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REMETRIZATION RESULTS FOR POSSIBLY INFINITE SELF-SIMILAR SYSTEMS

Radu Miculescu — Alexandru Mihail

Abstract. In this paper we introduce a concept of possibly infinite self-similar system which generalizes the attractor of a possibly infinite iterated function system whose constitutive functions are φ-contractions. We prove that for a uniformly possibly infinite self-similar system there exists a remetrization which makes contractive all its constitutive functions. Then, based on this result, we show that for such a system there exist a comparison function φ and a remetrization of the system which makes φ-contractive all its constitutive functions. Finally we point out that in the case of a finite set of constitutive functions our concept of a possibly infinite self-similar system coincides with Kameyama's concept of a topological self-similar system.

1. Introduction

In order to generalize the notion of the attractor of an iterated function system A. Kameyama (see [10]) introduced the concepts of topological self-similar set and self-similar topological system as follows:

Definition 1.1. A compact Hausdorff topological space K is called a topological self-similar set if there exist continuous functions $f_1, \ldots, f_N \colon K \to K$, where $N \in \mathbb{N}^* = \{1, 2, \ldots\}$, and a continuous surjection $\pi \colon \Lambda \to K$, where

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$\Lambda = \{1, \ldots, N\}^{\mathbb{N}^*}$, such that the diagram

$$\begin{tikzcd} \Lambda \arrow[r, "\tau_i"] \arrow[d, "\pi"'] & \Lambda \arrow[d, "\pi"] \\ K \arrow[r, "f_i"'] & K \end{tikzcd}$$

commutes for all $i \in \{1, \ldots, N\}$, where

$$\tau_i(\omega_1 \ldots \omega_m \omega_{m+1} \ldots) = i\omega_1 \ldots \omega_m \omega_{m+1} \ldots \quad \text{for each } \omega_1 \ldots \omega_m \omega_{m+1} \ldots \in \Lambda.$$

We say that $(K, \{f_i\}_{i \in \{1,\ldots,N\}})$, a topological self-similar set together with a set of continuous maps as above, is a topological self-similar system.

He asked the following fundamental question (see [10]): *Given a topological self-similar system* $(K, \{f_i\}_{i \in \{1,\ldots,N\}})$*, does there exist a metric on* K *compatible to the topology such that all the functions* f_i *are contractions?* Such a metric is called a self-similar metric. L. Janoš ([8] and [9]) settles the case $N = 1$.

On the one hand, Kameyama provided a topological self-similar set which does not admit a self-similar metric and, on the other hand, he proved that every totally disconnected self-similar set and every non-recurrent finitely ramified self-similar set have a self-similar metric.

R. Atkins, M. Barnsley, A. Vince and D. Wilson [1] gave an affirmative answer to the above question for self-similar sets derived from affine transformations on $\mathbb{R}^m$ (see also [12] for a generalization of this result for a Banach space $(X, \|\cdot\|)$ instead of the Banach space $\mathbb{R}^m$ and for an arbitrary set I instead of the set $\{1, \ldots, N\}$), M. Barnsley and A. Vince [4] for projectives functions and A. Vince [14] for Möbius transformations.

The problem of the existence of a self-similar metric on a self-similar set was also studied by K. Hveberg [7], M. Barnsley and K. Igudesman [3], T. Banakh, W. Kubiś, N. Novosad, M. Nowak and F. Strobin [2].

In [13], we modified Kameyama's question (which, as we have seen, has a negative answer for an arbitrary topological self-similar system) by weakening the requirement that the functions in the topological self-similar system are contractions to requiring that they are φ-contractions. More precisely, we gave an affirmative answer to the following question: *Given a topological self-similar system* $(K, \{f_i\}_{i \in \{1,\ldots,N\}})$*, does there exist a metric* δ *on* K *which is compatible with the original topology and a comparison function* φ *such that* $f_i \colon (K, \delta) \to (K, \delta)$ *is* φ*-contraction for each* $i \in \{1, \ldots, N\}$*?*

In this paper we study the case of a possibly infinite family of functions $(f_i)_{i \in I}$. We introduce the concept of possibly infinite self-similar system which generalizes the notion of the attractor of a possibly infinite iterated function system whose constitutive functions are φ-contractions (see Proposition 3.7).

We prove that for a uniformly possibly infinite self-similar system there exists a remetrization which makes contractive all its constitutive functions (see Theorem 4.1). Then, based on this result, we show that for such a system there exist a comparison function φ and a remetrization of the system which makes φ-contractive all its constitutive functions (see Theorem 5.5). Finally we point out that when the set I is finite the concepts of a possibly infinite self-similar system and a topological self-similar system coincide. Consequently we obtain a generalization of the above mentioned affirmative answer to modified Kameyama's question.

