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PULLBACK ATTRACTORS FOR A NON-AUTONOMOUS SEMILINEAR DEGENERATE PARABOLIC EQUATION

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Abstract. In this paper, we consider the pullback attractors for a non-autonomous semilinear degenerate parabolic equation $u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) = g(x,t)$ defined on a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We provide that the usual $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor indeed can attract the $\mathscr{D}_\lambda$-class in $L^{2+\delta}(\Omega)$, where $\delta \in [0,\infty)$ can be arbitrary.

1. Introduction

In this paper, we consider the following non-autonomous degenerate parabolic equation:

$$
\begin{cases}
u_t - \operatorname{div}(\sigma(x)\nabla u) + f(u) = g(x,t) & \text{in } \Omega \times (\tau, +\infty),\\
u = 0 & \text{on } \partial\Omega \times (\tau, +\infty),\\
u|_{t=\tau} = u_\tau \in L^2(\Omega),
\end{cases}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, the diffusion coefficient σ, the nonlinearity f and the external force g satisfying the following conditions:

(C1) $\sigma(x)$ is a non-negative measurable function such that $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$.

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(C2) The function $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies, for any $s \in \mathbb{R}$,

$$\alpha_1|s|^p - \alpha_2 \le f(s)s \le \alpha_3|s|^p + \alpha_4, \quad p \ge 2, \tag{1.2}$$

$$f(0) = 0, \quad f^{'}(s) \ge -l, \tag{1.3}$$

where α_i, $i = 1,2,3,4$ are positive constants.

(C3) $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\int_{-\infty}^{0} e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds < +\infty, \tag{1.4}$$

where $\lambda > 0$ is the first eigenvalue of the operator $-\mathrm{div}(\sigma(x)\nabla\,\cdot\,)$ in Ω with the homogeneous Dirichlet boundary condition.

Assumption (C1) indicates that the function $\sigma(\,\cdot\,)$ may be extremely irregular, for example, $\sigma(\,\cdot\,)$ could be non-smooth, such as $\sigma(x) = |x-z|^\alpha$ for $\alpha \in (0,2)$ and every $z \in \overline{\Omega}$. The physical motivation of assumption on the diffusion variable $\sigma(\,\cdot\,)$ is to model the "perfect insulator" or "perfect conductor" of the media somewhere, see [1], [2], [4], [9], [10] for detailed discussions.

For equation (1.1) with degeneracy, the existence and uniqueness of solutions have been studied extensively, see for example, [4], [5], [14], [15] for the elliptic case and [8], [15], [18] for the parabolic problem.

The main purpose of this paper is to consider the dynamics of the dissipative dynamical systems, using the so-called pullback attractor ([6], [7], [11]), generated by the weak solutions of (1.1).

Before we continue with the setting of the problem, let us introduce a notation that will be used in the sequel.

Let R_λ be the set of all functions $\rho\colon \mathbb{R} \to [0,\infty)$ such that

$$e^{\lambda\tau}\rho^2(\tau) \to 0 \quad \text{as } \tau \to -\infty,$$

where $\lambda > 0$ is the first eigenvalue of the operator $-\mathrm{div}(\sigma(x)\nabla\,\cdot\,)$ in Ω with the homogeneous Dirichlet boundary condition; and the attraction universe

$$\begin{aligned} &\mathscr{D}_\lambda \text{ be the class of all families } \widehat{D} = \{D(t) : t \in \mathbb{R},\ D(t) \subset L^2(\Omega)\}, \\ &\text{such that } D(t) \subset \{u \in L^2(\Omega) : \|u\|_{L^2(\Omega)} \le \rho_{\widehat{D}}(t)\} \text{ for some } \rho_{\widehat{D}} \in R_\lambda. \end{aligned} \tag{1.5}$$

Under assumptions (C1)–(C3), the existence of a pullback $\mathscr{D}_\lambda$-attractor as well as analysis of its properties in the phase space $L^2(\Omega)$ for problem (1.1) has been studied extensively. Let us recall some typical results among them.

In [1], Anh and Bao proved that under assumptions (C1)–(C3), there exists an $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor for the process generated by the weak solutions of (1.1), and then, they also proved that such attractor can attract in $\mathscr{D}^1_0(\Omega,\sigma) \cap L^p(\Omega)$-norm (where the power p comes from (1.2)) if

$g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ satisfies

$$\int_{-\infty}^{0} e^{\lambda s}(\|g(s)\|^2_{L^2(\Omega)} + \|g'(s)\|^2_{L^2(\Omega)})ds < +\infty. \tag{1.6}$$

Furthermore, the authors in [2] show that the pullback attractor obtained in [1] is in fact an $(L^2(\Omega),\ \mathscr{D}^2_0(\Omega,\sigma) \cap L^{2p-2}(\Omega))$-pullback attractor if g satisfies the additional condition:

$$\int_{-\infty}^{0} e^{\lambda s}\|g^{'}(s)\|^{m'_k}_{L^{m_k}(\Omega)}\, ds < \infty,$$

where $m_k = 2\beta^{k+2}/(2\beta^{k+2} + 1 - 2\beta^{k+1})$ and $m'_k = 2\beta^{k+1}$ with $\beta = N/(N-2+\alpha)$ and $k \in \mathbb{N}$ satisfies $0 \le k \le \log_\beta(p-1)+1$. Note that, in the previously mentioned two papers [1], [2], the authors used essentially the method of differentiating the equation with respect to time t to get some estimates about higher-order integrability of u_t, and then obtained the higher-order integrability of $u(t)$ to obtain the attraction in $L^p(\Omega)$ and $L^{2p-2}(\Omega)$.

Recently, for a stochastic version of equation (1.1), under the same assumptions (C1)–(C2), Yang and Kloeden in [19] proved the existence of a random attractor in $L^2(\Omega)$, and then Zhao in [20] proved that the random attractor obtained in [19] indeed can attract in $\mathscr{D}^1_0(\Omega,\sigma) \cap L^p(\Omega)$-norm, where the power p is the same as in (1.2). On the other hand, for an autonomous equation (1.1) with the same assumptions (1.2)–(1.3) on nonlinearity, but with a different assumption on the degenerate function σ, Li, Ma and Zhong in [12] established the existence of a global attractor in $L^2(\Omega)$, and the attraction can be the $\mathscr{D}^1_0(\Omega,\sigma) \cap L^p(\Omega)$-norm too, where the power p is also the same as in (1.2).

