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SUBNORMALITY, ANALYTICITY AND PERTURBATIONS

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ABSTRACT. The question of joint subnormality of analytic operator-valued functions on open subsets of normed spaces is studied with special emphasis laid on the role played by sets of uniqueness in recovering the joint subnormality. Minimal normal extensions of such functions are characterized via coefficients of their Taylor series expansions. Investigating perturbations of unitary operators and subnormal partial isometries gives rise to numerous illustrative examples. An explicit matrix construction of normal extensions of specific perturbations of subnormal partial isometries is supplied.

1. Introduction. Given a class of bounded Hilbert space operators it is tempting to know when linear combinations of its members remain within this class. If the class of subnormal or hyponormal operators is taken into consideration, then the above question has an intimate connection with commutativity, cf. [**7, 11**]. Surprisingly, in the case of subnormal operators even the commutativity of two operators does not ensure their sum to be hyponormal [**1, 24**], see also [**25**, Problem]. On the other hand, there exists a pair of noncommuting subnormal operators whose all linear combinations are subnormal, cf. [**7**, Example 3.2] and its reformulated version in Example 31. This, however, is impossible for normal operators, even if linear combinations are replaced by (ranges of) analytic operator-valued functions, cf. [**16**, Theorem] and its generalization formulated in Theorem 9. Roughly speaking, normality and analyticity implies joint normality (this is not the case if "analyticity" is replaced by "real-analyticity," cf. Example 21). In this paper, inter alia, we try to figure out to what extent this remains true for subnormal operators. As shown in [**7**, Example 3.2] analyticity of subnormal-operator-valued functions does not help

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us establish their joint subnormality. In fact, unless Question 3 posed in Section 8 is answered in the affirmative, one cannot expect that the general criterion for joint subnormality stated in Theorem 4 may find substantially simpler shape in the case of analytic operator-valued functions. It turns out that "joint subnormality" is a proper substitute for "normality" as long as analogues of [**16**, Theorem] are concerned. It is shown in Proposition 11 that an analytic operator-valued function (defined on a region) whose restriction to a set of uniqueness is jointly subnormal must necessarily be jointly subnormal. This phenomenon does not occur if the set of uniqueness has empty interior and "joint subnormality" is replaced by "normality," cf. Example 22.

When exploring the topic of joint subnormality of an analytic operator-valued function it is natural to investigate some basic questions like analyticity of minimal normal extensions, cf. Proposition 11, and the relationship between the function and its Taylor coefficients subject to both joint subnormality and minimality of normal extensions, cf. Theorem 12. Sections 6 and 7 are devoted to a study of specific perturbations of unitary operators and subnormal partial isometries, respectively. This is accompanied with the detailed discussion which has important consequences for families of normal or subnormal operators. Section 7 also contains an explicit matrix construction of normal extensions of perturbations being considered. Furthermore, some open questions are raised in Section 8. Regarding analytic operator-valued functions defined on open subsets of normed spaces we follow the terminology of the monograph [**8**], see also [**20**] for an alternative approach.

2. Prerequisites. Denote by $\mathbf{N}$ the additive semi-group of all nonnegative integers. As usual $\mathbf{R}$ and $\mathbf{C}$ stand for the fields of real and complex numbers, respectively. We adhere to the convention that *all linear spaces considered in this article are complex unless otherwise stated.* In what follows, $\mathcal{H}$ and $\mathcal{K}$ represent complex Hilbert spaces. We always use $\langle \cdot, - \rangle$ for denoting inner product. We write $\mathbf{B}(\mathcal{H})$ for the C^*-algebra of all bounded linear operators on $\mathcal{H}$ and $I = I_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. Unless otherwise stated, convergence of sequences and series with entries in $\mathbf{B}(\mathcal{H})$ as well as continuity and differentiability of functions taking values in $\mathbf{B}(\mathcal{H})$ refers to the operator norm topology. Let $\mathfrak{F}$ be a subset of $\mathbf{B}(\mathcal{H})$. Denote by $\overline{\mathfrak{F}}$ the closure of $\mathfrak{F}$ with respect to the operator norm topology. We shall abbreviate the strong, respectively weak, operator topology to SOT, respectively WOT, and write $\overline{\mathfrak{F}}^{\mathrm{SOT}}$ for the SOT-closure of $\mathfrak{F}$. We denote by $\operatorname{alg}\mathfrak{F}$, respectively $C^*(\mathfrak{F})$, $W^*(\mathfrak{F})$ the algebra, respectively C^*-algebra, W^*-algebra, generated by the set $\mathfrak{F} \cup \{I\}$. In view of Fuglede's theorem, cf. [**14**], $W^*(\mathfrak{F})$ is a commutative set of normal operators whenever $\mathfrak{F}$ is so. This fact will be of tacit and frequent use. Given a subset $\mathcal{E}$ of $\mathcal{H}$, we denote by $\mathfrak{F}\mathcal{E}$ the set $\cup_{T\in\mathfrak{F}} T(\mathcal{E})$. For $A \in \mathbf{B}(\mathcal{H})$, we write $A \geqslant 0$ in case $\langle Ah, h\rangle \geqslant 0$ for all $h \in \mathcal{H}$. Given $A, B \in \mathbf{B}(\mathcal{H})$, we set $[A, B] = AB - BA$; $[A, B]$ is called the *commutator* of operators A and B. If $A : \mathcal{H} \to \mathcal{K}$ is a bounded linear operator, then $|A|$ stands for the square root $\sqrt{A^*A}$ of A^*A. As usual, "lin" is an abbreviation for "linear span." Given a normed space $\mathcal{X}$ and real $r > 0$, we denote by $\mathbb{B}_{\mathcal{X}}(r)$ the open ball $\{x \in \mathcal{X} : \|x\| < r\}$; by convention, we set $\mathbb{B}_{\mathcal{X}}(\infty) = \mathcal{X}$.

The proof of the following fact is left to the reader.

Lemma 1. *If $\varphi: \Omega \to \mathbf{B}(\mathcal{H})$ is an analytic function defined on a nonempty open subset of $\mathbf{C}$, then $\varphi^{(n)}(\Omega) \subseteq \overline{\operatorname{lin}\varphi(\Omega)}$ for all integers $n \geqslant 0$, where $\varphi^{(n)}$ stands for the derivative (with respect to operator norm topology) of order n of φ.*

A nonempty subset E of a normed space $\mathcal{X}$ is said to be a *set of uniqueness* (*in* $\mathcal{X}$) if for every connected open set $\Omega \subseteq \mathcal{X}$ containing E, each analytic function $f: \Omega \to \mathbf{C}$ vanishing on E is equal to 0 on Ω. It is well known that every nonempty open subset of $\mathcal{X}$ is a set of uniqueness, cf. [**8**, Theorem 12.9]. In the single-variable case each infinite compact subset of $\mathbf{C}$ is a set of uniqueness. The proof of the following fact is based on an idea from [**16**].

Lemma 2. *Let Ω be a nonempty connected open subset of a normed space $\mathcal{X}$ and $\varphi: \Omega \to \mathbf{B}(\mathcal{H})$ an analytic function. Then*

$$\overline{\operatorname{lin}\varphi(\Omega)} = \overline{\operatorname{lin}\varphi(E)},\ C^*(\varphi(\Omega)) = C^*(\varphi(E)) \tag{1}$$

and

$$W^*(\varphi(\Omega)) = W^*(\varphi(E))$$

for every set of uniqueness $E \subseteq \Omega$.

