Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 195–217
DOI: 10.12775/TMNA.2016.002

EXISTENCE RESULTS FOR A CLASS OF HEMIVARIATIONAL INEQUALITIES INVOLVING THE STABLE (g, f, α)-QUASIMONOTONICITY

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Abstract. In this paper, by introducing a new concept of the stable (g, f, α)-quasimonotonicity and applying the properties of Clarke's generalized gradient and KKM technique, we show the existence results of solutions for hemivariational inequalities when the constraint set is compact, bounded and unbounded, respectively, which extends and improves several well-known results in many respects. In the last section, we also give an example to present the our main result.

1. Introduction

As an important and useful generalization of variational inequalities, hemivariational inequalities were first introduced by Panagiotopoulos (see [15], [16]) as the variational formulation of an important class of unilateral or inequality problems in mechanics. It is based on the notation of Clarke's generalized gradient for a class of locally Lipschitz functions. Hemivariational inequalities appear in a variety of mechanical problems, for example, the unilateral contact problems

2010 *Mathematics Subject Classification.* 47H04, 49J52.

Key words and phrases. Existence results; hemivariational inequalities; Clarke's generalized gradient; KKM principle; stable (g, f, α)-quasimonotonicity.

The first author was supported by NNSF of China Grant Nos. 11271087, 61263006, NSF of Guangxi Grant No. 2014GXNSFDA118002 and Special Funds of Guangxi Distinguished Experts Construction Engineering.

The second author was supported by the Innovation Project of Guangxi University for Nationalities No. gxun-chx2014098 and Science Research Project 2014 of the China-ASEAN Study Center (Guangxi Science Experiment Center) of Guangxi University for Nationalities.

in nonlinear elasticity, the problems describing the adhesive and friction effects, the nonconvex semipermeability problems, the masonry structures, and the delamination problems in multilayered composites (one can see [13], [14], [17]). In the last few years many kinds of hemivariational inequalities have been studied (see [2], [3], [9]–[12], [20]) and the study of hemivariational inequalities has emerged as a new and interesting branch of applied mathematics.

Very recently, many authors studied the existence results for some types of hemivariational inequalities (see [18], [19], [21]). In 2011, Zhang and He ([21]) study a kind of hemivariational inequalities of the Hartman–Stampacchia type by introducing the concept of stable quasimonotonicity. They considered that the constraint set is a bounded (or unbounded), closed and convex subset in a reflexive Banach space. The authors gave sufficient conditions for the existence and boundedness of solutions. In 2013, Tang and Huang ([18]) generalized the result of [21], by introducing the concept of stable ϕ-quasimonotonicity. By applying the stable ϕ-quasimonotonicity and the properties of Clarke's generalized directional derivative and generalized gradient, they obtained some existence theorems when the constrained set is nonempty, bounded (or unbounded), closed and convex in a reflexive Banach space. In the same year, Wangkeeree and Preechasilp ([19]) generalized the results of [18] and [21], by introducing the concept of stable f-quasimonotonicity. By applying the stable f-quasimonotonicity, they obtained some existence theorems similar to [18].

The aim of this paper is to study the existence of solutions for generalized problems of hemivariational inequalities in a reflexive Banach space. To establish our results, we introduce a new concept of stable (g, f, α)-quasimonotonicity and use the properties of Clarke's generalized directional derivative, generalized gradient, and KKM technique. Our results extend and improve some results in [18], [19], [21] in many respects.

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries and necessary materials. In Section 3, we introduce some kinds of generalized monotonicity of a mapping. In Section 4, we are devoted to proving our main results. We show the existence of solutions in the case when the constraint set is compact, bounded and unbounded in Theorems 4.1, 4.2 and 4.6, respectively. Theorem 4.8 provides a sufficient condition to the boundedness of the solution set. In Section 5, we give an example to present the generalized monotonicity and our main result.

2. Preliminaries

Let E be a real Banach space with the norm denoted by $\|\cdot\|_E$. Denote by E^* its dual space and by $\langle\cdot\,,\cdot\rangle_E$ the duality pairing between E^* and E. Let $F\colon K(\subseteq E) \rightrightarrows E^*$ be a multivalued mapping, $g\colon K\times K\to E$ be a mapping.

Let $f\colon K \times K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be a mapping satisfying the set $\mathcal{D}(f) = \{u \in K : f(u,v) \neq -\infty, \text{ for all } v \in K\} \neq \emptyset$. Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$ and $\partial\Omega$ be its boundary. Let X be Ω or $\partial\Omega$, $T\colon E \to L^p(X;\mathbb{R}^k)$ a linear continuous operator, where $k \geq 1$, $1 < p < \infty$. Let $j^\circ(x,y;w)$ be the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x,\cdot)\colon \mathbb{R}^k \to \mathbb{R}$ at the point $y \in \mathbb{R}^k$ with respect to direction $w \in \mathbb{R}^k$, where $x \in X$. In this paper, we discuss the following hemivariational inequalities involving a multivalued mapping and a nonlinear term:

$$(2.1)\qquad \begin{cases} \text{Find } u \in \mathcal{D}(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, g(u,v)\rangle + f(u,v) + \displaystyle\int_X j^\circ(x,\widehat{u}(x);\widehat{g}(u,v)(x))\,dx \geq 0, \\ \text{for all } v \in K, \end{cases}$$

where $\widehat{u} := Tu$, $\widehat{g}(u,v) := Tg(u,v)$.

By introducing the concept of stable quasimonotonicity, Zhang and He ([21]) considered the following hemivariational inequalities of the Hartman–Stampacchia type:

$$(2.2)\qquad \begin{cases} \text{Find } u \in K \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v-u\rangle + \displaystyle\int_\Omega j^\circ(x,\widehat{u}(x);\widehat{v}(x)-\widehat{u}(x))\,dx \geq 0, \\ \text{for all } v \in K, \end{cases}$$

where K is a bounded (or unbounded), closed and convex subset in a reflexive Banach space E.

By introducing the concept of stable ϕ-quasimonotonicity with respect to a certain subset U of E^*, Tang and Huang ([18]) generalized the result of [21], with the following variational-hemivariational inequalities:

$$(2.3)\qquad \begin{cases} \text{Find } u \in K \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v-u\rangle + \phi(v) - \phi(u) + \displaystyle\int_\Omega j^\circ(x,\widehat{u}(x);\widehat{v}(x)-\widehat{u}(x))\,dx \geq 0, \\ \text{for all } v \in K, \end{cases}$$

where $\phi\colon E \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous function such that $K_\phi := K \cap \operatorname{dom}\phi \neq \emptyset$.

By introducing the concept of stable f-quasimonotonicity with respect to a certain subset U of E^*, Wangkeeree and Preechasilp [19] generalized the results of [18] and [21], with the following variational-hemivariational inequalities:

$$(2.4)\qquad \begin{cases} \text{Find } u \in K \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v-u\rangle + f(u,v) + \displaystyle\int_\Omega j^\circ(x,\widehat{u}(x);\widehat{v}(x)-\widehat{u}(x))\,dx \geq 0, \\ \text{for all } v \in K, \end{cases}$$

where $f\colon K \times K \to \overline{\mathbb{R}}$ is such that the set $\mathcal{D}(f) = \{u \in K : f(u,v) \neq -\infty$, for all $v \in K\}$, is nonempty.

For a suitable choice of F, g, f and T one can obtain a wide class of inequality problems, including mixed variational inequalities and Stampacchia variational inequalities. For example, by choosing $g(u,v) = v - u$, one easily sees that problems (2.2)–(2.4) in [18], [19], [21] are special cases of (2.1). Moreover, our new concept of the stable (g,f,α)-quasimonotonicity is more general than that of the stable f-quasimonotonicity in [19]. Therefore, our results extend and improve some results in [18], [19], [21].