In this paper, we will extend the known results, without any additional conditions except (C1)–(C3), by showing that the known $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor indeed can attract the $\mathscr{D}_\lambda$-class in $L^{2+\delta}(\Omega)$-norm for any $\delta \in [0,\infty)$. That is,

THEOREM 1.1. *Under assumptions* (C1)–(C3)*, let $U(t,\tau)$ be the process generated by the weak solutions of* (1.1) *and $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ be the $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor. Then, for any $\delta \in [0,\infty)$ and any $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$, the following properties hold*:

(a) *$\widehat{\mathscr{A}}$ can attract the $\mathscr{D}_\lambda$-class in $L^{2+\delta}$-norm, that is,*

$$\lim_{\tau\to-\infty} \mathrm{dist}_{L^{2+\delta}}(U(t,\tau)D(\tau), \mathscr{A}(t)) = 0 \quad \textit{for all } t \in \mathbb{R}; \tag{1.7}$$

(b) *for any complete trajectory $v(t) \in \mathscr{A}(t)$ $(t \in \mathbb{R})$ of $U(t,\tau)$, there exist two sequences $T(t,\delta,\widehat{D},\widehat{\mathscr{A}})$ and $M_\delta(t)$ (which depend only on t,δ,N, $\int_{-\infty}^{t} e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\, ds$ and the L^2-size of $\mathscr{A}(t)$), such that*

$$\int_\Omega |U(t,\tau)u_\tau - v(t)|^{2+\delta}\, dx \le M_\delta(t) \quad \textit{for any } t-\tau \ge T(t,\delta,\widehat{D},\widehat{\mathscr{A}}). \tag{1.8}$$

Remark 1.2. Theorem 1.1 (a) implies immediately both the $(L^2(\Omega), L^p(\Omega))$-pullback attractor obtained in [1] and $(L^2(\Omega), L^{2p-2}(\Omega))$-pullback attractor obtained in [2]. Theorem 1.1 (b) implies that $\mathscr{A}(t)-v(t)$ is bounded in $L^{2+\delta}(\Omega)$ for any $\delta \in [0,\infty)$ despite the fact that we do not know whether $\mathscr{A}(t)$ is bounded in $L^{2+\delta}(\Omega)$ or not.

Remark 1.3. Note that our external forcing term g satisfies only the L^2-integrability (C3), thus we cannot obtain the L^∞-estimate for the solution; at the same time, from (1.4), even for the non-degenerate equation (e.g. as that in [13]), the solution of (1.1) at most belongs to $W^{2,2p-2}(\Omega)$, however, since our spatial dimension $N \geq 3$ is arbitrary, in general we do not have $W^{2,2p-2}(\Omega) \hookrightarrow L^\infty(\Omega)$. Therefore, the $L^{2+\delta}(\Omega)$-attraction for any $\delta \in [0,\infty)$ is not trivial, and seems un-expectable to some extent.

The rest of this article consists of three sections. In Section 2, we recall the main concepts and results about function spaces, solutions of (1.1) and abstract results about pullback attractors we will use in this paper. In Section 3, we recall and give an outline of proof about the existence of an $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor. Finally, we prove the main result, Theorem 1.1, in Section 4.

2. Notation and abstract results

In this section, we will firstly introduce the function spaces and recall the existence and uniqueness of the solution to equation (1.1), and then, we will recall some abstract results related to pullback attractors which will be used later.

2.1. Weighted Sobolev spaces. We recall some properties of weighted Sobolev spaces with weight function $\sigma(x)$.

Definition 2.1. Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ and $\sigma(x)$ is a non-negative measurable function such that $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,2)$, $\liminf\limits_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$. The Hilbert space $\mathscr{D}^1_0(\Omega,\sigma)$ is defined as the closure of $C^\infty_0(\Omega)$ with the norm

$$\|u\|_{\mathscr{D}^1_0(\Omega,\sigma)} := \bigg(\int_\Omega \sigma(x)|\nabla u|^2\,dx\bigg)^{1/2},$$

and the product

$$\langle u,v\rangle_{\mathscr{D}^1_0(\Omega,\sigma)} := \int_\Omega \sigma(x)\nabla u \nabla v\,dx.$$

The following lemma refers to the continuous and compact inclusions of $\mathscr{D}^1_0(\Omega,\sigma)$.

Lemma 2.2 ([2], [4], [5]). *Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, and $\sigma(x)$ satisfies* (C1). *Then the following embeddings hold*:

(a) $\mathscr{D}_0^1(\Omega,\sigma)$ *is continuously embedded into* $L^{2^*_\alpha}(\Omega)$;
(b) $\mathscr{D}_0^1(\Omega,\sigma)$ *is compactly embedded into* $L^p(\Omega)$ *as* $p\in[1,2^*_\alpha)$, *where* $2^*_\alpha = 2N/(N-2+\alpha)$ *and* α *comes from* (1.1).

Remark 2.3. The exponent 2^*_α plays the role of the critical exponent in the classical Sobolev embedding. $2^*_\alpha>2$ when $\alpha\in(0,2)$.

Under condition (C1), the operator $\mathbf{A}:=-\mathrm{div}(\sigma(x)\nabla\,\cdot\,)$ is positive and self-adjoint with the domain of definition,

$$\mathrm{Dom}(\mathbf{A})=\{u\in\mathscr{D}_0^1(\Omega,\sigma):\mathbf{A}u\in L^2(\Omega)\}.$$

The space $\mathrm{Dom}(\mathbf{A})$ is a Hilbert space endowed with the usual scalar product. Moreover, there exists a complete orthonormal system of eigenvectors $(e_j,\lambda_j)_{j\in\mathbb{N}}$ such that

$$(e_j,\lambda_j)=\delta_{ij}\quad\text{and}\quad -\mathrm{div}(\sigma(x)\nabla e_j)=\lambda_j e_j,\quad i,j=1,2,\ldots,$$
$$0<\lambda_1<\lambda_2\le\ldots,\quad \lambda_j\to+\infty,\quad j\to+\infty.$$

Noting that

$$\lambda=\lambda_1=\inf\left\{\frac{\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}}{\|u\|^2_{L^2(\Omega)}}:u\in\mathscr{D}_0^1(\Omega,\sigma),\ u\neq0\right\},\tag{2.1}$$

we have

$$\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\ge\lambda\|u\|^2_{L^2(\Omega)}\quad\text{for all } u\in\mathscr{D}_0^1(\Omega,\sigma).\tag{2.2}$$

Lemma 2.4 ([2]). *Assume that* Ω *is a bounded domain in* $\mathbb{R}^N$ $(N\ge2)$, *and* $\sigma(x)$ *satisfies* (C1), *then the following estimate holds*:

$$\lambda\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\le\int_\Omega|\mathbf{A}u|^2\,dx,$$

where λ *is the positive constant given in* (2.1).

2.2. Solutions for equation (1.1)**.** For the readers' convenience, in this subsection we will recall the definitions of solutions of equation (1.1), see [1], [2], [12] for more details.

Definition 2.5 (Weak solution). A function $u(x,t)$ is called a weak solution of (1.1) on $[\tau,T]$ if and only if

$$u\in C([\tau,T];L^2(\Omega))\cap L^2(\tau,T;\mathscr{D}_0^1(\Omega,\sigma))\cap L^p(\tau,T;L^p(\Omega))$$

and $u|_{t=\tau}=u_\tau$ almost everywhere in Ω such that

$$\int_\tau^T\int_\Omega\left(\frac{\partial u}{\partial t}\phi+\sigma\nabla u\nabla\varphi+f(u)\varphi\right)dx\,dt=\int_\tau^T\int_\Omega g\varphi\,dx\,dt$$

holds for all test functions $\varphi\in L^2(\tau,T;\mathscr{D}_0^1(\Omega,\sigma))\cap L^p(\tau,T;L^p(\Omega))$.

