Topological **M**ethods in **N**onlinear **A**nalysis
Volume 47, No. 1, 2016, 43–54
DOI: 10.12775/TMNA.2015.086

ON THE ASYMPTOTIC RELATION OF TOPOLOGICAL AMENABLE GROUP ACTIONS

Wojciech Bułatek — Brunon Kamiński — Jerzy Szymański

Abstract. For a topological action Φ of a countable amenable orderable group G on a compact metric space we introduce a concept of the asymptotic relation $\mathbf{A}(\Phi)$ and we show that $\mathbf{A}(\Phi)$ is non-trivial if the topological entropy $h(\Phi)$ is positive. It is also proved that if the Pinsker σ-algebra $\pi_\mu(\Phi)$ is trivial, where μ is an invariant measure with full support, then $\mathbf{A}(\Phi)$ is dense. These results are generalizations of those of Blanchard, Host and Ruette ([3]) that concern the asymptotic relation for $\mathbb{Z}$-actions. We give an example of an expansive G-action ($G = \mathbb{Z}^2$) with $\mathbf{A}(\Phi)$ trivial which shows that the Bryant–Walters classical result ([3]) fails to be true in general case.

1. Introduction

One of important characteristics of topological dynamical systems with $\mathbb{Z}$ as the group of time is the asymptotic relation. Let $\mathbf{A}(T)$ denote the asymptotic relation of a dynamical system (X,T). It is known ([10]) that $\mathbf{A}(T)$ is trivial (i.e. equals the diagonal relation Δ) for deterministic systems in the sense of [10], in particular for distal systems. On the other hand, $\mathbf{A}(T)$ is non-trivial for expansive T (cf. [3]) and also for systems with positive topological entropy $h(T)$ (cf. [2]). An interesting characterization of systems with zero topological entropy by use of $\mathbf{A}(T)$ is given in [6].

2010 *Mathematics Subject Classification.* Primary: 37B05, 37B40; Secondary: 28D20.
Key words and phrases. Topological G-action; entropy; asymptotic relation; generalized Pinsker formula; Pinsker σ-algebra.

If T admits an invariant probability measure μ with full support such that T is a K-automorphism with respect to μ, then $\mathbf{A}(T)$ is dense in $X \times X$ ([2], [10]).

The aim of this paper is to extend the concept of asymptoticity to topological actions of countable amenable orderable groups.

First we show that if the topological entropy $h(\Phi)$ is positive then $\mathbf{A}(\Phi)$ is non-trivial (Corollary 4.5).

Next we prove that if Φ satisfies a stronger condition, namely if the Pinsker σ-algebra $\pi_\mu(\Phi)$ is trivial for an invariant measure μ with full support then $\mathbf{A}(\Phi)$ is dense in $X \times X$ (Proposition 4.6).

In order to show these results we first prove that for any invariant measure μ there exists a measurable partition η with properties analogous to those of the Rokhlin extreme partitions (cf. [16]) and such that any pair of points from the same atom of η belongs to $\mathbf{A}(\Phi)$.

We also give an example of an expansive $\mathbb{Z}^2$-action ($\mathbb{Z}^2$ is equipped with the lexicographical order) with $\mathbf{A}(\Phi)$ trivial.

2. Preliminaries

Let (X, d) be a compact metric space and suppose μ is a Borel probability measure on X.

We assume X is equipped with the σ-algebra $\mathcal{B}$ being the completion of the Borel σ-algebra with respect to μ. The extension of μ to $\mathcal{B}$ will be also denoted by μ.

We denote by $\mathcal{M}(X)$ the lattice of measurable partitions of $(X, \mathcal{B}, \mu)$. For the definition and basic properties of $\mathcal{M}(X)$ we refer the reader to [16] (see also [12]).

Let $\mathcal{F}(X) \subset \mathcal{M}(X)$ denote the set of finite partitions.

For any $\xi \in \mathcal{M}(X)$ we denote by $R_\xi \subset X \times X$ the equivalence relation determined by ξ and by $\widehat{\xi}$ the σ-algebra of ξ-sets, i.e. measurable unions of elements of ξ. We denote by $\mathcal{N}$ the σ-algebra corresponding to the trivial partition ν_X of X.

Let $\xi, \eta \in \mathcal{M}(X)$. The relation $\xi \prec \eta$ means that any atom of η is included in some atom of ξ.

If $\xi \prec \eta$ then obviously $\widehat{\xi} \subset \widehat{\eta}$.

For a countable family $\{\xi_t;\ t \in T\} \subset \mathcal{M}(X)$ we denote by $\bigvee_{t \in T} \xi_t$ its join. It is known ([16]) that $\bigvee_{t \in T} \xi_t \in \mathcal{M}(X)$. Moreover, if the elements of ξ_t, $t \in T$, are Borel sets then the elements of $\bigvee_{t \in T} \xi_t$ are so.

