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CORRIGENDUM TO "THE SPLITTING LEMMAS FOR NONSMOOTH FUNCTIONALS ON HILBERT SPACES II. THE CASE AT INFINITY" (TOPOL. METHODS NONLINEAR ANAL. 44 (2014), 277–335)

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Abstract. We show how to correct errors in [1, § 4] caused by the incorrect inequality [1, (4.2)].

Here we only point out main corrected points and refer readers to [2, § 4] for a completely rewritten version of [1, § 4]. After removing the incorrect inequality [1, (4.2)] some corrections to the arguments in [1, § 4] should be made.

- The original (q_1^*) and (q_3^*) should be replaced by the following slightly stronger ones:

(q_1^*) There exist constants $c_1 > 0$, $r \in (0,1)$ and a function $E \in L^2(\Omega)$ such that $|q(x,t)| \le E(x) + c_1|t|^r$ for almost $x \in \Omega$ and for all $t \in \mathbb{R}$.

(q_3^*) For almost every $x \in \Omega$ the function $\mathbb{R} \ni t \mapsto q(x,t)$ is differentiable and $\Omega \times \mathbb{R} \ni (x,t) \mapsto q_t(x,t) := \frac{\partial q}{\partial t}(x,t)$ is a Carthéodory function. There exist $s \in (n/2, \infty)$, $\ell \in L^s(\Omega)$, and a bounded measurable $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(t) \to \hbar \in \mathbb{R}$ as $|t| \to \infty$ and $|q_t(x,t)| \le \ell(x)h(t)$ for almost every $x \in \Omega$ and for all $t \in \mathbb{R}$.

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By the latter, $s \in (n/2, \infty)$, and so $s/(s-1) < n/(n-2)$ for $n > 2$. Set

$$\xi(s,n) = \begin{cases} \dfrac{s}{s-1} + \dfrac{n}{n-2} & \text{if } n > 2, \\ \dfrac{3s}{s-1} & \text{if } n = 2, \end{cases} \tag{0.1}$$

and

$$\eta(s,n) = \begin{cases} \dfrac{s}{2}\dfrac{2sn-2s-n}{s^2-s-n} & \text{if } n > 2, \\ \dfrac{3s}{s-1} & \text{if } n = 2. \end{cases} \tag{0.2}$$

Note that $H = H_0^1(\Omega) \hookrightarrow L^{\xi(s,n)}(\Omega)$. Let $c(s,n,\Omega) > 0$ be the best constant such that

$$\|u\|_{L^{\xi(s,n)}} \le c(s,n,\Omega)\|\nabla u\|_{L^2} = c(s,n,\Omega)\|u\|_H \quad \text{for all } u \in H. \tag{0.3}$$

• Two lines above Proposition 4.2 of [1] should be changed into:

Since $|q_t(x,t)| \le \ell(x)h(t)$ by (q_3^*), $1/s + 1/\eta(s,n) + 2/\xi(s,n) = 1$, $\eta(s,n) > 1$, and $2s/(s-1) < \xi(s,n) < 2n/(n-2)$ for $n > 2$, using the generalized Hölder inequality and Sobolev embedding theorem, we deduce

$$\begin{aligned} \left|\int_\Omega q_t(x,u(x))v(x)w(x)\,dx\right| &\le \int_\Omega |\ell(x)|\cdot|h(u(x))|\cdot|v(x)|\cdot|w(x)|\,dx \\ &\le \|\ell\|_{L^s}\|v\|_{L^{\xi(s,n)}}\|w\|_{L^{\xi(s,n)}}\left(\int_\Omega |h(u(x))|^{\eta(s,n)}\,dx\right)^{1/\eta(s,n)} \\ &\le (c(s,n,\Omega))^2\|\ell\|_{L^s}\|v\|_H\|w\|_H\left(\int_\Omega |h(u(x))|^{\eta(s,n)}\,dx\right)^{1/\eta(s,n)} \qquad (0.4) \\ &\le (c(s,n,\Omega))^2\|\ell\|_{L^s}\|v\|_H\|w\|_H|\Omega|^{1/\eta(s,n)}\sup h \qquad (0.5) \end{aligned}$$

for any $u, v, w \in H$. It follows that $B(u) \in L_s(H)$.