Definition 2.6 (Strong solution). A function $u(x,t)$ is called a strong solution of (1.1) on $[\tau,T]$ if and only if

$$u \in C([\tau,T];\mathscr{D}_0^1(\Omega,\sigma)) \cap L^2(\tau,T;\mathrm{Dom}(\mathbf{A})) \cap L^\infty(\tau,T;L^p(\Omega))$$

and the three equations in (1.1) are satisfied almost everywhere in their corresponding domains.

The following two lemmas refer to the existence and uniqueness of the global weak solution and strong solution for the degenerate parabolic equation (1.1), which can be obtained by the Faedo–Galerkin method (see [8], [15], [17], [18]). Here we only state them as follows.

Lemma 2.7. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\sigma(x)$ satisfies* (C1), *$g \in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ and f satisfies* (C2). *Then for any initial data $u_\tau \in L^2(\Omega)$ and any $T > \tau$, there exists a unique weak solution u of equation* (1.1) *which satisfies*

$$u \in C([\tau,T];L^2(\Omega)) \cap L^2(\tau,T;\mathscr{D}_0^1(\Omega,\sigma)) \cap L^p(\tau,T;L^p(\Omega)),$$

and the mapping $u_\tau \to u(t)$ is continuous in $L^2(\Omega)$.

Lemma 2.8. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\sigma(x)$ satisfies* (C1), *$g \in L^2_{\mathrm{loc}}(\mathbb{R};L^2(\Omega))$ and f satisfies* (C2). *If $u_\tau \in \mathscr{D}_0^1(\Omega,\sigma) \cap L^p(\Omega)$ then there exists a unique strong solution u of equation* (1.1) *which satisfies*

$$u \in C([\tau,T];\mathscr{D}_0^1(\Omega,\sigma)) \cap L^2(\tau,T;\mathrm{Dom}(\mathbf{A})) \cap L^\infty(\tau,T;L^p(\Omega)).$$

2.3. Abstract results on pullback attractors. In this subsection, we recall some results about the existence of pullback attractors and their properties, see [6], [7], [11] for more details.

Let us consider a process U on a metric space X, i.e. a family $\{U(t,\tau) : -\infty < \tau < +\infty\}$ of continuous mappings $U(t,\tau) : X \to X$, such that

(a) $U(\tau,\tau) = \mathrm{Id}$;
(b) $U(t,\tau) = U(t,r)U(r,\tau)$ for all $\tau \le r \le t$.

Let $\mathscr{P}(X)$ denote all bounded sets of X, $\mathscr{D}_\lambda$ be a class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathscr{P}(X)$.

Definition 2.9. A family $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ is said to be a pullback $\mathscr{D}_\lambda$-attractor for the process $\{U(t,\tau)\}$ if

(a) $\mathscr{A}(t)$ is compact in X for all $t \in \mathbb{R}$;
(b) $\widehat{\mathscr{A}}$ is $\mathscr{D}_\lambda$-pullback attracting in X, i.e.

$$\lim_{\tau\to-\infty} \mathrm{dist}_X(U(t,\tau)D(\tau),\mathscr{A}(t)) = 0, \quad \text{for all } D \in \mathscr{D}_\lambda \text{ and all } t \in \mathbb{R};$$

(c) $\widehat{\mathscr{A}}$ is invariant, i.e.

$$U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t), \quad \text{for all } \infty < \tau < t < +\infty.$$

We call $\mathscr{A}$ minimal if for any family $\mathscr{C} = \{C(t) : t \in \mathbb{R}\} \subset \mathscr{D}_\lambda$ of closed sets such that $\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), C(t)) = 0$, we have $\mathscr{A}(t) \subset C(t)$.

Denote by $\mathscr{K}$ the collection of all complete trajectories of $U(t,\tau)$, that is,

$$\mathscr{K} := \{\widehat{u} = \{u(t) : t \in \mathbb{R}\} : U(t,\tau)u(\tau) = u(t) \text{ for any } \infty < \tau \le t < \infty\}.$$

The following result gives conclusions about the construction of attractors (see e.g. [11]).

Lemma 2.10. *Let $U(t,\tau)$ be a process on Banach space X, and $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$ be the pullback $\mathscr{D}_\lambda$-attractor of $U(t,\tau)$. Then, for any $t \in \mathbb{R}$,*

$$\mathscr{A}(t) = \bigcup_{\widehat{u}\in\mathscr{K}\cap\mathscr{D}_\lambda} u(t),$$

consequently, there exists at least one complete trajectory $\widehat{v}$ of $U(t,\tau)$ which satisfies $\widehat{v} \in \mathscr{D}_\lambda$.

We also need the following abstract result given in [16] to get the higher-order attraction.

Lemma 2.11. *Let $Z \hookrightarrow Y \hookrightarrow X$ be the three Banach spaces with continuous embeddings, $U(\,\cdot\,,\,\cdot\,)$ be a process defined on X and $\mathscr{D}_\lambda$ be a class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathscr{P}(X)$. Moreover, assume that*

(a) *$U(\,\cdot\,,\,\cdot\,)$ has a pullback $\mathscr{D}_\lambda$-attractor $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ in X, and $\widehat{\mathscr{A}} \in \mathscr{D}_\lambda$;*
(b) *$\widehat{v} = \{v(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$ is a complete trajectory of $U(t,\tau)$;*
(c) *$W(t,\tau)$ $(-\infty < \tau \le t < \infty)$ are a family of operators defined on X satisfying*

$$U(t,\tau)\,\cdot\, = v(t) + W(t,\tau)(\,\cdot\, - v(\tau)) \quad \text{for all } \tau \le t;$$

(d) *there exists $\widehat{\mathscr{B}_0} = \{B_0(t) : t \in \mathbb{R}\}$ with $B_0(t)$ is bounded in Z for each $t \in \mathbb{R}$, satisfying that for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathscr{D}_\lambda$, there exists $\tau_0 = \tau_0(t,\widehat{D}) \le t$ such that*

$$W(t,\tau)(D(\tau) - v(\tau)) \subset B_0(t) \quad \text{for all } \tau \le \tau_0. \tag{2.3}$$

Then, the following hold:

(i) *$\widehat{\mathscr{B}} = \{v(t)\}_{t\in\mathbb{R}} + \widehat{\mathscr{B}}_0 := \{B(t) = v(t) + B_0(t) : t \in \mathbb{R}\}$ is a $\mathscr{D}_\lambda$-absorbing set in X for the process $U(\,\cdot\,,\,\cdot\,)$;*
(ii) *$\operatorname{dist}_X(\widehat{\mathscr{A}}, \widehat{\mathscr{B}}) = 0$, i.e.*

$$\operatorname{dist}_X(\mathscr{A}, v(t) + B_0(t)) = \operatorname{dist}_X(\mathscr{A} - v(t), B_0(t)) = 0 \quad \text{for all } t \in \mathbb{R}; \tag{2.4}$$