Let $\langle G, \cdot \rangle$ be a countable amenable group equipped with a set $\Gamma \subset G$ called an algebraic past satisfying the following conditions:

- $\Gamma \cap \Gamma^{-1} = \emptyset$,
- $\Gamma \cup \Gamma^{-1} \cup \{e\} = G$,

- $\Gamma \cdot \Gamma \subset \Gamma$,
- $g\Gamma g^{-1} \subset \Gamma$,

where e is the identity of G, $g \in G$.

For a finite set $A \subset G$ we denote by $|A|$ the number of elements of A.

It is well known that the amenability of G is equivalent to the existence of a Følner sequence (A_n) of finite subsets of G, i.e. a sequence satisfying the condition

$$\lim_{n\to\infty} \frac{|g \cdot A_n \cap A_n|}{|A_n|} = 1 \quad \text{for any } g \in G.$$

It is also known (cf. [14]) that every countable amenable group has a Følner sequence (A_n) such that

$$A_n^{-1} = A_n, \quad A_n \subset A_{n+1}, \quad n \geq 1, \quad \bigcup_{n=1}^{\infty} A_n = G.$$

Such sequences will be called summing ones (cf. [8]).

The existence of an algebraic past in G is equivalent to the fact that G is orderable, i.e. there exists in G a linear order $<$ compatible with the group operation. We have $\Gamma = \{g \in G;\ g < e\}$.

It is well-known that all free groups are orderable and abelian groups are orderable iff they are torsion free ([7]).

Let $\mathcal{H}(X)$ be the group of all homeomorphisms of X and let Φ be a topological action of G on X, i.e. a homomorphism of G into $\mathcal{H}(X)$.

For $g \in G$ we denote by Φ^g the homeomorphism corresponding to g.

Let $h(\Phi)$ be the topological entropy of Φ. We denote by $\mathcal{P}(X, \Phi)$ the set of all Φ-invariant probability measures. Given a measure $\mu \in \mathcal{P}(X, \Phi)$ we use $h_\mu(\Phi)$ and $\pi_\mu(\Phi)$ for the entropy and the Pinsker σ-algebra of Φ, respectively.

The generalized variational principle ([15], [18]) says that

$$h(\Phi) = \sup\{h_\mu(\Phi);\ \mu \in \mathcal{P}(X, \Phi)\}.$$

3. Generalized Pinsker formula

Let $\mu \in \mathcal{P}(X, \Phi)$. From now up to the proof of Corollary 4.4 we will omit subscript μ in the notation of entropies H_μ and h_μ.

For a partition $\xi \in \mathcal{M}(X)$ and a set $A \subset G$ we put

$$\xi(A) = \bigvee_{g \in A} \Phi^g \xi.$$

Let $\xi^- = \xi(\Gamma)$, $\xi_\Phi = \xi(G)$. Let $\sigma \in \mathcal{M}(X)$ be totally invariant, i.e. $\sigma_\Phi = \sigma$.

Proceeding in the same manner as Safonov in the proof of Theorem 1 ([17]) we obtain the following relative version of that theorem. Namely, we have

Proposition 3.1. *For any Følner sequence (A_n) in G and any $\xi \in \mathcal{F}(X)$ it holds that*
$$\lim_{n\to\infty} \frac{1}{|A_n|} H(\xi(A_n)|\widehat{\sigma}) = H(\xi|\widehat{\xi}^- \vee \widehat{\sigma}).$$

Taking $\sigma = \nu_X$ (i.e. $\widehat{\sigma} = \mathcal{N}$) we put
$$h(\xi,\Phi) = \lim_{n\to\infty} \frac{1}{|A_n|} H(\xi(A_n)) = H(\xi|\widehat{\xi}^-)$$
and we call this limit the mean entropy of ξ w.r. to Φ.

Lemma 3.2. *If $A \subset G$ is finite, then*
$$h(\xi(A),\Phi) = h(\xi,\Phi).$$

Proof. Let (A_n) be a Følner sequence. It is easy to see that
$$H([\xi(A)](A_n)) = H\bigg(\bigvee_{g\in A_n} \Phi^g \xi(A)\bigg) = H(\xi(A_n \cdot A))$$
for any $n \geq 1$. It is also easy to check that $(A_n \cdot A)_{n\geq 1}$ is a Følner sequence and
$$\lim_{n\to\infty} \frac{|A_n \cdot A|}{|A_n|} = 1,$$
which implies our result. □

Next three results are generalizations of facts well-known in the case of $\mathbb{Z}$-actions (see for example [5]).