Now, we introduce some basic preliminaries. For a nonempty, closed and convex subset K of E and every $r > 0$, we define

$$B_r := \{u \in K : \|u\|_E \leq r\}.$$

Let us recall the notations related to the Clarke's generalized directional derivative and generalized gradient for a locally Lipschitz function $h\colon E \to \mathbb{R}$ (see [1], [4]). Denote by $h^0(u;v)$ the Clarke's generalized directional derivative of h at the point $u \in E$ in the direction $v \in E$, that is

$$h^0(u;v) := \limsup_{\lambda \to 0^+,\, \zeta \to u} \frac{h(\zeta + \lambda v) - h(\zeta)}{\lambda}.$$

Recall also that the Clarke's subdifferential or generalized gradient of h at $u \in E$, denoted by $\partial h(u)$, is a subset of E^* given by

$$\partial h(u) := \{u^* \in E^* : h^0(u;v) \geq \langle u^*, v\rangle_E, \text{ for all } v \in E\}.$$

Let $j\colon X \times \mathbb{R}^k \to \mathbb{R}$ be a function and the mapping

$$j(\,\cdot\,, y)\colon X \to \mathbb{R} \text{ be measurable, for every } y \in \mathbb{R}^k. \tag{2.5}$$

We assume that at least one of the following conditions holds: either there exists $l \in L^q(X;\mathbb{R})$ $(1/p + 1/q = 1)$ such that

$$|j(x,y_1) - j(x,y_2)| \leq l(x)|y_1 - y_2|, \quad \text{for all } x \in X, \text{ for all } y_1, y_2 \in \mathbb{R}^k, \tag{2.6}$$

or

$$\text{the mapping } j(x,\,\cdot\,) \text{ is locally Lipschitz, for all } x \in X, \tag{2.7}$$

and there exists $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \quad \text{for all } x \in X, \text{ for all } z \in \partial j(x,y). \tag{2.8}$$

Let $J\colon L^p(X;\mathbb{R}^k) \to \mathbb{R}$ be an arbitrary locally Lipschitz functional. For each $u, v \in E$ there exists $z_u \in \partial J(\widehat{u})$ such that

$$J^0(\widehat{u};\widehat{v}) = \langle z_u, \widehat{v}\rangle_{L^p} = \max\{\langle \omega, \widehat{v}\rangle_{L^p} : \omega \in \partial J(\widehat{u})\}.$$

Denoting by $T^*\colon L^q(X;\mathbb{R}^k) \to E^*$ the adjoint operator of T, we define the subset $U(J,T)$ of E^* as follows:

$$U(J,T) = \{-z_u^* : u \in K, z_u^* = T^* z_u\}. \tag{2.9}$$

From Remark 2.2 of [21] it is easy to deduce that

$$J^0(\widehat{u};\widehat{g}(u,v)) = \langle z_u^*, g(u,v)\rangle_E. \tag{2.10}$$

LEMMA 2.1 ([1, Proposition 2.1.1]). *Let $h\colon K \to \mathbb{R}$ be locally Lipschitz of rank L_u near u. Then*:

(a) *the function $v \mapsto h^0(u;v)$ is finite, positively homogeneous, subadditive on E, and satisfies $|h^0(u;v)| \le L_u\|v\|$;*
(b) *$h^0(u;v)$ is upper semicontinuous as a function of (u,v), and as a function of v alone, is Lipschitz of rank L_u near u on E;*
(c) *$h^0(u;-v) = (-h)^0(u;v)$.*

LEMMA 2.2 ([1, Theorem 2.7.5]). *If $J(\varphi) = \int_X j(x,\varphi(x))\,dx$, and j satisfies conditions* (2.5) *and* (2.6) *or* (2.5) *and* (2.7)–(2.8), *then J is Lipschitz on bounded subsets, and one has*

$$\partial J(\varphi) \subset \int_X \partial j(x,\varphi(x))\,dx.$$

Furthermore, if j is regular at $(x,\varphi(x))$ then J is regular at φ and equality holds.

In order to obtain the main result of this paper, we need the following lemma.

LEMMA 2.3 ([5, Theorem 2.7.5]). *Let K be a nonempty subset of a Hausdorff topological vector space E and let $G\colon K \rightrightarrows E$ be a multivalued mapping satisfying the following properties*:

(a) *G is a* KKM *mapping, i.e.*

$$\operatorname{conv}\{x_1,\dots,x_n\} \subset \bigcup_{i=1}^n G(x_i), \quad \text{for } x_i \in K,\ i = 1,\dots,n;$$

(b) *$G(x)$ is closed in E for every $x \in K$;*
(c) *$G(x_0)$ is compact in E for some $x_0 \in K$.*

Then $\bigcap_{x\in K} G(x) \neq \emptyset$.

We also need the following definition (see [8]).

DEFINITION 2.4. The multivalued mapping $F\colon K \rightrightarrows E^*$ is said to be

(a) *lower semicontinuous* (l.s.c.) at x_0, if for any $x_0^* \in F(x_0)$ and any sequence $\{x_n\}_{n\ge1} \subset K$ with $x_n \to x_0$, there exists a sequence $x_n^* \in F(x_n)$ which converges to x_0^*;
(b) *lower hemicontinuous* (l.h.c.), if the restriction of F to every line segment of K is lower semicontinuous with respect to the weak topology in E^*.

Now, we recall some concepts of monotonicity which can be founded in [21], [18], [19].

Definition 2.5. Let $\phi\colon K \to \mathbb{R}$ be a function, and $F\colon K \rightrightarrows E^*$ a multivalued mapping, $f\colon K \times K \to \mathbb{R}$ a bifunction. Then F is said to be

(a) *monotone*, if for each $u, v \in K$,

$$\langle v^* - u^*, v - u\rangle \geq 0, \quad \text{for all } u^* \in F(u) \text{ and } v^* \in F(v);$$

(b) *pseudomonotone*, if for each $u, v \in K$,

$$\langle u^*, v - u\rangle \geq 0 \Rightarrow \langle v^*, v - u\rangle \geq 0, \quad \text{for all } u^* \in F(u) \text{ and } v^* \in F(v);$$

(c) *quasimonotone*, if for each $u, v \in K$,

$$\langle u^*, v - u\rangle > 0 \Rightarrow \langle v^*, v - u\rangle \geq 0, \quad \text{for all } u^* \in F(u) \text{ and } v^* \in F(v);$$

(d) *stably pseudomonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are pseudomonotone for every $\zeta \in U$;

(e) *stably quasimonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are quasimonotone for every $\zeta \in U$;

(f) $\phi(\,\cdot\,)$*-pseudomonotone* (ϕ*-pseudomonotone*), if for each $u, v \in K$,

$$\langle u^*, v - u\rangle + \phi(v) - \phi(u) \geq 0 \Rightarrow \langle v^*, v - u\rangle + \phi(v) - \phi(u) \geq 0,$$

for all $u^* \in F(u)$ and $v^* \in F(v)$;

(g) $\phi(\,\cdot\,)$*-quasimonotone* (ϕ*-quasimonotone*), if for each $u, v \in K$,

$$\langle u^*, v - u\rangle + \phi(v) - \phi(u) > 0 \Rightarrow \langle v^*, v - u\rangle + \phi(v) - \phi(u) \geq 0,$$

for all $u^* \in F(u)$ and $v^* \in F(v)$;

(h) *stably* ϕ*-pseudomonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are ϕ-pseudomonotone for every $\zeta \in U$;

(i) *stably* ϕ*-quasimonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are ϕ-quasimonotone for every $\zeta \in U$;

(j) $f(\,\cdot\,,\,\cdot\,)$*-pseudomonotone* (f*-pseudomonotone*), if for each $u, v \in K$,

$$\langle u^*, v - u\rangle + f(u, v) \geq 0 \Rightarrow \langle v^*, v - u\rangle + f(u, v) \geq 0,$$

for all $u^* \in F(u)$ and $v^* \in F(v)$;

(k) $f(\,\cdot\,,\,\cdot\,)$*-quasimonotone* (f*-quasimonotone*), if for each $u, v \in K$,

$$\langle u^*, v - u\rangle + f(u, v) > 0 \Rightarrow \langle v^*, v - u\rangle + f(u, v) \geq 0,$$

for all $u^* \in F(u)$ and $v^* \in F(v)$;

(l) *stably* f*-pseudomonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are f-pseudomonotone for every $\zeta \in U$;

(m) *stably* f*-quasimonotone with respect to the set* $U \subset E^*$, if F and $F(\,\cdot\,) - \zeta$ are f-quasimonotone for every $\zeta \in U$.

3. Generalized monotonicity

In this section, we introduce the concept of stable (g,f,α)-quasimonotonicity with respect to the set $U\subset E^*$ which is useful for establishing the existence theorems for the main results.