(iii) *if* $B_0(t)$ *is closed in* X *for all* $t \in \mathbb{R}$, *then*

$$\mathscr{A}(t) - v(t) \subset B_0(t) \quad \text{for all } t \in \mathbb{R}; \tag{2.5}$$

further, we assume that the space Y *satisfies* $\|\cdot\|_Y \leq C\|\cdot\|_X^{\theta}\|\cdot\|_Z^{1-\theta}$ *for some* $\theta \in (0,1]$ *and constant* C, *then for any* $\widehat{D} \in \mathscr{D}_\lambda$ *and any* $t \in \mathbb{R}$,

$$\operatorname{dist}_Y(U(t,\tau)D(\tau), \mathscr{A}(t)) \to 0 \quad \text{as } \tau \to -\infty. \tag{2.6}$$

In the sequel we shall need the following lemma belonging to the family of Gronwall type lemmas, see [13].

LEMMA 2.12. *Let for some* $\lambda > 0$, $\tau \in \mathbb{R}$ *and, for* $s > \tau$,

$$y'(s) + \lambda y(s) \leq h(s),$$

where the functions y, y', h *are assumed to be locally integrable and* y, h *nonnegative on the interval* $t < s < t + r$, *for some* $t \geq \tau$. *Then*

$$y(t+r) \leq e^{-\lambda r/2}\frac{2}{r}\int_t^{t+r/2} y(s)\,ds + e^{-\lambda(t+r)}\int_t^{t+r} e^{\lambda s}h(s)\,ds.$$

3. The existence of pullback attractor in $L^2(\Omega)$

Thanks to Lemma 2.7, under assumptions (C1), (C2), $g \in L^2_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$, we can define the bi-parametric family of maps

$$U(t,\tau) : L^2(\Omega) \mapsto L^2(\Omega), \text{ with } \tau \leqslant t, \text{ given by } U(t,\tau)u_\tau = u(t), \tag{3.1}$$

where $u(t) = u(t;\tau,u_\tau)$ is the unique weak solution of problem (1.1), which forms a process on $L^2(\Omega)$.

We start with the existence of pullback attractors in $L^2(\Omega)$. The following result about the existence of pullback attractors in $L^2(\Omega)$ can be deduced directly from the arguments given in [1, 19], here we recall it, and for the later application, we give an outline of its proof.

LEMMA 3.1. *Under assumptions* (C1)–(C3), *the process* $U(t,\tau)$ *generated by the weak solution of* (1.1) *has an* $(L^2(\Omega), L^2(\Omega))$ *pullback* $\mathscr{D}_\lambda$*-attractor* $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$, *that is,* $\widehat{\mathscr{A}} \in \mathscr{D}_\lambda$ *and satisfying*:

(a) $\mathscr{A}(t)$ *is compact in* $L^2(\Omega)$ *for all* $t \in \mathbb{R}$;
(b) $\widehat{\mathscr{A}(t)}$ *is* $L^2(\Omega)$*-pullback* $\mathscr{D}_\lambda$*-attracting, that is, for any* $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$,

$$\lim_{\tau \to -\infty} \operatorname{dist}_{L^2(\Omega)}(U(t,\tau)D(\tau), \mathscr{A}(t)) = 0 \quad \text{for all } t \in \mathbb{R}; \tag{3.2}$$

(c) $\widehat{\mathscr{A}}$ *is invariant, i.e.*

$$U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t) \quad \text{for any } \infty < \tau \leq t < \infty. \tag{3.3}$$

Proof. According to the general criterion (e.g. see [6], [7], [11]) about the existence of pullback attractor, it is sufficient to prove the below two claims. Moreover, since the estimates are standard, we only present it by some formal arguments, which can be justified by means of the approximation procedure as that for the existence of solutions (e.g. see [8], [12], [18]).

Claim 1. There exists a pullback $\mathscr{D}_\lambda$-absorbing set in $L^2(\Omega)$.

Let $u_\tau \in L^2(\Omega)$ and $u(t) = U(t,\tau)u_\tau$. Multiplying the first equation in (1.1) by u and integrating over Ω, we deduce that

$$\begin{aligned}(3.4)\quad \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\mathscr{D}^1_0(\Omega,\sigma)} + \alpha_1\|u\|^p_{L^p(\Omega)}& \\ \le \frac{2}{\lambda}\|g(t)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} + \alpha_2|\Omega|,&\end{aligned}$$

where we have used (1.2) and Young's inequality. Applying the inequality $\|u\|^2_{\mathscr{D}^1_0(\Omega,\sigma)} \ge \lambda\|u\|^2_{L^2(\Omega)}$, we get

$$(3.5)\qquad \frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|u\|^2_{\mathscr{D}^1_0(\Omega,\sigma)} + 2\alpha_1\|u\|^p_{L^p(\Omega)} \le \frac{4}{\lambda}\|g(t)\|^2_{L^2(\Omega)} + 2\alpha_2|\Omega|$$

and

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} + \alpha_1\|u\|^p_{L^p(\Omega)} \le \frac{2}{\lambda}\|g(t)\|^2_{L^2(\Omega)} + \alpha_2|\Omega|.$$

Using the Gronwall inequality, then we have that

$$\begin{aligned}(3.6)\quad \|u(t)\|^2_{L^2(\Omega)} \le e^{-\lambda(t-\tau)}\|u(\tau)\|^2_{L^2(\Omega)}& \\ + 2\alpha_2|\Omega|\lambda^{-1} + 4\lambda^{-1}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds.&\end{aligned}$$

For each $t \in \mathbb{R}$, we define $R(t)$ as a positive number given by

$$R^2(t) = 1 + 2\alpha_2|\Omega|\lambda^{-1} + 4\lambda^{-1}e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds,$$

then, from (1.4) we know that $R(t) < \infty$ for every $t \in \mathbb{R}$.

Define $\widehat{D_0} := \{D(\tau) : D(\tau) := \{v \in L^2(\Omega) \mid \|v\|_{L^2(\Omega)} \le R(\tau)\},\ \tau \in \mathbb{R}\}$, then, according to the definition of $\mathscr{D}_\lambda$ given in (1.5), we know that $\widehat{D_0}$ is a pullback $\mathscr{D}_\lambda$-absorbing set for the process $U(t,\tau)$, that is, for any $\widehat{D} = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathscr{D}_\lambda$ and any $t \in \mathbb{R}$, there exists a $\widehat{T} = T(t,\widehat{D}) > 0$ satisfying

$$\begin{aligned}(3.7)\quad U(t,\tau)D(\tau) \subset \widehat{D_0} = \{u \in L^2(\Omega) : \|u(t)\|_{L^2(\Omega)} \le R(t)\}& \\ \text{for all } \tau \in \mathbb{R} \text{ satisfies } t-\tau \le \widehat{T}.&\end{aligned}$$

Claim 2. The process $U(t,\tau)$ defined in (3.1) is $\mathscr{D}_\lambda$-pullback asymptotic compact in $L^2(\Omega)$.