Proof. Take a continuous linear functional ξ on $\mathbf{B}(\mathcal{H})$ which vanishes on $\operatorname{lin}\varphi(E)$. Since $\xi \circ \varphi$ is an analytic function vanishing on a set of uniqueness E, the function $\xi \circ \varphi$ must vanish on Ω, or equivalently $\xi = 0$ on $\operatorname{lin}\varphi(\Omega)$. As each closed linear subspace of a normed space is determined by the family of all continuous linear functionals vanishing on this subspace, cf. [**9**, Theorem III.6.13], we get $\overline{\operatorname{lin}\varphi(\Omega)} = \overline{\operatorname{lin}\varphi(E)}$. This implies (1). □

An operator $S \in \mathbf{B}(\mathcal{H})$ is said to be *subnormal* if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a normal operator $N \in \mathbf{B}(\mathcal{K})$ such that $Sh = Nh$ for all $h \in \mathcal{H}$; N is called a *normal extension* of S. A normal extension $N \in \mathbf{B}(\mathcal{K})$ of S is said to be *minimal* if $\mathcal{K}$ is the only closed linear subspace of $\mathcal{K}$ containing $\mathcal{H}$ and reducing N. This in turn is equivalent to $\mathcal{K} = \vee\{N^{*m}\mathcal{H}\colon m \geqslant 0\}$ (for this and other facts concerning subnormal operators we refer the reader to [**10, 19**]). If $S \in \mathbf{B}(\mathcal{H})$ is a subnormal operator, then there exists a unique Borel semi-spectral measure F on the closed interval $[0,\infty)$ taking values in $\mathbf{B}(\mathcal{H})$ such that[1]

$$S^{*n}S^n = \int_0^\infty t^n \mathrm{d}F(t), \quad n \geqslant 0. \tag{2}$$

The uniqueness of F is guaranteed by the fact that the growth of the sequence $\{\|S^{*n}S^n\|\}_{n=0}^\infty$ is of polynomial type (see [**33**, Theorem 2] or the survey article [**15**]). To justify the existence of such an F, take a minimal normal extension $N \in \mathbf{B}(\mathcal{K})$ of S. Let E be the spectral measure of N. Define the spectral measure E_ϱ on $[0,\infty)$ by

$$E_\varrho(\sigma) = E(\varrho^{-1}(\sigma)), \quad \sigma\text{-Borel subset of} \quad [0,\infty), \tag{3}$$

where $\varrho : \mathbf{C} \to [0,\infty)$ is given by $\varrho(z) = |z|^2$ for $z \in \mathbf{C}$. Applying the measure transport theorem, cf. [**18**, Theorem C, page 163], we obtain

$$\begin{aligned}\langle S^{*n}S^n h, h\rangle = \|N^n h\|^2 &= \int_{\mathbf{C}} \varrho(z)^n \langle E(\mathrm{d}z)h, h\rangle \\ &= \int_0^\infty t^n \langle E_\varrho(\mathrm{d}t)h, h\rangle, \quad h \in \mathcal{H},\end{aligned}$$

for every $n \geqslant 0$. Hence, the semi-spectral measure

$$F(\cdot) \stackrel{\mathrm{def}}{=} PE_\varrho(\cdot)|_{\mathcal{H}} \tag{4}$$

satisfies (2); here P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. On the other hand, if an operator $S \in \mathbf{B}(\mathcal{H})$ fulfills (2) with some Borel semi-spectral measure on $[0, \infty)$, then S is subnormal (cf. [**12, 22**]; see also [**4, 5, 13**] for related characterizations of subnormality based on the truncated complex moment problem).

Below "supp F" and "Sp (A)" abbreviate the closed support of a semi-spectral measure[2] F and the spectrum of an operator A, respectively. On account of Theorem 5 of [**33**], $\operatorname{supp} F \subseteq [0, \|S\|^2]$ whenever S and F satisfy (2).

Lemma 3. *Let $S \in \mathbf{B}(\mathcal{H})$ be a subnormal operator, $N \in \mathbf{B}(\mathcal{K})$ a minimal normal extension of S and F a (unique) Borel semi-spectral measure on $[0, \infty)$ satisfying (2). Then*

$$\operatorname{supp} F = \operatorname{Sp}(N^*N) = \{|z|^2 : z \in \operatorname{Sp}(N)\}, \tag{5}$$

$$\operatorname{Sp}(N) \subseteq \{z \in \mathbf{C} : |z|^2 \in \operatorname{supp} F\}. \tag{6}$$

Proof. Let E be the spectral measure of N, and let E_ϱ be as in (3). Since by the measure transport theorem

$$N^*N = \int_{\mathbf{C}} \varrho(z) E(\mathrm{d}z) = \int_0^\infty t\, E_\varrho(\mathrm{d}t),$$

we see that E_ϱ is the spectral measure of N^*N. It follows from [**27**, (5.4.14), (5.4.18)] that $\varrho(\operatorname{Sp}(N)) = \operatorname{Sp}(N^*N)$. We show that $\operatorname{supp} F = \operatorname{Sp}(N^*N)$. It suffices to prove that $\operatorname{supp} F = \operatorname{supp} E_\varrho$. Let σ be a Borel subset of $[0, \infty)$ such that $F(\sigma) = 0$. Then by (4) we have

$$\|E_\varrho(\sigma)h\|^2 = \langle E_\varrho(\sigma)h, h\rangle = \langle F(\sigma)h, h\rangle = 0, \quad h \in \mathcal{H}.$$

Hence $E_\varrho(\sigma)|_{\mathcal{H}} = 0$. Since N^* commutes with E, we get

$$E_\varrho(\sigma)(N^{*m}h) = N^{*m}E_\varrho(\sigma)h = 0, \quad h \in \mathcal{H}, \quad m \geqslant 0.$$

By the minimality of N, we must have $E_\varrho(\sigma) = 0$. That $E_\varrho(\sigma) = 0$ implies $F(\sigma) = 0$ is evident due to (4). This justifies the equality $\operatorname{supp} F = \operatorname{supp} E_\varrho$ and consequently completes the proof of (5). The inclusion (6) is a direct consequence of (5). $\square$

An operator $T \in \mathbf{B}(\mathcal{H})$ is called *hyponormal* if $[T^*, T] \geqslant 0$. Every subnormal operator is hyponormal but not conversely, cf. [**10, 19**]. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *cosubnormal*, respectively *cohyponormal*, if T^* is subnormal, respectively hyponormal. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *quasinormal* if T commutes with $|T|$ or equivalently T commutes with $|T|^2$. Every quasinormal operator is subnormal but not conversely, cf. [**6**], [**19**, Problem 154]. Finally, an operator which is hyponormal or cohyponormal is called *semi-normal*.

Following [**21**], we say that an operator-valued function $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ defined on a set Ω is *jointly subnormal* if there exists a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (isometric embedding) and a function $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ such that $\Phi(\omega)$, $\omega \in \Omega$, are commuting normal operators and $\varphi(\omega) = \Phi(\omega)|_{\mathcal{H}}$ for all $\omega \in \Omega$; such a Φ is called a *normal extension* of φ. If, moreover, $\mathcal{K}$ is a unique closed linear subspace of $\mathcal{K}$ containing $\mathcal{H}$ and reducing each operator $\Phi(\omega)$, $\omega \in \Omega$, then we say that Φ is a *minimal normal extension* of φ. Given a normal extension $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ of φ, we set

$$\mathfrak{M}(\Phi, \mathcal{H}) = \bigvee_{n=1}^{\infty} \{\Phi(\omega_1)^{*\alpha_1} \dots \Phi(\omega_n)^{*\alpha_n}\mathcal{H}\colon \omega_1, \dots, \omega_n \in \Omega, \\ \alpha_1, \dots, \alpha_n \in \mathbf{N}\}.$$

One can check that

$$\mathfrak{M}(\Phi, \mathcal{H}) = \overline{\operatorname{lin} C^*(\Phi(\Omega))\mathcal{H}} = \overline{\operatorname{lin} W^*(\Phi(\Omega))\mathcal{H}}. \tag{7}$$

