^{-c} u_0^\theta v_0^\delta \, dx.$$

Since $\theta/p + \delta/q > 1$, we have $\lim_{t\to\infty} J(t^{1/p}u_0, t^{1/q}v_0) = -\infty$. Thus, there exists $t_0 > \max\{\rho^p, \rho^q\}$ large enough, such that $J(t_0^{1/p}u_0, t_0^{1/q}v_0) < 0$. The result follows by considering $e = (t_0^{1/p}u_0, t_0^{1/q}v_0)$. $\square$

Using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2], without the Palais–Smale condition (see [18]), there exists a sequence $\{(u_n, v_n)\} \subset E$, satisfying

$$J(u_n, v_n) \to c_\lambda \quad \text{and} \quad J'(u_n, v_n) \to 0, \qquad \text{in } E^{-1} \text{ (dual of } E),$$

as $n \to +\infty$, where $c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ and $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$.

Lemma 5.3. *If* (M1), (M2) *and* (F2) *hold, then* $\lim_{\lambda\to+\infty} c_\lambda = 0$.

Proof. Define $\overline{u}_0 = t_0^{1/p}u_0$ and $\overline{v}_0 = t_0^{1/q}v_0$, where (u_0, v_0) is given by Lemma 5.2. Since from Lemmas 5.1 and 5.2 the functional J has the Mountain Pass geometry, it follows that there exists t_λ verifying $J(t_\lambda^{1/p}\overline{u}_0, t_\lambda^{1/q}\overline{v}_0) = \max_{t\geq 0} J(t^{1/p}\overline{u}_0, t^{1/q}\overline{v}_0)$. Using (3.1) and (F2), we obtain

$$\begin{aligned} 0 = J'\big(t_\lambda^{1/p}\overline{u}_0, t_\lambda^{1/q}\overline{v}_0\big)\left(\frac{1}{p}t_\lambda^{1/p}\overline{u}_0, \frac{1}{q}t_\lambda^{1/q}\overline{v}_0\right) &\leq \frac{\xi_1}{p^2}\, m_0 t_\lambda \|\overline{u}_0\|_A^p + \frac{\xi_2}{q^2} m_0 t_\lambda \|\overline{v}_0\|_B^q \\ - \lambda \overline{C} t_\lambda^{\theta/p+\delta/q} \int_\Omega |x|^{-c} \overline{u}_0^\theta \overline{v}_0^\delta \, dx &- (\alpha+\gamma) t_\lambda^{\alpha/p+\gamma/q} \int_\Omega |x|^{-\beta} \overline{u}_0^\alpha \overline{v}_0^\gamma \, dx. \end{aligned}$$

Consider $C_0 = \max\{t_0^{1/p}, t_0^{1/q}\}$. Since $\|(u_0, v_0)\| = 1$, we have $\|u_0\|_A^p, \|v_0\|_B^q \leq 1$, and so

$$\begin{aligned} \left(\frac{\xi_1}{p^2} + \frac{\xi_2}{q^2}\right) C_0 m_0 t_\lambda &\geq \lambda \overline{C} t_\lambda^{\theta/p+\delta/q} \int_\Omega |x|^{-c} \overline{u}_0^\theta \overline{v}_0^\delta \, dx \\ &+ (\alpha+\gamma) t_\lambda^{\alpha/p+\gamma/q} \int_\Omega |x|^{-\beta} \overline{u}_0^\alpha \overline{v}_0^\gamma \, dx \geq (\alpha+\gamma) t_\lambda^{\alpha/p+\gamma/q} \int_\Omega |x|^{-\beta} \overline{u}_0^\alpha \overline{v}_0^\gamma \, dx. \end{aligned}$$