2. Preliminaries

In the sequel, by $\mathbb{N}$ we mean the set $\{0,1,\dots\}$ and by $\mathbb{N}^*$ the set $\{1,2,\dots\}$. Let I be an arbitrary set. By $\Lambda(I)$ we mean the set $I^{\mathbb{N}^*}$ and by $\Lambda_n(I)$ we mean the set $I^{\{1,\dots,n\}}$. The elements of $\Lambda(I)$ are written as $\omega=\omega_1\dots\omega_m\omega_{m+1}\dots$ and the elements of $\Lambda_n(I)$ are written as words $\omega=\omega_1\dots\omega_n$, where $\omega_i\in I$. Hence $\Lambda(I)$ is the set of infinite words with letters from the alphabet I and $\Lambda_n(I)$ is the set of words of length n with letters from the alphabet I. By $\Lambda^*(I)$ we denote the set of all finite words, i.e. $\Lambda^*(I)=\bigcup_{n\in\mathbb{N}^*}\Lambda_n(I)\cup\{\lambda\}$, where by λ we mean the empty word. If $\omega=\omega_1\dots\omega_m\omega_{m+1}\dots\in\Lambda(I)$ or if $\omega=\omega_1\dots\omega_n\in\Lambda_n(I)$, where $m,n\in\mathbb{N}^*$, $n\geq m$, then the word $\omega_1\dots\omega_m$ is denoted by $[\omega]_m$. By $|\omega|$ we mean the length of ω. For two words $\alpha=\alpha_1\dots\alpha_n\in\Lambda_n(I)$ and $\beta=\beta_1\dots\beta_m\in\Lambda_m(I)$ or $\beta=\beta_1\dots\beta_m\beta_{m+1}\dots\in\Lambda(I)$, by $\alpha\beta$ we mean the concatenation of the words α and β, i.e. $\alpha\beta=\alpha_1\dots\alpha_n\beta_1\dots\beta_m$ and respectively, $\alpha\beta=\alpha_1\dots\alpha_n\beta_1\dots\beta_m\beta_{m+1}\dots$ On $\Lambda(I)$, we consider the metric

$$d_\Lambda(\alpha,\beta)=\sum_{k=1}^{\infty}\frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k},\quad \text{where } \delta_x^y=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{if } x\neq y,\end{cases}$$

$\alpha=\alpha_1\dots\alpha_{n+1}\alpha_{n+2}\dots$ and $\beta=\beta_1\dots\beta_{n+1}\beta_{n+2}\dots$

Let (X,d) be a metric space and $f_i\colon X\to X$, $i\in I$. For $\omega=\omega_1\dots\omega_m\in\Lambda_m(I)$, we consider $f_\omega=f_{\omega_1}\circ\dots\circ f_{\omega_m}$ and, for a subset H of X, $H_\omega=f_\omega(H)$. We also consider $f_\lambda=\mathrm{Id}$ and $H_\lambda=H$.

For a subset A of a metric space (X,d), we denote by $\operatorname{diam}(A)$ the diameter of A (or, if necessary, $\operatorname{diam}_d(A)$).

3. Possibly infinite self-similar systems

A possibly infinite self-similar system generalizes the concept of the attractor of an infinite iterated function system containing φ-contractions (see [5] and [15]), as Proposition 3.7 points out.

Definition 3.1. A possibly infinite self-similar system (PISSS for short) consists of a complete and bounded metric space (A,d) and a family of continuous functions $(f_i)_{i\in I}$, where $f_i\colon A\to A$, such that:

(a) $A=\overline{\bigcup_{i\in I} A_i}$;

(b) $\lim_{n\to\infty} \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}(A_\omega)=0$.

We denote it by $\mathcal{S}=((A,d),(f_i)_{i\in I})$. If, in addition, the family of functions $(f_i)_{i\in I}$ is equicontinuous, then $\mathcal{S}$ is called uniformly possibly infinite self-similar system (UPISSS for short).

Definition 3.2. Let (X,d) be a metric space. A family of functions $(f_i)_{i\in I}$, $f_i\colon X\to X$, is called bounded if the set $\bigcup_{i\in I} f_i(A)$ is bounded, for every bounded subset A of X.

Definition 3.3. A function $\varphi\colon[0,\infty)\to[0,\infty)$ is called a comparison function if it satisfies the following three properties:

(a) φ is increasing;

(b) $\varphi(t)<t$ for any $t>0$;

(c) φ is right-continuous.

Remark 3.4. Note that $\varphi(0)=0$ for each comparison function φ.

Remark 3.5 (see Remark 1 from [11]). Any function $\varphi\colon[0,\infty)\to[0,\infty)$ satisfying (b) and (c) from the above definition has the following property:

$$\lim_{n\to\infty}\varphi^{[n]}(t)=0$$

for any $t>0$, where by $\varphi^{[n]}$ we mean the composition of φ by itself n times.