Notice that Φ is a minimal normal extension of φ if and only if $\mathfrak{M}(\Phi, \mathcal{H}) = \mathcal{K}$. By [**21**, Theorem 2] any two minimal normal extensions $\Phi_1\colon \Omega \to \mathbf{B}(\mathcal{K}_1)$ and $\Phi_2\colon \Omega \to \mathbf{B}(\mathcal{K}_2)$ of φ are *$\mathcal{H}$-unitarily equivalent*, which means that there exists a (unique) unitary isomorphism $U\colon \mathcal{K}_1 \to \mathcal{K}_2$ such that $U|_{\mathcal{H}} = I_{\mathcal{H}}$ and $U\Phi_1(\omega) = \Phi_2(\omega)U$ for all $\omega \in \Omega$. It is plain that if $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is jointly subnormal, then $\varphi(\Omega)$ is commutative. To make the terminology complete, we say that a set $\mathfrak{F} \subseteq \mathbf{B}(\mathcal{H})$ is *jointly subnormal* if the identity embedding $\mathfrak{F} \hookrightarrow \mathbf{B}(\mathcal{H})$ is jointly subnormal; by convention, a (*minimal*) *normal extension of* $\mathfrak{F}$ is understood as a (minimal) normal extension of $\mathfrak{F} \hookrightarrow \mathbf{B}(\mathcal{H})$. It is easily seen that a function $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is jointly subnormal if and only if the set $\varphi(\Omega)$ is jointly subnormal. Moreover, if $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension of φ, then there exists a unique function

$\Psi\colon \varphi(\Omega) \to \mathbf{B}(\mathcal{K})$ such that $\Psi \circ \varphi = \Phi$; such Ψ is automatically a minimal normal extension of the embedding $\varphi(\Omega) \hookrightarrow \mathbf{B}(\mathcal{H})$.

A simple example of a jointly subnormal analytic operator-valued function is given by the formula $\Omega \ni z \mapsto (zI_{\mathcal{H}} - S)^{-1} \in \mathbf{B}(\mathcal{H})$, where $S \in \mathbf{B}(\mathcal{H})$ is a subnormal operator and Ω is the resolvent set of S. If $N \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of S, then the function $\Omega \ni z \mapsto (zI_{\mathcal{K}} - N)^{-1} \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\Omega \ni z \mapsto (zI_{\mathcal{H}} - S)^{-1} \in \mathbf{B}(\mathcal{H})$.

3. Joint subnormality revised. The following theorem provides basic tools for further investigations. It generalizes and subsumes some of Itô's results, cf. [**21**]. The condition (B) of Theorem 4 may be viewed as a noncommutative Halmos-Bram-Itô condition, cf. [**5, 17, 21**].

Theorem 4. *Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ be a function defined on a nonempty set Ω. Then the following conditions are equivalent*:

(A) *φ is jointly subnormal,*

(B) *for every integer $n \geqslant 1$, for all n-sequences $\{h_\alpha\}_{\alpha\in\mathbf{N}^n} \subseteq \mathcal{H}$ with finite number of nonzero entries and for all n-tuples $(\omega_1, \ldots, \omega_n) \in \Omega^n$*

$$\sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbf{N}^n\\ \beta=(\beta_1,\ldots,\beta_n)\in\mathbf{N}^n}} \langle \varphi(\omega_1)^{\alpha_1}\cdots\varphi(\omega_n)^{\alpha_n} h_\beta, \varphi(\omega_1)^{\beta_1}\cdots\varphi(\omega_n)^{\beta_n} h_\alpha\rangle \geqslant 0,$$

(C) *the algebra $\overline{\operatorname{alg}\varphi(\Omega)}^{\mathrm{SOT}}$ is jointly subnormal.*

If $\mathfrak{A} \stackrel{\mathrm{def}}{=} \overline{\operatorname{alg}\varphi(\Omega)}^{\mathrm{SOT}}$ is jointly subnormal and $\Theta\colon \mathfrak{A} \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathfrak{A}$, then

(a) *$\Phi \stackrel{\mathrm{def}}{=} \Theta \circ \varphi$ is a minimal normal extension of φ,*

(b) *Θ is an isometric algebra homomorphism such that $\Theta(I_{\mathcal{H}}) = I_{\mathcal{K}}$,*

(c) *$\Theta^{-1}(A) = A|_{\mathcal{H}}$ for all $A \in \Theta(\mathfrak{A})$,*

(d) *$\Theta|_{\mathfrak{F}}\colon \mathfrak{F} \to \Theta(\mathfrak{F})$ is an* SOT*-homeomorphism and a* WOT*-homeomorphism for all bounded subsets $\mathfrak{F}$ of $\mathfrak{A}$,*

(e) *$\Theta(\mathfrak{A})$ is an* SOT*-closed subalgebra of $\mathbf{B}(\mathcal{K})$,*
(f) *$\Theta^{-1}\colon \Theta(\mathfrak{A}) \to \mathfrak{A}$ is* SOT*-continuous and* WOT*-continuous,*

(g) *$W^*(\Phi(\Omega)) = W^*(\Theta(\mathfrak{A}))$.*

Proof. Implication (A) $\Rightarrow$(B) is well known.

(B) $\Rightarrow$ (C). By Theorem 3.2 of [**31**], $\varphi(\Omega)$ is commutative. The collection Γ of all functions $\gamma\colon \Omega \to \mathbf{N}$ for which the set $\{\omega \in \Omega\colon \gamma(\omega) \neq 0\}$ is finite becomes a commutative semigroup with pointwise defined addition as a semi-group operation and the zero function as a neutral element. Set $A_\gamma = \prod_{\omega\in\Omega} \varphi(\omega)^{\gamma(\omega)}$ for $\gamma \in \Gamma$. It is clear that $\{A_\gamma\colon \gamma \in \Gamma\}$ is an operator representation of Γ, and (B) is equivalent to the positive definiteness of $\{A_\gamma\colon \gamma \in \Gamma\}$ in the sense of Definition 1 of [**21**]. Applying Theorems 1 and 7 of [**21**], we get the joint subnormality of $\mathfrak{A}$.

Implication (C) $\Rightarrow$ (A) is evident.

Suppose now that $\mathfrak{A}$ is jointly subnormal and $\Theta\colon \mathfrak{A} \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathfrak{A}$. Applying Lemma 3 a) of [**21**] to the operator representation $\mathfrak{A} \hookrightarrow \mathbf{B}(\mathcal{H})$ of $\mathfrak{A}$ we see that Θ is an algebra homomorphism such that $\Theta(I_\mathcal{H}) = I_\mathcal{K}$. In turn, by Theorem 2 of [**21**], Θ is an isometry. This gives us (b) and (c). Since $\mathfrak{A} = \overline{\mathrm{alg}\,\{A_\gamma\colon \gamma \in \Gamma\}}^{\mathrm{SOT}}$, we infer from Theorem 7 of [**21**] that $W^*(\Theta(\mathfrak{A})) = W^*(\{\Theta(A_\gamma)\colon \gamma \in \Gamma\})$ and the function $\Gamma \ni \gamma \mapsto \Theta(A_\gamma) \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\{A_\gamma\colon \gamma \in \Gamma\}$. Employing the fact that Θ preserves multiplication, we get both (a) and (g).

(d) and (f). Take a net $\{T_\sigma\colon \sigma \in \Sigma\} \subseteq \mathfrak{F}$ which is SOT-convergent to $T \in \mathfrak{F}$. It follows from the commutativity of $W^*(\Theta(\mathfrak{A}))$ that $\Theta(T)f = \lim_\sigma \Theta(T_\sigma)f$ for every vector f in $\mathcal{L} \stackrel{\mathrm{def}}{=} \mathrm{lin}\,\{\Theta(A)^*h\colon h \in \mathcal{H},\, A \in \mathfrak{A}\}$. Since by (b), $\sup_{\sigma\in\Sigma} \|\Theta(T_\sigma)\| = \sup_{\sigma\in\Sigma} \|T_\sigma\| < \infty$, and by the minimality of Θ, $\overline{\mathcal{L}} = \mathcal{K}$, we conclude that $\Theta(T)f = \lim_\sigma \Theta(T_\sigma)f$ for every $f \in \mathcal{K}$, which means that the mapping $\Theta|_\mathfrak{F}\colon \mathfrak{F} \to \Theta(\mathfrak{F})$ is SOT-continuous. A similar argument yields the WOT-continuity of $\Theta|_\mathfrak{F}$. As (f) is a consequence of (c), (d) follows immediately.