Since $\alpha/p + \gamma/q > \alpha/p^* + \gamma/q^* = 1$, we have that $\{t_\lambda\}$ is a bounded sequence. Thus there exist a sequence $\{\lambda_n\}$ and $\beta_0 \geq 0$ such that $\lambda_n \to +\infty$ and $t_{\lambda_n} \to \beta_0$, as $n \to \infty$. Consequently, there exists $D > 0$ such that

$$\left(\frac{\xi_1}{p^2} + \frac{\xi_2}{q^2}\right) C_0 m_0 t_{\lambda_n} \leq D, \quad \text{for all } n \in \mathbb{N},$$

and so

$$\lambda_n \overline{C} t_{\lambda_n}^{\theta/p+\delta/q} \int_\Omega |x|^{-c} \overline{u}_0^{\theta} \overline{v}_0^{\delta}\, dx + (\alpha+\gamma) t_{\lambda_n}^{\alpha/p+\gamma/q} \int_\Omega |x|^{-\beta} \overline{u}_0^{\alpha} \overline{v}_0^{\gamma}\, dx \leq D, \tag{5.2}$$

for all $n \in \mathbb{N}$. If $\beta_0 > 0$, we obtain

$$\lim_{n\to\infty} \left[\lambda_n \overline{C} t_{\lambda_n}^{\theta/p+\delta/q} \int_\Omega |x|^{-c} \overline{u}_0^{\theta} \overline{v}_0^{\delta}\, dx + (\alpha+\gamma) t_{\lambda_n}^{\alpha/p+\gamma/q} \int_\Omega |x|^{-\beta} \overline{u}_0^{\alpha} \overline{v}_0^{\gamma}\, dx\right] = +\infty,$$

which contradicts with (5.2). Thus we conclude that $\beta_0 = 0$. Now, let us consider the path $\gamma_*(t) = (t^{1/p}\overline{u}_0, t^{1/q}\overline{v}_0)$, for $t \in [0,1]$, which belongs to Γ, to get the following estimate:

$$0 < c_\lambda \leq \max_{t\in[0,1]} J(\gamma_*(t)) = J\big(t_\lambda^{1/p}\overline{u}_0, t_\lambda^{1/q}\overline{v}_0\big) \leq \left(\frac{\xi_1}{p^2} + \frac{\xi_2}{q^2}\right) C_0 m_0 t_\lambda.$$

In this way, observing that $\{c_\lambda\}$ is a monotonous sequence, we conclude that

$$\lim_{\lambda\to+\infty} c_\lambda = 0. \qquad \square$$

REMARK 5.4. Due to Lemma 5.3, there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\begin{aligned} c_\lambda &< \left(\frac{1}{p}\, m_0 - \frac{1}{\xi_1}\, M_1(t_1)\right) t_1, \quad \text{for all } \lambda > \lambda_1,\\ c_\lambda &< \left(\frac{1}{q}\, m_0 - \frac{1}{\xi_2}\, M_2(t_2)\right) t_2, \quad \text{for all } \lambda > \lambda_2. \end{aligned}$$

LEMMA 5.5. *Suppose that* $\lambda > \lambda_3 = \max\{\lambda_1, \lambda_2\}$ *and* (M1), (M2), (F2) *and* (F3) *hold. Let* $\{(u_n, v_n)\} \subset E$ *be a sequence such that*

$$J(u_n, v_n) \to c_\lambda \quad \textit{and} \quad J'(u_n, v_n) \to 0, \tag{5.3}$$

as $n \to +\infty$. *Then, for all* $n \in \mathbb{N}$, *we have*

$$\|u_n\|_A^p \leq t_1 \quad \textit{and} \quad \|v_n\|_B^q \leq t_2.$$

PROOF. We claim that $\{(u_n, v_n)\}$ is a bounded sequence. Indeed, by (5.3), (F3) and (3.1), we obtain

$$\begin{aligned} c_\lambda + o_n(1)\|(u_n, v_n)\| &\geq J(u_n, v_n) - J'(u_n, v_n) \cdot \left(\frac{1}{\xi_1} u_n, \frac{1}{\xi_2} v_n\right)\\ &\geq \left[\frac{m_0}{p} - \frac{1}{\xi_1} M_{t_1}(t_1)\right] \|u_n\|_A^p + \left[\frac{m_0}{q} - \frac{1}{\xi_2} M_{t_2}(t_2)\right] \|v_n\|_B^q, \end{aligned}$$

which implies that $\{(u_n, v_n)\}$ is a bounded sequence. Thus, from (5.3) we obtain

$$|J'(u_n, v_n) \cdot (u_n, v_n)| \leq |J'(u_n, v_n)| \cdot \|(u_n, v_n)\| \to 0,$$

as $n \to +\infty$. Which implies that

$$\begin{aligned} c_\lambda &= J(u_n, v_n) - J'(u_n, v_n) \cdot \left(\frac{1}{\xi_1} u_n, \frac{1}{\xi_2} v_n\right) + o_n(1)\\ &\geq \left[\frac{m_0}{p} - \frac{1}{\xi_1} M_{t_1}(\|u_n\|_A^p)\right] \|u_n\|_A^p + \left[\frac{m_0}{q} - \frac{1}{\xi_2} M_{t_2}(\|v_n\|_B^q)\right] \|v_n\|_B^q + o_n(1). \end{aligned}$$