Definition 3.1. Let $g\colon K\times K\to E$, $f\colon K\times K\to\overline{\mathbb{R}}$, $\alpha\colon E\to\mathbb{R}$, and $F\colon K\rightrightarrows E^*$. Then F is said to be

(a) (g,f,α)-*pseudomonotone*, if for each $u,v\in K$,

$$\langle u^*,g(u,v)\rangle+f(u,v)\geq-\alpha(u-v)\Rightarrow\langle v^*,g(u,v)\rangle+f(u,v)\geq-\alpha(u-v),$$

for all $u^*\in F(u)$, $v^*\in F(v)$;

(b) (g,f,α)-*quasimonotone*, if for each $u,v\in K$,

$$\langle u^*,g(u,v)\rangle+f(u,v)>-\alpha(u-v)\Rightarrow\langle v^*,g(u,v)\rangle+f(u,v)\geq-\alpha(u-v),\tag{3.1}$$

for all $u^*\in F(u)$ and $v^*\in F(v)$;

(c) *stably* (g,f,α)*-pseudomonotone with respect to the set* $U\subset E^*$, if F and $F(\,\cdot\,)-\zeta$ are ϕ-pseudomonotone for every $\zeta\in U$;

(d) *stably* (g,f,α)*-quasimonotone with respect to the set* $U\subset E^*$, if F and $F(\,\cdot\,)-\zeta$ are ϕ-quasimonotone for every $\zeta\in U$.

Remark 3.2. (a) It is easy to verify that the (g,f,α)-quasimonotonicity is weaker than the (g,f,α)-pseudomonotonicity.

(b) If $g(u,v)=v-u$, $\alpha\equiv0$, then (3.1) becomes

$$\langle u^*,v-u\rangle+f(u,v)>0\Rightarrow\langle v^*,v-u\rangle+f(u,v)\geq0,$$

for all $u^*\in F(u)$ and $v^*\in F(v)$ almost everywhere, F is f-quasimonotone. But in most instances, $\langle u^*,g(u,v)\rangle+f(u,v)>0$ is false. So we have necessary to use $\langle u^*,g(u,v)\rangle+f(u,v)>-\alpha(u-v)$ for some α. This is one of motivation to introduce the generalized monotonicity above.

Remark 3.3. We represent the implications between monotonicity and some kinds of generalized monotonicity through the following two diagrams (see [18], [19], [21]):

$$\text{(D1)}\qquad\begin{array}{ccccc}\text{monotone} & \Rightarrow & \begin{array}{c}\text{stably pseudomonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \text{pseudomonotone}\\ & & \Downarrow & & \Downarrow\\ & & \begin{array}{c}\text{stably quasimonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \text{quasimonotone}\end{array}$$

$$
\text{(D2)}\qquad
\begin{array}{ccccc}
\text{monotone} & \Rightarrow & \begin{array}{c}\text{stably } \phi\text{-pseudomonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \phi\text{-pseudomonotone}\\
 & & \Downarrow & & \Downarrow\\
 & & \begin{array}{c}\text{stably } \phi\text{-quasimonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \phi\text{-quasimonotone}
\end{array}
$$

$$
\text{(D3)}\qquad
\begin{array}{ccccc}
\text{monotone} & \Rightarrow & \begin{array}{c}\text{stably } f\text{-pseudomonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & f\text{-pseudomonotone}\\
 & & \Downarrow & & \Downarrow\\
 & & \begin{array}{c}\text{stably } f\text{-quasimonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & f\text{-quasimonotone}
\end{array}
$$

$$
\text{(D4)}\qquad
\begin{array}{ccccc}
\text{monotone} & \Rightarrow & \begin{array}{c}\text{stably } \alpha\text{-pseudomonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \alpha\text{-pseudomonotone}\\
 & & \Downarrow & & \Downarrow\\
 & & \begin{array}{c}\text{stably } \alpha\text{-quasimonotone}\\ \text{with respect to the set } U\end{array} & \Rightarrow & \alpha\text{-quasimonotone}
\end{array}
$$

where α above stands for (g, f, α).

The inverse direction of each implication relationship mentioned by two diagrams above does not hold in general. Example 2 in [6] had shown that a stably pseudomonotone mapping with respect to a closed line segment is not necessarily monotone; Example 3.1 in [7] had shown that a quasimonotone mapping may not be pseudomonotone; Example 4.1 in [21] had shown that a stably quasimonotone mapping is not necessarily stable pseudomonotone; Example 3.1 in [18] had shown that stably ϕ-pseudomonotone mapping may not be monotone, while Example 4.1 in the same paper had shown that a stably ϕ-quasimonotone mapping is not necessarily stably ϕ-pseudomonotone and ϕ-quasimonotone mapping may not be ϕ-pseudomonotone. Example 3.4 in [19] had shown that stably ϕ-pseudomonotone mapping may not be f-quasimonotone, while Example 3.5 in the same paper has shown that a stably f-pseudomonotone mapping is not necessarily stably ϕ-quasimonotone.

Let A, A_ϕ and A_f denote the sets consisting of generalized monotonicity in diagrams (D1)–(D3), respectively. Example 3.1 in [18] had shown that each of generalized monotonicity taken from the set A is independent of any one taken from A_ϕ if $\phi \neq I_K$. Also, Examples 3.4 and 3.5 in [19] had shown that each of generalized monotonicity taken from the set A_ϕ is independent of any one taken from A_f if $f(u, v) \neq \phi(v) - \phi(u)$ for all $u, v \in K$.

The motivation of introducing the notions of stable (g, f, α)-pseudomonotonicity and stable (g, f, α)-quasimonooonicity is based on using some relatively weak conditions to solve the problems of hemivariational inequalities like (2.1). If $g(u,v)=v-u$ for all $u,v\in K$, and A_α denotes the set consisting of generalized monotonicity in diagram (D4), we give the following two examples to illustrate that each of generalized monotonicity taken from the set A_f is independent of any one taken from A_α if $\alpha\not\equiv 0$.

Example 3.4. Let $E=\mathbb{R}^2$ and $K=[3,5]\times\{0\}$. Let $g\colon K\times K\to E$, $f\colon K\times K\to\overline{\mathbb{R}}$, $\alpha\colon K\to\mathbb{R}$, $F\colon K\rightrightarrows E^*$ defined by, respectively,

$$g(u,v)=v-u,\quad f(u,v)=v_1^2-u_1^2,\quad \alpha(u)=4u_1,\quad F(u)=[-5,1]\times\{0\},$$

where $u=(u_1,0),\ v=(v_1,0)$. Then we get that F is stably f-pseudomonotone with respect to the set $V:=\{(0,m):m\in\mathbb{R}\}\subset\mathbb{R}^2$ but not (g,f,α)-quasimonotone.

At first, we show that F is f-pseudomonotone on K. If

$$0\le\langle u^*,v-u\rangle+f(u,v)=u_1^*(v_1-u_1)+v_1^2-u_1^2=(u_1^*+v_1+u_1)(v_1-u_1),$$

since $u_1^*\in[-5,1]$ and $u_1,v_1\in[3,5]$, we get that $u_1^*+v_1+u_1>0$. It implies that $v_1-u_1\ge 0$. Thus we have

$$\langle v^*,v-u\rangle+f(u,v)=(v_1^*+v_1+u_1)(v_1-u_1)\ge 0.$$

Hence F is f-pseudomonotone on K.

Next, we show that $F(\,\cdot\,)-\zeta$ is stably f-pseudomonotone for each $\zeta=(0,m)$ in V. If

$$0\le\langle u^*-\zeta,v-u\rangle+f(u,v)=u_1^*(v_1-u_1)+v_1^2-u_1^2=(u_1^*+v_1+u_1)(v_1-u_1),$$

since $u_1^*\in[-5,1]$ and $u_1\ ,v_1\in[3,5]$, we get that $u_1^*+v_1+u_1>0$. It implies that $v_1-u_1\ge 0$. Thus we have

$$\langle v^*-\zeta,v-u\rangle+f(u,v)=(v_1^*+v_1+u_1)(v_1-u_1)\ge 0.$$

Hence F is stably f-pseudomonotone with respect to the set V.

However, if we take $u_1=3$, $v_1=4$, $u_1^*=1$, $v_1^*=-5$, then we have

$$\begin{aligned}\langle u^*,v-u\rangle+f(u,v)+\alpha(u-v)&=u_1^*(v_1-u_1)+v_1^2-u_1^2+4(u_1-v_1)\\&=(u_1^*+v_1+u_1-4)(v_1-u_1)=(1+4+3-4)(4-3)=4>0\end{aligned}$$

but

$$\begin{aligned}\langle v^*,v-u\rangle+f(u,v)+\alpha(u-v)&=v_1^*(v_1-u_1)+v_1^2-u_1^2+4(u_1-v_1)\\&=(v_1^*+v_1+u_1-4)(v_1-u_1)=(-5+4+3-4)(4-3)=-2<0.\end{aligned}$$

Hence F is not (g,f,α)-quasimonotone on K.