We will prove that there exists a family $\widehat{B}$ of pullback $\mathscr{D}_\lambda$-absorbing sets which are bounded in $D_0^1(\Omega,\sigma)$, from which the pullback asymptotical compactness in $L^2(\Omega)$ follows immediately by the compact embedding (see Lemma 2.2).

Firstly, we multiply the first equation in (1.1) by $-\mathrm{div}(\sigma(x)\nabla u)$ and integrate over Ω to deduce that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}+\int_\Omega|\mathbf{A}u_n|^2\,dx+(f(u),-\mathrm{div}(\sigma(x)\nabla u))=(g(t),-\mathrm{div}(\sigma(x)\nabla u)).$$

Applying (1.3) and the embedding $\lambda\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\le\int_\Omega|\mathbf{A}u|^2\,dx$, we get

$$\frac{d}{dt}\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}+\lambda\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\le 2l\|u\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}+\|g(t)\|^2_{L^2(\Omega)},$$

then apply Lemma 2.12 and the above inequality with $r=1$ to get

$$\begin{aligned}\|u(t+1)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}&\le 2e^{-\lambda/2}\int_t^{t+1/2}\|u(s)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\,ds\\&\quad+e^{-\lambda(t+1)}\int_t^{t+1}e^{\lambda s}\big(2l\|u(s)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}+\|g(s)\|^2_{L^2(\Omega)}\big)\,ds\end{aligned}\tag{3.8}$$

for every $t\ge\tau$.

Now, we estimate the right-hand side in terms of the data using the energy inequality (3.5). Integrating (3.5) from t to $t+1/2$ and using (3.6), we get

$$\begin{aligned}\int_t^{t+1/2}&\|u(s)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\,ds\\&\le\|u(t)\|^2_{L^2(\Omega)}+\alpha_2|\Omega|+4\lambda^{-1}\int_t^{t+1/2}\|g(s)\|^2_{L^2(\Omega)}\,ds\\&\le\|u(t)\|^2_{L^2(\Omega)}+\alpha_2|\Omega|+4\lambda^{-1}e^{-\lambda t}\int_t^{t+1}e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds\\&\le e^{-\lambda(t-\tau)}\|u(\tau)\|^2_{L^2(\Omega)}+\alpha_2(2\lambda^{-1}+1)|\Omega|\\&\quad+4\lambda^{-1}e^{-\lambda t}\int_{-\infty}^{t+1}e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds.\end{aligned}\tag{3.9}$$

Integrating (3.5) from t to $t+1$ and using (3.6), we get

$$\begin{aligned}e^{-\lambda(t+1)}\int_t^{t+1}e^{\lambda s}\|u(s)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\,ds&\le\int_t^{t+1}\|u(s)\|^2_{\mathscr{D}_0^1(\Omega,\sigma)}\,ds\\&\le\|u(t)\|^2_{L^2(\Omega)}+2\alpha_2|\Omega|+4\lambda^{-1}\int_t^{t+1}\|g(s)\|^2_{L^2(\Omega)}\,ds\\&\le\|u(t)\|^2_{L^2(\Omega)}+2\alpha_2|\Omega|+4\lambda^{-1}e^{-\lambda t}\int_t^{t+1}e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds\\&\le e^{-\lambda(t-\tau)}\|u(\tau)\|^2_{L^2(\Omega)}+2\alpha_2(\lambda^{-1}+1)|\Omega|\\&\quad+4\lambda^{-1}e^{-\lambda t}\int_{-\infty}^{t+1}e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds.\end{aligned}\tag{3.10}$$

Applying (3.9) and (3.10) to (3.8), we conclude that

$$\|u(t+1)\|^2_{\mathscr{D}^1_0(\Omega,\sigma)} \le \widetilde{C}\bigg(1 + e^{-\lambda t}\int_{-\infty}^{t+1} e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds\bigg),$$

uniformly with respect to all initial conditions $u(\tau) \in D(\tau)$ for all $\tau \in \mathbb{R}$ satisfying $t - \tau \le \widehat{T}$ (where $\widehat{T}$ comes from (3.7) which corresponds to $\widehat{D}$), with $\widetilde{C} = \widetilde{C}(|\Omega|, \alpha_2, \lambda, l)$. This proves the existence of the $\mathscr{D}_\lambda$-absorbing sets in $D^1_0(\Omega,\sigma)$. $\square$

4. The $(L^2(\Omega), L^{2+\delta}(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor

In this section, our main propose is to obtain the existence of the pullback $\mathscr{D}_\lambda$-attractor $(L^2(\Omega), L^{2+\delta}(\Omega))$. So we obtain a maximum principle about the strong solutions and $L^{2+\delta}$-type estimate for the weak solutions.

4.1. A maximum principle about the strong solutions. The purpose of this subsection is to establish, applying the Stampacchia's truncation method, some a priori L^∞ estimates for the strong solutions with initial data $(u_\tau, g) \in (\mathscr{D}^1_0(\Omega,\sigma) \cap L^\infty(\Omega)) \times L^\infty(\Omega \times (\tau, T))$, which will guarantee the test functions that we used in next subsection to make sense.

Lemma 4.1. *Assume* (C1)–(C2). *For any $-\infty < \tau \le T < \infty$ and any initial data $(u_\tau, g) \in (\mathscr{D}^1_0(\Omega,\sigma) \cap L^\infty(\Omega)) \times L^\infty(\Omega \times (\tau, T))$, the unique strong solution u of* (1.1) *belongs to $L^\infty(\Omega \times (\tau, T))$.*

Proof. We will use the Stampacchia's truncation method in Brezis [3], fixing a function $H(\,\cdot\,) \in C^1(\mathbb{R})$ such that

(a) $|H'(s)| \le M < \infty$, for all $s \in \mathbb{R}$;
(b) H is strictly increasing on $(0, \infty)$;
(c) $H(s) = 0$, for all $s \le 0$;

and defining

$$G(s) := \int_0^s H(\delta)\,d\delta.$$

Denote $K' := \max\{\|u_\tau\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega\times(\tau,T))}\}$. Under assumption (1.2), there exists a positive constant M which depends only on β, α_1 and K' such that

$$f(s) \ge K' \quad \text{as } s \ge M \quad \text{and} \quad f(s) \le -K' \quad \text{as } s \le -M. \tag{4.1}$$

Then, we define $K := \max\{K', M\} + 1$.