Lemma 3.3 (generalized Pinsker formula). *For any $\xi, \eta \in \mathcal{F}(X)$ we have*
$$h(\xi \vee \eta, \Phi) = h(\xi,\Phi) + H(\eta|\widehat{\eta}^- \vee \widehat{\xi}_\Phi).$$

Proof. Let (A_n) be a summing sequence and let $\xi_n = \xi(A_n)$, $\eta_n = \eta(A_n)$, for $n \geq 1$. We have
$$H(\xi_n \vee \eta_n) = H(\xi_n) + H(\eta_n|\xi_n) \geq H(\xi_n) + H(\eta_n|\widehat{\xi}_\Phi) \quad \text{for all } n \geq 1.$$
Hence, dividing both sides of the above inequality by $|A_n|$, taking the limit as $n \to \infty$ and applying Proposition 3.1, we get
$$h(\xi \vee \eta, \Phi) \geq h(\xi,\Phi) + H(\eta|\widehat{\eta}^- \vee \widehat{\xi}_\Phi).$$

To prove the converse inequality we take n_0 such that $e \in A_{n_0}$ and let $n > n_0$. Applying Proposition 3.1, Lemma 3.2 and simple properties of entropy we have
$$\begin{aligned}h(\xi \vee \eta, \Phi) &\leq h(\xi_n \vee \eta, \Phi) = H(\xi_n \vee \eta|\widehat{\xi}_n^- \vee \widehat{\eta}^-)\\ &= H(\xi_n|\widehat{\xi}_n^- \vee \widehat{\eta}^-) + H(\eta|\widehat{\xi}_n \vee \widehat{\xi}_n^- \vee \widehat{\eta}^-) \leq H(\xi_n|\widehat{\xi}_n^-) + H(\eta|\widehat{\eta}^- \vee \widehat{\xi}_n)\\ &= h(\xi_n,\Phi) + H(\eta|\widehat{\eta}^- \vee \widehat{\xi}_n) = h(\xi,\Phi) + H(\eta|\widehat{\eta}^- \vee \widehat{\xi}_n).\end{aligned}$$

Since $A_n \nearrow G$ we have $\xi_n \nearrow \xi_\Phi$ and so taking the limit in the above inequality as $n \to \infty$ we get

$$h(\xi \vee \eta, \Phi) \leq h(\xi, \Phi) + H(\eta | \widehat{\eta}^- \vee \widehat{\xi}_\Phi)$$

which completes the proof. □

Let $(\mathcal{A}_g)_{g \in G}$ be a net of sub-σ-algebras of $\mathcal{B}$. We denote by $\bigvee_{g \in G} \mathcal{A}_g$ the smallest σ-algebra containing all $\mathcal{A}_g$ and by $\bigcap_{g \in G} \mathcal{A}_g$ the intersection of all $\mathcal{A}_g$, $g \in G$. We say that $(\mathcal{A}_g)_{g \in g}$ is increasing (decreasing) if for any $g_1, g_2 \in G$ such that $g_1 < g_2$ we have $\mathcal{A}_{g_1} \subset \mathcal{A}_{g_2}$ ($\mathcal{A}_{g_1} \supset \mathcal{A}_{g_2}$).

Theorem 3.4 (Martingale Convergence Theorem). *If the net $(\mathcal{A}_g)_{g \in G}$ of sub-σ-algebras of $\mathcal{B}$ is increasing (decreasing), then for every $f \in L^2(X, \mu)$ it holds*

$$\lim_{g \in G} E(f | \mathcal{A}_g) = E\Big(f \,\Big|\, \bigvee_{g \in G} \mathcal{A}_g\Big) \qquad \Big(E\Big(f \,\Big|\, \bigcap_{g \in G} \mathcal{A}_g\Big)\Big)$$

in the L^2-norm.

One can show this theorem applying standard methods of the theory of projections of Hilbert space (cf. [13]).

In the proof of the next proposition we need the following corollary of the above theorem.

Corollary 3.5. *If the net $(\mathcal{A}_g)_{g \in G}$ is increasing (decreasing), then for every partition $\xi \in \mathcal{F}(X)$ we have*

$$\lim_{g \in G} H(\xi | \mathcal{A}_g) = H\Big(\xi \,\Big|\, \bigvee_{g \in G} \mathcal{A}_g\Big) \qquad \Big(H\Big(\xi \,\Big|\, \bigcap_{g \in G} \mathcal{A}_g\Big)\Big).$$

The proof of this corollary is based on the fact that the convergence in the L^2-norm implies the convergence in measure μ and on the natural generalization of the Lebesgue dominated convergence theorem for the nets of functions indexed by G.