Now, suppose that either $\|u_n\|_A^p > t_1$ or $\|v_n\|_B^q > t_2$. Without loss of generality, suppose $\|u_n\|_A^p > t_1$. Since M_{t_2} is increasing, from (3.1), we have

$$M_{t_2}(\|v_n\|_B^q) \leq M_{t_2}(t_2) < \frac{\xi_2}{q}\, m_0,$$

which implies

$$\frac{1}{q}\, m_0 - \frac{1}{\xi_2} M_{t_2}(\|v_n\|_B^q) > 0.$$

Moreover, since $\|u_n\|^p > t_1$, we have $M_{t_1}(\|u_n\|_A^p) = M_{t_1}(t_1)$. Thus, we obtain

$$c_\lambda > \left[\frac{m_0}{p} - \frac{1}{\xi_1} M_{t_1}(t_1)\right] t_1 + o_n(1).$$

Passing to the limit as $n \to +\infty$, we obtain

$$c_\lambda \geq \left[\frac{m_0}{p} - \frac{1}{\xi_1} M_{t_1}(t_1)\right] t_1, \quad \text{for all } \lambda > \lambda_3,$$

which contradicts with Remark 5.4. The same occurs if we suppose $\|v_n\|_B^q > t_2$. This concludes the proof. □

Proof of Theorem 1.1. It follows from Lemma 5.3 that there exists $\lambda_4 > 0$ such that

$$c_\lambda < \left(\frac{\alpha}{\xi_1} + \frac{\gamma}{\xi_2} - 1\right) \frac{m_0}{\alpha + \gamma} K_{r_1, r_2}, \quad \text{for all } \lambda > \lambda_4. \tag{5.4}$$

Set $\lambda^* = \max\{\lambda_3, \lambda_4\}$. Fix $\lambda \geq \lambda^*$. From Lemmas 5.1 and 5.2, there exists a bounded sequence $\{(u_n, v_n)\} \subset E$ such that $J(u_n, v_n) \to c_\lambda$ and $J'(u_n, v_n) \to 0$ in E^{-1}, as $n \to \infty$. Since (5.4) holds, it follows from Lemma 4.1 that, up to a subsequece, $(u_n, v_n) \to (u, v)$ strongly in E. Thus (u, v) is a weak solution to problem (3.2). Moreover, by Lemma 5.5 we conclude that (u, v) is a weak solution to problem (1.1). □

6. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We remind that we suppose that $p = q$ to get a result for all $\lambda > 0$. Moreover, we are considering $a_1 = a_2$, what implies that $A = B$.

We observe that if $p = q$, we also can apply Lemmas 5.1 and 5.2 to show that the functional J has the Mountain Pass geometry. So, using a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2] without the Palais-Smale condition (see [18]), one can find a sequence $\{(u_n, v_n)\} \subset E$ satisfying

$$J(u_n, v_n) \to c_\lambda \quad \text{and} \quad J'(u_n, v_n) \to 0, \quad \text{in } E^{-1} \text{ (dual of } E),$$

as $n \to +\infty$, where $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$, $\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = (t_0 u_0, t_0 v_0)\}$, and $(u_0, v_0) \in E$ is such that $u_0 > 0$ and $v_0 > 0$.

