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TOPOLOGICAL STRUCTURE OF THE SOLUTION SET OF SINGULAR EQUATIONS WITH SIGN CHANGING TERMS UNDER DIRICHLET BOUNDARY CONDITION

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Abstract. In this paper we establish existence of connected components of positive solutions of the equation $-\Delta_p u = \lambda f(u)$ in Ω, under Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian, and $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

1. Introduction

In this paper we establish existence of a continuum of positive solutions of

$$(\mathrm{P})_\lambda \qquad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, Δ_p is the p-Laplacian, $1 < p < \infty$, $\lambda > 0$ is a real parameter, $f\colon (0,\infty) \to \mathbb{R}$ is a continuous function which may blow up to $\pm\infty$ at the origin.

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Definition 1.1. By a solution of $(\mathrm{P})_\lambda$ we mean a function $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, with $u > 0$ in Ω, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \int_\Omega f(u)\varphi \, dx, \quad \varphi \in W_0^{1,p}(\Omega). \tag{1.1}$$

Definition 1.2. The solution set of $(\mathrm{P})_\lambda$ is

$$\mathcal{S} := \{(\lambda, u) \in (0,\infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (\mathrm{P})_\lambda\}. \tag{1.2}$$

In the pioneering work [5], Crandall, Rabinowitz and Tartar employed topological methods, Schauder Theory, and Maximum Principles to prove existence of an unbounded connected subset in $\mathbb{R} \times C_0(\overline{\Omega})$ of positive solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -Lu = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is a linear second order uniformly elliptic operator,

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$$

and $g\colon \overline{\Omega} \times (0,\infty) \to (0,\infty)$ is a continuous function satisfying $g(x,t) \xrightarrow{t \to 0^+} 0$ uniformly for $x \in \overline{\Omega}$. A typical example is $g(x,t) = t^\gamma$, where $\gamma > 0$.

Several techniques have been employed in the study of (P_λ). In [11], Giacomoni, Schindler and Takac employed variational methods to investigate the problem

$$\begin{cases} -\Delta_p u = \dfrac{\lambda}{u^\delta} + u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $p-1 < q < p^*-1$, $\lambda > 0$ and $0 < \delta < 1$ with $p^* = Np/(N-p)$ if $1 < p < N$, $p^* \in (N,\infty)$ if $p = N$, and $p^* = \infty$ if $p > N$. Several results were shown in that paper, among them existence, multiplicity and regularity of solutions.

In the present work we exploit the topological structure of the solution set of (P_λ) and our main assumptions are:

(f_1) $f\colon (0,\infty) \to \mathbb{R}$ is continuous and

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = 0,$$

(f_2) there are positive numbers a, β, A with $\beta < 1$ such that

(i) $f(u) \geq a/u^\beta$ for $u > A$,

(ii) $\limsup_{u\to 0} u^\beta |f(u)| < \infty$.

The main result of this paper is:

THEOREM 1.3. *Assume* (f_1)–(f_2). *Then there is a number* $\lambda_0 > 0$ *and a connected subset* Σ *of* $[\lambda_0, \infty) \times C(\overline{\Omega})$ *satisfying*

$$\Sigma \subset \mathcal{S}, \tag{1.3}$$

$$\Sigma \cap (\{\lambda\} \times C(\overline{\Omega})) \neq \emptyset, \quad \lambda_0 \le \lambda < \infty. \tag{1.4}$$

There is a broad literature on singular problems and we further refer the reader to Lazer and McKenna in [16], Diaz, Morel and Oswald [8], Gerghu and Radulescu [10], Goncalves, Rezende and Santos [13], Hai [14, 15], Mohammed [19], Shi and Yao [21], Hoang Loc and Schmitt [18], Carl and Perera [4], and their references.

Our result includes examples such as

$$\begin{aligned}
&u^q - \frac{1}{u^\beta}, \quad \beta > 0,\ 0 < q < p-1,\\
&\frac{1}{u^\beta} - \frac{1}{u^\alpha}, \quad 0 < \beta < \alpha < 1,\\
&\ln(u).
\end{aligned}$$

In the proof of our Theorem 1.3 we shall employ topological arguments to construct a suitable connected component of the solution set $\mathcal{S}$ of $(\mathrm{P})_\lambda$. More precisely, we shall use in a nontrivial way Theorem 2.1 from Sun and Song [23] whose proof is based on the famous lemma of Whyburn, (cf. [26, Theorem 9.3]). At first some notations:

Let $M = (M, d)$ be a metric space and denote by $\{\Sigma_n\}$ a sequence of connected components of M. The *upper limit* of $\{\Sigma_n\}$ is defined by

$$\overline{\lim}\, \Sigma_n = \left\{ u \in M \;\middle|\; \text{there is } (u_{n_i}) \subseteq \bigcup \Sigma_n \text{ with } u_{n_i} \in \Sigma_{n_i} \text{ and } u_{n_i} \to u \right\}.$$

REMARK 1.4. $\overline{\lim}\, \Sigma_n$ is a closed subset of M.

THEOREM 1.5. *Let* M *be a metric space and* $\{\alpha_n\}, \{\beta_n\} \in \mathbb{R}$ *be sequences satisfying* $\ldots < \alpha_n < \ldots < \alpha_1 < \beta_1 < \ldots < \beta_n < \ldots$ *with* $\alpha_n \to -\infty$ *and* $\beta_n \to \infty$. *Assume that* $\{\Sigma_n^*\}$ *is a sequence of connected subsets of* $\mathbb{R} \times M$ *satisfying*:

(a) $\Sigma_n^* \cap (\{\alpha_n\} \times M) \neq \emptyset$,
(b) $\Sigma_n^* \cap (\{\beta_n\} \times M) \neq \emptyset$,

for each n. *For each* $\alpha, \beta \in (-\infty, \infty)$ *with* $\alpha < \beta$,

(c) $\left(\bigcup \Sigma_n^*\right) \cap ([\alpha, \beta] \times M)$ *is a relatively compact subset of* $\mathbb{R} \times M$.