(e) Take a net $\{T_\sigma\colon \sigma \in \Sigma\} \subseteq \mathfrak{A}$ such that $\{\Theta(T_\sigma)\colon \sigma \in \Sigma\}$ is SOT-convergent to an operator $N \in \overline{\Theta(\mathfrak{A})}^{\mathrm{SOT}}$. It follows from (c) that

$$\langle T_\sigma f, g\rangle = \langle \Theta(T_\sigma) f, g\rangle \longrightarrow \langle N f, g\rangle, \quad f \in \mathcal{H},\ g \in \mathcal{K}. \tag{8}$$

Thus $Nf \in \mathcal{K} \ominus (\mathcal{K} \ominus \mathcal{H}) = \mathcal{H}$ for every $f \in \mathcal{H}$, which means that $N(\mathcal{H}) \subseteq \mathcal{H}$. Applying again (8), we deduce that the net $\{T_\sigma\colon \sigma \in \Sigma\}$ is SOT-convergent to $N|_\mathcal{H} \in \mathfrak{A}$. Since $N, \Theta(N|_\mathcal{H}) \in \overline{\Theta(\mathfrak{A})}^{\mathrm{SOT}}$, $N|_\mathcal{H} =$

$\Theta(N|_{\mathcal{H}})|_{\mathcal{H}}$ and $\overline{\Theta(\mathfrak{A})}^{\mathrm{SOT}}$ is commutative, we infer from [**21**, Lemma 3 a)] that $N = \Theta(N|_{\mathcal{H}}) \in \Theta(\mathfrak{A})$. □

In view of Theorem 4 a set $\mathfrak{F} \subseteq \mathbf{B}(\mathcal{H})$ is jointly subnormal if and only if each of its finite subsets is jointly subnormal. Put $\mathfrak{F}' = \{T \in \mathbf{B}(\mathcal{H})\colon TA = AT \text{ for all } A \in \mathfrak{F}\}$.

Corollary 5. *Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ be a jointly subnormal function defined on a nonempty set Ω, and let $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ be a minimal normal extension of φ. Set $\mathfrak{A} = \overline{\operatorname{alg}\varphi(\Omega)}^{\mathrm{SOT}}$ and $\mathfrak{B} = \overline{\operatorname{alg}\varphi(\Omega)}$. Then*

1° *there exists a unique mapping $\Theta\colon \mathfrak{A} \to \Phi(\Omega)'$ such that $\Theta(A)|_{\mathcal{H}} = A$ for all $A \in \mathfrak{A}$,*

2° *there exists a unique continuous[3] algebra homomorphism $\Psi\colon \mathfrak{B} \to \mathbf{B}(\mathcal{K})$ such that $\Psi(I_{\mathcal{H}}) = I_{\mathcal{K}}$ and $\Psi(\varphi(\omega)) = \Phi(\omega)$ for all $\omega \in \Omega$.*

Moreover, $\Psi(A) = \Theta(A)$ for all $A \in \mathfrak{B}$, Ψ is a minimal normal extension of $\mathfrak{B}$ and consequently Θ is a minimal normal extension of $\mathfrak{A}$.

Proof. 1°. The uniqueness of Θ is a direct consequence of [**21**, Lemma 3 a)]. Since $W^*(\Phi(\Omega)) \subseteq \Phi(\Omega)'$, we infer from Theorem 4 and $\mathcal{H}$-unitary equivalence of minimal normal extensions that there exists a minimal normal extension $\Theta\colon \mathfrak{A} \to \Phi(\Omega)'$ of $\mathfrak{A}$ which necessarily satisfies $\Theta(A)|_{\mathcal{H}} = A$ for all $A \in \mathfrak{A}$.

2°. The uniqueness of Ψ is straightforward. By Theorem 4 and $\mathcal{H}$-unitary equivalence of minimal normal extensions, we see that $\Psi \stackrel{\mathrm{def}}{=} \Theta|_{\mathfrak{B}}$ is a minimal normal extension of $\mathfrak{B}$ which has all the desired properties. □

Corollary 6. *Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ be a function defined on a nonempty set Ω. Suppose that Ω_0 is a nonempty subset of Ω such that $\varphi|_{\Omega_0}$ is jointly subnormal and $\varphi(\Omega) \subseteq \overline{\operatorname{alg}\varphi(\Omega_0)}^{\mathrm{SOT}}$. Then φ is jointly subnormal. Moreover, if $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension of φ, then $\Phi|_{\Omega_0}$ is a minimal normal extension of $\varphi|_{\Omega_0}$.*

Proof. Set $\mathfrak{A}_0 = \overline{\operatorname{alg}\varphi(\Omega_0)}^{\mathrm{SOT}}$ and observe that $\mathfrak{A}_0 = \overline{\operatorname{alg}\varphi(\Omega)}^{\mathrm{SOT}}$. The joint subnormality of φ is a consequence of part (C) of Theorem 4. Let $\Theta_0: \mathfrak{A}_0 \to \mathbf{B}(\mathcal{K})$ be a minimal normal extension of $\mathfrak{A}_0$. By Theorem 4, $\Theta_0 \circ \varphi|_{\Omega_0}$ is a minimal normal extension of $\varphi|_{\Omega_0}$. As a consequence, $\Theta_0 \circ \varphi$ is a minimal normal extension of φ. Since minimality determines normal extensions uniquely up to the $\mathcal{H}$-unitary equivalence, the proof is complete. □

Remark 7. Regarding Theorem 4, consider a jointly subnormal operator-valued function $\varphi: \Omega \to \mathbf{B}(\mathcal{H})$. Let Θ and Φ be as in Theorem 4. Since Θ is an isometric algebra homomorphism, the mapping Φ enjoys many properties of φ such as continuity, differentiability or summability, i.e., the series $\sum_{n=0}^{\infty} c_n \varphi(\omega_n)$ and $\sum_{n=0}^{\infty} c_n \Phi(\omega_n)$ are simultaneously convergent and the norms of their sums are equal, $\{c_n\}_{n=0}^{\infty} \subseteq \mathbf{C}$, $\{\omega_n\}_{n=0}^{\infty} \subseteq \Omega$. The same goes for algebraic properties of φ, i.e., if φ is a representation of a semi-group, respectively a group, a vector space, an algebra, then so is Φ.

According to the Bishop theorem, cf. [**4**], the set of all subnormal operators on $\mathcal{H}$ is equal to the SOT-closure of the set of all normal operators on $\mathcal{H}$. Our next goal is to adapt this to the context of jointly subnormal functions.

A jointly subnormal function $\varphi: \Omega \to \mathbf{B}(\mathcal{H})$ is said to be *flat* if $\dim \mathcal{H} = \dim \mathcal{K}$ for every (equivalently, for at least one[4]) minimal normal extension $\Phi: \Omega \to \mathbf{B}(\mathcal{K})$ of φ ("dim" is an abbreviation for orthogonal dimension).

Theorem 8. *Let $\varphi: \Omega \to \mathbf{B}(\mathcal{H})$ be a function on a nonempty set Ω.*

(i) *If the function φ is jointly subnormal and flat, then there exists a net $\{\varphi_\sigma: \Omega \to \mathbf{B}(\mathcal{H}) \mid \sigma \in \Sigma\}$ such that*

(i) (a) *$\varphi_\sigma(\Omega)$ consists of commuting normal operators for every $\sigma \in \Sigma$,*

(i) (b) $\varphi(\omega) = \mathrm{SOT} - \lim_{\sigma \in \Sigma} \varphi_\sigma(\omega)$ *for every $\omega \in \Omega$.*

(ii) *If a net $\{\varphi_\sigma: \Omega \to \mathbf{B}(\mathcal{H}) \mid \sigma \in \Sigma\}$ satisfies* (i) (a) *and* (i) (b), *then φ is jointly subnormal.*

Proof. (i) Denote by Σ the set of all finite dimensional linear subspaces of $\mathcal{H}$ partially ordered by inclusion. Let $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ be a minimal normal extension of φ. Since $\dim \mathcal{H} = \dim \mathcal{K}$, for every $\sigma \in \Sigma$ there exists a unitary isomorphism $U_\sigma\colon \mathcal{H} \to \mathcal{K}$ such that $U_\sigma(f) = f$ for all $f \in \sigma$. For each $\sigma \in \Sigma$ we define the function $\varphi_\sigma\colon \Omega \to \mathbf{B}(\mathcal{H})$ by

$$\varphi_\sigma(\omega) = U_\sigma^* \Phi(\omega) U_\sigma, \quad \omega \in \Omega. \tag{9}$$

Since $U_\sigma^*(f) = f$ for all $f \in \sigma$ and $\sigma \in \Sigma$, we conclude that (i) (a) and (i) (b) are valid.