• (b) of [1, Proposition 4.2] should be replaced by

(b) Under the assumption (q_3^*), J is C^2 and $J''(u) := D(\nabla J)(u) = B(u)$ for all $u \in H$. Moreover, if $a = \lambda_m$ it holds with the constant $c(s,n,\Omega)$ in (0.4) that

$$\begin{aligned} \|g''(z+u)\|_{\mathcal{L}(H)} \le\; &(c(s,n,\Omega))^2\|\ell\|_{L^s}\|h\circ(z+u) - \hbar\|_{L^{\eta(s,n)}} \\ &+ (c(s,n,\Omega))^2|\Omega|^{1/\eta(s,n)}\|\ell\|_{L^s}\hbar \end{aligned} \tag{0.6}$$

for any $z \in H^0_\infty = \mathrm{Ker}(B(\infty))$ and $u \in H^\pm_\infty := (H^0_\infty)^\perp$.

• The last two lines on [1, 325] (or the equalities [1, (4.13)]) should be removed. And [1, Claim 4.4] should be replaced by:

CLAIM 4.4. *For given numbers $\rho > 0$ and $\varepsilon > 0$ there exists $R_0 > 0$ such that*

$$\|h(z+u) - \hbar\|_{L^{\eta(s,n)}} + \hbar|\Omega|^{1/\eta(s,n)} < \varepsilon + \hbar|\Omega|^{1/\eta(s,n)}$$

for any $u \in \overline{B}_{H^{\pm}_\infty}(\theta,\rho)$ *and* $z \in H^0_\infty$ *with* $\|z\|_H \geq R_0$. *Here* $\eta(s,n)$ *is given by* (0.2).

- (c) of [1, Proposition 4.6] should be replaced by

(c) If $a = \lambda_m$, then $C_k(J,\infty;\mathbb{K}) = 0$ for all $k \notin [m^- - 1, m^+]$ provided that

$$(c(s,n,\Omega))^2|\Omega|^{1/\eta(s,n)}\|\ell\|_{L^s}\hbar < \begin{cases} \dfrac{\lambda_2-\lambda_1}{\lambda_2} & \text{for } m=1,\\ \min\left\{\dfrac{\lambda_m-\lambda_{m^--1}}{\lambda_{m^--1}}, \dfrac{\lambda_{m^++1}-\lambda_2}{\lambda_{m^++1}}\right\} & \text{for } m>1 \end{cases} \tag{0.7}$$

and

$$(c(s,n,\Omega))^2|\Omega|^{1/\eta(s,n)}\|\ell\|_{L^s}\sup h < 1. \tag{0.8}$$

- The main result in [1, § 4], Theorem 4.7, should be restated as:

THEOREM 4.7. *Suppose that assumptions* (p^*) *and* (q_1^*)–(q_3^*) *are satisfied.*

(a) *If* a_0 *is not an eigenvalue of* $-\triangle$ *then* (4.1) (*of* [1]) *has at least one nontrivial solution provided that for some* $m \in \mathbb{N}$, (0.7)–(0.8) *hold and either* $a_0 < \lambda_m < a$ *or* $a < \lambda_m < a_0$.

(b) *If* $a_0 = \lambda_m$ *is an eigenvalue but* (0.7)–(0.8) *and* (q_4^+) *hold in addition, then* (4.1) (*of* [1]) *has at least one nontrivial solution provided that either* $a < a_0$ *or* $a_0 < \lambda_k < a$ *for some* $k > m$ *and* (0.7)–(0.8) *hold with* $m = k$.

(c) *If* $a_0 = \lambda_m$ *is an eigenvalue but* (0.7)–(0.8) *and* (q_4^-) *hold in addition, then* (4.1) (*of* [1]) *has at least one nontrivial solution provided that either* $a_0 < a$ *or* $a < \lambda_k < a_0$ *for some* $k < m$ *and* (0.7) *holds with* $m = k$.