Example 3.5. Let $E=\mathbb{R}^2$ and $K=[-1,2]\times\{0\}$. Let $g\colon K\times K\to E$, $f\colon K\times K\to\overline{\mathbb{R}}$, $\alpha\colon K\to\mathbb{R}$, $F\colon K\rightrightarrows E^*$ defined respectively by

$$\begin{aligned} g(u,v)&=v-u, & f(u,v)&=v_1^2-u_1v_1-4(v_1-u_1),\\ \alpha(u)&=-4u_1, & F(u)&=[3/4,3]\times\{0\}, \end{aligned}$$

where $u=(u_1,0)$, $v=(v_1,0)$. Then we get that F is stably (g,f,α)-pseudomonotone with respect to the set $V:=\{(0,m):m\in\mathbb{R}\}\subset\mathbb{R}^2$ but not f-quasimonotone.

At first, we show that F is (g,f,α)-pseudomonotone on K. If

$$\begin{aligned} 0&\le\langle u^*,v-u\rangle+f(u,v)+\alpha(u-v)\\ &=u_1^*(v_1-u_1)+v_1^2-u_1v_1-4(v_1-u_1)-4(u_1-v_1)=(u_1^*+v_1)(v_1-u_1), \end{aligned}$$

since $u_1^*+v_1>0$, we have $v_1-u_1>0$. It implies that $v_1-u_1\ge 0$. Thus we have

$$\langle v^*,v-u\rangle+f(u,v)=(v_1^*+v_1)(v_1-u_1)\ge 0.$$

Hence F is (g,f,α)-pseudomonotone on K.

Next, we show that $F(\,\cdot\,)-\zeta$ is stably (g,f,α)-pseudomonotone for each $\zeta=(0,m)\in V$. If

$$\begin{aligned} 0\le,&\langle u^*-\zeta,v-u\rangle+f(u,v)\\ &=u_1^*(v_1-u_1)+v_1(v_1-u_1)\\ &=(u_1^*+v_1)(v_1-u_1), \end{aligned}$$

since $u_1^*+v_1>0$, we have $v_1-u_1>0$. It implies that $v_1-u_1\ge 0$. Thus we have

$$\langle v^*-\zeta,v-u\rangle+f(u,v)=(v_1^*+v_1)(v_1-u_1)\ge 0.$$

Hence F is stably (g,f,α)-pseudomonotone with respect to the set V.

However, if we take $u_1=1$, $v_1=2$, $u_1^*=3$, $v_1^*=4/3$, then we have

$$\begin{aligned} \langle u^*,v-u\rangle+f(u,v)&=u_1^*(v_1-u_1)+v_1^2-u_1v_1-4(v_1-u_1)\\ &=(u_1^*+v_1-4)(v_1-u_1)=(3+2-4)(2-1)=1>0, \end{aligned}$$

but

$$\begin{aligned} \langle v^*,v-u\rangle+f(u,v)&=v_1^*(v_1-u_1)+v_1^2-u_1v_1-4(v_1-u_1)\\ &=(v_1^*+v_1-4)(v_1-u_1)=\left(\frac{4}{3}+2-4\right)(2-1)=-\frac{2}{3}<0. \end{aligned}$$

Hence F is not f-quasimonotone on K.

4. Existence theorems

Now, we give some existence results for our problem. At the first, we give the follow existence result when K is a compact and convex subset of E.

Theorem 4.1. *Let K be a nonempty, compact and convex subset of a real Banach space E. Assume that*:

- $g\colon K \times K \to E$ *is a mapping satisfying the following conditions*:
  (a) $g(u,u) = 0$ *for all* $u \in K$,
  (b) *for all* $v \in K$, $g(\,\cdot\,, v)$ *is continuous,*
  (c) *for all* $u \in K$, $n \in \mathbb{N}$, $\lambda_j \in [0,1]$, $j = 1, \dots, n$ *such that* $\sum_{j=1}^{n} \lambda_j = 1$, $g(u, \cdot\,)$ *satisfies*
  $$g\bigg(u, \sum_{j=1}^{n} \lambda_j v_j\bigg) = \sum_{j=1}^{n} \lambda_j g(u, v_j);$$
- $f\colon K \times K \to \overline{\mathbb{R}}$ *is a mapping satisfying the following conditions*:
  (d) $\mathcal{D}(f) = \{u \in K : f(u,v) \neq -\infty$ *for all* $v \in K\}$ *is nonempty,*
  (e) $f(u,u) = 0$ *for all* $u \in K$,
  (f) *for all* $v \in K$, $f(\,\cdot\,, v)$ *is upper semicontinuous,*
  (g) *for all* $u \in K$, $f(u, \cdot\,)$ *is convex;*
- $J\colon L^p(X;\mathbb{R}^k) \to \mathbb{R}$ *is the mapping*
  $$J(\varphi) = \int_X j(x, \varphi(x))\,dx,$$
  where j satisfies conditions (2.5) *and* (2.6) *or* (2.5) *and* (2.7)–(2.8);
- $T\colon E \to L^p(X;\mathbb{R}^k)$ *is a linear continuous operator*;
- $F\colon K \rightrightarrows E^*$ *is l.s.c. with respect to the weak* topology of E^*.*

Then problem (2.1) *has at least one solution.*

The proof is similar to Theorem 3.1 of [3]. So we omit it here.

We point out the fact that in the above case when K is a compact convex subset of E we do not impose any monotonicity conditions on F, nor we assume E to be a reflexive space. However, in applications, most problems lead to an inequality whose solution is sought in a closed and convex subset of the space E. Weakening the hypotheses on K by assuming that K is only bounded, closed and convex, we need to impose certain monotonicity properties on F and assume in addition that E is reflexive. We shall use the generalized monotonicity in Section 3. Furthermore, we need to suppose that T is compact and for all $v \in K$, $f(\,\cdot\,, v)$ is weakly upper semicontinuous.

Now, we consider the existence of solutions of problem (2.1) when K is a bounded, closed and convex subset of E.

THEOREM 4.2. *Let K be a nonempty, bounded, closed and convex subset of a real reflexive Banach space E. Assume that*:

- $g\colon K \times K \to E$ *is a mapping satisfying the following conditions*:
  - (a) $g(u,v) + g(v,u) = 0$ *for all* $u,v \in K$,
  - (b) *for all* $v \in K$, $g(\cdot,v)$ *is continuous,*
  - (c) *for all* $u \in K$, $n \in \mathbb{N}$, $\lambda_j \in [0,1]$, $j = 1,\dots,n$ *such that* $\sum_{j=1}^{n} \lambda_j = 1$, $g(u,\,\cdot\,)$ *satisfies*
  $$g\bigg(u, \sum_{j=1}^{n} \lambda_j v_j\bigg) = \sum_{j=1}^{n} \lambda_j g(u,v_j);$$
- $f\colon K \times K \to \overline{\mathbb{R}}$ *is a mapping satisfying the following conditions*:
  - (d) $\mathcal{D}(f) = \{u \in K : f(u,v) \neq -\infty,$ *for all* $v \in K\} \neq \emptyset$,
  - (e) $f(u,u) = 0$ *for all* $u \in K$,
  - (f) $f(u,v) + f(v,u) \geq 0$ *for all* $u,v \in K$,
  - (g) *for all* $v \in K$, $f(\,\cdot\,,v)$ *is weakly upper semicontinuous,*
  - (h) *for all* $u \in K$, $f(u,\,\cdot\,)$ *is convex*;
- $J\colon L^p(X;\mathbb{R}^k) \to \mathbb{R}$ *is the mapping*
  $$J(\varphi) = \int_X j(x,\varphi(x))\,dx,$$
  where j satisfies conditions (2.5) *and* (2.6) *or* (2.5) *and* (2.7)–(2.8);
- $T\colon E \to L^p(X;\mathbb{R}^k)$ *is a linear compact operator*;
- $\alpha\colon E \to R$ *is a convex and weakly upper semicontinuous functional such that*
  $$\lim_{t\downarrow 0} \frac{\alpha(tu)}{t} = 0, \quad \textit{for all } u \in E;$$
- $F\colon K \rightrightarrows E^*$ *is a l.h.c. multivalued mapping and stably (g,f,α)-quasi-monotone with respect to the set $U(J,T)$ defined as* (2.9).