In order to obtain the level c_λ below the level given by Lemma 4.1, we will give some estimates. We define the Sobolev space

$$W^{1,p}_{a_1,b_1}(\Omega) = \{u \in L^{p^*}(\Omega, |x|^{-b_1 p^*}) : |\nabla u| \in L^p(\Omega, |x|^{-a_1 p})\},$$

and endow it with the norm

$$\|u\|_{W^{1,p}_{a_1,b_1}(\Omega)} = \|u\|_{p^*,b_1 p^*} + \|\nabla u\|_{p,a_1 p}.$$

We consider the best constant of the weighted Caffarelli–Kohn–Nirenberg type given by

$$\widetilde{S}_{a_1,p} = \inf_{u \in W^{1,p}_{a_1,b_1}(\mathbb{R}^N)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-a_1 p} |\nabla u|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{-b_1 p^*} |u|^{p^*} dx \right)^{p/p^*}} \right\}.$$

We also set $R^{1,p}_{a_1,b_1}(\Omega)$ as the subspace of $W^{1,p}_{a_1,b_1}(\Omega)$ of the radial functions, more precisely,

$$R^{1,p}_{a_1,b_1}(\Omega) = \big\{u \in W^{1,p}_{a_1,b_1}(\Omega) : u(x) = u(|x|)\big\},$$

endowed with the induced norm

$$\|u\|_{R^{1,p}_{a_1,b_1}(\Omega)} = \|u\|_{W^{1,p}_{a_1,b_1}(\Omega)}.$$

Horiuchi [13] has proved that

$$\widetilde{S}_{a_1,p,R} = \inf_{u \in R^{1,p}_{a_1,b_1}(\mathbb{R}^N)\setminus\{0\}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-a_1 p} |\nabla u|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{-b_1 p^*} |u|^{p^*} dx \right)^{p/p^*}} \right\}$$

is achieved by the functions of the form $u_\varepsilon(x) = k_{a_1,p}(\varepsilon) v_\varepsilon(x)$ for all $\varepsilon > 0$, where

$$k_{a_1,p}(\varepsilon) = c\varepsilon^{(N-d_1 p)/d_1 p^2}$$

and

$$v_\varepsilon(x) = \big(\varepsilon + |x|^{(d_1 p(N-p-a_1 p))/((p-1)(N-d_1 p))}\big)^{-(N-d_1 p)/(d_1 p)}.$$

Moreover, u_ε satisfies

$$\int_{\mathbb{R}^N} |x|^{-a_1 p} |\nabla u_\varepsilon|^p \, dx = \int_{\mathbb{R}^N} |x|^{-b_1 p^*} |u_\varepsilon|^{p^*} dx = (\widetilde{S}_{a_1,p,R})^{p^*/(p^*-p)}. \tag{6.1}$$

From (6.1) we obtain

$$\int_{\mathbb{R}^N} |x|^{-a_1 p} |\nabla v_\varepsilon|^p \, dx = [k_{a_1,p}(\varepsilon)]^{-p} (\widetilde{S}_{a_1,p,R})^{p^*/(p^*-p)} \tag{6.2}$$

and

$$\int_{\mathbb{R}^N} |x|^{-b_1 p^*} |v_\varepsilon|^{p^*} dx = [k_{a_1,p}(\varepsilon)]^{-p^*} (\widetilde{S}_{a_1,p,R})^{p^*/(p^*-p)}. \tag{6.3}$$

Let R_0 be a positive constant and set $\Psi \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \le \Psi(x) \le 1$, $\Psi(x) = 1$, for all $|x| \le R_0$, and $\Psi(x) = 0$, for all $|x| \ge 2R_0$. Set

$$\widetilde{v}_\varepsilon(x) = \Psi(x) v_\varepsilon(x), \tag{6.4}$$

for all $x \in \mathbb{R}^N$ and $\varepsilon > 0$. Without loss of generality we can consider $B(0;2R_0) \subset \Omega$.

Lemma 6.1.

$$\lim_{\varepsilon\to 0^+} \frac{\|\widetilde{v}_\varepsilon\|_A^p}{\left(\displaystyle\int_\Omega |x|^{-b_1 p^*} |\widetilde{v}_\varepsilon|^{p^*} dx\right)^{p/p^*}} = 0.$$