Then there is a number $\lambda_0 > 0$ *and a connected component* Σ^* *of* $\overline{\lim}\, \Sigma_n^*$ *such that* $\Sigma^* \cap (\{\lambda\} \times M) \neq \emptyset$ *for each* $\lambda \in (\lambda_0, \infty)$.

2. Some auxiliary results

We gather below a few technical results. For completeness, a few proofs will be provided in the appendix. The Euclidean distance from $x \in \Omega$ to $\partial\Omega$ is

$$d(x) = \operatorname{dist}(x, \partial\Omega).$$

The result below derives from Gilbarg and Trudinger [12], and Vàzquez [25].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Then*:

(a) *$d \in \operatorname{Lip}(\overline{\Omega})$ and d is C^2 in a neighbourhood of $\partial\Omega$,*

(b) *if ϕ_1 denotes a positive eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ one has*

$$\phi_1 \in C^{1,\alpha}(\overline{\Omega}) \quad \text{with } 0 < \alpha < 1, \qquad \frac{\partial \phi_1}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

and there are positive constants C_1, C_2 such that

$$C_1 d(x) \le \phi_1(x) \le C_2 d(x), \quad x \in \Omega.$$

The result below is due to Crandall, Rabinowitz and Tartar [5], Lazer and McKenna [16] in the case $p = 2$ and Giacomoni, Schindler and Takac [11] in the case $1 < p < \infty$.

Lemma 2.2. *Let $\beta \in (0,1)$ and $m > 0$. Then the problem*

$$\begin{cases} -\Delta_p u = \dfrac{m}{u^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

admits only a weak solution $u_m \in W_0^{1,p}(\Omega)$. Moreover, $u_m \ge \varepsilon_m \phi_1$ in Ω for some constant $\varepsilon_m > 0$.

Remark 2.3. By the results in [17], [11], there is $\alpha \in (0,1)$ such that $u_m \in C^{1,\alpha}(\overline{\Omega})$.

The result below, which is crucial in this work, and whose proof is provided in the appendix, is basically due to Hai [15].

Lemma 2.4. *Let $g \in L^\infty_{\mathrm{loc}}(\Omega)$. Assume that there is $\beta \in (0,1)$ and $C > 0$ such that*

$$|g(x)| \le \frac{C}{d(x)^\beta}, \quad x \in \Omega. \tag{2.2}$$

Then there is only a weak solution $u \in W_0^{1,p}(\Omega)$ of

$$\begin{cases} -\Delta_p u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In addition, there exist constants $\alpha \in (0,1)$ and $M > 0$, with M depending only on C, β, Ω such that $u \in C^{1,\alpha}(\overline{\Omega})$ and $\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le M$.

REMARK 2.5. The solution operator associated to (2.3) is: let

$$\mathcal{M}_{\beta,\infty} = \left\{ g \in L^{\infty}_{loc}(\Omega) \;\middle|\; |g(x)| \leq \frac{C}{d(x)^{\beta}},\ x \in \Omega \right\},$$
$$S \colon \mathcal{M}_{\beta,\infty} \to W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \quad S(g) := u.$$

Notice that $\|S(g)\|_{C^{1,\alpha}(\overline{\Omega})} \leq M$, for all $g \in \mathcal{M}_{C,d,\beta,\infty}$, with M depending only on C, β, Ω.

COROLLARY 2.6. *Let $g, \widetilde{g} \in L^{\infty}_{\rm loc}(\Omega)$ with $g \geq 0$, $g \neq 0$ satisfying* (2.2). *Then, for each $\varepsilon > 0$, the problem*

$$\begin{cases} -\Delta_p u_\varepsilon = g\, \chi_{\{d>\varepsilon\}} + \widetilde{g}\ \chi_{\{d<\varepsilon\}} & \text{in } \Omega; \\ u_\epsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

admits only a solution $u_\varepsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. In addition, there is $\varepsilon_0 > 0$ such that

$$u_\varepsilon \geq \frac{u}{2} \quad \text{in } \Omega \text{ for each } \varepsilon \in (0, \varepsilon_0),$$

where u is the solution of (2.3).

A proof of the corollary above will be included in the appendix.

3. Lower and upper solutions

In this section we present two results, due to Hai [15, Theorem 2.1], on existence of lower and upper solutions of $(\mathrm{P})_\lambda$. At first some definitions.

DEFINITION 3.1. A function $\underline{u} \in W^{1,p}_0(\Omega)$ with $\underline{u} > 0$ in Ω such that

$$\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \cdot \nabla \varphi \, dx \leq \lambda \int_\Omega f(\underline{u}) \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega),\ \varphi \geq 0,$$

is a lower solution of $(\mathrm{P})_\lambda$.

DEFINITION 3.2. A function $\overline{u} \in W^{1,p}_0(\Omega)$ with $\overline{u} > 0$ in Ω such that

$$\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla \varphi \, dx \geq \lambda \int_\Omega f(\overline{u}) \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega),\ \varphi \geq 0,$$

is an upper solution of $(\mathrm{P})_\lambda$.

We establish the existence of a lower solution.

THEOREM 3.3. *Assume* (f_1)–(f_2). *Then there exist $\lambda_0 > 0$ and a non-negative function $\psi \in C^{1,\alpha}(\overline{\Omega})$, with $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$, $\alpha \in (0,1)$ such that for each $\lambda \in [\lambda_0, \infty)$, $\underline{u} = \lambda^r \psi$ with $r = 1/(p + \beta - 1)$, is a lower solution of* $(\mathrm{P})_\lambda$.

PROOF OF THEOREM 3.3. See Hai [15, p. 622]. □

By Lemma 2.2, there are both a function $\phi \in C^{1,\alpha}(\overline{\Omega})$, with $\alpha \in (0,1)$, such that

$$\begin{cases} -\Delta_p \phi = \dfrac{1}{\phi^\beta} & \text{in } \Omega, \\ \phi > 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and a constant $C_1 > 0$ such that $\phi \geq C_1 d$ in Ω.