(ii) Take $\omega_1, \dots, \omega_n \in \Omega$, and set $T_j = \varphi(\omega_j)$ and $T_{j,\sigma} = \varphi_\sigma(\omega_j)$ for $j = 1, \dots, n$ and $\sigma \in \Sigma$. Denote by $E_{j,\sigma}$ the spectral measure of $T_{j,\sigma}$. It follows from (i) (a) that

(10) the spectral measures $E_{1,\sigma}, \dots, E_{n,\sigma}$ commute for every $\sigma \in \Sigma$.

Put $G = \mathbb{B}_{\mathbf{C}}(r)$ with $r \stackrel{\text{def}}{=} \max_{1 \leqslant j \leqslant n} \|T_j\| + 1$. Since the spectrum of each operator T_j is contained in G, we infer from [**10**, Lemma II.1.18] that

$$\text{SOT-}\lim_{\sigma \in \Sigma} E_{j,\sigma}(G) = I_{\mathcal{H}}, \quad j = 1, \dots, n. \tag{11}$$

Set $\widetilde{T}_{j,\sigma} = E_{j,\sigma}(G) T_{j,\sigma}$ for $j = 1, \dots, n$ and $\sigma \in \Sigma$. Notice that

$$\widetilde{T}_{j,\sigma} = \int_G z E_{j,\sigma}(\mathrm{d}z), \quad j = 1, \dots, n, \ \sigma \in \Sigma. \tag{12}$$

By (11), (i) (b) and (12) we have $\text{SOT-}\lim_{\sigma \in \Sigma} \widetilde{T}_{j,\sigma} = T_j$ and $\sup_{\sigma \in \Sigma} \|\widetilde{T}_{j,\sigma}\| \leqslant r$ for all $j = 1, \dots, n$. This leads to

$$\text{SOT-}\lim_{\sigma \in \Sigma} \widetilde{T}_{1,\sigma}^{\alpha_1} \cdots \widetilde{T}_{n,\sigma}^{\alpha_n} = T_1^{\alpha_1} \cdots T_n^{\alpha_1}, \quad \alpha_1, \dots, \alpha_n \in \mathbf{N}. \tag{13}$$

It follows from (10) and (12) that $\widetilde{T}_{1,\sigma}, \dots, \widetilde{T}_{n,\sigma}$ are commuting normal operators for every $\sigma \in \Sigma$. This combined with (13) enables us to show that the condition (B) of Theorem 4 is satisfied. This completes the proof. □

4. Analytic normal-operator-valued functions. The theorem of [**16**] can be generalized to the case of analytic functions defined on open subsets of normed spaces as follows, see also [**3**, Lemma 2] for $\Omega \subseteq \mathbf{C}^d$.

Theorem 9. *Let Ω be a nonempty connected open subset of a normed space $\mathcal{X}$ and $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ an analytic function. Suppose Ω_0 is a nonempty open subset of Ω and $\varphi(\Omega_0)$ consists of normal operators. Then $W^*(\varphi(\Omega))$ consists of commuting normal operators.*

Proof. We can assume, by diminishing Ω_0 if necessary, that Ω_0 is convex. Take two points $a, b \in \Omega_0$ and define the analytic mapping $\psi_{a,b}\colon \mathbf{C} \to \mathcal{X}$ by

$$\psi_{a,b}(z) = a + z(b-a), \quad z \in \mathbf{C}.$$

Then $\Omega_{a,b} \overset{\text{def}}{=} \psi_{a,b}^{-1}(\Omega_0)$ is an open and convex neighborhood of the closed interval $[0,1]$. Hence the function $\varphi_{a,b} \overset{\text{def}}{=} \varphi \circ \psi_{a,b}|_{\Omega_{a,b}}$ is analytic and the set $\varphi_{a,b}(\Omega_{a,b})$ being a subset of $\varphi(\Omega_0)$ consists of normal operators. Since $\Omega_{a,b}$ is connected, we infer from [**16**, Theorem] that the set $\varphi_{a,b}(\Omega_{a,b})$ is commutative. This means that the operators $\varphi(a) = \varphi_{a,b}(0)$ and $\varphi(b) = \varphi_{a,b}(1)$ commute. Summarizing, we have proved that the set $\varphi(\Omega_0)$ consists of commuting normal operators. Applying Lemma 2 to $E = \Omega_0$ completes the proof. $\square$

Our next goal is to formulate an infinite-dimensional version of [**16**, Lemma].

Proposition 10. *Let Ω be a nonempty connected open subset of a normed space $\mathcal{X}$ and $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ an analytic function. Suppose that $\sum_{n=0}^{\infty} P_n(x - x_0)$ is a Taylor series expansion of φ at a point $x_0 \in \Omega$. Then the following conditions are equivalent*:

(i) *the set $\varphi(\Omega)$ consists of normal operators,*

(ii) *the family $\{P_n(x)\colon x \in \mathcal{X},\ n \in \mathbf{N}\}$ consists of commuting normal operators.*

Proof. (i) $\Rightarrow$ (ii). By Theorem 9, $\overline{\operatorname{lin} \varphi(\Omega)}$ consists of commuting normal operators. Since $P_n(x) = (1/n!)(\mathrm{d}^n/\mathrm{d}z^n)\varphi(x_0 + zx)|_{z=0}$, Lemma 1 implies that $P_n(x) \in \overline{\operatorname{lin} \varphi(\Omega)}$ for all $x \in \mathcal{X}$ and $n \in \mathbf{N}$.

(ii) $\Rightarrow$ (i). Set $\mathcal{P} = \{P_n(x) : x \in \mathcal{X},\ n \in \mathbf{N}\}$. By (ii) and Fuglede's theorem, $\overline{\operatorname{lin}\mathcal{P}}$ consists of commuting normal operators. Let $r > 0$ be such that $\Omega_0 \stackrel{\text{def}}{=} x_0 + \mathbb{B}_{\mathcal{X}}(r) \subseteq \Omega$ and the series $\sum_{n=0}^{\infty} P_n(x - x_0)$ converges to $\varphi(x)$ for every $x \in \Omega_0$. Hence $\varphi(\Omega_0) \subseteq \overline{\operatorname{lin}\mathcal{P}}$, which, together with Lemma 2, shows that the set $\varphi(\Omega)$ consists of commuting normal operators. $\square$

5. Jointly subnormal analytic functions. The following proposition shows that joint subnormality of an analytic function can be retrieved from that of its restriction to a set of uniqueness.

Proposition 11. *Let $\varphi : \Omega \to \mathbf{B}(\mathcal{H})$ be an analytic function defined on a nonempty open subset Ω of a normed space $\mathcal{X}$.*

(i) *If φ is jointly subnormal, then any minimal normal extension $\Phi : \Omega \to \mathbf{B}(\mathcal{K})$ of φ is analytic; moreover, there exists a net $\{\varphi_\sigma : \Omega \to \mathbf{B}(\mathcal{H}) \mid \sigma \in \Sigma\}$ of analytic functions satisfying the conditions* (i) (a) *and* (i) (b) *of Theorem* 8.