Proof. By a straightforward computation we obtain

$$\|\widetilde{v}_\varepsilon\|_A^p \le [k_{a_1,p}(\varepsilon)]^{-p} (\widetilde{S}_{a_1,p,R})^{p^*/p^*-p} + O(1) \tag{6.5}$$

and

$$\int_\Omega |x|^{-b_1 p^*} |\widetilde{v}_\varepsilon|^{p^*} dx = \varepsilon^{-(N-d_1 p/(d_1 p) p^*} \cdot O(1), \quad \text{for all } \varepsilon \in (0,1), \tag{6.6}$$

where $O(1)$ denotes a positive constant. Therefore, for all $\varepsilon \in (0,1)$, from (6.5) and (6.6) we obtain

$$\begin{aligned}
\frac{\|\widetilde{v}_\varepsilon\|_A^p}{\left(\displaystyle\int_\Omega |x|^{-b_1 p^*} |\widetilde{v}_\varepsilon|^{p^*} dx\right)^{p/p^*}} &\le \frac{[k_{a_1,p}(\varepsilon)]^{-p} (\widetilde{S}_{a_1,p,R})^{p^*/(p^*-p)} + O(1)}{\left(\varepsilon^{-(N-d_1 p)/(d_1 p) p^*} \cdot O(1)\right)^{p/p^*}} \\
&= \frac{c^{-p} (\widetilde{S}_{a_1,p,R})^{p^*/(p^*-p)} \varepsilon^{(N-d_1 p)/(d_1 p)(p-1)} + O(1) \varepsilon^{(N-d_1 p)/(d_1 p p)}}{O(1)}.
\end{aligned}$$

Since $p > 1$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\|\widetilde{v}_\varepsilon\|_A^p}{\left(\displaystyle\int_\Omega |x|^{-b_1 p^*} |\widetilde{v}_\varepsilon|^{p^*} dx\right)^{p/p^*}} = 0. \qquad \square$$

Lemma 6.2. *Suppose $p = q$. Assume* (M1), (M2), (F1) *and* (F2) *hold. Set*

$$\begin{aligned}
l^* = \min\bigg\{ &\left(\frac{1}{p} m_0 - \frac{1}{\xi_1} M_1(t_1)\right) t_1, \left(\frac{1}{p} m_0 - \frac{1}{\xi_1} M_2(t_2)\right) t_2, \\
&\left(\frac{\alpha}{\xi_1} + \frac{\gamma}{\xi_2} - 1\right) \frac{m_0}{p^*} \left(\frac{m_0}{2p^* C}\right)^{p/(p^*-p)} \bigg\}.
\end{aligned}$$

Then, there exists $\varepsilon_1 \in (0,1)$ such that $\sup_{t\ge 0} J(t(\widetilde{v}_\varepsilon, \widetilde{v}_\varepsilon)) < l^*$, for all $\varepsilon \le \varepsilon_1$.

Proof. Let $0 < \varepsilon < 1$ and $\widetilde{v}_\varepsilon$ be as in (6.4). Since from Lemmas 5.1 and 5.2 the functional J satisfies the Mountain Pass geometry, there exists $t_\varepsilon > 0$ such that

$$\sup_{t \ge 0} J(t(\widetilde{v}_\varepsilon, \widetilde{v}_\varepsilon)) = J(t_\varepsilon(\widetilde{v}_\varepsilon, \widetilde{v}_\varepsilon)).$$

Since $p = q$, we have

$$\begin{aligned}\sup_{t\geq 0} J(t(\widetilde{v}_\varepsilon,\widetilde{v}_\varepsilon)) &= \frac{1}{p}\widehat{M}_{t_1}(\|t_\varepsilon\widetilde{v}_\varepsilon\|_A^p) + \frac{1}{p}\widehat{M}_{t_2}(\|t_\varepsilon\widetilde{v}_\varepsilon\|_A^p) \\ &\quad - \lambda\int_\Omega |x|^{-c}F(x,t_\varepsilon\widetilde{v}_\varepsilon,t_\varepsilon\widetilde{v}_\varepsilon)\,dx - \int_\Omega |x|^{-\beta}t_\varepsilon^{p^*}|\widetilde{v}_\varepsilon|^{p^*}\,dx \\ &\leq \frac{\xi_1+\xi_2}{p^2}\, m_0 t_\varepsilon^p\|\widetilde{v}_\varepsilon\|_A^p - t_\varepsilon^{p^*}\int_\Omega |x|^{-\beta}|\widetilde{v}_\varepsilon|^{p^*}\,dx.\end{aligned}$$

Now we consider the function $g\colon \mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$, given by

$$g(s) = \bigg(\frac{\xi_1+\xi_2}{p^2}m_0\|\widetilde{v}_\varepsilon\|_A^p\bigg)s^p - \bigg(\int_\Omega |x|^{-\beta}|\widetilde{v}_\varepsilon|^{p^*}\,dx\bigg)s^{p^*}.$$