Proposition 3.6. *For any $\xi, \eta, \zeta \in \mathcal{F}(X)$ with $\xi \preceq \eta$ we have*

$$\lim_{g \in \Gamma} H(\xi | \widehat{\eta}^- \vee \Phi^g \widehat{\zeta}^-) = H(\xi | \widehat{\eta}^-).$$

Proof. First we consider the case $\xi = \eta$. By Lemma 3.3 and the invariance of μ w.r. to Φ, we have

$$h(\xi \vee \Phi^g \zeta, \Phi) = h(\xi, \Phi) + H(\Phi^g \zeta | \Phi^g \widehat{\zeta}^- \vee \widehat{\xi}_\Phi) = h(\xi, \Phi) + H(\zeta | \widehat{\zeta}^- \vee \widehat{\xi}_\Phi), \quad g \in \Gamma.$$

On the other hand,

$$\begin{aligned} h(\xi\vee\Phi^g\zeta,\Phi) &= H(\xi\vee\Phi^g\zeta|\widehat{\xi}^-\vee\Phi^g\widehat{\zeta}^-)\\ &= H(\Phi^g\zeta|\widehat{\xi}^-\vee\Phi^g\widehat{\zeta}^-)+H(\xi|\widehat{\xi}^-\vee\Phi^g(\widehat{\zeta}\vee\widehat{\zeta}^-))\\ &= H(\zeta|\widehat{\zeta}^-\vee\Phi^{g^{-1}}\widehat{\xi}^-)+H(\xi|\widehat{\xi}^-\vee\Phi^g(\widehat{\zeta}\vee\widehat{\zeta}^-)). \end{aligned}$$

Combining the two above equalities, we get

$$\begin{aligned} h(\xi,\Phi) &= H(\xi|\widehat{\xi}^-\vee\Phi^g(\widehat{\zeta}\vee\widehat{\zeta}^-))+H(\zeta|\widehat{\zeta}^-\vee\Phi^{g^{-1}}\widehat{\xi}^-)-H(\zeta|\widehat{\zeta}^-\vee\widehat{\xi}_\Phi)\\ &\le H(\xi|\widehat{\xi}^-\vee\Phi^g\widehat{\zeta}^-)+H(\zeta|\widehat{\zeta}^-\vee\Phi^{g^{-1}}\widehat{\xi}^-)-H(\zeta|\widehat{\zeta}^-\vee\widehat{\xi}_\Phi). \end{aligned}$$

From Corollary 3.5 we get

$$\lim_{g\in\Gamma}H(\zeta|\widehat{\zeta}^-\vee\Phi^{g^{-1}}\widehat{\xi}^-)=H(\zeta|\widehat{\zeta}^-\vee\widehat{\xi}_\Phi).$$

Therefore

$$\lim_{g\in\Gamma}H(\xi|\widehat{\xi}^-\vee\Phi^g\widehat{\zeta}^-)\ge h(\xi,\Phi)=H(\xi|\widehat{\xi}^-).$$

Since the converse inequality is obvious we obtain the desired equality.

Now, let $\xi\preceq\eta$. Thus we have

$$\begin{aligned} H(\xi|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-) &= H(\xi\vee\eta|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-)-H(\eta|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-\vee\widehat{\xi})\\ &= H(\eta|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-)-H(\eta|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-\vee\widehat{\xi})\\ &\ge H(\eta|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-)-H(\eta|\widehat{\eta}^-\vee\widehat{\xi}). \end{aligned}$$

By the first part of the proof, we get

$$\begin{aligned} \lim_{g\in\Gamma}H(\xi|\widehat{\eta}^-\vee\Phi^g\widehat{\zeta}^-) &\ge H(\eta|\widehat{\eta}^-)-H(\eta|\widehat{\eta}^-\vee\widehat{\xi})\\ &= H(\xi\vee\eta|\widehat{\eta}^-)-H(\eta|\widehat{\eta}^-\vee\widehat{\xi})=H(\xi|\widehat{\eta}^-). \end{aligned}$$

Since the converse inequality is clear we obtain the result. □

4. Asymptotic relation

Definition 4.1. For a given topological G-action Φ on X the relation

$$\mathbf{A}(\Phi)=\Big\{(x,x')\in X\times X;\ \lim_{g\in\Gamma^{-1}}d(\Phi^gx,\Phi^gx')=0\Big\}$$

is said to be the asymptotic relation of Φ.

The limit in the above definition has the following meaning:

$$\forall\,\varepsilon>0\quad\exists\,g_0\in\Gamma^{-1}\quad\forall\,g>g_0\qquad d(\Phi^gx,\Phi^gx')<\varepsilon.$$

It is clear that $\mathbf{A}(\Phi)$ is an equivalence relation.