Definition 3.6. Let (X,d) be a metric space and a function $\varphi\colon[0,\infty)\to[0,\infty)$. A function $f\colon X\to X$ is called φ-contraction if

$$d(f(x),f(y))\le\varphi(d(x,y)),\quad\text{for every } x,y\in X.$$

Proposition 3.7. *Given a complete metric space (X,d) and a comparison function $\varphi\colon[0,\infty)\to[0,\infty)$, if a bounded family of functions $(f_i)_{i\in I}$, where $f_i\colon X\to X$, is such that each function f_i is φ-contraction, then there exists a unique bounded and closed subset A of X such that $A=\overline{\bigcup_{i\in I} A_i}$ and $((A,d),(f_i\mid A)_{i\in I})$ is a* UPISSS.

Proof. We have:

(a) For the existence of the set A see Theorem 2.5 from [5].

(b) For each $n\in\mathbb{N}$, we have

$$\sup_{\omega\in\Lambda_n(I)}\operatorname{diam}(A_\omega)\le\varphi^{[n]}(\operatorname{diam}(A)).$$

Consequently, taking into account Remarks 3.4 and 3.5, we get that

$$\lim_{n\to\infty} \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}(A_\omega) = 0.$$

(c) Using Remark 3.4, we infer that $d(f_i(x), f_i(y)) \le \varphi(d(x,y)) \le d(x,y)$, for each $x, y \in A$ and $i \in I$, and we conclude that the family of functions $(f_i)_{i\in I}$ is equicontinuous. $\square$

The above proposition provides a large class of UPISSSs. In particular, as the functions τ_i have Lipschitz constant less or equal to $1/3$, $((\Lambda(I), d_\Lambda), (\tau_i)_{i\in I})$ is a UPISSS having the property that $(\Lambda(I), d_\Lambda)$ is not compact, in case that I is infinite.

The next two propositions emphasize a connection between the points of $\Lambda(I)$ and the elements of A.

PROPOSITION 3.8. *Let* $\mathcal{S} = ((A,d),(f_i)_{i\in I})$ *be a* PISSS. *Then, for each* $\omega \in \Lambda(I)$*, the set* $\bigcap_{n\in\mathbb{N}^*} \overline{A_{[\omega]_n}}$ *has exactly one element.*

PROOF. Note that $A_{[\omega]_{n+1}} \subseteq A_{[\omega]_n}$, so $\overline{A_{[\omega]_{n+1}}} \subseteq \overline{A_{[\omega]_n}}$ for each $n \in \mathbb{N}^*$ and

$$\lim_{n\to\infty} \operatorname{diam}(\overline{A_{[\omega]_n}}) = \lim_{n\to\infty} \operatorname{diam}(A_{[\omega]_n}) = 0.$$

Then, since A is a complete metric space, basing on Cantor's intersection theorem, we conclude that $\bigcap_{n\in\mathbb{N}^*} \overline{A_{[\omega]_n}}$ has one point. $\square$

We denote by a_ω the element of $\bigcap_{n\in\mathbb{N}^*} \overline{A_{[\omega]_n}}$, so $\{a_\omega\} = \bigcap_{n\in\mathbb{N}^*} \overline{A_{[\omega]_n}}$.

PROPOSITION 3.9. *Let* $\mathcal{S} = ((A,d),(f_i)_{i\in I})$ *be a* PISSS. *Then, in the framework of the previous proposition, for each* $a \in A$ *and each* $\omega \in \Lambda$*, we have*

$$\lim_{n\to\infty} f_{[\omega]_n}(a) = a_\omega.$$

Moreover, the convergence is uniform with respect to a *and* ω*, i.e. for each* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}^*$ *such that the inequality* $d(f_{[\omega]_m}(a), a_\omega) < \varepsilon$ *is valid for each* $n \in \mathbb{N}^*$*,* $n \ge n_\varepsilon$*,* $a \in A$ *and* $\omega \in \Lambda(I)$*.*

PROOF. Since, for every $n \in \mathbb{N}^*$,

$$d(f_{[\omega]_n}(a), a_\omega) \le \operatorname{diam}(\overline{A_{[\omega]_n}}) = \operatorname{diam}(A_{[\omega]_n}) \le \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}(A_\omega),$$

and $\lim\limits_{n\to\infty} \sup\limits_{\omega\in\Lambda_n} \operatorname{diam}(A_\omega) = 0$, we infer that for every $a \in A$ and every $\omega \in \Lambda(I)$, we have $\lim\limits_{n\to\infty} f_{[\omega]_n}(a) = a_\omega$ uniformly with respect to $a \in A$ and $\omega \in \Lambda(I)$. $\square$

4. A remetrization that makes contractive all the constitutive functions of a UPISSS

In this section, given a UPISSS $((A,d),(f_i)_{i\in I})$, we construct a metric ρ, which is equivalent to d, having the property that all the functions $f_i\colon (A,\rho)\to (A,\rho)$ are contractive.