Next, we establish the existence of an upper solution.

Theorem 3.4. *Assume* (f_1)–(f_2) *and take* $\Lambda > \lambda_0$ *with* λ_0 *as in Theorem* 3.3. *Then for each* $\lambda \in [\lambda_0, \Lambda]$, $(\mathrm{P})_\lambda$ *admits an upper solution* $\overline{u} = \overline{u}_\lambda = M\phi$ *where* $M > 0$ *is a constant and* ϕ *is given by* (3.1).

Proof of Theorem 3.4. See Hai in [15, p. 623]. □

4. Further technical results

At first we introduce some notations, remarks and lemmas. Take $\Lambda > \lambda_0$ and set $I_\Lambda := [\lambda_0, \Lambda]$. For each $\lambda \in I_\Lambda$, by Theorem 3.3,

$$\underline{u} = \underline{u}_\lambda = \lambda^r \psi$$

is a lower solution of $(\mathrm{P})_\lambda$. Pick $M = M_\Lambda \geq \Lambda^r \delta^{1/(p-1)}$. By Theorem 3.4,

$$\overline{u} = \overline{u}_\lambda = M_\Lambda \phi$$

is an upper solution of $(\mathrm{P})_\lambda$. It follows that

$$\underline{u} = \lambda^r \psi \leq \Lambda^r \delta^{1/(p-1)} \phi \leq M\phi = \overline{u}. \tag{4.1}$$

The convex, closed subset of $I_\Lambda \times C(\overline{\Omega})$, defined by

$$\mathcal{G}_\Lambda := \{(\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) \mid \lambda \in I_\Lambda,\ \underline{u} \leq u \leq \overline{u} \text{ and } u = 0 \text{ on } \partial\Omega\}$$

will play a key role in this work.

For each $u \in C(\overline{\Omega})$ define

$$f_\Lambda(u) = \chi_{S_1} f(\underline{u}) + \chi_{S_2} f(u) + \chi_{S_3} f(\overline{u}), \quad x \in \Omega, \tag{4.2}$$

where $S_1 := \{x \in \Omega \mid u(x) < \underline{u}(x)\}$, $S_2 := \{x \in \Omega \mid \underline{u}(x) \leq u(x) \leq \overline{u}(x)\}$, $S_3 := \{x \in \Omega \mid \overline{u}(x) < u(x)\}$, and χ_{S_i} is the characteristic function of S_i.

Lemma 4.1. *For each* $u \in C(\overline{\Omega})$, $f_\Lambda(u) \in L^\infty_{\mathrm{loc}}(\Omega)$ *and there are* $C > 0$, $\beta \in (0,1)$ *such that*

$$|f_\Lambda(u)(x)| \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega. \tag{4.3}$$

PROOF. Indeed, let $\mathcal{K} \subset \Omega$ be a compact subset. Then both $\underline{u}$ and $\overline{u}$ achieve a positive maximum and a positive minimum on $\mathcal{K}$. Since f is continuous in $(0, \infty)$ then $f_\Lambda(u) \in L^\infty_{\mathrm{loc}}(\Omega)$.

Verification of (4.3): Since $\Omega = \bigcup_{i=1}^{3} S_i$ it is enough to show that

$$|f(u(x))| \le \frac{C}{d(x)^\beta}, \quad x \in S_i,\ i = 1, 2, 3.$$

At first, by (f_2)(ii) there are $C, \delta > 0$ such that

$$|f(s)| \le \frac{C}{s^\beta}, \quad 0 < s < \delta.$$

Let $\Omega_\delta = \{x \in \Omega \mid d(x) < \delta\}$. Recalling that $\underline{u} \in C^1(\overline{\Omega})$, let

$$D = \max_{\overline{\Omega}} d(x), \qquad \nu_\delta := \min_{\overline{\Omega^c_\delta}} d(x), \qquad \nu^\delta := \max_{\overline{\Omega^c_\delta}} d(x),$$

and notice that both $0 < \nu_\delta \le \nu^\delta \le D < \infty$ and $f([\nu_\delta, \nu^\delta])$ are compact.

On the other hand, applying Theorems 3.3, 3.4, Lemmas 2.1 and 2.2 we infer that $0 < \lambda_0^r \psi \le \lambda^r \psi = \underline{u} \le \overline{u} = M\phi$ in Ω and

$$\frac{1}{\underline{u}^\beta}, \frac{1}{\overline{u}^\beta} \le \frac{1}{(\lambda_0^r \psi(x))^\beta} \le \frac{C}{d(x)^\beta}, \quad x \in \Omega_\delta.$$

To finish the proof, we distinguish three cases:

(1) $x \in S_1$. In this case, $f_\Lambda(u(x)) = f(\underline{u}(x))$. If $x \in S_1 \cap \Omega_\delta$ we infer that

$$|f_\Lambda(u(x))| \le \frac{C}{\underline{u}(x)^\beta} \le \frac{C}{d(x)^\beta}.$$

If $x \in S_1 \cap \Omega^c_\delta$ pick positive numbers d_i, $i = 1, 2$, such that $d_1 \le \underline{u}(x) \le d_2$, $x \in \Omega^c_\delta$. Hence

$$|f_\Lambda(u(x))| \le \frac{C}{d(x)^\beta}, \quad x \in \Omega.$$

(2) $x \in S_2$. In this case, $0 < \lambda_0^r \psi \le u \le M\phi$ and, as a consequence,

$$|f(u(x))| \le \frac{C}{u(x)^\beta}, \quad x \in \Omega_\delta.$$

Hence, there is a positive constant $\widetilde{C}$ such that $|f(u(x))| \le \widetilde{C}$, $x \in \overline{\Omega^c_\delta}$. Thus

$$|f(u(x))| \le \begin{cases} \widetilde{C} & \text{if } x \in \overline{\Omega^c_\delta}, \\ \dfrac{C}{d(x)^\beta} & \text{if } x \in \Omega_\delta. \end{cases}$$