Then problem (2.1) *has at least one solution.*

PROOF. For any $v \in K$ define a multivalued mapping $G\colon K \rightrightarrows E$ as follows:
$$G(v) := \Big\{u \in K : \inf_{v^* \in F(v)} \langle v^*, g(u,v)\rangle + f(u,v) + J^\circ(\widehat{u}; \widehat{g}(u,v)) \geq -\alpha(u-v)\Big\}.$$
$G(v)$ is nonempty since $v \in G(v)$ for each $v \in K$. Consider two cases of G: G is not a KKM mapping, and G is a KKM mapping.

Case 1. If G is not a KKM mapping, then there exist $u_i \in K$ and $\lambda_i \in [0,1]$, $i = 1,\dots,N$, with $\sum_{i=1}^{N} \lambda_i = 1$ such that $u_0 = \sum_{i=1}^{N} \lambda_i u_i \notin \bigcup_{i=1}^{N} G(u_i)$, that is
$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, g(u_0,u_i)\rangle + f(u_0,u_i) + J^\circ(\widehat{u_0}; \widehat{g}(u_0,u_i)) + \alpha(u_0 - u_i) < 0, \tag{4.1}$$
forall $i \in \{1,\dots,N\}$.

We claim that there exists a neighbourhood U of u_0 such that for all $v \in U \cap K$,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, g(v, u_i)\rangle + f(v, u_i) + J^\circ(\widehat{v}; \widehat{g}(v, u_i)) + \alpha(v - u_i) < 0,$$

for all $i \in \{1, \dots, N\}$. If not, for any neighbourhood U of u_0, there exists $v_0 \in U \cap K$ and $i_0 \in \{1, \dots, N\}$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, g(v_0, u_{i_0})\rangle + f(v_0, u_{i_0}) + J^\circ(\widehat{v_0}; \widehat{g}(v_0, u_{i_0})) + \alpha(v_0 - u_{i_0}) \geq 0.$$

Putting $U = B(u_0, 1/n)$, so there exists $v_n \in B(u_0, 1/n) \cap K$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, g(v_n, u_{i_0})\rangle + f(v_n, u_{i_0}) + J^\circ(\widehat{v_n}; \widehat{g}(v_n, u_{i_0})) + \alpha(v_n - u_{i_0}) \geq 0.$$

By (b) of Lemma 2.1, $v_n \to u_0$, (b), (g) of the assumptions and the weak upper semicontinuity of α, passing to the superior limit, we obtain that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, g(u_0, u_{i_0})\rangle + f(u_0, u_{i_0}) + J^\circ(\widehat{u_0}; \widehat{g}(u_0, u_{i_0})) + \alpha(u_0 - u_{i_0}) \geq 0,$$

which is a contradiction with (4.1), hence we have the claim.

From (2.10) and (4.1), there exists a neighbourhood U of u_0 such that for all $v \in U \cap K$,

$$\begin{aligned}
&\inf_{u_i^* \in F(u_i)} \langle u_i^*, g(v, u_i)\rangle + f(v, u_i) + \langle z_v^*, g(v, u_i)\rangle + \alpha(v - u_i) \\
&\qquad = \inf_{u_i^* \in F(u_i)} \langle u_i^*, g(v, u_i)\rangle + f(v, u_i) + J^\circ(\widehat{v}; \widehat{g}(v, u_i)) + \alpha(v - u_i) < 0,
\end{aligned}$$

for all $i \in \{1, \dots, N\}$, which can be rewritten as

$$\inf_{u_i^* \in F(u_i)} \langle u_i^* - (-z_v^*), g(v, u_i)\rangle + f(v, u_i) < -\alpha(v - u_i), \quad \text{for all } i \in \{1, \dots, N\}.$$

By the stable (g, f, α)-quasimonotonicity of F with respect to the set $U(J, T)$, we get that

$$\sup_{v^* \in F(v)} \langle v^* - (-z_v^*), g(v, u_i)\rangle + f(v, u_i) \leq -\alpha(v - u_i), \quad \text{for all } i \in \{1, \dots, N\},$$

which, for all $i \in \{1, \dots, N\}$, is equivalent to

$$\sup_{v^* \in F(v)} \langle v^*, g(v, u_i)\rangle + f(v, u_i) + J^\circ(\widehat{v}; \widehat{g}(v, u_i)) + \alpha(v - u_i) \leq 0.$$

From (a) of Lemma 2.1, (c), (h) of the assumptions and the linearity of T, we have that

$$
\begin{aligned}
(4.2)\qquad &\sup_{v^*\in F(v)} \langle v^*,g(v,u_0)\rangle + f(v,u_0) + J^\circ(\widehat{v};\widehat{g}(v,u_0)) + \alpha(v-u_0) \\
&= \sup_{v^*\in F(v)} \bigg\langle v^*, g\bigg(v, \sum_{i=1}^N \lambda_i u_i\bigg)\bigg\rangle + f\bigg(v, \sum_{i=1}^N \lambda_i u_i\bigg) \\
&\quad + J^\circ\bigg(\widehat{v}; \widehat{g}\bigg(v, \sum_{i=1}^N \lambda_i u_i\bigg)\bigg) + \alpha\bigg(v - \sum_{i=1}^N \lambda_i u_i\bigg) \\
&\le \sum_{i=1}^N \lambda_i \Big[\sup_{v^*\in F(v)} \langle v^*, g(v,u_i)\rangle \\
&\quad + f(v,u_i) + J^\circ(\widehat{v};\widehat{g}(v,u_i)) + \alpha(v-u_i)\Big] \le 0,
\end{aligned}
$$

By (a) of Lemma 2.1, we have that

$$
J^\circ(\widehat{v};\widehat{g}(u_0,v)) + J^\circ(\widehat{v};\widehat{g}(v,u_0)) \ge J^\circ(\widehat{v};0) = 0.
$$

From (a), (f) of the assumptions, combining the last inequality with (4.2), we obtain that

$$
(4.3)\qquad \inf_{v^*\in F(v)} \langle v^*, g(u_0,v)\rangle + f(u_0,v) + J^\circ(\widehat{v};\widehat{g}(u_0,v)) - \alpha(v-u_0) \ge 0,
$$

for all $v \in U\cap K$. Let $v' \in K$ be any element and define

$$
u_m = \frac{1}{m} v' + \bigg(1 - \frac{1}{m}\bigg) u_0, \quad m \ge 1.
$$

Thus, $u_m \to u_0$ as $m \to \infty$ and hence there exists $M \in \mathbb{N}$ such that $u_m \in U\cap K$ for all $m > M$.

For any given $u_0^* \in F(u_0)$, since F is l.h.c., there exists a sequence $\{u_m^*\}$ in $F(u_m)$ converging weakly star to u_0^*. It follows from (4.3) that for any $m > M$,

$$
\langle u_m^*, g(u_0,u_m)\rangle + f(u_0,u_m) + J^\circ(\widehat{u_m};\widehat{g}(u_0,u_m)) - \alpha(u_m-u_0) \ge 0,
$$

for all $v \in U\cap K$.

By (a) of Lemma 2.1, (a), (c), (e) and (h) of the assumptions, and the linearity of T, we have that

$$
\begin{aligned}
0 &\le \bigg\langle u_m^*, g\bigg(u_0, u_0 + \frac{1}{m}(v'-u_0)\bigg)\bigg\rangle + f\bigg(u_0, u_0 + \frac{1}{m}(v'-u_0)\bigg) \\
&\quad + J^\circ\bigg(\widehat{u_m}; \widehat{g}\bigg(u_0, u_0 + \frac{1}{m}(v'-u_0)\bigg)\bigg) - \alpha\bigg(\frac{1}{m}(v'-u_0)\bigg) \\
&\le \frac{1}{m}\big[\langle u_m^*, g(u_0,v')\rangle + f(u_0,v') + J^\circ(\widehat{u_m};\widehat{g}(u_0,v'))\big] - \alpha\bigg(\frac{1}{m}(v'-u_0)\bigg).
\end{aligned}
$$

Multiplying the last inequality by m and letting $m \to \infty$, from (b) of Lemma 2.1 and the assumption of α, we obtain that

$$\langle u_0^*, g(u_0, v')\rangle + f(u_0, v') + J^\circ(\widehat{u_0}; \widehat{g}(u_0, v')) \geq 0, \quad \text{for all } v' \in K. \tag{4.4}$$

Since $J(\varphi) = \int_X j(x, \varphi(x))\,dx$ and j satisfies conditions (2.5) and (2.6) or (2.5) and (2.7)–(2.8), by Lemma 2.2, we get that

$$\int_X j^\circ(x, \widehat{u}(x); \widehat{g}(u,v)(x))\,dx \geq J^\circ(\widehat{u}; \widehat{g}(u,v)), \quad \text{for all } u, v \in K.$$

It follows from (4.4) that

$$\langle u_0^*, g(u_0, v')\rangle + f(u_0, v') + \int_X j^\circ(x, \widehat{u_0}(x); \widehat{g}(u_0, v')(x))\,dx \geq 0, \quad \text{for all } v' \in K.$$

Obviously, we have $u_0 \in \mathcal{D}(f)$. Hence problem (2.1) has at least one solution.