Theorem 4.1. *Let* $((A,d),(f_i)_{i\in I})$ *be a* UPISSS. *Then there exists a metric* ρ *on* A *having the following three properties:*

(a) $\rho(f_i(x),f_i(y))\le \rho(x,y)$, *for each* $i\in I$ *and each* $x,y\in A$; *consequently* $\rho(f_\omega(x),f_\omega(y))\le \rho(x,y)$, *for each* $x,y\in A$ *and each* $\omega\in\Lambda^*(I)$.
(b) ρ *is equivalent to* d.
(c) *The metric space* (A,ρ) *is complete and bounded.*

Proof. Define $\rho\colon A\times A\to[0,\infty)$ by

$$\rho(x,y)=\sup_{\omega\in\Lambda^*(I)} d(f_\omega(x),f_\omega(y)),\quad \text{for every } x,y\in A.$$

The function ρ attains finite values since $d(f_\omega(x),f_\omega(y))\le \operatorname{diam}(A)$ for every $\omega\in\Lambda^*(I)$ and every $x,y\in A$. Obviously, ρ is a bounded metric in A, satisfies (a) and $d\le\rho$.

To establish (b) we only have to prove that if $(a_n)_{n\in\mathbb{N}}$ is a sequence of elements from A and $l\in A$ is such that $\lim\limits_{n\to\infty} d(a_n,l)=0$, then $\lim\limits_{n\to\infty}\rho(a_n,l)=0$.

Indeed, as $\lim\limits_{n\to\infty}\sup\limits_{\omega\in\Lambda_n(I)}\operatorname{diam}(A_\omega)=0$, for every $\varepsilon>0$ there exists $m_\varepsilon\in\mathbb{N}$ such that $\sup\limits_{\omega\in\Lambda_n(I)}\operatorname{diam}(A_\omega)<\varepsilon/2$ for every $n\in\mathbb{N}$, $n\ge m_\varepsilon$, so

$$d(f_\omega(a_n),f_\omega(l))\le \operatorname{diam}(A_\omega)\le \sup_{\omega'\in\Lambda_{|\omega|}(I)}\operatorname{diam}(A_{\omega'})<\varepsilon/2$$

for every $n\in\mathbb{N}$ and every $\omega\in\Lambda^*(I)$ with $|\omega|\ge m_\varepsilon$. Since the family of functions $(f_i)_{i\in I}$ is equicontinuous, the family $(f_\omega)_{\omega\in\Lambda^*(I),\,|\omega|<m_\varepsilon}$ has the same property, so, for every $\varepsilon>0$ there exists $n_\varepsilon\in\mathbb{N}$ such that the inequality $d(f_\omega(a_n),f_\omega(l))<\varepsilon/2$ is valid for every $n\in\mathbb{N}$, $n\ge n_\varepsilon$ and every $\omega\in\Lambda^*(I)$ such that $|\omega|<m_\varepsilon$. We showed that for every $\varepsilon>0$ there exists $n_\varepsilon\in\mathbb{N}$ such that

$$\rho(a_n,l)=\sup_{\omega\in\Lambda^*(I)} d(f_\omega(a_n),f_\omega(l))\le\varepsilon/2<\varepsilon$$

for every $n\in\mathbb{N}$, $n\ge n_\varepsilon$. Hence $\lim\limits_{n\to\infty}\rho(a_n,l)=0$.

Now we prove (c). The boundedness of (A,ρ) is obvious as $\rho(x,y)\le\operatorname{diam}(A)$ for every $x,y\in A$. We claim that (A,ρ) is complete.

Indeed, if $(a_n)_{n\in\mathbb{N}}$ is a ρ-Cauchy sequence of elements from A, then $(a_n)_{n\in\mathbb{N}}$ is also a d-Cauchy sequence. As (A,d) is complete, there exists $l\in A$ such that $\lim\limits_{n\to\infty} d(a_n,l)=0$ and therefore $\lim\limits_{n\to\infty}\rho(a_n,l)=0$. $\square$

Proposition 4.2. *In the above framework* $((A,\rho),(f_i)_{i\in I})$ *is a* PISSS.

Proof. According to Theorem 4.1 (c), (A, ρ) is complete and bounded.

As the metrics d and ρ are equivalent, the function $f_i\colon (A,\rho) \to (A,\rho)$ is continuous for each $i \in I$ (since $f_i\colon (A,d) \to (A,d)$ is continuous) and the equality $A = \overline{\bigcup_{i\in I} A_i}$, which is valid for d, is also true for ρ.