Note that u is a strong solution (see Lemma 2.8) and $u \in C([\tau,T]; \mathscr{D}^1_0(\Omega,\sigma))$, we have that

$$H(u(t) - K) \in \mathscr{D}^1_0(\Omega,\sigma) \quad \text{for any } t \in [\tau, T], \tag{4.2}$$

$$H(u(t) - K) \in L^2(\tau, T; \mathscr{D}^1_0(\Omega,\sigma)). \tag{4.3}$$

Hence, from the definition of strong solutions, we have that

$$\begin{aligned}(4.4)\quad \int_\tau^t \int_\Omega u'(x,s)H(u(s)-K)\,dx\,ds \\ + \int_\tau^t \int_\Omega \sigma(x)H'(u(s)-K)|\nabla u(s)|^2\,dx\,ds \\ = -\int_\tau^t \int_\Omega f(u(x,s))H(u(s)-K)\,dx\,ds + \int_\tau^t \int_\Omega g(x,s)H(u(s)-K)\,dx\,ds.\end{aligned}$$

In the following, we will deal with the terms in (4.4) one by one.

Firstly, it is obvious that

$$\int_\tau^t \int_\Omega \sigma(x)H'(u(s)-K)|\nabla u(s)|^2\,dx\,ds \geq 0. \tag{4.5}$$

Secondly, from (4.3) and the definition of K', we have that

$$0 \leq \int_\tau^t \int_\Omega K'H(u(s)-K)\,dx\,ds < \infty,$$

and, combining with (4.1) and the definition of K, we know that

$$\begin{aligned}-\int_\tau^t \int_\Omega (f(u(x,s))-K')H(u(s)-K)\,dx\,ds &\leq 0,\\ \int_\tau^t \int_\Omega (g(x,s)-K')H(u(s)-K)\,dx\,ds &\leq 0.\end{aligned}$$

Inserting the above estimates into (4.4), we obtain that

$$\int_\tau^t \int_\Omega u'(x,s)H(u(s)-K)\,dx\,ds \leq 0,$$

which implies that

$$\int_\Omega G(u(x,t)-K)\,dx - \int_\Omega G(u(x,\tau)-K)\,dx \leq 0 \quad \text{for all } t \in [\tau,T],$$

consequently, we can obtain

$$u(x,t) \leq K \quad \text{a.e. on } \Omega \text{ for all } t \in [\tau,T]. \tag{4.6}$$

Similarly, defining $\widetilde{H}(s) = H(-s)$ and replacing $H(u(s)-K)$ by $\widetilde{H}(u(s)+K)$ in (4.4), we can deduce that

$$u(x,t) \geq -K \quad \text{a.e. on } \Omega \text{ for all } t \in [\tau,T]. \tag{4.7}$$

From (4.6) and (4.7), we can finish our proof immediately. $\square$

4.2. $L^{2+\delta}$-type estimate for the weak solution. Throughout this section, we denote by (Lemmas 3.1 and 2.10 guarantee its existence)

$$\widehat{v} := \{v(t) : t \in \mathbb{R}\} \quad \text{with } v(t) \in \mathscr{A}(t) \text{ for all } t \in \mathbb{R} \tag{4.8}$$

a fixed complete trajectory of $U(t, \tau)$, that is

$$U(t,\tau)v(\tau) = v(t) \quad \text{for any } \infty < \tau \le t < +\infty.$$

To make our proof rigorous, let us consider the approximation of solutions. Let $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \in \mathscr{D}_\lambda$ and $u(t) = U(t,\tau)u_\tau$, $u_\tau \in D(\tau)$. For any (fixed) $\tau \in \mathbb{R}$ and $T > \tau$, we know that there exist two sequences $\{(u_{\tau m}, g_m)\}$ and $\{(v_{\tau m}, g_m)\}$ satisfying

$$u_{\tau m}, v_{\tau m} \in \mathscr{D}_0^1(\Omega,\sigma) \cap L^\infty(\Omega) \quad \text{and} \quad g_m \in L^\infty(\Omega \times (\tau, T)) \tag{4.9}$$

such that

$$u_{\tau m} \to u_\tau,\ v_{\tau m} \to v_\tau \in L^2(\Omega) \quad \text{and} \quad g_m \to g \in L^2(\tau, T; L^2(\Omega)) \tag{4.10}$$

as $m \to \infty$, and

$$u_m \to u \quad \text{and} \quad v_m \to v \in C([\tau, T]; L^2(\Omega)), \tag{4.11}$$

where u_m and v_m are the unique strong solutions of (1.1) corresponding to the regular data $(u_{\tau m}, g_m)$ and $(v_{\tau m}, g_m)$ respectively; note that (4.11) can be deduced by a similar proof for the uniqueness of weak solutions, here we omit it.

Without loss of generality, from (4.10) we can assume that

$$\|u_{\tau m}\|^2_{L^2(\Omega)} \le \|u_\tau\|^2_{L^2(\Omega)} + 1 \quad \text{and} \quad \|v_{\tau m}\|^2_{L^2(\Omega)} \le \|v_\tau\|^2_{L^2(\Omega)} + 1, \tag{4.12}$$

for all $m = 1, 2, \dots$ Denote $w_m(t) = u_m(t) - v_m(t)$, then, $w_m(t)$ is the unique strong solution of the following equation:

$$\begin{cases} w_{m,t} - \operatorname{div}(\sigma(x)\nabla w_m) + f(u_m(t)) - f(v_m(t)) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in \Omega \times (0,T), \\ w_m|_{\partial\Omega} = 0, \\ w_m|_{t=\tau} = u_m(\tau) - v_m(\tau). \end{cases} \tag{4.13}$$

Applying Lemma 4.1, we know that $u_m(t), v_m(t) \in L^\infty(\Omega \times (\tau, T))$ for each $m = 1, 2, \dots$, so

$$w_m(t) = u_m(t) - v_m(t) \in C([\tau, T]; \mathscr{D}_0^1(\Omega,\sigma)) \cap L^\infty(\Omega \times (\tau, T))$$

and, for any $0 \le \theta < \infty$,

$$|w_m|^\theta \cdot w_m \in L^2(\tau, T; \mathscr{D}_0^1(\Omega,\sigma)) \cap L^\infty(\Omega \times (\tau, T)), \tag{4.14}$$

consequently, we can multiply (4.13) by $|w_m|^\theta \cdot w_m$ for any $0 \le \theta < \infty$.