On the other hand,

$$\frac{1}{D^\beta} \le \frac{1}{d(x)^\beta}, \quad x \in \overline{\Omega^c_\delta},$$

and therefore there is a constant $C > 0$ such that

$$|f(u(x))| \leq \begin{cases} \dfrac{C}{D^\beta} & \text{if } x \in \overline{\Omega^c_\delta}, \\ \dfrac{C}{d(x)^\beta} & \text{if } x \in \Omega_\delta. \end{cases}$$

Therefore,

$$|f(u(x))| \leq \frac{C}{d(x)^\beta}, \quad x \in S_2,\ u \in \mathcal{G}_\Lambda.$$

(3) $x \in S_3$. In this case $f_\Lambda(u(x)) = f(\overline{u}(x))$. If $x \in S_3 \cap \Omega_\delta$ we infer that

$$|f_\Lambda(u(x))| \leq \frac{C}{\overline{u}(x)^\beta} \leq \frac{C}{d(x)^\beta}.$$

If $x \in S_3 \cap \Omega^c_\delta$. Pick positive numbers d_i, $i = 1, 2$, such that $d_1 \leq \overline{u}(x)) \leq d_2$, $x \in \Omega^c_\delta$. Hence

$$|f_\Lambda(u(x))| \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega.$$

This ends the proof of Lemma 4.1. □

REMARK 4.2. By Lemmas 2.4, 4.1 and Remark 2.5, for each $v \in C(\overline{\Omega})$ and $\lambda \in I_\Lambda$,

$$\lambda f_\Lambda(v) \in L^\infty_{\text{loc}}(\Omega) \quad \text{and} \quad |\lambda f_\Lambda(v)| \leq \frac{C_\Lambda}{d^\beta(x)} \quad \text{in } \Omega, \tag{4.4}$$

where $C_\Lambda > 0$ is a constant independent of v and $\beta \in (0, 1)$. So for each v,

$$\begin{cases} -\Delta_p u = \lambda f_\Lambda(v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

admits only a solution $u = S(\lambda f_\Lambda(v)) \in W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Set $F_\Lambda(u)(x) = f_\Lambda(u(x))$, $u \in C(\overline{\Omega})$, and consider the operator

$$T \colon I_\Lambda \times C(\overline{\Omega}) \to W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}),$$
$$T(\lambda, u) = S(\lambda F_\Lambda(u)) \quad \text{if } \lambda_0 \leq \lambda \leq \Lambda,\ u \in C(\overline{\Omega}).$$

Notice that if $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ satisfies $u = T(\lambda, u)$ then u is a solution of

$$\begin{cases} -\Delta_p u = \lambda f_\Lambda(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

LEMMA 4.3. *If $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ and $u = T(\lambda, u)$ then $(\lambda, u) \in \mathcal{G}_\Lambda$.*

PROOF. Indeed, let $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ such that $T(\lambda, u) = u$. Then

$$\int_\Omega |\nabla u|^{p-2} \nabla u . \nabla \varphi \, dx = \lambda \int_\Omega f_\Lambda(u) \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega).$$

We claim that $u \geq \underline{u}$. Assume on the contrary, that $\varphi := (\underline{u} - u)^+ \not\equiv 0$. Then

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx &= \int_{u<\underline{u}} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx \\ &= \lambda\int_{u<\underline{u}} f_\Lambda(u)\cdot\varphi\,dx = \lambda\int_{u<\underline{u}} f(\underline{u})\cdot\varphi\,dx \\ &\geq \int_{u<\underline{u}} |\nabla\underline{u}|^{p-2}\nabla\underline{u}\cdot\nabla\varphi\,dx = \int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\cdot\nabla\varphi\,dx.\end{aligned}$$

Hence

$$\int_\Omega [|\nabla u|^{p-2}\nabla u - |\nabla\underline{u}|^{p-2}\nabla\underline{u}]\cdot\nabla(u-\underline{u})\,dx \leq 0.$$

It follows, by Lemma 1.2, that $\int_\Omega |\nabla(u-\underline{u})|^p\,dx \leq 0$, contradicting $\varphi \not\equiv 0$. Thus, $(\underline{u} - u)^+ = 0$, that is, $\underline{u} - u \leq 0$, and so $\underline{u} \leq T(\lambda, u)$.

We claim that $\overline{u} \geq u$. Assume on the contrary that $\varphi := (u - \overline{u})^+ \not\equiv 0$. We have

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx &= \int_{\overline{u}<u} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,dx \\ &= \lambda\int_{\overline{u}<u} f_\Lambda(u)\cdot\varphi\,dx = \lambda\int_{\overline{u}<u} f(\overline{u})\cdot\varphi\,dx \\ &\leq \int_{\overline{u}<u} |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi\,dx = \int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi\,dx.\end{aligned}$$

Therefore,

$$\int_\Omega [|\nabla u|^{p-2}\nabla u - |\nabla\overline{u}|^{p-2}\nabla\overline{u}]\cdot\nabla(u-\overline{u})\,dx \leq 0,$$

contradicting $\varphi \not\equiv 0$. Thus $(u - \overline{u})^+ = 0$ so that $u - \overline{u} \leq 0$, which gives $\overline{u} \geq T(\lambda, u)$. As a consequence of the arguments above $u \in \mathcal{G}_\Lambda$, showing Lemma 4.3.□

REMARK 4.4. By the definitions of f_Λ and $\mathcal{G}_\Lambda$, for each $(\lambda, u) \in \mathcal{G}_\Lambda$

$$f_\Lambda(u) = f(u), \quad x \in \Omega. \tag{4.6}$$

REMARK 4.5. By Remark 2.5, there is $R_\Lambda > 0$ such that $\mathcal{G}_\Lambda \subset B(0, R_\Lambda) \subset C(\overline{\Omega})$ and

$$T(I_\Lambda \times \overline{B(0, R_\Lambda)}) \subseteq B(0, R_\Lambda).$$

Notice that, by (4.6) and Lemma 4.3, if $(\lambda, u) \in I_\Lambda \times C(\overline{\Omega})$ satisfies $u = T(\lambda, u)$ then (λ, u) is a solution of $(\mathrm{P})_\lambda$. By Remark 4.4, to solve $(\mathrm{P})_\lambda$ it suffices to look for fixed points of T.