Moreover, for every $x, y \in A$, $n \in \mathbb{N}$ and $\omega \in \Lambda_n(I)$, we have

$$\begin{aligned}\rho(f_\omega(x), f_\omega(y)) &= \sup_{\theta\in\Lambda^*(I)} d(f_\theta(f_\omega(x)), f_\theta(f_\omega(y))) = \sup_{\theta\in\Lambda^*(I)} d(f_{\theta\omega}(x)), f_{\theta\omega}(y)) \\ &\le \sup_{\theta\in\Lambda^*(I)} \operatorname{diam}_d(A_{\theta\omega}) \le \sup_{\theta\in\Lambda^*(I)} \operatorname{diam}_d(A_{[\theta\omega]_n}) \le \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}_d(A_\omega),\end{aligned}$$

so

$$\sup_{\omega\in\Lambda_n(I)} \operatorname{diam}_\rho(A_\omega) \le \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}_d(A_\omega), \quad \text{for every } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}_d(A_\omega) = 0$, from the previous inequality it follows that $\lim_{n\to\infty} \sup_{\omega\in\Lambda_n(I)} \operatorname{diam}_\rho(A_\omega) = 0$. We conclude that $((A,\rho),(f_i)_{i\in I})$ is a PISSS. $\square$

Remark 4.3. According to Propositions 3.9 and 4.2, for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that the inequality $\rho(f_{[\omega]_n}(a), a_\omega) < \varepsilon$ is valid for every $n \in \mathbb{N}^*$, $n \ge n_\varepsilon$, $a \in A$ and $\omega \in \Lambda(I)$.

Using the method of mathematical induction, we get a strictly increasing sequence of natural numbers $(m_k)_{k\in\mathbb{N}^*}$ such that the inequality

$$\rho(f_{[\omega]_n}(a), a_\omega) < \frac{5^{k-1}}{2^{4k}}$$

is valid for every $k \in \mathbb{N}^*$, $n \in \mathbb{N}^*$, $n \ge m_k$, $a \in A$ and $\omega \in \Lambda(I)$.

Note that, using the triangle inequality, we get that

$$\rho(f_{[\omega]_n}(a_1), f_{[\omega]_n}(a_2)) < \frac{5^{k-1}}{2^{4k-1}}$$

for each $k \in \mathbb{N}^*$, $n \in \mathbb{N}^*$, $n \ge m_k$, $a_1, a_2 \in A$ and $\omega \in \Lambda(I)$.

5. A remetrization that makes φ-contractions all the constitutive functions of a UPISSS

In this section, given a UPISSS $((A,d),(f_i)_{i\in I})$, we construct a comparison function φ and a metric δ, which is equivalent to d, such that all the functions $f_i : (A,\delta) \to (A,\delta)$ are φ-contractions.

We mention that in the sequel:

- By L we mean $\lim_{n\to\infty} L_n$, where $L_n = \prod_{k=1}^{n+1}(1+2^{-k})$ for every $n \in \mathbb{N}$. Note that, since

$$\ln L_n = \ln \prod_{k=1}^{n+1}(1+2^{-k}) = \sum_{k=1}^{n+1} \ln(1+2^{-k}) \le \sum_{k=1}^{n+1} 2^{-k} < 1$$

for every $n \in \mathbb{N}$, the sequence $(L_n)_{n\in\mathbb{N}}$ is bounded. As it is clear that it is also increasing, we infer that it is convergent.

- $x_k = 5^{k-1}/2^{4k-1}$ for every $k \in \mathbb{N}^*$.
- $(m_k)_{k\in\mathbb{N}^*}$ is the sequence from Remark 4.3 and $y_k = L_{m_k}/L_{m_k} + 1 = 2^{m_k+2}/2^{m_k+2} + 1$ for every $k \in \mathbb{N}^*$.

Given a UPISSS $((A,d),(f_i)_{i\in I})$, we consider the function $\delta\colon A\times A \to [0,\infty]$ given by

$$\delta(x,y) = \sup_{\omega\in\Lambda^*(I)} L_{|\omega|}\rho(f_\omega(x), f_\omega(y)),$$

for every $x, y \in A$, where ρ is the metric introduced in Theorem 4.1.

Proposition 5.1. *In the above framework, the inequality*

$$\frac{3}{2}\rho(x,y) \le \delta(x,y) \le L\rho(x,y),$$

is valid for every $x, y \in A$.

Proof. On the one hand, for every $x, y \in A$, we have

$$\frac{3}{2}\rho(x,y) = L_{|\lambda|}\rho(f_\lambda(x), f_\lambda(y)) \le \delta(x,y).$$

On the other hand, since by Theorem 4.1 (a) the inequality

$$L_{|\omega|}\rho(f_\omega(x), f_\omega(y)) \le L_{|\omega|}\rho(x,y) \le L\rho(x,y)$$

is valid for every $\omega \in \Lambda^*(I)$, $x, y \in A$, we get that

$$\delta(x,y) = \sup_{\omega\in\Lambda^*(I)} L_{|\omega|}\rho(f_\omega(x), f_\omega(y)) \le L\rho(x,y),$$

for every $x, y \in A$. □

Hence $\delta\colon A\times A \to [0,\infty)$ and it is a metric which is equivalent to ρ, so to d, as the reader can routinely verify.

Proposition 5.2. *In the above framework, the inequality*

$$\delta(f_i(x), f_i(y)) \le \delta(x,y),$$

is valid for every $x, y \in A$ and every $i \in I$.