(ii) *If Ω is connected and $E \subseteq \Omega$ is a set of uniqueness, then $\varphi|_E$ is jointly subnormal if and only if φ is jointly subnormal; moreover, if $\Phi : \Omega \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension of φ, then $\Phi|_E$ is a minimal normal extension of $\varphi|_E$.*

Proof. (i) Due to Theorem 4, there exists an isometric linear isomorphism $\Theta : \overline{\operatorname{lin}\varphi(\Omega)} \to \overline{\operatorname{lin}\Phi(\Omega)}$ such that $\Phi = \Theta \circ \varphi$. It is now clear that Φ is an analytic function, see [**8**, Exercise 12D]. This guarantees that the formula (9) defines a net of analytic functions with the desired properties.

(ii) Since by Lemma 2 $\overline{\operatorname{lin}\varphi(\Omega)} = \overline{\operatorname{lin}\varphi(E)}$, the conclusion is a direct consequence of Corollary 6. $\square$

Given a normed space $\mathcal{X}$, an integer $n \geqslant 1$ and an n-homogeneous polynomial $P : \mathcal{X} \to \mathbf{B}(\mathcal{H})$, we denote by $\check{P}$ the unique symmetric n-linear map $\check{P} : \mathcal{X}^n \to \mathbf{B}(\mathcal{H})$ such that $P(x) = \check{P}(x, \dots, x)$ for all $x \in \mathcal{X}$. If P is a polynomial of degree 0, then we put $\check{P} \stackrel{\text{def}}{=} P$. We now relate joint subnormality of an analytic function φ to that of coefficients of Taylor series expansion of φ at a given point. Our approach deals with a more general situation where functions under consideration are not necessarily analytic sums of power series.

Theorem 12. *Let $\mathcal{X}$ be a normed space and $P_n\colon \mathcal{X} \to \mathbf{B}(\mathcal{H})$ a continuous n-homogeneous polynomial for every $n \in \mathbf{N}$. Fix $r \in (0,\infty]$.*

(I) *If the function $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto P_n(x) \in \mathbf{B}(\mathcal{H})$ is jointly subnormal and the function $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto Q_n(x) \in \mathbf{B}(\mathcal{K})$ is its minimal normal extension, then*

(I) (i) *$Q_n\colon \mathcal{X} \to \mathbf{B}(\mathcal{K})$ is a continuous n-homogeneous polynomial for all $n \in \mathbf{N}$,*

(I) (ii) *the series $\sum_{n=0}^{\infty} \lambda_n P_n(x)$ and $\sum_{n=0}^{\infty} \lambda_n Q_n(x)$ are simultaneously convergent and the norms of their sums are equal for all $x \in \mathcal{X}$ and $\{\lambda_n\}_{n=0}^{\infty} \subseteq \mathbf{C}$,*

(I) (iii) *the radii of uniform convergence of $\sum_{n=0}^{\infty} P_n$ and $\sum_{n=0}^{\infty} Q_n$ are equal,*

(I) (iv) *if $\varphi\colon \mathbb{B}_{\mathcal{X}}(r) \to \mathbf{B}(\mathcal{H})$ is of the form[5] $\varphi(x) = \sum_{n=0}^{\infty} P_n(x)$, $x \in \mathbb{B}_{\mathcal{X}}(r)$, then φ is jointly subnormal and the function $\Phi\colon \mathbb{B}_{\mathcal{X}}(r) \to \mathbf{B}(\mathcal{K})$ defined by $\Phi(x) = \sum_{n=0}^{\infty} Q_n(x)$, $x \in \mathbb{B}_{\mathcal{X}}(r)$, is a minimal normal extension of φ.*

(II) *If $\varphi\colon \mathbb{B}_{\mathcal{X}}(r) \to \mathbf{B}(\mathcal{H})$ is given by $\varphi(x) = \sum_{n=0}^{\infty} P_n(x)$, $x \in \mathbb{B}_{\mathcal{X}}(r)$, and φ is jointly subnormal, then any minimal normal extension $\Phi\colon \mathbb{B}_{\mathcal{X}}(r) \to \mathbf{B}(\mathcal{K})$ of φ is of the form $\Phi(x) = \sum_{n=0}^{\infty} Q_n(x)$, $x \in \mathbb{B}_{\mathcal{X}}(r)$, where $Q_n\colon \mathcal{X} \to \mathbf{B}(\mathcal{K})$ are continuous n-homogeneous polynomials; moreover, the function $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto P_n(x) \in \mathbf{B}(\mathcal{H})$ is jointly subnormal and the function $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto Q_n(x) \in \mathbf{B}(\mathcal{K})$ is its minimal normal extension.*

Proof. (I) Suppose that $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto Q_n(x) \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathbf{N} \times \mathcal{X} \ni (n,x) \mapsto P_n(x) \in \mathbf{B}(\mathcal{H})$. Due to Theorem 4, there exists an isometric linear isomorphism

$$\Theta\colon \overline{\operatorname{lin}\{P_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}} \longrightarrow \overline{\operatorname{lin}\{Q_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}}$$

such that $\Theta^{-1}(T) = T|_{\mathcal{H}}$ for every $T \in \overline{\operatorname{lin}\{Q_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}}$, and

$$\Theta(P_n(x)) = Q_n(x), \quad x \in \mathcal{X}, \quad n \in \mathbf{N}. \tag{14}$$

(I) (i) Fix $n \geqslant 1$. By the polarization formula, cf. [**8**, Theorem 4.6],

$$\check{P}_n(\mathcal{X}^n) \subseteq \operatorname{lin}\{P_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}.$$

Hence the map $\widetilde{Q}_n \overset{\text{def}}{=} \Theta \circ \check{P}_n\colon \mathcal{X}^n \to \mathbf{B}(\mathcal{K})$ is well-defined, continuous, symmetric and n-linear. Since by (14)

$$\widetilde{Q}_n(x, \dots, x) = \Theta \check{P}_n(x, \dots, x) = \Theta P_n(x) = Q_n(x), \quad x \in \mathcal{X},$$

we conclude that Q_n is a continuous n-homogeneous polynomial. The case $n = 0$ is obvious due to (14).

Since Θ is a linear isometry, (14) and the Cauchy-Hadamard formula for the radius of uniform convergence, cf. [**8**, Theorem 11.5], imply (I) (ii) and (I) (iii).

(I) (iv) Notice first that in view of (I) (ii) the power series $\sum_{n=0}^{\infty} Q_n(x)$ converges for all $x \in \mathbb{B}_{\mathcal{X}}(r)$, which legitimizes the definition of $\Phi(x)$ as well as the inclusion "$\subseteq$" in the following equality

$$\overline{\operatorname{lin} \Phi(\mathbb{B}_{\mathcal{X}}(r))} = \overline{\operatorname{lin}\{Q_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}}. \tag{15}$$

The opposite inclusion can be deduced from Lemma 1 and the equality[6]

$$Q_n(x) = \frac{1}{n!}\, \frac{\mathrm{d}^n}{\mathrm{d}z^n} \Phi(zx)|_{z=0},$$

which is valid for all $x \in \mathcal{X}$ and $n \in \mathbf{N}$. By (15) and Fuglede's theorem, $\Phi(\mathbb{B}_{\mathcal{X}}(r))$ consists of commuting normal operators, and

$$\Phi(x)|_{\mathcal{H}} = \Theta^{-1}\Phi(x) = \sum_{n=0}^{\infty} \Theta^{-1} Q_n(x) \overset{(14)}{=} \sum_{n=0}^{\infty} P_n(x) = \varphi(x),\ x \in \mathbb{B}_{\mathcal{X}}(r).$$

This means that φ is jointly subnormal and Φ is its normal extension. By (7) and (15), Φ is minimal.