- Lemma 4.9 in [1] and its proof below [1, Lemma 4.8] should be replaced by:

LEMMA 4.9. *For* $a = \lambda_m$, *if either* $\hbar = 0$ *or* $\hbar > 0$ *and* (0.7)–(0.8) *are satisfied, then taking* $\rho_{\nabla J}$ *as any positive number* ρ *there exists* $R_1 > 0$ *such that the conditions of* [1, Corollary 1.6] *are satisfied.*

PROOF. It suffices to check that conditions (c)–(d) of [1, Corollary 1.6] are satisfied. Firstly, we claim that condition (c) holds. In fact, since $Q(\infty)v = -aKv$ by [1, (4.7)], we deduce from [1, (4.5)] and (0.5) that

$$\begin{aligned}(B(u)v - Q(\infty)v, v)_H &= (v,v)_H - \int_\Omega q_t(x,u(x))(v(x))^2\,dx\\ &\geq (v,v)_H - (c(s,n,\Omega))^2|\Omega|^{1/\eta(s,n)}\|\ell\|_{L^s}(\sup h)\cdot\|v\|_H^2\\ &\geq (1-(c(s,n,\Omega))^2|\Omega|^{1/\eta(s,n)}\|\ell\|_{L^s}\sup h)\|v\|_H^2.\end{aligned}$$

This and (0.8) lead to the desired conclusion.

Next, we prove condition (d) holds. By [1, (4.12)], $\|\nabla J(z)\|_H = o(\|z\|_H)$ as $z \in H^0_\infty$ and $\|z\|_H \to \infty$. Hence

$$M(A) = M(\nabla J) = \lim_{R\to\infty} \sup\{\|(I - P^0_\infty)\nabla J(z)\|_H : z \in H^0_\infty, \|z\|_H \ge R\} = 0.$$

By [1, Lemma 4.8] and (0.7), we may take a small $\varepsilon > 0$ such that

$$(c(s,n,\Omega))^2\|\ell\|_s(\varepsilon + \hbar|\Omega|^{1/\eta(s,n)}) < 1/C^\infty_1.$$

For this $\varepsilon > 0$ and a given number $\rho > 0$, by Claim 4.4, there exists $R_0 > 0$ such that

$$\|h(z+u) - \hbar\|_{L^{\eta(s,n)}} + \hbar|\Omega|^{1/\eta(s,n)} < \varepsilon + \hbar|\Omega|^{1/\eta(s,n)}$$

for any $u \in \overline{B}_{H^\pm_\infty}(\theta,\rho)$ and $z \in H^0_\infty$ with $\|z\|_H \ge R_0$. These and (0.6) lead to

$$\begin{aligned}\|(I - P^0_\infty)[B(z+u) - B(\infty)]|_{H^\pm_\infty}\|_{\mathcal{L}(H^\pm_\infty)} &\le \|B(z+u) - B(\infty)\|_{L(H)} \\ &= \|g''(z+u)\|_{\mathcal{L}(H)} \le (c(s,n,\Omega))^2\|\ell\|_s\left(\varepsilon + \hbar|\Omega|^{\frac{1}{\eta(s,n)}}\right) < \frac{1}{\kappa C^\infty_1}\end{aligned}$$

for any $u \in \bar{B}_{H^\pm_\infty}(\theta,\rho)$ and $z \in H^0_\infty$ with $\|z\|_H \ge R_0$, and for some $\kappa > 1$. □

Finally, we provide the correct proof of [1, Proposition 4.2 (a)]. Other proofs should be corrected in a similar manner, see [2, § 4] for details.

- The proof of [1, Proposition 4.2 (a)] should be changed as follows:

It suffices to prove that the functional g in [1, (4.10)] is C^1. By (q^*_1),

$$|q(x,t_1+t_2)| \le E(x) + c_1(1+|t_1|+|t_2|)^r \le E(x) + c_1 + c_1|t_1| + c_1|t_2|$$

for almost every $x \in \Omega$ and any $t_1, t_2 \in \mathbb{R}$, and so

$$\begin{aligned}(0.9)\quad |Q(x,u(x)+v(x)) - Q(x,u(x))| &\le \sup_{\tau\in[0,1]} |q(x,u(x)+\tau v(x))|\cdot|v(x)| \\ &\le (E(x) + c_1 + c_1|u(x)| + c_1|v(x)|)\cdot|v(x)|\end{aligned}$$

for all $u, v \in H$,

$$\begin{aligned}(0.10)\qquad |g(u+v) - g(u)| &\le \int_\Omega \big(E(x) + c_1 + c_1|u(x)| + c_1|v(x)|\big)\cdot|v(x)|\,dx \\ &\le (\|E\|_{L^2} + c_1|\Omega|^{1/2} + c_1\|u\|_{L^2} + c_1\|v\|_{L^2})\|v\|_{L^2}.\end{aligned}$$