With the preparation above, we will prove the following main result of this subsection:

Lemma 4.2. *Let $\widehat{D} \in \mathscr{D}_\lambda$, $\widehat{v}$ be the fixed complete trajectory given in (4.8) and T be a fixed time. Assume that $g \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ satisfies (1.4), f satisfies (1.2)–(1.3) and $u_{\tau m}, v_{\tau m}$ satisfy (4.9)–(4.12). Then, for each $t \in (\tau, T)$ and each $k = 0, 1, \ldots$, there exist two sequences $\widetilde{T_k}(t, \widehat{D}, \widehat{v})$ (which depends only on k, t, λ, $\widehat{D}$ and $\widehat{v}$) and $\overline{M}_k(t)$ (which depends only on k, t, λ, N and $\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)} ds$), such that for any $m = 1, 2, \ldots$, the solution w_m of (1.1) satisfies*

$$(\mathrm{A}_k) \qquad \int_\Omega |w_m(t)|^{2(N/(N-2+\alpha))^k}\, ds \le \overline{M}_k(t) \quad \text{for any } t - \tau \ge \widetilde{T_k}(t, \widehat{D}, \widehat{v}),$$

$$(\mathrm{B}_k) \qquad \int_s^{s+1} \bigg(\int_\Omega |w_m(\varsigma)|^{2(N/(N-2+\alpha))^{k+1}}\, dx \bigg)^{(N-2+\alpha)/N} d\varsigma \le \overline{M}_k(t)$$

for any $s - \tau \ge \widetilde{T_k}(t, \widehat{D}, \widehat{v})$.

Proof. Since u_m is a strong solution, similar to (3.5) we have that

$$\frac{d}{ds}\|u_m\|^2_{L^2(\Omega)} + \|u_m\|^2_{\mathscr{D}^1_0(\Omega,\sigma)} + 2\alpha_1\|u_m\|^p_{L^p} \le \frac{4}{\lambda}\|g_m(s)\|^2_{L^2(\Omega)} + 2\alpha_2|\Omega|, \tag{4.15}$$

then, combining with the embedding (2.2), (4.15) implies that

$$\|u_m(t)\|^2_{L^2(\Omega)} \le e^{-\lambda(t-\tau)}\|u_m(\tau)\|^2_{L^2(\Omega)} + \frac{4}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g_m(s)\|^2_{L^2(\Omega)}\, ds + \frac{2\alpha_2}{\lambda}|\Omega|$$

for all $t \in (\tau, T)$. Similarly, we also have

$$\|v_m(t)\|^2_{L^2(\Omega)} \le e^{-\lambda(t-\tau)}\|v_m(\tau)\|^2_{L^2(\Omega)} + \frac{4}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g_m(s)\|^2_{L^2(\Omega)}\, ds + \frac{2\alpha_2}{\lambda}|\Omega|$$

for all $t \in (\tau, T)$. Therefore, we obtain that

$$\begin{aligned} \|w_m(t)\|^2_{L^2(\Omega)} \le{}& 2e^{-\lambda(t-\tau)}(\|u_m(\tau)\|^2_{L^2(\Omega)} + \|v_m(\tau)\|^2_{L^2(\Omega)}) \\ &+ \frac{8}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g_m(s)\|^2_{L^2(\Omega)}\, ds + \frac{4\alpha_2}{\lambda}|\Omega| \end{aligned} \tag{4.16}$$

for all $t \in (\tau, T)$, consequently, there exists $\widetilde{T_0}(t, \widehat{D}, \widehat{v})$ such that

$$\begin{aligned} \|w_m(t)\|^2_{L^2(\Omega)} \le{}& 2e^{-\lambda(t-\tau)}(\|u_m(\tau)\|^2_{L^2(\Omega)} + \|v_m(\tau)\|^2_{L^2(\Omega)}) \\ &+ \frac{8}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g_m(s)\|^2_{L^2(\Omega)}\, ds + \frac{4\alpha_2}{\lambda}|\Omega| \\ \le{}& 4e^{-\lambda(t-\tau)}(\|u_\tau\|^2_{L^2(\Omega)} + \|v_\tau\|^2_{L^2(\Omega)} + 1) + \overline{M'}_0(t) \\ \le{}& 1 + \overline{M'}_0(t) \end{aligned} \tag{4.17}$$

for any $t - \tau \ge \widetilde{T}_0(t, \widehat{D}, \widehat{v})$, where we have used (4.12), and set

$$\overline{M'}_0(t) := \frac{8}{\lambda}e^{-\lambda t}\int_\tau^t e^{\lambda s}\|g_m(s)\|^2_{L^2(\Omega)}\, ds + \frac{4\alpha_2}{\lambda}|\Omega|.$$

Now, multiplying (4.13) by w_m and integrating with both space and time, we have that

$$\int_s^{s+1}\int_\Omega \sigma(x)|\nabla w_m(\varrho)|^2\,dx\,d\varrho \le (l+1)(\overline{M'}_0(t)+1) \tag{4.18}$$

for any $s-\tau \ge \widetilde{T}_0(t,\widehat{D},\widehat{v})$. From the embedding $\mathscr{D}_0^1(\Omega,\sigma)\hookrightarrow L^{2N/(N-2+\alpha)}(\Omega)$, we have that

$$\int_s^{s+1} \|w_m(\varrho)\|^2_{L^{2N/(N-2+\alpha)}(\Omega)}\,dx\,d\varrho \le C_N^2(l+1)(\overline{M'}_0(t)+1) \tag{4.19}$$

for any $s-\tau \ge \widetilde{T}_0(t,\widehat{D},\widehat{v})$, where C_N is a constant depending only on the domain Ω and the spatial dimension N.

Set $\overline{M}_0(t)=(1+C_N^2(l+1))(\overline{M'}_0(t)+1)$. From (4.17) and (4.19) we know that (A_0) and (B_0) hold.

In the following, we assume that (A_k) and (B_k) hold for $k\ge 0$, by induction, we will show that (A_{k+1}) and (B_{k+1}) hold. Multiplying (4.13) by $|w_m|^{2(N/(N-2+\alpha))^{k+1}-2}w_m$, we deduce that

$$\begin{aligned} &\frac{1}{2}\left(\frac{N}{N-2+\alpha}\right)^{k+1}\frac{d}{dt}\|w_m\|^{2(N/(N-2+\alpha))^{k+1}}_{L^{2(N/(N-2+\alpha))^{k+1}}(\Omega)} \\ &\quad+\frac{2((N-2+\alpha)/N)^{k+1}-1}{(N/(N-2+\alpha))^{2(k+1)}}\int_\Omega \sigma(x)|\nabla|w_m(t)|^{(N/(N-2+\alpha))^{k+1}}|^2\,dx \\ &\qquad\qquad\le l\|w_m\|^{2(N/(N-2+\alpha))^{k+1}}_{L^{2(N/(N-2+\alpha))^{k+1}}(\Omega)} \end{aligned} \tag{4.20}$$

for almost every $t\in(\tau,T)$, and we have that

$$\frac{d}{dt}\|w_m\|^{2(N/(N-2+\alpha))^{k}}_{L^{2(N/(N-2+\alpha))^{k+1}}(\Omega)} \le 2l\left(\frac{N}{N-2+\alpha}\right)^{k}\|w_m\|^{2(N/(N-2+\alpha))^{k}}_{L^{2(N/(N-2+\alpha))^{k+1}}(\Omega)} \tag{4.21}$$

for almost every $t\in(\tau,T)$. Applying the uniform Gronwall lemma to (4.21) and (B_k), we can deduce that

$$\int_\Omega |w_m(t)|^{2(N/(N-2+\alpha))^{k+1}}\,dx \le C_{\overline{M'}_k(t),l,N,k} \tag{4.22}$$

for any $t-\tau \ge \widetilde{T}_k(t,\widehat{D},\widehat{v})+1$. Now, having (4.22), by integrating (4.20) over $[s,s+1]$, we can obtain that

$$\int_s^{s+1}\int_\Omega |\sigma(x)|\nabla|w_m(\varrho)|^{(N/(N-2+\alpha))^{k+1}}|^2\,dx\,d\varrho \le C_{\overline{M'}_k(t),l,N,k} \tag{4.23}$$

for any $s-\tau \ge \widetilde{T}_k(t,\widehat{D},\widehat{v})+1$.