Proof. We have

$$\begin{aligned} L_{|\omega|}\rho(f_\omega(f_i(x)), f_\omega(f_i(y))) &= L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \\ &\le L_{|\omega i|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \le \sup_{\omega\in\Lambda^*(I)} L_{|\omega|}\rho(f_\omega(x), f_\omega(y)) = \delta(x,y) \end{aligned}$$

for every $x, y \in A$, $i \in I$ and $\omega \in \Lambda^*(I)$, so

$$\delta(f_i(x), f_i(y)) = \sup_{\omega\in\Lambda^*(I)} L_{|\omega|}\rho(f_\omega(f_i(x)), f_\omega(f_i(y))) \le \delta(x,y),$$

for every $x, y \in A$ and every $i \in I$. □

PROPOSITION 5.3. *In the above framework, the inequality*

$$\delta(f_i(x), f_i(y)) \leq \max\left\{ \sup_{\omega \in \Lambda^*(I),\, |\omega| < m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), Lx_k \right\}$$

is valid for every $x, y \in A$, $i \in I$ and $k \in \mathbb{N}^*$.

PROOF. We have

$$\delta(f_i(x), f_i(y)) = \sup_{\omega \in \Lambda^*(I)} L_{|\omega|}\rho(f_\omega(f_i(x)), f_\omega(f_i(y)))$$

$$= \max\left\{ \sup_{\omega \in \Lambda^*(I), |\omega| < m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), \sup_{\omega \in \Lambda^*(I), |\omega| \geq m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)) \right\}$$

$$\overset{\text{Remark 4.3}}{\leq} \max\left\{ \sup_{\omega \in \Lambda^*(I), |\omega| < m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), Lx_k \right\},$$

for every $x, y \in A$, $i \in I$ and $k \in \mathbb{N}^*$. □

PROPOSITION 5.4. *In the above framework, for every $k \in \mathbb{N}$, $x, y \in A$ and $i \in I$, we have $\delta(f_i(x), f_i(y)) \leq y_k \delta(x, y)$, provided that $Lx_k < \delta(f_i(x), f_i(y))$.*

PROOF. Let us consider $k \in \mathbb{N}$, $x, y \in A$ and $i \in I$ such that $Lx_k < \delta(f_i(x), f_i(y))$. Then, taking into account Proposition 5.3, we have

$$Lx_k < \delta(f_i(x), f_i(y)) \leq \max\left\{ \sup_{\omega \in \Lambda^*(I), |\omega| < m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)), Lx_k \right\},$$

so,

$$\delta(f_i(x), f_i(y)) \leq \sup_{\omega \in \Lambda^*(I), |\omega| < m_k} L_{|\omega|}\rho(f_{\omega i}(x), f_{\omega i}(y)).$$

Then, for every $\varepsilon > 0$ there exists $\omega_\varepsilon \in \Lambda^*(I)$, $|\omega_\varepsilon| < m_k$ such that

$$\delta(f_i(x), f_i(y)) - \varepsilon < L_{|\omega_\varepsilon|}\rho(f_{\omega_\varepsilon i}(x), f_{\omega_\varepsilon i}(y))$$

and consequently, as the sequence $(L_n/L_{n+1})_{n \in \mathbb{N}^*}$ is increasing, we get

$$\delta(f_i(x), f_i(y)) - \varepsilon < L_{|\omega_\varepsilon i|}\rho(f_{\omega_\varepsilon i}(x), f_{\omega_\varepsilon i}(y)) \frac{L_{|\omega_\varepsilon|}}{L_{|\omega_\varepsilon i|}} \leq \frac{L_{|\omega_\varepsilon|}}{L_{|\omega_\varepsilon i|}} \delta(x, y) \leq y_k \delta(x, y),$$

for every $\varepsilon > 0$. Therefore $\delta(f_i(x), f_i(y)) \leq y_k \delta(x, y)$. □

THEOREM 5.5. *Let $((A, d), (f_i)_{i \in I})$ be a* UPISSS. *Then there exist a comparison function φ and a metric δ, which is equivalent to d, such that*

$$\delta(f_i(x), f_i(y)) \leq \varphi(\delta(x, y)),$$

for every $x, y \in A$ and $i \in I$, i.e. the function $f_i \colon (A, \delta) \to (A, \delta)$ is φ-contraction for every $i \in I$.

Proof. Note that, in the above framework, the strictly decreasing sequence $(x_k)_{k\in\mathbb{N}^*}$ of positive reals converges to 0 and the strictly increasing sequence $(y_k)_{k\in\mathbb{N}^*}$ of reals greater or equal to $1/2$ converges to 1.