Theorem 4.2. *There exists a partition $\eta\in\mathcal{M}(X)$ with*

(a) $\Phi^g\eta\preceq\eta$, $g\in\Gamma$,

(b) $\bigvee_{g \in G} \Phi^g \widehat{\eta} = \mathcal{B}$,

(c) $\bigcap_{g \in G} \Phi^g \widehat{\eta} \subset \pi_\mu(\Phi)$,

(d) $R_\eta \subset \mathbf{A}(\Phi)$,

where R_η denotes the equivalence relation associated with η.

Proof. Let $(\alpha_n) \subset \mathcal{F}(X)$ be a sequence of Borel measurable partitions such that

$$\alpha_n \preceq \alpha_{n+1}, \ n \in \mathbb{N} \quad \text{and} \quad \operatorname{diam} \alpha_n \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

It is well-known (cf. [1]) that α_n, $n \geq 1$, generate the Borel σ-algebra.

Now we modify (α_n), applying a technique similar to that of Rokhlin from [16], to get a new sequence $(\xi_p) \subset \mathcal{F}(X)$ with

$$H(\xi_p | \widehat{\xi_p^-}) - H(\xi_p | \widehat{\xi_{p+t}^-}) < \frac{1}{p} \quad \text{for any } p, t \geq 1. \tag{4.2}$$

For a sequence $(g_k) \subset G$ with $g_k < g_{k+1}$, $k \in \mathbb{N}$, we put

$$\xi_p = \bigvee_{k=1}^{p} \Phi^{g_k^{-1}} \alpha_k, \quad p \geq 1.$$

Now we shall choose (g_k) in such a way that (4.2) holds. Let $g_1 \in G$ be arbitrary. Suppose that $g_1, \ldots, g_{j-1}$ are defined. Applying Proposition 3.6, we choose $g_j > g_{j-1}$ such that

$$H(\xi_i | \widehat{\xi_{j-1}^-}) - H(\xi_i | \widehat{\xi_j^-}) < \frac{1}{i} \cdot \frac{1}{2^{j-i}}, \quad 1 \leq i \leq j-1.$$

Now let $p, t \geq 1$ be arbitrary. We have

$$H(\xi_p | \widehat{\xi_p^-}) - H(\xi_p | \widehat{\xi_{p+t}^-}) = \sum_{j=p}^{p+t-1} (H(\xi_p | \widehat{\xi_j^-}) - H(\xi_p | \widehat{\xi_{j+1}^-})) < \frac{1}{p} \cdot \sum_{j=1}^{t-1} \frac{1}{2^j} < \frac{1}{p}.$$

We put

$$\xi = \bigvee_{p=1}^{\infty} \xi_p, \quad \eta = \xi^-.$$

Taking in (4.2) the limit as $t \to \infty$ we get

$$H(\xi_p | \widehat{\xi_p^-}) - H(\xi_p | \widehat{\eta}) < \frac{1}{p}, \quad p \geq 1. \tag{4.3}$$

It is clear that η satisfies (a).

In order to prove (b) observe that taking any $h \in \Gamma$ we get

$$\begin{aligned} \bigvee_{g \in G} \Phi^g \widehat{\eta} &= \bigvee_{g \in G} \bigvee_{h \in \Gamma} \bigvee_{p=1}^{\infty} \Phi^{g \cdot h} \widehat{\xi_p} \supset \bigvee_{g \in G} \bigvee_{p=1}^{\infty} \Phi^{g \cdot h} \widehat{\xi_p} \\ &\supset \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{p} \Phi^{g_k \cdot h^{-1} \cdot h \cdot g_k^{-1}} \widehat{\alpha}_k = \bigvee_{p=1}^{\infty} \widehat{\alpha}_p, \end{aligned}$$

i.e. (b) is satisfied. Since $(\widehat{\xi}_p)_\Phi$ contains $\widehat{\alpha}_p$ for all $p \geq 1$, we have

$$\bigvee_{p=1}^{\infty} (\widehat{\xi}_p)_\Phi = \mathcal{B}. \tag{4.4}$$

Now we shall show that (c) is also true. Indeed, let $\alpha \in \mathcal{F}(X)$ be measurable w.r.t. to $\bigcap_{g\in G} \Phi^g \widehat{\eta}$ and let $p \in \mathbb{N}$. Applying Lemma 3.3 we have

$$h(\alpha \vee \xi_p, \Phi) = h(\alpha, \Phi) + H(\xi_p | \widehat{\xi}_p^- \vee \widehat{\alpha}_\Phi) = h(\xi_p, \Phi) + H(\alpha | \widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi).$$