LEMMA 4.6. *$T\colon I_\Lambda \times \overline{B(0, R_\Lambda)} \to \overline{B(0, R_\Lambda)}$ is continuous and compact.*

PROOF. Let $\{(\lambda_n, u_n)\} \subseteq I_\Lambda \times \overline{B(0, R_\Lambda)}$ be a sequence such that

$$\lambda_n \to \lambda \quad \text{and} \quad u_n \xrightarrow{C(\overline{\Omega})} u, \quad \text{as } n \to \infty.$$

Set $v_n = T(\lambda_n, u_n)$ and $v = T(\lambda, u)$ so that $v_n = S(\lambda_n F_\Lambda(u_n))$ and $v = S(\lambda F_\Lambda(u))$. It follows that

$$\int_\Omega (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\cdot\nabla(v_n - v)\,dx = \lambda_n \int_\Omega (f_\Lambda(u_n) - f_\Lambda(u))(v_n - v)\,dx \le C\int_\Omega |f_\Lambda(u_n) - f_\Lambda(u)|\,dx.$$

Since

$$|f_\Lambda(u_n) - f_\Lambda(u)| \le \frac{C}{d(x)^\beta} \in L^1(\Omega) \quad \text{and} \quad f_\Lambda(u_n(x)) \to f_\Lambda(u(x)) \quad \text{a.e. } x \in \Omega,$$

as $n \to \infty$, it follows by Lebesgue's theorem that

$$\int_\Omega |f_\Lambda(u_n) - f_\Lambda(u)|\,dx \to 0, \quad \text{as } n \to \infty.$$

Therefore $v_n \to v$, as $n \to \infty$ in $W_0^{1,p}(\Omega)$. On the other hand, since $u_n \xrightarrow{C(\overline{\Omega})} u$, as $n \to \infty$, by the proof of Lemma 4.1,

$$\lambda_n f_\Lambda(u_n) \in L^\infty_{\mathrm{loc}}(\Omega) \quad \text{and} \quad |\lambda_n f_\Lambda(u_n)| \le \frac{C_\Lambda}{d^\beta(x)} \quad \text{in } \Omega.$$

By Lemma 2.4, there is a constant $M > 0$ such that $\|v_n\|_{C^{1,\alpha}(\overline{\Omega})} \le M$ so that $v_n \xrightarrow{C(\overline{\Omega})} v$. This shows that $T\colon I_\Lambda \times \overline{B(0,R_\Lambda)} \to \overline{B(0,R_\Lambda)}$ is continuous.

The compactness of T follows from the arguments in the five lines above. $\square$

5. Bounded connected sets of solutions of (P_λ)

By applying the previous technical results and the Leray–Schauder Continuation theorem (see [6]) which we state below regarding the use of its notations, we get

Theorem 5.1. *Let D be an open bounded subset of the Banach space X. Let $a, b \in \mathbb{R}$ with $a < b$ and assume that $T\colon [a,b]\times\overline{D} \to X$ is compact and continuous. Consider $\Phi\colon [a,b]\times\overline{D} \to X$ defined by $\Phi(t,u) = u - T(t,u)$. Assume that*

(a) $\Phi(t,u) \ne 0$, $t \in [a.b]$, $u \in \partial D$,

(b) $\deg(\Phi(t,\cdot\,), D, 0) \ne 0$ *for some* $t \in [a,b]$,

and set $\mathcal{S}_{a,b} = \{(t,u) \in [a,b]\times\overline{D} \mid \Phi(t,u) = 0\}$. Then, there is a connected compact subset $\Sigma_{a,b}$ of $\mathcal{S}_{a,b}$ such that

$$\Sigma_{a,b} \cap (\{a\}\times D) \ne \emptyset \quad \textit{and} \quad \Sigma_{a,b} \cap (\{b\}\times D) \ne \emptyset.$$

We will be able to show the following auxiliary result.

Theorem 5.2. *Assume* (f_1)–(f_2). *Then there is a number $\lambda_0 > 0$ and for each $\Lambda > \lambda_0$ there is a connected set $\Sigma_\Lambda \subset ([\lambda_0, \Lambda]\times C(\overline{\Omega})$ satisfying*:

$$\Sigma_\Lambda \subset \mathcal{S}, \qquad \Sigma_\Lambda \cap (\{\lambda_0\}\times C(\overline{\Omega})) \ne \emptyset, \qquad \Sigma_\Lambda \cap (\{\Lambda\}\times C(\overline{\Omega})) \ne \emptyset.$$

Proof of Theorem 5.2. At first, some notations and technical results are needed. The Leray–Schauder theorem above will be applied to the operator T in the settings of Section 4. Remember that T is continuous, compact and $T(I_\Lambda \times \overline{B(0,R_\Lambda)}) \subset B(0,R_\Lambda)$.

Consider $\Phi\colon I_\Lambda \times \overline{B(0,R)} \to \overline{B(0,R)})$ defined by $\Phi(\lambda,u) = u - T(\lambda,u)$.

LEMMA 5.3. Φ *satisfies*:

(a) $\Phi(\lambda,u) \neq 0$ $(\lambda,u) \in I_\Lambda \times \partial B(0,R_\Lambda)$,

(b) $\deg(\Phi(\lambda,\,\cdot\,), B(0,R_\Lambda), 0) \neq 0$ *for each* $\lambda \in I_\Lambda$.

PROOF. The verification of (a) is straightforward since $T(I_\Lambda \times \overline{B(0,R_\Lambda)}) \subset B(0,R_\Lambda)$.