With the notation $z_k = 2Lx_k$, one can easily check that, for every $k \in \mathbb{N}^*$, $z_k \le z_{k-1}/2$ and $z_k y_{k+1} \le z_{k-1} y_k$. Moreover, we have

$$(*)\ \ \forall\, i \in I \quad \forall\, k \in \mathbb{N}^* \quad \forall\, x, y \in A \quad \delta(x,y) > z_k \ \Rightarrow\ \delta(f_i(x), f_i(y)) \le y_k \delta(x,y).$$

Indeed, if $\delta(f_i(x), f_i(y)) \le z_k/2$, then we have $\delta(f_i(x), f_i(y)) \le z_k/2 < \delta(x,y)/2 \le y_k\delta(x,y)$ and if $\delta(f_i(x), f_i(y)) > z_k/2$, we just use Proposition 5.4.

Now we define the function $\varphi\colon [0,\infty) \to [0,\infty)$ in the following way:

$$\varphi(0) = 0, \quad \varphi(t) = t - z_1(1 - y_2)$$

for $t \in (z_1, \infty)$ and

$$\varphi(t) = \left(\frac{t - z_k}{z_{k-1} - z_k}\right) z_{k-1} y_k + \left(\frac{z_{k-1} - t}{z_{k-1} - z_k}\right) z_k y_{k+1},$$

for every $k \in \mathbb{N}$, $k \ge 2$ and every $t \in (z_k, z_{k-1}]$. It is clear that φ is a comparison function.

Now we prove that $\delta(f_i(x), f_i(y)) \le \varphi(\delta(x,y))$, for every $x, y \in A$ and $i \in I$. Since the above inequality is obvious if $\delta(x,y) = 0$, we shall treat the following two cases:

(i) $\delta(x,y) \in (z_1, \infty)$;
(ii) $\delta(x,y) \in (z_k, z_{k-1}]$ for some $k \in \mathbb{N}$, $k \ge 2$.

In the first case, from $(*)$, we infer that $\delta(f_i(x), f_i(y)) \le y_1\delta(x,y)$ for every $i \in I$. As $z_1 < \delta(x,y)$ and $y_1 \le y_2$, we obtain

$$y_1\delta(x,y) \le \delta(x,y) - z_1(1 - y_2) = \varphi(\delta(x,y))$$

and we conclude that $\delta(f_i(x), f_i(y)) \le \varphi(\delta(x,y))$ for every $i \in I$.

In the second case, using again $(*)$, we get $\delta(f_i(x), f_i(y)) \le y_k\delta(x,y)$ for every $i \in I$. As $z_k < \delta(x,y) \le z_{k-1}$, we obtain

$$y_k\delta(x,y) \le \left(\frac{\delta(x,y) - z_k}{z_{k-1} - z_k}\right) z_{k-1} y_k + \left(\frac{z_{k-1} - \delta(x,y)}{z_{k-1} - z_k}\right) z_k y_{k+1} = \varphi(\delta(x,y)),$$

and we conclude that $\delta(f_i(x), f_i(y)) \le \varphi(\delta(x,y))$ for every $i \in I$. □

Definition 5.6. Given a metric space (X,d), a possibly infinite iterated function system is a pair $\mathcal{S} = ((X,d), (f_i)_{i\in I})$, where $f_i\colon X \to X$ is continuous for every $i \in I$.

Definition 5.7. Given a comparison function $\varphi\colon [0,\infty) \to [0,\infty)$, a possibly infinite iterated function system $\mathcal{S} = ((X,d), (f_i)_{i\in I})$ is called φ-hyperbolic if there exists a metric δ on X, equivalent to d, such that the function $f_i\colon (X,\delta) \to (X,\delta)$ is φ-contraction for every $i \in I$.

Now, Theorem 5.5 could be restated in the following way:

THEOREM 5.8. *Let* $((A,d),(f_i)_{i\in I})$ *be a* UPISSS. *Then there exists a comparison function* φ *such that the possibly infinite iterated function system* $\mathcal{S} = ((A,d),(f_i)_{i\in I})$ *is* φ*-hyperbolic.*

REMARK 5.9. Taking into account Proposition 3.7, which states that each possibly infinite iterated function system whose constitutive functions form a bounded family of φ-contractions generates a uniformly possibly infinite self-similar system, and Theorem 5.8, that says that for each uniformly possibly infinite self-similar system there exists a comparison function φ such that it becomes a φ-hyperbolic possibly infinite iterated function system, we conclude that there exists a strong connection between uniformly possibly infinite self-similar systems and φ-hyperbolic possibly infinite iterated function systems.