Hence

$$h(\alpha, \Phi) = h(\xi_p, \Phi) - H(\xi_p | \widehat{\xi}_p^- \vee \widehat{\alpha}_\Phi) + H(\alpha | \widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi).$$

Since the σ-algebra $\bigcap_{g\in G} \Phi^g \widehat{\eta}$ is Φ-invariant, we have $\widehat{\alpha}_\Phi \subset \bigcap_{g\in G} \Phi^g \widehat{\eta} \subset \widehat{\eta}$. Therefore, applying the inequality (4.3), we get

$$\begin{aligned} h(\alpha, \Phi) &\leq h(\xi_p, \Phi) - H(\xi_p | \widehat{\xi}_p^- \vee \widehat{\eta}) + H(\alpha | \widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi) \\ &= H(\xi_p | \widehat{\xi}_p^-) - H(\xi_p | \widehat{\eta}) + H(\alpha | \widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi) \\ &< \frac{1}{p} + H(\alpha | \widehat{\alpha}^- \vee (\widehat{\xi}_p)_\Phi). \end{aligned} \tag{4.5}$$

Hence taking in (4.5) the limit as $p \to \infty$ and applying (4.4) we get

$$h(\alpha, \Phi) = 0, \tag{4.6}$$

i.e. α is measurable w.r.t. to $\pi_\mu(\Phi)$, which proves (c).

Now we shall check that $R_\eta(x) \subset \mathbf{A}(\Phi)(x)$, for any $x \in X$. Indeed, let $y \in R_\eta(x)$, $g \in \Gamma^{-1}$, $\varepsilon > 0$ be arbitrary. We take $p \in \mathbb{N}$ with $\operatorname{diam} \alpha_p < \varepsilon$. From the definition of η we have $y \in R_{\Phi^{g^{-1}}\xi}(x)$. The relation $\xi \succeq \xi_p \succeq \Phi^{g_p^{-1}} \alpha_p$ gives $y \in R_{\Phi^{(g_p \cdot g)^{-1}} \alpha_p}(x)$. This means that $(\Phi^{g_p \cdot g} x, \Phi^{g_p \cdot g} y) \in R_{\alpha_p}$, and so $d(\Phi^{g_p \cdot g} x, \Phi^{g_p \cdot g} y) < \varepsilon$. In other words, $d(\Phi^g x, \Phi^g y) < \varepsilon$ for all $g > g_p$, i.e. $y \in \mathbf{A}(\Phi)(x)$. □

In the proof of the next corollary we shall use the following.

Remark 4.3. In order to show (4.6) it is enough to assume $\widehat{\alpha}_\Phi \subset \widehat{\eta}$. The relation $\widehat{\alpha}_\Phi \subset \bigcap_{g\in G} \Phi^g \widehat{\eta}$ is not necessary.

Let (X, Φ), (Y, Ψ) be topological G-actions and let (Y, Ψ) be a factor of (X, Φ) given by a continuous surjection $\varphi \colon X \to Y$. We denote by R_φ the relation $\{(x_1, x_2) \in X \times X;\ \varphi(x_1) = \varphi(x_2)\}$. It is clear that R_φ is a closed equivalence relation in X.

The following result is a generalization of Proposition 2 of [2].

Corollary 4.4. *If (X, Φ), (Y, Ψ) are topological G-actions, (Y, Ψ) is a factor of (X, Φ) with a factor map $\varphi \colon X \to Y$ such that $\mathbf{A}(\Phi) \subset R_\varphi$, then $h(\Psi) = 0$.*

Proof. Let $\nu \in \mathcal{P}(Y, \Psi)$ be arbitrary. We shall show that $h_\nu(\Psi) = 0$. Applying standard methods ([4, Proposition 3.11]) one can find $\mu \in \mathcal{P}(X, \Phi)$ such that $\nu = \mu \circ \varphi^{-1}$.

Take $\eta \in \mathcal{M}(X)$ given by Theorem 4.2 for the measure μ. Hence $R_\eta \subset \mathbf{A}(\Phi)$ and so, by our assumption $R_\eta \subset R_\varphi$. This means that

$$\eta \succeq \varphi^{-1}(\varepsilon_Y). \tag{4.7}$$

Let $\alpha \in \mathcal{F}(Y)$ be arbitrary. We have $\varphi^{-1}\alpha \preceq \varphi^{-1}(\varepsilon_Y)$. Since $\varphi^{-1}(\varepsilon_Y)$ is Φ-invariant $(\varphi^{-1}\alpha)_\Phi \preceq \varphi^{-1}(\varepsilon_Y)$. By (4.7) $(\varphi^{-1}\alpha)_\Phi \preceq \eta$, i.e. $\widehat{(\varphi^{-1}\alpha)}_\Phi \subset \widehat{\eta}$ and so applying Remark 4.3 we get

$$h_\nu(\alpha, \Psi) = h_\mu(\varphi^{-1}\alpha, \Phi) = 0.$$

Therefore $h_\nu(\Psi) = 0$. Using the variational principle ([15], [18]) we receive $h(\Psi) = 0$. □

Applying the above result for φ being the identity we obtain at once the following.