To prove (b) set $R = R_\Lambda$, take $\lambda \in I_\Lambda$ and consider the homotopy

$$\Psi_\lambda(t,u) = u - tT(\lambda,u), \quad (t,u) \in [0,1] \times \overline{B(0,R)}.$$

It follows that $0 \notin \Psi_\lambda(I \times \partial B(0,R))$. By the invariance under homotopy property of the Leray–Schauder degree

$$\deg(\Psi_\lambda(t,\,\cdot\,), B(0,R), 0) = \deg(\Psi_\lambda(0,\,\cdot\,), B(0,R), 0) = 1, \quad t \in [0,1].$$

Setting $\Phi(\lambda,u) = u - T(\lambda,u)$, $(\lambda,u) \in I_\Lambda \times \overline{B(0,R)}$, we also have

$$\deg(\Phi(\lambda,\,\cdot\,), B(0,R), 0) = 1, \quad \lambda \in I_\Lambda.$$

Set $\mathcal{S}_\Lambda = \{(\lambda,u) \in I_\Lambda \times \overline{B(0,R)} \mid \Phi(\lambda,u) = 0\} \subset \mathcal{G}_\Lambda$. By the Leray–Schauder Continuation theorem, there is a connected component $\Sigma_\Lambda \subset \mathcal{S}_\Lambda$ such that

$$\Sigma_\Lambda \cap (\{\lambda_*\} \times \overline{B(0,R)}) \neq \emptyset \quad \text{and} \quad \Sigma_\Lambda \cap (\{\Lambda\} \times \overline{B(0,R)}) \neq \emptyset.$$

We point out that $\mathcal{S}_\Lambda$ is the solution set of the auxiliary problem

$$\begin{cases} -\Delta_p u = \lambda f_\Lambda(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and, since $\Sigma_\Lambda \subset \mathcal{S}_\Lambda \subset \mathcal{G}_\Lambda$, it follows using the definition of f_Λ that

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for $(\lambda,u) \in \Sigma_\Lambda$, showing that $\Sigma_\Lambda \subset \mathcal{S}$. This ends the proof of Theorem 5.2. $\square$

6. Proof of Theorem 1.3

Consider Λ as introduced in Section 5 and take a sequence $\{\Lambda_n\}$ such that $\lambda_0 < \Lambda_1 < \Lambda_2 < \ldots$ with $\Lambda_n \to \infty$. Set $\beta_n = \Lambda_n$ and take a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\alpha_n \to -\infty$ and $\ldots < \alpha_n < \ldots < \alpha_1 < \lambda_0$.

Following the notations of Section 4, consider the sequence of intervals $I_n = [\lambda_0, \Lambda_n]$. Set $M = C(\overline{\Omega})$ and let

$$\mathcal{G}_{\Lambda_n} := \{(\lambda, u) \in I_n \times \overline{B}_{R_n} \mid \underline{u} \le u \le \overline{u},\ u = 0 \text{ on } \partial\Omega\},$$

where $R_n = R_{\Lambda_n}$. Consider the sequence of compact operators

$$T_n \colon [\lambda_0, \Lambda_n] \times \overline{B}_{R_n} \to \overline{B}_{R_n}$$

defined by

$$T_n(\lambda, u) = S(\lambda F_{\Lambda_n}(u))) \quad \text{if } \lambda_0 \le \lambda \le \Lambda_n,\ u \in \overline{B}_{R_n}.$$

Next consider the extension of T_n, namely $\widetilde{T}_n \colon \mathbb{R} \times \overline{B}_{R_n} \to \overline{B}_{R_n}$, defined by

$$\widetilde{T}_n(\lambda, u) = \begin{cases} T_n(\lambda_0, u) & \text{if } \lambda \le \lambda_0, \\ T_n(\lambda, u) & \text{if } \lambda_0 \le \lambda \le \Lambda_n, \\ T_n(\Lambda_n, u) & \text{if } \lambda \ge \Lambda_n. \end{cases}$$

Notice that $\widetilde{T}_n$ is continuous and compact.

Applying Theorem 5.1 to $\widetilde{T}_n \colon [\alpha_n, \beta_n] \times \overline{B}_{R_n} \to \overline{B}_{R_n}$, we get a compact connected component Σ_n^* of $\mathcal{S}_n = \{(\lambda, u) \in [\alpha_n, \beta_n] \times \overline{B}_{R_n} \mid \Phi_n(\lambda, u) = 0\}$, where

$$\Phi_n(\lambda, u) = u - \widetilde{T}_n(\lambda, u).$$

Notice that Σ_n^* is also a connected subset of $\mathbb{R} \times M$. By Theorem 1.5, there is a connected component Σ^* of $\overline{\lim}\, \Sigma_n^*$ such that

$$\Sigma^* \cap (\{\lambda\} \times M) \ne \emptyset \quad \text{for each } \lambda \in \mathbb{R}.$$

Set $\Sigma = ([\lambda_*, \infty) \times M) \cap \Sigma^*$. Then $\Sigma \subset \mathbb{R} \times M$ is connected and

$$\Sigma \cap (\{\lambda\} \times M) \ne \emptyset, \quad \lambda_0 \le \lambda < \infty.$$

We claim that $\Sigma \subset \mathcal{S}$. Indeed, at first notice that

$$\widetilde{T}_{n+1}\big|_{([\lambda_0, \Lambda_n] \times \overline{B}_{R_n})} = \widetilde{T}_n\big|_{([\lambda_0, \Lambda_n] \times \overline{B}_{R_n})} = T_n. \tag{6.1}$$

If $(\lambda, u) \in \Sigma$ with $\lambda > \lambda_0$, there is a sequence $(\lambda_{n_i}, u_{n_i}) \in \bigcup \Sigma_n^*$ with $(\lambda_{n_i}, u_{n_i}) \in \Sigma_{n_i}^*$ such that $\lambda_{n_i} \to \lambda$ and $u_{n_i} \to u$, as $n_i \to \infty$. Then $u \in B_{R_N}$ for some integer $N > 1$.