6. Kameyama's topological self-similar systems are particular cases of possibly infinite self-similar systems

PROPOSITION 6.1. *In the framework of Definition* 1.1*, we have*

$$K = \overline{\bigcup_{i=1}^{N} K_i}.$$

PROOF. Indeed, for each $x \in K = \pi(\Lambda)$ there exists $\omega = \omega_1 \dots \omega_m \omega_{m+1} \dots$ in Λ such that $x = \pi(\omega) = \pi(\omega_1 \omega') = f_{\omega_1}(\pi(\omega')) \in K_{\omega_1}$, where $\omega' = \omega_2 \dots \omega_{m+1} \dots$, so $x \in \bigcup_{i=1}^{N} K_i$. Thus $K \subseteq \bigcup_{i=1}^{N} K_i \subseteq K$, so $K = \bigcup_{i=1}^{N} K_i$. As K is compact, we infer that $K = \overline{\bigcup_{i=1}^{N} K_i}$. □

THEOREM 6.2 (see [10, Theorem 5.1]). *A topological self-similar set is metrizable.*

PROPOSITION 6.3 (see [10, Lemma 1.6]). *Let* $(K, \{f_i\}_{i\in\{1,\dots,N\}})$ *be a topological self-similar system and* d *any metric on* K *which is compatible with the original topology. Then*

$$\lim_{n\to\infty} \left(\max_{\omega\in\Lambda_n(\{1,\dots,N\})} \operatorname{diam}(K_\omega) \right) = 0.$$

REMARK 6.4. From Proposition 6.1 and Proposition 6.3, we infer that if $(K, \{f_i\}_{i\in\{1,\dots,N\}})$ is a topological self-similar system and d any metric on K which is compatible with the original topology, then $\mathcal{S} = ((K,d),(f_i)_{i\in I})$, where $I = \{1,\dots,N\}$, is a PISSS. In addition, since the functions f_i are continuous and the set I is finite, $\mathcal{S}$ is a UPISSS. Consequently Kameyama's topological self-similar systems are particular cases of possibly infinite self-similar systems and Theorem 5.5 is a generalization of our result from [13] stating that given a topological self-similar system $(K,(f_i)_{i\in\{1,\dots,N\}})$ there exist a metric δ on K which is compatible with the original topology and a comparison function φ such that $f_i\colon (K,\delta)\to(K,\delta)$ is φ-contraction for every $i\in\{1,\dots,N\}$.

Now let us consider a PISSS $\mathcal{S}=((A,d),(f_i)_{i\in I})$ for which the set I is finite.

Proposition 6.5. *In the above framework, (A,d) is a compact Hausdorff topological space.*

Proof. From the definition of a PISSS we have $A=\overline{\bigcup_{i\in I} A_i}$, so

$$A_j=f_j\bigg(\bigcup_{i\in I}\overline{A_i}\bigg)=\bigcup_{i\in I} f_j(\overline{A_i})\overset{f_j\text{ continuous}}{\subseteq}\bigcup_{i\in I}\overline{f_j(A_i)}=\bigcup_{i,j\in I}\overline{A_{ji}}$$

for every $j\in I$. Hence $A=\overline{\bigcup_{j\in I} A_j}\subseteq\bigcup_{i,j\in I}\overline{A_{ji}}\subseteq A$, so $A=\bigcup_{\omega\in\Lambda_2(I)}\overline{A_\omega}$. In a similar way we can prove that

$$(*)\qquad A=\bigcup_{\omega\in\Lambda_n(I)}\overline{A_\omega}\quad\text{for every } n\in\mathbb{N}.$$

As $\lim_{n\to\infty}\sup_{\omega\in\Lambda_n(I)}\operatorname{diam}(\overline{A_\omega})=\lim_{n\to\infty}\sup_{\omega\in\Lambda_n(I)}\operatorname{diam}(A_\omega)=0$ and $\Lambda_n(I)$ is finite, from $(*)$ we infer that A is totally bounded. Since it is also complete, we conclude that it is compact. $\square$

Theorem 5.5 assures us that there exist a comparison function φ and a metric δ, which is equivalent to d, such that all the functions $f_i\colon(A,\delta)\to(A,\delta)$ are φ-contractions. Since $A=\bigcup_{i\in I} f_i(A)$, we come to the conclusion that the attractor of the iterated function system $((A,\delta),(f_i)_{i\in I})$ is A. Note that, taking into account Proposition 3.8, we can consider the function $\pi\colon\Lambda(I)\to A$ given by

$$\pi(\omega)=a_\omega\quad\text{for every }\omega\in\Lambda(I).$$

Then, from the standard properties of such iterated function systems (see, for example, [6], where the more general case of iterated function systems consisting of Meir–Keeler functions is treated) we obtain the following result:

Proposition 6.6. *In the above framework, the function π has the following properties*:

(a) *it is onto*;
(b) $\pi\circ\tau_i=f_i\circ\pi$ *for every* $i\in I$;
(c) *it is continuous.*

Remark 6.7. From Propositions 6.5 and 6.6, we conclude that a PISSS $\mathcal{S}=((A,d),(f_i)_{i\in I})$ for which the set I is finite is a topological self-similar system, hence the concepts of PISSS and topological self-similar system coincide for finite sets I.

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RADU MICULESCU AND ALEXANDRU MIHAIL
Faculty of Mathematics and Computer Science
University of Bucharest
Str. Academiei 14
010014 Bucharest, ROMANIA

E-mail address: miculesc@yahoo.com, mihail_alex@yahoo.com