(II) Following the argument used in the proof of (15) we get

$$\overline{\operatorname{lin} \varphi(\mathbb{B}_{\mathcal{X}}(r))} = \overline{\operatorname{lin}\{P_n(x)\colon x \in \mathcal{X},\, n \in \mathbf{N}\}}.$$

By part (C) of Theorem 4, the function $\mathbf{N} \times \mathcal{X} \ni (n, x) \mapsto P_n(x) \in \mathbf{B}(\mathcal{H})$ is jointly subnormal. Let $\mathbf{N} \times \mathcal{X} \ni (n, x) \mapsto Q_n(x) \in \mathbf{B}(\mathcal{K})$ be its minimal normal extension. Applying (I) we conclude that Q_n are continuous n-homogeneous polynomials and the function $\Phi: \mathbb{B}_{\mathcal{X}}(r) \to \mathbf{B}(\mathcal{K})$ defined by $\Phi(x) = \sum_{n=0}^{\infty} Q_n(x)$, $x \in \mathbb{B}_{\mathcal{X}}(r)$, is a minimal normal extension of φ. Any other minimal normal extension of φ being $\mathcal{H}$-unitarily equivalent to Φ is of the same form. Since $\mathbf{N} \times \mathcal{X} \ni (n, x) \mapsto Q_n(x) \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathbf{N} \times \mathcal{X} \ni (n, x) \mapsto P_n(x) \in \mathbf{B}(\mathcal{H})$ and the $\mathcal{H}$-unitary equivalence preserves minimal normal extensions the proof is complete. $\square$

Specifying Theorem 12 for polynomials we get

Corollary 13. *Let $\mathcal{X}$ be a normed space and $k \in \mathbf{N}$. Assume that $\varphi: \mathcal{X} \to \mathbf{B}(\mathcal{H})$ is a continuous polynomial of degree k which is jointly subnormal and $\Phi: \mathcal{X} \to \mathbf{B}(\mathcal{K})$ is its minimal normal extension. Then Φ is a continuous polynomial of degree k. Moreover, if φ is k-homogeneous, then so is Φ.*

The reader may formulate other variants of Theorem 12 and Corollary 13, e.g., not assuming polynomials in question to be continuous.

Corollary 14. *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence in $\mathbf{B}(\mathcal{H})$ and $\varphi: \mathbb{B}_{\mathbf{C}}(r) \to \mathbf{B}(\mathcal{H})$ a function of the form $\varphi(z) = \sum_{n=0}^{\infty} z^n A_n$, $z \in \mathbb{B}_{\mathbf{C}}(r)$, where r is a positive real number. If the function $\mathbf{N} \ni n \mapsto \varphi(z_n) \in \mathbf{B}(\mathcal{H})$ is jointly subnormal for some sequence $\{z_n\}_{n=0}^{\infty} \subseteq \mathbb{B}_{\mathbf{C}}(r)$ with an accumulation point in $\mathbb{B}_{\mathbf{C}}(r)$, then the functions $\varphi: \mathbb{B}_{\mathbf{C}}(r) \to \mathbf{B}(\mathcal{H})$ and $\mathbf{N} \ni n \mapsto A_n \in \mathbf{B}(\mathcal{H})$ are jointly subnormal.*

Proof. Combine part (ii) of Proposition 11 with Theorem 12. $\square$

We conclude this section with a characterization of joint subnormality of analytic functions in several complex variables. We leave it to the reader to check that the technique of the proof of Theorem 12 works equally well in the present case (Proposition 11 may be helpful as well). Below we use the standard multi-index notation.

Proposition 15. *Let $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ be an analytic function defined on a nonempty connected open subset Ω of $\mathbf{C}^d$, $d \geqslant 1$, and let $\varphi(z) = \sum_{\alpha\in\mathbf{N}^d}(z-w)^\alpha A_\alpha$ be the power series expansion of φ at a point $w \in \Omega$. Then the function φ is jointly subnormal if and only if the function $\mathbf{N}^d \ni \alpha \mapsto A_\alpha \in \mathbf{B}(\mathcal{H})$ is jointly subnormal. Moreover, if $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ is a minimal normal extension[7] of φ and $\Phi(z) = \sum_{\alpha\in\mathbf{N}^d}(z-w)^\alpha B_\alpha$ is the power series expansion of Φ at w, then $\mathbf{N}^d \ni \alpha \mapsto B_\alpha \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathbf{N}^d \ni \alpha \mapsto A_\alpha \in \mathbf{B}(\mathcal{H})$. Conversely, if $\mathbf{N}^d \ni \alpha \mapsto B_\alpha \in \mathbf{B}(\mathcal{K})$ is a minimal normal extension of $\mathbf{N}^d \ni \alpha \mapsto A_\alpha \in \mathbf{B}(\mathcal{H})$, then there exists a minimal normal extension $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ of φ such that $\Phi(z) = \sum_{\alpha\in\mathbf{N}^d}(z-w)^\alpha B_\alpha$ is the power series expansion of Φ at w.*

6. Perturbing unitary operators. In this section we investigate the question under what circumstances semi-normality is preserved when a unitary operator U is perturbed by an operator T which is algebraically tied up with U. This algebraic relationship is elucidated below in Lemma 16 which provides a background for discussing the analytic polynomial $S_z = U + zT$ in one complex variable z.

Lemma 16. *Let $U \in \mathbf{B}(\mathcal{H})$ be a unitary operator and $T \in \mathbf{B}(\mathcal{H})$ ($\mathcal{H}$ is a real or complex Hilbert space). Then the following conditions are equivalent*

(i) $T^*U = T^*T$,

(ii) $TU^* = TT^*$,

(iii) $T = UP$, *where $P \in \mathbf{B}(\mathcal{H})$ is an orthogonal projection*,

(iv) $T^* = U^*Q$, *where $Q \in \mathbf{B}(\mathcal{H})$ is an orthogonal projection.*

If any of the above conditions holds, then

(a) *T is a partial isometry with the initial space $T^*(\mathcal{H}) = P(\mathcal{H})$ and the final space $T(\mathcal{H}) = Q(\mathcal{H})$,*

(b) *$[T, U] = 0$ if and only if $T^*(\mathcal{H}) = T(\mathcal{H})$,*

(c) *$[(U + zT)^*, U + zT] = (|1 + z|^2 - 1)(P - Q)$ for all scalars z,*

(d) *$U + zT$ is unitary for all scalars z such that $|1 + z| = 1$,*

(e) *if $|1+z| \neq 1$, then $U+zT$ is normal if and only if $T^*(\mathcal{H}) = T(\mathcal{H})$;*

if this is so, then $\mathcal{M} \stackrel{\text{def}}{=} T(\mathcal{H})$ *reduces* U *and* $U + zT = ((1+z)U|_{\mathcal{M}}) \oplus U|_{\mathcal{M}^\perp}$.

Remark 17. Any pair (U, T) satisfying condition (i) of Lemma 16, where U is a unitary operator, can be constructed as follows. Take two closed liner subspaces $\mathcal{K}$ and $\mathcal{L}$ of $\mathcal{H}$ such that $\dim \mathcal{K} = \dim \mathcal{L}$ and $\dim \mathcal{K}^\perp = \dim \mathcal{L}^\perp$, and consider any unitary operator $U \in \mathbf{B}(\mathcal{H})$ such that $U(\mathcal{K}) = \mathcal{L}$ and $U(\mathcal{K}^\perp) = \mathcal{L}^\perp$. Put $T = UP$, where $P \in \mathbf{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. Then the pair (U, T) satisfies condition (i) of Lemma 16, $T^*(\mathcal{H}) = \mathcal{K}$ and $T(\mathcal{H}) = \mathcal{L}$. Notice that the case in which $\mathcal{K} \nsubseteq \mathcal{L}$ and $\mathcal{L} \nsubseteq \mathcal{K}$ is not excluded.

Proof of Lemma 16. (i) $\Rightarrow$ (iii). It follows from $T^*U = T^*T$ that $T = UT^*T$, hence $T^* = T^*TU^*$, and finally $T^*T = (T^*T)^2$, which means that $P \stackrel{\text{def}}{=} T^*T$ is an orthogonal projection such that $T = UP$.

The implication (iii) $\Rightarrow$ (i) is clear.

Applying (i) $\Leftrightarrow$ (iii) to (T^*, U^*) in place of (T, U) we obtain (ii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (iv). Setting $Q = UPU^*$ and taking adjoints in $T = UP$, we obtain

$$T^* = PU^* = U^*(UPU^*) = U^*Q.$$

Applying (iii) $\Rightarrow$ (iv) to (T^*, U^*) in place of (T, U), we get (iv) $\Rightarrow$ (iii).