As $H \hookrightarrow L^2(\Omega)$, g is continuous. We also need to prove that g has a bounded linear Gâteaux derivative $Dg(u)$ at every point $u \in H$ and that $H \ni u \mapsto Dg(u) \in H^*$ is continuous. For $u, v \in H = H^1_0(\Omega)$, $\tau \in (-1,1)\setminus\{0\}$ and almost every $x \in \Omega$ we get

$$\left|\frac{Q(x,u(x)+\tau v(x)) - Q(x,u(x))}{\tau}\right| \le \big(E(x) + c_1 + c_1|u(x)| + c_1|v(x)|\big)\cdot|v(x)|$$

by (0.9). From this and the Lebesgue dominated convergence theorem, we derive

$$Dg(u)[v] = \frac{d}{d\tau}\bigg|_{\tau=0} g(u+\tau v) = -\int_\Omega q(x,u(x))\cdot v(x)\,dx.$$

That is, g is Gâteaux differentiable. Clearly, $Dg(u) \in H^*$ since we have as above

$$|Dg(u)[v]| = \bigg| \int_\Omega q(x, u(x)) \cdot v(x)\, dx \bigg| \le (\|E\|_{L^2} + c_1|\Omega|^{1/2} + c_1\|u\|_{L^2})\|v\|_{L^2}.$$

Let $u_1, u_2, v \in H$. By (q_3^*), the functions $\mathbb{R} \ni t \mapsto q(x,t)$ and $\mathbb{R} \ni t \mapsto q_t(x,t)$ are continuous for almost every $x \in \Omega$. The calculus fundamental theorem and (0.5) lead to

$$\begin{aligned}
&\bigg| \int_\Omega [q(x, u_2(x)) - q(x, u_1(x))] \cdot v(x)\, dx \bigg| \\
&= \bigg| \int_0^1 \bigg[\int_\Omega q_t(x, u_1(x) + \tau(u_2(x) - u_1(x)))(u_2(x) - u_1(x)) \cdot v(x)\, dx \bigg] d\tau \bigg| \\
&\le \int_0^1 \bigg[\int_\Omega |q_t(x, u_1(x) + \tau(u_2(x) - u_1(x)))| \cdot |u_2(x) - u_1(x)| \cdot |v(x)|\, dx \bigg] d\tau \\
&\le (c(n,s,\Omega))^2 \|\ell\|_{L^s} |\Omega|^{1/\eta(s,n)} (\sup h) \cdot \|u_2 - u_1\|_H \|v\|_H
\end{aligned}$$

and hence

$$\|Dg(u_1) - Dg(u_2)\|_{H^*} \le (c(n,s,\Omega))^2 \|\ell\|_{L^s} |\Omega|^{1/\eta(s,n)} (\sup h) \cdot \|u_2 - u_1\|_H.$$

It follows that J is $C^{1,1}$.

The expression of ∇J is clear. It remains to prove [1, (4.10)]. Since $|Q(x,t)| \le |t|E(x) + c_1|t|^{r+1}$ by (q_1^*), $r \in (0,1)$ and $H \hookrightarrow L^{r+1}$, for some constant $C_r > 0$ we have

$$\begin{aligned}
|g(u)| &\le \int_\Omega |Q(x, u(x))|\, dx \le \int_\Omega (E(x)|u(x)| + c_1|u(x)|^{r+1})\, dx \\
&\le \|E\|_{L^2}\|u\|_{L^2} + c_1\|u\|_{L^{r+1}}^{r+1} \le C_r(\|E\|_{L^2}\|u\|_H + c_1\|u\|_H^{r+1})
\end{aligned}$$

for all $u \in H$.

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