Corollary 4.5. *If Φ is a topological G-action with $\mathbf{A}(\Phi) = \Delta$ then the topological entropy $h(\Phi) = 0$.*

Proposition 4.6. *If (X, Φ) posesses a measure $\mu \in \mathcal{P}(X, \Phi)$ with full support such that $\pi_\mu(\Phi) = \mathcal{N}$ then $\mathbf{A}(\Phi)$ is dense in $X \times X$.*

Proof. Let μ satisfy our assumption and let η be the partition given by Theorem 4.2. It follows from (a), (c) that

$$\bigcap_{g \in \Gamma} \Phi^g \widehat{\eta} = \bigcap_{g \in G} \Phi^g \widehat{\eta} = \mathcal{N}.$$

For $g \in \Gamma$ let λ_g be the following relative product:

$$\lambda_g = \mu \underset{\Phi^g \widehat{\eta}}{\times} \mu.$$

Applying Theorem 3.4 and proceeding in the same way as in the proof of Lemma 5 (iv) ([2]) we see that $\mu \times \mu$ is the weak limit

$$\mu \times \mu = \mu \underset{\mathcal{N}}{\times} \mu = \lim_{g \in \Gamma} \lambda_g.$$

Therefore and since we deal with a closed set, we get

$$(\mu \times \mu)(\overline{\mathbf{A}(\Phi)}) \geq \limsup_{g \in \Gamma} \lambda_g(\overline{\mathbf{A}(\Phi)}). \tag{4.8}$$

By Lemma 6 of [2] we have

$$(\mu \underset{\widehat{\eta}}{\times} \mu)(R_\eta) = 1$$

and so, by the inclusion $R_\eta \subset \mathbf{A}(\Phi) \subset \overline{\mathbf{A}(\Phi)}$ we obtain

$$(\mu \underset{\widehat{\eta}}{\times} \mu)(\overline{\mathbf{A}(\Phi)}) = 1.$$

Hence the $\Phi \times \Phi$-invariance of $\mathbf{A}(\Phi)$ implies $\lambda_g(\overline{\mathbf{A}(\Phi)}) = 1$, $g \in G$, and therefore the inequality (4.8) implies $(\mu \times \mu)(\overline{\mathbf{A}(\Phi)}) = 1$, i.e. $\operatorname{Supp} \mu \times \mu \subset \overline{\mathbf{A}(\Phi)}$.

By our assumption $\operatorname{Supp} \mu = X$ and so $\operatorname{Supp} \mu \times \mu = X \times X$ which implies $\overline{\mathbf{A}(\Phi)} = X \times X$, i.e. $\mathbf{A}(\Phi)$ is dense in $X \times X$. □

It is known (cf. [3]) that for any expansive homeomorphism T of X the asymptotic relation $\mathbf{A}(T)$ is nontrivial. We shall show that if we take $G = \mathbb{Z}^2$ and we equip it with the lexicographical order $\overset{\star}{\succ}$ then we can obtain the trivial relation $\mathbf{A}(\Phi)$ for an expansive action Φ.

Example 4.7. We consider the group $(Y = \{0,1\}^{\mathbb{Z}^2}, +)$ where $+$ is the coordinatewise addition mod 2. The set Y is equipped with the metric

$$d(x, x') = \sum_{g \in \mathbb{Z}^2} \frac{|x(g) - x'(g)|}{2^{\|g\|}},$$

where $x, x' \in Y$, $\|g\| = |m| + |n|$, $g = (m,n) \in \mathbb{Z}^2$. It is clear that $(Y, +)$ is a compact metric abelian group.

Let Φ be the shift $\mathbb{Z}^2$-action on Y, i.e.

$$(\Phi^h x)(g) = x(g+h), \quad x \in Y,\ g, h \in \mathbb{Z}^2.$$

We put $F = \{(-1,-1), (0,0), (1,0), (0,1), (1,1)\}$ and $F_g = F + g$, $g \in \mathbb{Z}^2$. We define a continuous homomorphism $\varphi\colon Y \to Y$ by

$$\varphi(x)(g) = \sum_{u \in F_g} x(u), \quad x \in Y,\ g \in \mathbb{Z}^2.$$

It is clear that φ commutes with the action of Φ. Hence the set $X = \ker \varphi$ is Φ-invariant (the identity element of Y is a fixed point for Φ-action) and obviously compact. From now on Φ shall denote the restriction of the Φ-action to the set X. We claim that Φ is expansive and $\mathbf{A}(\Phi) = \Delta$. The expansiveness of Φ is obvious.