Case 2. If G is a KKM mapping. For any $v \in K$, we consider the following mapping:

$$u \mapsto \inf_{v^* \in F(v)} \langle v^*, g(u,v)\rangle + f(u,v) + J^\circ(\widehat{u}; \widehat{g}(u,v)) + \alpha(u - v).$$

We claim that the above mapping is weakly upper semicontinuous. For a given a sequence $\{\mu_n\} \subset K$ such that $\mu_n \rightharpoonup \mu_0$, it follows from the linearity and compactness of T that $T\mu_n \to T\mu_0$; that is $\widehat{\mu_n} \to \widehat{\mu_0}$ as $n \to \infty$. Since $g(\,\cdot\,, y)$ is continuous and $f(\,\cdot\,, y)$ is weakly upper semicontinuous for all $y \in K$ and by (b) of Lemma 2.1, we have that

$$\begin{aligned}
&\limsup_{n\to\infty}\Big[\inf_{v^*\in F(v)} \langle v^*, g(\mu_n, v)\rangle + f(\mu_n, v) + J^\circ(\widehat{\mu_n}; \widehat{g}(\mu_n, v)) + \alpha(\mu_n - v)\Big] \\
&\quad \leq \limsup_{n\to\infty}\Big(\inf_{v^*\in F(v)} \langle v^*, g(\mu_n, v)\rangle\Big) + \limsup_{n\to\infty} f(\mu_n, v) \\
&\qquad + \limsup_{n\to\infty} J^\circ(\widehat{\mu_n}; \widehat{g}(\mu_n, v)) + \limsup_{n\to\infty} \alpha(\mu_n - v) \\
&\quad \leq \inf_{v^*\in F(v)} \langle v^*, g(\mu_0, v)\rangle + f(\mu_0, v) + J^\circ(\widehat{\mu_0}; \widehat{g}(\mu_0, v)) + \alpha(\mu_0 - v).
\end{aligned}$$

Thus we have the claim. Then $G(v)$ is weakly closed. It follows from the convexity, boundedness and closedness of a subset K in a reflexive Banach space E, we have that K is weakly compact. Since $G(v) \subset K$, we get that $G(v)$ is weakly compact for each $v \in K$. Thus, all conditions of Lemma 2.3 are satisfied in the weak topology and hence we obtain that $\bigcap_{v\in K} G(v) \neq \emptyset$. Taking $u_0 \in \bigcap_{v\in K} G(v)$, we have

$$\inf_{v^*\in F(v)} \langle v^*, g(u_0, v)\rangle + f(u_0, v) + J^\circ(\widehat{u_0}; \widehat{g}(u_0, v)) + \alpha(u_0 - v) \geq 0, \tag{4.5}$$

for all $v \in K$. Let $v_0 \in K$ be any element and define

$$u_n = \frac{1}{n} v_0 + \Big(1 - \frac{1}{n}\Big) u_0, \quad n \geq 1.$$

Then $u_n \in K$ for all $n \geq 1$. For any given $u_0^* \in F(u_0)$, since F is l.h.c., there exists a sequence $\{u_n^*\}$ in $F(u_n)$ converging weakly star to u_0^*. From (4.5), for any $n \geq 1$, we have

$$\langle u_n^*, g(u_0, u_n)\rangle + f(u_0, u_n) + J^\circ(\widehat{u_n}; \widehat{g}(u_0, u_n)) + \alpha(u_0 - u_n) \geq 0.$$

By (a) of Lemma 2.1, (a), (c), (e) and (h) of the assumptions, and the linearity of T, we have that

$$\begin{aligned} 0 \leq &\left\langle u_n^*, g\left(u_0, u_0 + \frac{1}{n}(v_0 - u_0)\right)\right\rangle + f\left(u_0, u_0 + \frac{1}{n}(v_0 - u_0)\right) \\ &+ J^\circ\left(\widehat{u_n}; \widehat{g}\left(u_0, u_0 + \frac{1}{n}(v_0 - u_0)\right)\right) + \alpha\left(\frac{1}{n}(u_0 - v_0)\right) \\ \leq &\frac{1}{n}\big[\langle u_n^*, g(u_0, v_0)\rangle + f(u_0, v_0) + J^\circ(\widehat{u_n}; \widehat{g}(u_0, v_0))\big] + \alpha\left(\frac{1}{n}(u_0 - v_0)\right), \end{aligned}$$

Multiplying the last inequality by n and letting $n \to \infty$, from (b) of Lemma 2.1 and the assumption of α, we obtain that

$$\langle u_0^*, g(u_0, v_0)\rangle + f(u_0, v_0) + J^\circ(\widehat{u_0}; \widehat{g}(u_0, v_0)) \geq 0, \quad \text{for all } v_0 \in K. \tag{4.6}$$

Since $J(\varphi) = \int_X j(x, \varphi(x))\,dx$ and j satisfies conditions (2.5) and (2.6) or (2.5) and (2.7)–(2.8), by Lemma 2.2, we get that

$$\int_X j^\circ(x, \widehat{u}(x); \widehat{g}(u, v)(x))\,dx \geq J^\circ(\widehat{u}; \widehat{g}(u, v)), \quad \text{for all } u, v \in K.$$

It follows from (4.6) that

$$\langle u_0^*, g(u_0, v_0)\rangle + f(u_0, v_0) + \int_X j^\circ(x, \widehat{u_0}(x); \widehat{g}(u_0, v_0)(x))\,dx \geq 0, \quad \text{for all } v_0 \in K.$$

Obviously, we have $u_0 \in \mathcal{D}(f)$. Hence problem (2.1) has at least one solution. This completes the proof. □

Remark 4.3. (a) Taking $\alpha(u) = \langle \zeta, u\rangle^2 + C$, $\zeta \in E^*$, $C \in \mathbb{R}$, it is obvious that α satisfies the conditions in Theorem 4.2.

(b) Theorem 4.2 generalizes and improves some recent results. In fact,

- If $\alpha \equiv 0$, $g(u, v) = v - u$, then Theorem 4.2 reduces to Theorem 4.1 of [19].
- If $\alpha \equiv 0$, $g(u, v) = v - u$, $f \equiv 0$, then Theorem 4.2 reduces to Theorem 3.1 of [21].
- If $\alpha \equiv 0$, $g(u, v) = v - u$, $f(u, v) = \phi(v) - \phi(u)$, where $\phi\colon E \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous function such that $K_\phi = K \cap \operatorname{dom}\phi = \mathcal{D}(f)$, then Theorem 4.2 reduces to Theorem 4.1 of [21].

(c) We present an example in Section 5 to show that the assumptions regarding g, f, F, α are reasonable.

Next, we omit the boundedness of K in Theorem 4.2, we need to introduce the concept of f-coercivity.

PROPOSITION 4.4. *Consider the following f-coercivity conditions*:

(a) *There exists a nonempty subset V_0 contained in a weakly compact subset V_1 of K such that the set*

$$D = \Big\{u \in K : \inf_{v^* \in F(v)} \langle v^*, g(u, v)\rangle + f(u, v) + J^\circ(\widehat{u}; \widehat{g}(u, v)) \geq 0, \text{ for all } v \in V_0\Big\}$$

is weakly compact or empty.

(b) *There exists $n_0 \in \mathbb{N}$ such that for every $u \in K \setminus B_{n_0}$, there exists some $v \in K$ with $\|v\| < \|u\|$ such that*

$$\sup_{u^* \in F(u)} \langle u^*, g(u, v)\rangle + f(u, v) + J^\circ(\widehat{u}; \widehat{g}(u, v)) \leq 0.$$

(c) *There exists $n_0 \in \mathbb{N}$ such that for every $u \in K \setminus B_{n_0}$, there exists some $v \in K$ with $\|v\| < \|u\|$ such that*

$$\sup_{u^* \in F(u)} \langle u^*, g(u, v)\rangle + f(u, v) + \int_X j^\circ(x, \widehat{u}(x); \widehat{g}(u, v)(x))\, dx < 0.$$

Then we have:

(i) (a)$\Rightarrow$(b), *if F is stably $(g, f, 0)$-quasimonotone with respect to the set $U(J, T)$.*

(ii) (c)$\Rightarrow$(b), *if $J(\varphi) = \int_X j(x, \varphi(x))dx$, and j satisfies conditions* (2.5) *and* (2.6) *or* (2.5) *and* (2.7)–(2.8).