We can assume that $(\lambda_{n_i}, u_{n_i}) \in [\lambda_0, \Lambda_N] \times B_{R_N}$. On the other hand, by (6.1),

$$u_{n_i} = T_{n_i}(\lambda_{n_i}, u_{n_i}) = T_N(\lambda_{n_i}, u_{n_i}).$$

Passing to the limit we get $u = T_N(\lambda, u)$ which shows that $(\lambda, u) \in \Sigma_N$ and so

$$(\lambda, u) \in \mathcal{S} := \{(\lambda, u) \in (0, \infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (\mathrm{P})_\lambda\}.$$

This ends the proof of Theorem 1.3. $\square$

Appendix A

In this section we present proofs of Lemma 2.4, Corollary 2.6 and recall some results referred to in the paper. We begin with the Browder–Minty theorem, (cf. Deimling [6]). Let X be a real reflexive Banach space with dual space X^*. A map $F\colon X \to X^*$ is monotone if

$$\langle Fx - Fy, x - y\rangle \geq 0, \quad x, y \in X,$$

F is hemicontinuous if

$$F(x + ty) \overset{*}{\rightharpoonup} Fx \quad \text{as } t \to 0,$$

and F is coercive if

$$\frac{\langle Fx, x\rangle}{|x|} \to \infty \quad \text{as } |x| \to \infty.$$

THEOREM 1.1. *Let X be a real reflexive Banach space and let $F\colon X \to X^*$ be a monotone, hemicontinous and coercive operator. Then $F(X) = X^*$. Moreover, if F is strictly monotone then it is a homeomorphism.*

The inequality below, (cf. [22], [20]), is very useful when dealing with the p-Laplacian.

LEMMA 1.2. *Let $p > 1$. Then there is a constant $C_p > 0$ such that*

$$(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq \begin{cases} C_p|x-y|^p & \text{if } p \geq 2, \\ C_p\dfrac{|x-y|^p}{(1+|x|+|y|)^{2-p}} & \text{if } \ p \leq 2, \end{cases} \tag{A.1}$$

where $x, y \in \mathbb{R}^N$ and $(\,\cdot\,,\,\cdot\,)$ is the usual inner product of $\mathbb{R}^N$.

Recall the Hardy inequality (cf. Brézis [3]).

THEOREM 1.3. *There is a positive constant C such that*

$$\int_\Omega \left|\frac{u}{d}\right|^\beta dx \leq C\int_\Omega |\nabla u|^p, \quad u \in W_0^{1,p}(\Omega).$$

PROOF OF LEMMA 2.4. By the Hölder inequality,

$$\int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx \leq \|u\|_{1,p'}\|v\|_{1,p}, \tag{A.2}$$

where $1/p + 1/p' = 1$, and so the expression

$$\langle -\Delta_p u, v\rangle := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx, \quad u, v \in W_0^{1,p}(\Omega), \tag{A.3}$$

defines a continuous, bounded (nonlinear) operator, namely

$$\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \qquad u \mapsto \Delta_p u.$$

By (A.1), $-\Delta_p$ it is strictly monotone and coercive, that is

$$\langle -\Delta_p u - (-\Delta_p v), u - v\rangle > 0, \quad u, v \in W_0^{1,p}(\Omega),\ u \neq v$$

and
$$\frac{\langle -\Delta_p u, u\rangle}{\|u\|_{1,p}} \xrightarrow{\|u\|_{1,p}\to\infty} \infty.$$
By the Browder–Minty theorem, $\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a homeomorphism.

Consider
$$F_g(u) = \int_\Omega gu\,dx, \quad u \in W_0^{1,p}(\Omega).$$

Claim. $F_g \in W^{-1,p'}(\Omega)$.

Assume for a while the claim has been proved. Since $-\Delta_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a homeomorphism, there is only $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u = F_g$, that is
$$\langle -\Delta_p u, v\rangle = \int_\Omega gv\,dx, \quad v \in W_0^{1,p}(\Omega)$$

Verification of Claim. Let V be an open neighborhood of $\partial\Omega$ such that $0 < d(x) < 1$ for $x \in V$ so that
$$1 < \frac{1}{d(x)^\beta} < \frac{1}{d(x)}, \quad x \in V.$$
Now, if $v \in W_0^{1,p}(\Omega)$ we have
$$|F_g(v)| \le \int_\Omega |g||v|\,dx = \int_{V^c} |g||v|\,dx + \int_V |g||v|\,dx \le C\|v\|_{1,p} + \int_\Omega \left|\frac{v}{d}\right| dx.$$
Applying the Hardy inequality in the last term above, we get to
$$|F_g(v)| \le C\|v\|_{1,p},$$
showing that $F_g \in W^{-1,p'}(\Omega)$, proving the claim.

Regularity of u. At first we treat the case $p = 2$. By [5], there is a solution v of
$$\begin{cases} -\Delta v = 1/v^\beta & \text{in } \Omega,\\ v > 0 & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega,\end{cases}$$
which belongs to $C^1(\overline{\Omega})$ and by the Hopf theorem $\partial v/\partial\nu < 0$ on $\partial\Omega$. Since also $d \in C^1(\overline{\Omega})$ and $\partial d/\partial\nu < 0$ on $\partial\Omega$ there a constant $C > 0$ such that $v \le Cd$ in Ω. Moreover, $-\Delta v = 1/v^\beta \ge C/d^\beta$. Consider the problem
$$\begin{cases} -\Delta\widetilde{u} = |g| & \text{in } \Omega,\\ \widetilde{u} = 0 & \text{on } \partial\Omega.\end{cases}$$
By [11, theorem B.1], $\widetilde{u} \in C^{1,\alpha}(\overline{\Omega})$ and $\|\widetilde{u}\|_{C^{1,\alpha}(\overline{\Omega})} \le M_0$ for some positive constant M_0. By the Maximum Principle, $\widetilde{u} \le v \le Cd$ in Ω.

Setting $\overline{u} = u + \widetilde{u}$ we get $-\Delta\overline{u} = g + |g| \geq 0$ in Ω and by the arguments above, $\overline{u} \leq Cd$ in Ω. Thus, as a consequence of [11, Theorem B.1], there are $\alpha \in (0,1)$ and $M_0 > 0$ such that

$$\overline{u}, \widetilde{u} \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|\overline{u}\|_{C^{1,\alpha}(\overline{\Omega})}, \|\widetilde{u}\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_0,$$

ending the proof of Lemma 2.4 in the case $p = 2$.