Suppose $(x, y) \in \mathbf{A}(\Phi)$. There exists $g_0 = (m_0, n_0) \overset{\star}{\succ} (0,0)$ such that for all $g \overset{\star}{\succ} g_0$ we have $d(\Phi^g x, \Phi^g y) < 1$ and therefore

$$x(g) = \Phi^g x(0,0) = \Phi^g y(0,0) = y(g).$$

In particular,

$$x(m,n) = y(m,n) \quad \text{if } m \geq m_0 + 1. \tag{4.9}$$

Let $g = (m_0 + 1, n)$. Then, by definition of X,

$$\sum_{u \in F_g} x(u) = 0 = \sum_{u \in F_g} y(u).$$

Due to (4.9) four summands in above sums (corresponding to u's with first coordinate greater than m_0) are the same, hence

$$x(m_0, n-1) = y(m_0, n-1),$$

thus

$$x(m,n) = y(m,n) \quad \text{if } m \geq m_0$$

and induction gives $x(g) = y(g)$ for all $g \in \mathbb{Z}^2$, i.e. $(x,y) \in \Delta$.

Acknowledgements. The authors would like to thank the anonymous referee for fruitful remarks which allowed to improve the initial version of this paper.

References

[1] P. Billingsley, *Ergodic Theory and Information*, Wiley, New York 1965.

[2] F. Blanchard, B. Host and S. Ruette, *Asymptotic pairs in positive entropy systems*, Ergodic Theory Dynam. Systems **22** (2002), 671–686.

[3] B.F. Bryant and P. Walters, *Asymptotic properties of expansive homeomorphisms*, Math. System Theory **3** (1969), 60–66.

[4] M. Denker, Ch. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lectures Notes Math. **527**, Springer Verlag, Berlin, Heidelberg, New York, 1976.

[5] T. Downarowicz, *Entropy in dynamical systems*, Cambridge University Press, New Math. Monogr. **18** Cambridge, New York, Melbourne (2011).

[6] T. Downarowicz and Y. Lacroix, *Topological entropy zero and asymptotic pairs*, Israel J. Math. **189** (2012), 323–336.

[7] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford; Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto, Calif.-London (1963).

[8] F.P. Greenleaf, *Ergodic theorems and the construction of summing sequences in amenable locally compact groups*, Comm. Pure Appl. Math. **26** (1973), 29–46.

[9] W. Huang and X. Ye, *Devaney's chaos or* 2*-scattering implies Li–York's chaos*, Topology and its applications **117** (2002), 259–272.

[10] B. Kamiński, A. Siemaszko and J. Szymański, *On deterministic and Kolmogorov extensions for topological flows*, Topol. Methods Nonlinear Anal. **31** (2008), 191–204.

[11] J.C. Kieffer, *A generalized Shannon–McMillan theorem for the action of an amenable group on a probability space*, Ann. Probability **3** (1975), 1031–1037.

[12] N.F.G. Martin and J.W. England, *Mathematical theory of entropy*, Encyclopedia of Mathematics and its Applications **12**, Addison–Wesley Publishing Co., Reading, Mass. 1981.

[13] W. Mlak, *Hilbert spaces and operator theory*, PWN – Polish Scientific Publishers, Warszawa, Kluwer Academic Publishers, Dodrecht, 1991.

[14] I. Namioka, *Følner's conditions for amenable semi-groups*, Math. Scand. **15** (1964), 18–28.

[15] J.M. Ollagnier and D. Pinchon, *The variational principle*, Studia Math. **72** (1982), 151–159.

[16] V.A. Rokhlin, *On the fundamental ideas of measure theory*, Mat. Sb. **25** (67) (1949), 107–150.

[17] A. V. Safonov, *Informational pasts in groups*, Izv. Akad. Nauk. SSSR **47** (1983), 421–426.

[18] A.M. Stepin and A.T. Tagi-Zade, *Variational characterization of topological pressure of the amenable groups of transformations*, Dokl. Akad. Nauk SSSR **254** (1980), 545–549.

Manuscript received May 6, 2014
accepted October 23, 2015

Wojciech Bułatek and Jerzy Szymański
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18
87-100 Toruń, POLAND
E-mail address: bulatek@mat.umk.pl, jerzy@mat.umk.pl

Brunon Kamiński
Toruń School of Banking
ul. Młodzieżowa 31a
87-100 Toruń, POLAND
E-mail address: bkam@mat.umk.pl