PROOF. (i) If $D = \emptyset$, since V_0 is nonempty and contained in a weakly compact subset V_1 of K, then there exists a natural number $M < \infty$ such that $\|z\| < M$ for all $z \in V_0$. Taking $n_0 = M$, we obtain that for every $u \in K \setminus B_{n_0}$, there exists $v \in V_0 \neq \emptyset$ such that $v \in B_{n_0}$ and

$$\inf_{v^* \in F(v)} \langle v^*, g(u, v)\rangle + f(u, v) + J^\circ(\widehat{u}; \widehat{g}(u, v)) < 0. \tag{4.7}$$

If $D \neq \emptyset$, then D is weakly compact. Since $D \cup V_0 \subset D \cup V_1$, which is a weakly compact subset, we conclude that there exists a natural number $M < \infty$ such that $\|Z\| < M$ for all $z \in D \cup V_0$. Taking $n_0 = M$, we obtain that for every $u \in K \setminus B_{n_0}$ (4.7) holds. Hence, from the proofs of both case we can conclude that there exists $n_0 \in \mathbb{N}$ such that for every $u \in K \setminus B_{n_0}$, there exists some $v \in B_{n_0}$ such that (4.7) holds. Therefore,

$$\begin{aligned} 0 > & \inf_{v^* \in F(v)} \langle v^*, g(u, v)\rangle + f(u, v) + J^\circ(\widehat{u}; \widehat{g}(u, v)) \\ = & \inf_{v^* \in F(v)} \langle v^*, g(u, v)\rangle + f(u, v) + \langle z_u^*, g(u, v)\rangle, \end{aligned}$$

and hence

$$\inf_{v^* \in F(v)} \langle v^* - (-z_u^*), g(u, v)\rangle + f(u, v) < 0.$$

Since F is stably $(g,f,0)$-quasimonotone with respect to the set $U(J,T)$, we have

$$\sup_{u^*\in F(u)} \langle u^* - (-z_u^*), g(u,v)\rangle + f(u,v) \le 0,$$

and then

$$\sup_{u^*\in F(u)} \langle u^*, g(u,v)\rangle + f(u,v) + J^\circ(\widehat{u};\widehat{g}(u,v)) \le 0.$$

(ii) By Lemma 2.2, we get that

$$\int_X j^\circ(x,\widehat{u}(x);\widehat{g}(u,v)(x))dx \ge J^\circ(\widehat{u};\widehat{g}(u,v)), \quad \text{for all } u,v\in E.$$

Combining with (c), we get (b). □

Remark 4.5. (a) If $\alpha\equiv 0$, $g(u,v)=v-u$, then the f-coercivity conditions of Proposition 4.4 reduce to f-coercivity conditions of Proposition 4.3 of [19];

(b) If $\alpha\equiv 0$, $g(u,v)=v-u$, $f\equiv 0$, then the f-coercivity conditions of Proposition 4.4 reduce to coercivity conditions of Proposition 3.1 of [21];

(c) If $\alpha\equiv 0$, $g(u,v)=v-u$, $f(u,v)=\phi(v)-\phi(u)$ as (b) of Remark 4.3, then the f-coercivity conditions of Proposition 4.4 reduce to ϕ-coercivity conditions of Proposition 4.1 of [18].

Next, we present the result for the unbounded constrained set.

Theorem 4.6. *Let K be a nonempty, unbounded, closed and convex subset of a real reflexive Banach space E. Assume that g,f,J,T,α,F are the mappings satisfying the conditions as in Theorem* 4.2. *If condition* (b) *of Proposition* 4.4 *holds, then problem* (2.1) *has at least one solution.*

Proof. Take $m>n_0$. Since B_m is bounded and convex, from (4.4) or (4.6) in Theorem 4.2, we can conclude that there exists $u_m\in B_m\cap\mathcal{D}(f)$ and $u_m^*\in F(u_m)$ such that

$$\langle u_m^*, g(u_m,v)\rangle + f(u_m,v) + J^\circ(\widehat{u_m};\widehat{g}(u_m,v)) \ge 0, \quad \text{for all } v\in B_m\cap K. \tag{4.8}$$

Now, we consider two cases.

Case 1. If $\|u_m\|=m$, then $\|u_m\|>n_0$. Since condition (b) of Proposition 4.4 holds, there is some $v_0\in K$ with $\|v_0\|<\|u_m\|=m$ such that

$$\langle u_m^*, g(u_m,v_0)\rangle + f(u_m,v_0) + J^\circ(\widehat{u_m};\widehat{g}(u_m,v_0)) \le 0. \tag{4.9}$$

Let $v\in K$. Since $\|v_0\|<\|u_m\|=m$, there is $t\in(0,1)$ such that $v_t:=v_0+t(v-v_0)\in B_m\cap K$. Note that T is linear mapping and $f(x,\cdot)$ is a convex function. By (4.8), (4.9) and (a) of Lemma 2.1, it follows that

$$\begin{aligned}0 &\le \langle u_m^*, g(u_m, v_t)\rangle + f(u_m, v_t) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v_t))\\ &= \langle u_m^*, g(u_m, (1-t)v_0 + tv)\rangle\\ &\quad + f(u_m, (1-t)v_0 + tv) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, (1-t)v_0 + tv))\\ &\le (1-t)\big[\langle u_m^*, g(u_m, v_0)\rangle + f(u_m, v_0) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v_0))\big]\\ &\quad + t\big[\langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v))\big]\\ &\le t\big[\langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v))\big],\end{aligned}$$

for all $v \in K$. Dividing by t, we have that

$$\langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v)) \ge 0, \quad \text{for all } v \in K. \tag{4.10}$$

Case 2. If $\|u_m\| < m$, then for any $v \in K$, there is $t \in (0,1)$ such that $v_t' := u_m + t(v - u_m) \in B_m \cap K$. Note that T is linear mapping and $f(x, \cdot)$ is a convex function. By (4.10) and (a) of Lemma 2.1, it follows that

$$\begin{aligned}0 &\le \langle u_m^*, g(u_m, v_t')\rangle + f(u_m, v_t') + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v_t'))\\ &\le t\big[\langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v))\big],\end{aligned}$$

for all $v \in K$. Dividing by t, we have that (4.10) holds.

Since $J(\varphi) = \int_X j(x, \varphi(x))dx$ and j satisfies conditions (2.5) and (2.6) or (2.5) and (2.7)–(2.8), by Lemma 2.2, we get that

$$\int_X j^\circ(x, \widehat{u}(x); \widehat{g}(u, v)(x))\,dx \ge J^\circ(\widehat{u}; \widehat{g}(u, v)), \quad \text{for all } u, v \in E,$$

and hence

$$\begin{aligned}0 &\le \big[\langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + J^\circ(\widehat{u_m}; \widehat{g}(u_m, v))\big]\\ &\le \langle u_m^*, g(u_m, v)\rangle + f(u_m, v) + \int_X j^\circ(x, \widehat{u_m}(x); \widehat{g}(u_m, v)(x))\,dx,\end{aligned}$$

for all $v \in K$. This shows that problem (2.1) has at least one solution. □

REMARK 4.7. (a) If $\alpha \equiv 0$, $g(u, v) = v - u$, then Theorem 4.6 reduces to Theorem 4.5 of [19];

(b) If $\alpha \equiv 0$, $g(u, v) = v - u$, $f \equiv 0$, then then Theorem 4.6 reduces to Theorem 3.2 of [21];

(c) If $\alpha \equiv 0$, $g(u, v) = v - u$, $f(u, v) = \phi(v) - \phi(u)$ as (b) of Remark 4.3, then Theorem 4.6 reduces to Theorem 4.2 of [18].

If the constrained set K is bounded, then the solution set of problem (2.1) is obviously bounded. In the case of constrained set K is unbounded, the solution set of problem (2.1) may be unbounded. In the sequel, we provide a sufficient condition to guarantee the boundedness of the solution set of problem (2.1), when K is unbounded. The following result also generalized Theorem 4.5 of [18].