In what follows we treat the case $p > 1$. Let u be a solution of (2.3). It follows that

$$-\Delta_p u = g \leq \frac{C}{d^\beta} \quad \text{and} \quad -\Delta_p(-u) = (-1)^{p-1} g \leq \frac{C}{d^\beta}.$$

By Lemma 2.2, the problem

$$\begin{cases} -\Delta_p v = C/v^\beta & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

admits only a positive solution $v \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ with $v \leq Cd$ in Ω. Hence,

$$-\Delta_p(v) = \frac{C}{v^\beta} \geq \frac{1}{d^\beta} \quad \text{in } \Omega.$$

Therefore,

$$-\Delta_p |u| \leq \frac{C}{d^\beta} \leq -\Delta_p v.$$

By the Weak Comparison Principle, $|u| \leq v \leq Cd$ in Ω, showing that $u \in L^\infty(\Omega)$. Pick $w \in C^{1,\alpha}(\overline{\Omega})$ such that

$$-\Delta w = g \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

We have

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u - \nabla w) = 0 \quad \text{in } \Omega$$

in the weak sense. By Lieberman [17, Theorem 1] the proof of Lemma 2.4 ends. $\square$

PROOF OF COROLLARY 2.6. Existence of u_ε follows directly by Lemma 2.4. Moreover, there are $M > 0$ and $\alpha \in (0,1)$ such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})}, \|u_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} < M.$$

By Vázquez [25, Theorem 5], $\partial u/\partial\nu < 0$ on $\partial\Omega$ and recalling that $d \in C^1(\overline{\Omega})$ and $\partial d/\partial\nu < 0$ on $\partial\Omega$ it follows that

$$u \geq Cd \quad \text{in } \Omega. \tag{A.4}$$

Multiplying the equation

$$-\Delta_p u - (-\Delta_p u_\varepsilon) = g - (h\chi_{[d(x)>\varepsilon]} + \widetilde{g}\chi_{[d(x)<\varepsilon]})$$

by $u - u_\varepsilon$ and integrating we have

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) \cdot \nabla(u - u_\varepsilon)\, dx \leq 2M \int_{d(x)<\varepsilon} |g - \widetilde{g}|\, dx.$$

Using Lemma 1.2, we infer that $\|u-u_\varepsilon\|_{1,p}\to 0$ as $\varepsilon\to 0$. By the compact embedding $C^{1,\alpha}(\overline{\Omega})\hookrightarrow C^1(\overline{\Omega})$, it follows that

$$\|u-u_\varepsilon\|_{C^1(\overline{\Omega})}\leq \frac{C}{2}d,$$

and, using (A.4),

$$u_\varepsilon\geq u-\frac{C}{2}d\geq u-\frac{u}{2}=\frac{u}{2}. \qquad \square$$

References

[1] A. Anane, *Simplicité et isolation de la primiére valeur propre du p-Laplacien avec poids*, C.R. Acad. Sci. Paris Sér. I (1987), 725–728.

[2] L. Boccardo, F. Murat and J.P. Puel, *Résultats d'existence pour certains problémes elliptiques quasilinéaires*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **2** (1984), 213–235.

[3] H. Brézis, *Functional Analysis, Sobolev Spaces and partial differential equations*, Springer–Verlag, Berlin, (2011).

[4] S. Carl and K. Perera, *Generalized solutions of singular p-Laplacian problems in $\mathbb{R}^N$*, Nonlinear Stud. **18** (2011), 113–124.

[5] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), 193–222.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer–Verlag, Berlin, 1985.

[7] J.I. Diaz, J. Hernandez and F.J. Mancebo, *Branches of positive and free boundary solutions for some singular quasilinear elliptic problem*, J. Math. Anal. Appl. **352** (2009) 449–474.

[8] J.I. Diaz, J.M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), 1333–1344.

[9] E. DiBenedetto, *$C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850.

[10] M. Ghergu and V. Radulescu, *Sublinear singular elliptic problems with two parameters*, J. Differential Equations **195** (2003), 520–536.

[11] J. Giacomoni, I. Schindler and P. Takac, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), 117–158.

[12] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer–Verlag, New York, 1983.

[13] J.V. Goncalves, M.C. Rezende and C.A. Santos, *Positive solutions for a mixed and singular quasilinear problem*, Nonlinear Anal. **74** (2011), 132–140.

[14] D.D. Hai, *Singular boundary value problems for the p-Laplacian*, Nonlinear Anal. **73** (2010), 2876–2881.

[15] ———, *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. **383** (2011), 619–626.

[16] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc. **111** (1991), 721–730.

[17] G.M. Liebermann, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[18] N.H. Loc and K. Schmitt, *Boundary value problems for singular elliptic equations*, Rocky Mountain J. Math. **41** (2011), 555–572.

[19] A. MOHAMMED, *Positive solutions of the p-Laplace equation with singular nonlinearity*, J. Math. Anal. Appl. **352** (2009), 234–245.

[20] I. PERAL, *Multiplicity of Solutions for the p-Laplacian*, Second School on Nonlinear Functional Analysis and Applications to Differential Equations, Trieste, Italy, 1997.

[21] J. SHI AND M. YAO, *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A **138** (1998), 1389–1401.

[22] J. SIMON, *Regularité de la solution d'une equation non linéaire dans* $\mathbb{R}^N$, Lecture Notes in Mathematics, vol. 665, Springer–Verlag, New York, 1978.

[23] J.X. SUN AND F.M. SONG, *A property of connected components and its applications*, Topology Appl. **125** (2002), 553–560.

[24] P. TOLKSDORFF, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[25] J.L. VÁZQUEZ, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.

[26] G.T. WHYBURN, *Topological Analysis*, Princeton University Press, Princeton NJ, 1955.

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