Applying the embedding $\mathscr{D}_0^1(\Omega,\sigma)\hookrightarrow L^{2N/(N-2+\alpha)}(\Omega)$ again, from (4.23) we finally obtain that

$$\int_s^{s+1}\left(\int_\Omega |w_m(\varrho)|^{2(N/(N-2+\alpha))^{k+2}}\,dx\right)^{(N-2+\alpha)/N} d\varrho \le C_N^2 C_{\overline{M'}_k(t),l,N,k} \tag{4.24}$$

for any $s-\tau \ge \widetilde{T}_k(t,\widehat{D},\widehat{v})+1$. Therefore, set $\widetilde{T}_{k+1}(t,\widehat{D},\widehat{v}) = \widetilde{T}_k(t,\widehat{D},\widehat{v})+1$ and $\overline{M}_k(t) = \max\{C_{\overline{M'}_k(t),l,N,k}, C_N^2 C_{\overline{M'}_k(t),l,N,k}\}$, from (4.22) and (4.24) we know that (A_{k+1}) and (B_{k+1}) hold. □

As a consequence of Lemma 4.2 and Fatou's lemma, we have:

THEOREM 4.3. *Let $\widehat{D} \in \mathscr{D}_\lambda$, $\widehat{v}$ be the fixed complete trajectory given in* (4.8). *Assume* (C1)–(C3). *Then, for each $t \in \mathbb{R}$ and each $k = 0,1,\ldots$, there exist two sequences $T_k(t,\widehat{D},\widehat{v})$ (which depends only on $k,t,\lambda,\widehat{D}$ and $\widehat{v}$) and $\overline{M}_k(t)$ (which depends only on k,t,λ,N and $\int_{-\infty}^t e^{\lambda s}\|g(s)\|^2_{L^2(\Omega)}\,ds$) such that*

$$\int_\Omega |U(t,\tau)u_\tau - v(t)|^{2(N/(N-2+\alpha))^k}\,dx \le \overline{M}_k(t)$$

for any $t-\tau \ge T_k(t,\widehat{D},\widehat{v})$ and any $u_\tau \in D(\tau)$.

PROOF. For each fixed $t \in \mathbb{R}$ and $k \in \{0,1,\ldots\}$ we take

$$T_k(t,\widehat{D},\widehat{v}) = \widetilde{T}_k(t,\widehat{D},\widehat{v})+1,$$

where $\widetilde{T}_k(t,\widehat{D},\widehat{v})$ is the constant given in Lemma 4.2 corresponding to the pair t,k.

Set $T = t+1$ and for any (fixed) τ satisfying $t-\tau \ge T_k(t,\widehat{D},\widehat{v})$, we consider the approximation on the interval $[\tau,T]$. For the interval $[\tau,T]$ given above, choose two sequences $\{(u_{\tau m},g_m)\}$ and $\{(v_{\tau m},g_m)\}$ satisfying all conditions in (4.9)–(4.12). Then, from (4.11), we have that there exist two subsequences $\{u_{m_j}(t)\} \subset \{u_m(t)\}$ and $\{v_{m_j}(t)\} \subset \{v_m(t)\}$ satisfying that $u_{m_j}(t) \to u(t) = U(t,\tau)u_\tau$ and $v_{m_j}(t) \to v(t)$ almost everywhere on Ω as $j \to \infty$ (where the sequence m_j may depend on t). Then, applying Lemma 4.2, (A_k) and the Fatou's lemma, we can deduce that

$$\begin{aligned}\int_\Omega |U(t,\tau)u_\tau - v(t)|^{2(N/(N-2+\alpha))^k}\,dx \\ \le \liminf_{j\to\infty}\int_\Omega |u_{m_j}(t) - v_{m_j}(t)|^{2(N/(N-2+\alpha))^k}\,dx \le \overline{M}_k(t).\end{aligned}$$ □

4.3. Proof of Theorem 1.1. Now, we are able to prove Theorem 1.1 by checking the conditions of the abstract Lemma 2.11 in the case of our degenerate parabolic equation (1.1).

For each $\delta \in [0,\infty)$, we know that there exists a unique $k \in \{1,2,\ldots\}$ such that

$$2+\delta+1 \in \left(2\left(\frac{N}{N-2+\alpha}\right)^{k-1}, 2\left(\frac{N}{N-2+\alpha}\right)^k\right]. \tag{4.25}$$

Let

$$X = L^2(\Omega), \quad Y = L^{2+\delta}(\Omega) \quad \text{and} \quad Z = L^{2(N/(N-2+\alpha))^k}(\Omega).$$

Then, for X, Y and Z given above, we will check the conditions of Lemma 2.11 as followings:

(a) $\widehat{\mathscr{A}}$ is the $(L^2(\Omega), L^2(\Omega))$ pullback $\mathscr{D}_\lambda$-attractor in Lemma 3.1.

(b) $\widehat{v}$ is the fixed complete trajectory given in (4.8).

(c), (d) Applying Theorem 4.3, we know that

$$\int_\Omega |U(t,\tau)u_\tau - v(t)|^{2(N/(N-2+\alpha))^k}\,dx \le \overline{M}_k(t) \tag{4.26}$$

for any $t-\tau \ge T_k(t, \widehat{D}, \widehat{v})$, any $u_\tau \in D(\tau)$. Consequently, we can define

$$B_0(t) = \Big\{ w \in L^{2(N/(N-2+\alpha))^k}(\Omega) : \|w\|^{2(N/(N-2+\alpha))^k}_{L^{2(N/(N-2+\alpha))^k}(\Omega)} \le \overline{M}_k(t) \Big\}$$

for any $t \in \mathbb{R}$. Therefore, we see that all conditions in Lemma 2.11 are satisfied, and then, the $L^{2+\delta}(\Omega)$ pullback $\mathscr{D}_\lambda$-attraction (1.7) follows from conclusion (c) in Lemma 2.11, and (1.8) follows from (4.26) (where we take $T(t,\delta,\widehat{D},\widehat{\mathscr{A}}) := T_k(t,\widehat{D},\widehat{v})$ and $M_\delta(T) := \overline{M}_k(t)$). $\square$

Remark 4.4. Note that in the proof of Theorem 1.1, we used essentially only the interpolation, hence, our results and methods are applicable to the unbounded domain case such as that considered in [19], [20].

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