Theorem 4.8. *Let K be a nonempty, closed, unbounded and convex subset of a real reflexive Banach space E. Assume that g, f, J, T, α, F are the mappings satisfying the conditions as in Theorem* 4.2. *If condition* (c) *of Proposition* 4.4 *holds, then the solution set of problem* (2.1) *is nonempty and bounded.*

Proof. From Proposition 4.4, we have (c)$\Rightarrow$(b). By Theorem 4.6, we know that the solution set of problem (2.1) is nonempty. If the solution set is unbounded, then there exist $u_0 \in \mathcal{D}(f)$ and $u_0^* \in F(u_0)$ such that $\|u_0\| > n_0$ and

$$\langle u_0^*, g(u_0, v)\rangle + f(u_0, v) + \int_X j^\circ(x, \widehat{u_0}(x); \widehat{g}(u_0, v)(x))dx \geq 0, \tag{4.11}$$

for all $v \in K$. Since $\|u_0\| > n_0$, it follows from condition (c) of Proposition 4.4 that, there exists $v_0 \in K$ with $\|v_0\| < \|u_0\|$ such that

$$\sup_{u_0^{**} \in F(u_0)} \langle u_0^{**}, g(u_0, v_0)\rangle + f(u_0, v_0) + \int_X j^\circ(x, \widehat{u_0}(x); \widehat{g}(u_0, v_0)(x))\, dx < 0,$$

which is a contradiction with (4.11). $\square$

Remark 4.9. (a) If $\alpha \equiv 0$, $g(u, v) = v - u$, then Theorem 4.8 reduces to Theorem 4.7 of [19].

(b) If $\alpha \equiv 0$, $g(u, v) = v - u$, $f \equiv 0$, then then Theorem 4.8 reduces to Theorem 3.3 of [21].

(c) If $\alpha \equiv 0$, $g(u, v) = v - u$, $f(u, v) = \phi(v) - \phi(u)$ as (b) of Remark 4.3, then Theorem 4.8 reduces to Theorem 4.3 of [18].

5. Example

We consider the following example to present the generalized monotonicity in Definition 3.1 and Theorem 4.2.

Example 5.1. Let $a > 0$, $E = \mathbb{R}^3$ and $K = [-a, a] \times \{0\} \times [-a, a]$. Let $g\colon K \times K \to E$, $f\colon K \times K \to \overline{\mathbb{R}}$, $\alpha\colon K \to \mathbb{R}$, $F\colon K \rightrightarrows E^*$ defined by, respectively,

$$g(u, v) = v - u, \quad f(u, v) = v_1(v_1 - u_1), \quad \alpha(u) = u_1^2,$$
$$F(u) = \{(u_1^*, u_2^*, 0) \in R^3 : 3a \leq u_1^* \leq 4a,\ a \leq u_2^* \leq 2a\},$$

where $u = (u_1, 0, u_3)$, $v = (v_1, 0, v_3)$. Let $\Omega = (0, 1) \subset \mathbb{R}$ and $j\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$j(x, y) = |y_2|, \quad \text{for all } x \in \Omega \text{ and for all } y = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Let $J\colon L^2(\Omega, \mathbb{R}^3) \to \mathbb{R}$ be defined by

$$J(\varphi) = \int_\Omega j(x, \varphi(x))\, dx,$$

and $T\colon \mathbb{R}^3 \to L^2(\Omega, \mathbb{R}^3)$ be defined by

$$T(u) = T(u_1, u_2, u_3) = \varphi = (\varphi_1, \varphi_2, \varphi_3),$$

where $\varphi_k = u_k$ $(k = 1, 2, 3)$ for all $x \in \Omega$. Then

(i) F is (g, f, α)-quasimonotone, but not (g, f, α)-pseudomonotone;

(ii) F is stably (g, f, α)-quasimonotone with respect to the set $V := \{(0, m, 0) : m \in \mathbb{R}\} \subset \mathbb{R}^3$;

(iii) the set $U(J, T) \subseteq V$.

PROOF. (i) Firstly, we show that F is (g, f, α)-quasimonotone. In fact, let $u = (u_1, 0, u_3),\ v = (v_1, 0, v_3) \in K$ and $u^* = (u_1^*, u_2^*, 0) \in F(u)$ be such that

$$\langle u^*, v - u\rangle + f(u, v) > -\alpha(u - v).$$

Since

$$\langle u^*, v - u\rangle = \langle (u_1^*, u_2^*, 0), (v_1 - u_1, 0, v_3 - u_3\rangle = u_1^*(v_1 - u_1),$$
$$f(u, v) = v_1(v_1 - u_1), \qquad -\alpha(u - v) = -(u_1 - v_1)^2,$$

we have $(u_1^* + 2v_1 - u_1)(v_1 - u_1) > 0$. Since $u_1^* + 2v_1 - u_1 \geq 0$ we have and $u_1^* + 2v_1 - u_1 \neq 0$ and $v_1 - u_1 > 0$.

On the other hand,

$$\langle v^*, v - u\rangle = \langle (v_1^*, v_2^*, 0), (v_1 - u_1, 0, v_3 - u_3)\rangle = v_1^*(v_1 - u_1),$$

hence, $(v_1^* + 2v_1 - u_1)(v_1 - u_1) \geq 0$. Consequently, F is (g, f, α)-quasimonotone.

Next, we check that F is not (g, f, α)-pseudomonotone. Choose $u_0 = (a, 0, 0)$, $v_0 = (-a, 0, 0) \in K$, $u_0^* = (3a, a, 0) \in F(u_0)$, and $v_0^* = (4a, 2a, 0) \in F(v_0)$. We have

$$\begin{aligned}\langle u_0^*, v_0 - u_0\rangle + f(u_0, v_0) + \alpha(u_0 - v_0) &= 0,\\ \langle v_0^*, v_0 - u_0\rangle + f(u_0, v_0) + \alpha(u_0 - v_0) &= -a^2 < 0.\end{aligned}$$

This implies that F is not (g, f, α)-pseudomonotone.

(ii) Let $u = (u_1, 0, u_3),\ v = (v_1, 0, v_3) \in K,\ u^* = (u_1^*, u_2^*, 0) \in F(u),\ v^* = (v_1^*, v_2^*, 0) \in F(v)$ and $\zeta = (0, m, 0) \in V$ be such that

$$\langle u^* - \zeta, v - u\rangle + f(u, v) > -\alpha(u - v).$$

Since

$$\langle u^* - \zeta, v - u\rangle = \langle (u_1^*, u_2^* - m, 0), (v_1 - u_1, 0, v_3 - u_3\rangle = u_1^*(v_1 - u_1),$$
$$f(u, v) = v_1(v_1 - u_1), \qquad -\alpha(u - v) = -(u_1 - v_1)^2,$$

we have $(u_1^* + 2v_1 - u_1)(v_1 - u_1) > 0$. Since $u_1^* + 2v_1 - u_1 \geq 0$ we have and $u_1^* + 2v_1 - u_1 \neq 0$ and $v_1 - u_1 > 0$.

On the other hand,

$$\langle v^* - \zeta, v - u\rangle = \langle (v_1^*, v_2^* - m, 0), (v_1 - u_1, 0, v_3 - u_3\rangle = v_1^*(v_1 - u_1),$$

hence $(v_1^* + 2v_1 - u_1)(v_1 - u_1) \geq 0$, therefore,

$$\langle u^* - \zeta, v - u\rangle + f(u,v) \geq -\alpha(u-v),$$

which shows that $F - \zeta$ is (g, f, α)-quasimonotone for all $\zeta \in V$. Consequently, F is stably (g, f, α)-quasimonotone with respect to the set V.

(iii) In fact, it follows from (iii) of Example 4.1 in [21] that $U(J,T) \subseteq V$. Combining (ii) and (iii), we conclude that F is stably (g, f, α)-quasimonotone with respect to the set $U(J,T)$. $\square$

REMARK 5.2. It is easy to verify that g, f, α satisfy the conditions in Theorem 4.2. Then in this situation problem (2.1) has at least one solution.

6. Conclusions

In this paper, we establish some existence results for the a class of hemivariational inequalities problems in the case when the constraint set K is compact, bounded and unbounded, respectively. When the set K is bounded in a reflexive Banach space, we introduce the concept of stable (g, f, α)-quasimonotonicity and use the properties of Clarke's generalized directional derivative, Clarke's generalized gradient, and KKM technique. When the set K is unbounded, we give a coerciveness condition for the existence of the solution and a coerciveness condition for the existence and boundedness of solution. The results presented in this paper extend and improve some known results.

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Manuscript received July 19, 2014

accepted February 27, 2015

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