It is easy to see that

$$\overline{s} = \left(\frac{\dfrac{\xi_1+\xi_2}{p}m_0\|\widetilde{v}_\varepsilon\|_A^p}{p^*\displaystyle\int_\Omega |x|^{-\beta}|\widetilde{v}_\varepsilon|^{p^*}\,dx}\right)^{1/(p^*-p)}$$

is a maximum of g and we have

$$g(\overline{s}) = \bigg(\frac{1}{p}-\frac{1}{p^*}\bigg)\frac{\bigg(\dfrac{\xi_1+\xi_2}{p}\,m_0\bigg)^{p^*/(p^*-p)}}{(p^*)^{p/(p^*-p)}}\left(\frac{\|\widetilde{v}_\varepsilon\|_A^p}{\bigg(\displaystyle\int_\Omega |x|^{-\beta}|\widetilde{v}_\varepsilon|^{p^*}\,dx\bigg)^{p/p^*}}\right)^{p^*/(p^*-p)}.$$

So we have

$$\begin{aligned}\sup_{t\geq 0} J(t(\widetilde{v}_\varepsilon,\widetilde{v}_\varepsilon)) &\leq \bigg(\frac{1}{p}-\frac{1}{p^*}\bigg)\frac{\bigg(\dfrac{\xi_1+\xi_2}{p}\,m_0\bigg)^{p^*/(p^*-p)}}{(p^*)^{p/(p^*-p)}} \\ &\quad\cdot\left(\frac{\|\widetilde{v}_\varepsilon\|_A^p}{\bigg(\displaystyle\int_\Omega |x|^{-\beta}|\widetilde{v}_\varepsilon|^{p^*}\,dx\bigg)^{p/p^*}}\right)^{p^*/(p^*-p)}.\end{aligned}$$

It follows from Lemma 6.1 that there exists $0<\varepsilon_1<1$ such that

$$\sup_{t\geq 0} J(t(\widetilde{v}_\varepsilon,\widetilde{v}_\varepsilon)) < l^*, \quad \text{for all } \varepsilon\leq\varepsilon_1. \qquad \square$$

REMARK 6.3. Let us consider the path $\gamma_*(t) = t(t_0\widetilde{v}_{\varepsilon_1}, t_0\widetilde{v}_{\varepsilon_1})$, for $t\in[0,1]$, which belongs to Γ. It follows from Lemma 6.2 that we get the following estimate:

$$0 < c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \leq \sup_{s\geq 0} J(s(\widetilde{v}_{\varepsilon_1},\widetilde{v}_{\varepsilon_1})) < l^*.$$

LEMMA 6.4. *Suppose that $p = q$ and* (M1), (M2), (F2) *and* (F3) *hold. Let $\{(u_n,v_n)\}\subset E$ be a sequence such that*

$$J(u_n,v_n)\to c_\lambda \quad \textit{and} \quad J'(u_n,v_n)\to 0, \tag{6.7}$$

as $n \to +\infty$. Then, for all $n \in \mathbb{N}$, we have

$$\|u_n\|_A^p \le t_1 \quad \textit{and} \quad \|v_n\|_A^p \le t_2.$$

Proof. Due to Remark 6.3 the proof is essentially the same as in Lemma 5.5. ☐

Proof of Theorem 1.2. It follows from Remark 6.3 that

$$c_\lambda < \left(\frac{\alpha}{\xi_1} + \frac{\gamma}{\xi_2} - 1\right)\frac{m_0}{p^*}\left(\frac{m_0}{2p^*C}\right)^{p/(p^*-p)}. \tag{6.8}$$

From Lemmas 5.1 and 5.2, there exists a bounded sequence $\{(u_n, v_n)\} \subset E$ such that $J(u_n, v_n) \to c_\lambda$ and $J'(u_n, v_n) \to 0$ in E^{-1}, as $n \to \infty$. Since (6.8) holds and $p = q$, it follows from Lemma 4.1 that, up to a subsequence, $(u_n, v_n) \to (u, v)$ strongly in E. Thus (u, v) is a weak solution to problem (3.2). Moreover, by Lemma 6.4, we conclude that (u, v) is a weak solution to problem (1.1). ☐

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