The rest of the proof is straightforward. $\square$

Denote by $l^2_{\mathcal{H}}$ the Hilbert space of all square summable two-sided vector sequences $\{f_n\}_{n\in\mathbf{Z}} \subseteq \mathcal{H}$ ($\mathbf{Z}$ is the set of all integers). For bounded $\{\lambda_n\}_{n\in\mathbf{Z}} \subseteq \mathbf{C}$, the operator W defined on $l^2_{\mathcal{H}}$ by

$$W(\{h_n\}_{n\in\mathbf{Z}}) = \{\lambda_n h_{n-1}\}_{n\in\mathbf{Z}}, \quad \{h_n\}_{n\in\mathbf{Z}} \in l^2_{\mathcal{H}},$$

is called a *bilateral weighted shift* on $l^2_{\mathcal{H}}$ with weights $\{\lambda_n\}_{n\in\mathbf{Z}}$.

Lemma 18. *Let* $U \in \mathbf{B}(\mathcal{H})$ *be a unitary operator and* $T \in \mathbf{B}(\mathcal{H})$ *be such that* $T^*U = T^*T$. *Fix* $z \in \mathbf{C}$ *such that* $|1 + z| \neq 1$ *and set* $S_z = U + zT$. *Assume that* $T(\mathcal{H}) \subsetneq T^*(\mathcal{H})$, *respectively* $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$, *and set* $\Delta = T^*(\mathcal{H}) \ominus T(\mathcal{H})$, *respectively* $\widetilde{\Delta} = T(\mathcal{H}) \ominus T^*(\mathcal{H})$. *Then the* *closed linear space* $\mathcal{R} \overset{\text{def}}{=} \bigoplus_{n\in\mathbf{Z}} U^n(\Delta)$, *respectively* $\widetilde{\mathcal{R}} \overset{\text{def}}{=} \bigoplus_{n\in\mathbf{Z}} U^n(\widetilde{\Delta})$, *reduces* S_z *to an operator which is unitarily equivalent to the bilateral weighted shift on* l^2_Δ, *respectively* $l^2_{\widetilde{\Delta}}$, *with weights*

$$\lambda_n(z) = \begin{cases} 1 & \text{for } n \leqslant 0 \\ 1+z & \text{for } n \geqslant 1 \end{cases} \quad \left(\textit{resp. } \widetilde{\lambda}_n(z) = \begin{cases} 1+z & \text{for } n \leqslant 0 \\ 1 & \text{for } n \geqslant 1 \end{cases} \right),$$

and $S_z|_{\mathcal{R}^\perp}$, *respectively* $S_z|_{\widetilde{\mathcal{R}}^\perp}$, *is a normal operator.*

Proof. Consider first the case $T(\mathcal{H}) \subsetneq T^*(\mathcal{H})$. By Lemma 16 the space $\mathcal{M} \overset{\text{def}}{=} T^*(\mathcal{H})$ is invariant for U and $\Delta = \mathcal{M} \ominus U(\mathcal{M}) \neq \{0\}$. Applying the von Neumann-Wold decomposition to the isometry $U|_{\mathcal{M}}$, cf. [**27**, Theorem 4.7.1], we see that the space $\mathcal{M}_\infty \overset{\text{def}}{=} \bigcap_{n=0}^\infty U^n(\mathcal{M})$ reduces U and $\mathcal{M} = \mathcal{M}_\infty \oplus \bigoplus_{n=0}^\infty U^n(\Delta)$. Hence, $\mathcal{M}_\infty$ reduces T and consequently S_z. Since the space $\mathcal{M}_\infty \oplus \mathcal{R}$ reduces U and contains $\mathcal{M}$, we deduce that it reduces S_z. Thus $\mathcal{R}$ reduces S_z and

$$S_z|_{\mathcal{R}^\perp} = \big((1+z)U|_{\mathcal{M}_\infty}\big) \oplus U|_{(\mathcal{M}_\infty \oplus \mathcal{R})^\perp},$$

which means that $S_z|_{\mathcal{R}^\perp}$ is a normal operator. Define the unitary isomorphism $\Lambda\colon l^2_\Delta \to \mathcal{R}$ by

$$\Lambda(\{h_n\}_{n\in\mathbf{Z}}) = \sum_{n\in\mathbf{Z}} U^n h_n, \quad \{h_n\}_{n\in\mathbf{Z}} \in \ell^2_\Delta.$$

Then $\Lambda^{-1}(S_z|_{\mathcal{R}})\Lambda$ is the bilateral weighted shift on l^2_Δ with weights $\{\lambda_n(z)\}_{n\in\mathbf{Z}}$.

Consider now the case $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$. By Lemma 16 the space $\mathcal{N} \overset{\text{def}}{=} T(\mathcal{H})$ is invariant for U^* and $\widetilde{\Delta} = \mathcal{N} \ominus U^*(\mathcal{N}) \neq \{0\}$. Applying the previous paragraph to $(U^*, T^*, \bar{z})$ in place of (U, T, z) and noticing that $W = V^{-1} W_0^* V$, where W is the bilateral weighted shift on $l^2_{\widetilde{\Delta}}$ with weights $\{\widetilde{\lambda}_n(z)\}_{n\in\mathbf{Z}}$, W_0 is the bilateral weighted shift on $l^2_{\widetilde{\Delta}}$ with weights $\{\lambda_n(\bar{z})\}_{n\in\mathbf{Z}}$ and $V \in \mathbf{B}(\ell^2_{\widetilde{\Delta}})$ is the unitary operator defined by

$$V(\{h_n\}_{n\in\mathbf{Z}}) = \{h_{-n}\}_{n\in\mathbf{Z}}, \quad \{h_n\}_{n\in\mathbf{Z}} \in l^2_{\widetilde{\Delta}},$$

we conclude that $S_z|_{\widetilde{\mathcal{R}}}$ is unitarily equivalent to W and $S_z|_{\widetilde{\mathcal{R}}^\perp}$ is normal. $\square$

As in the scalar case we show that a bilateral weighted shift W on $l^2_{\mathcal{H}}$ with weights $\{\lambda_n\}_{n\in\mathbf{Z}}$ is hyponormal if and only if $|\lambda_n| \leqslant |\lambda_{n+1}|$ for all $n \in \mathbf{Z}$. Moreover, if W is hyponormal, then so is W^n for every $n \geqslant 0$.

Proposition 19. *Let $U \in \mathbf{B}(\mathcal{H})$ be a unitary operator, and let $T \in \mathbf{B}(\mathcal{H})$ be such that $T^*U = T^*T$. Fix $z \in \mathbf{C}$ such that $|1+z| \neq 1$ and set $S_z = U + zT$.*

(i) *S_z is hyponormal if and only if either $|1+z| > 1$ and $T(\mathcal{H}) \subseteq T^*(\mathcal{H})$ or $|1+z| < 1$ and $T^*(\mathcal{H}) \subseteq T(\mathcal{H})$.*

(ii) *If S_z is hyponormal, then for every $n \geqslant 1$, S_z^n is hyponormal.*

(iii) *S_{-1} is subnormal if and only if $T^*(\mathcal{H}) \subseteq T(\mathcal{H})$; if $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$, then $S_{-1} = N \oplus V$, where N is a normal operator and V is an operator which is unitarily equivalent to a unilateral shift of multiplicity $\dim T(\mathcal{H}) \ominus T^*(\mathcal{H})$.*

(iv) *If $z \neq -1$, then S_z is subnormal if and only if $T^*(\mathcal{H}) = T(\mathcal{H})$.*

Remark 20. Proposition 19 can be supplemented with the following three conditions (still assuming $|1+z| \neq 1$). The proof remains essentially the same.

(a) S_z is cohyponormal if and only if either $|1+z| < 1$ and $T(\mathcal{H}) \subseteq T^*(\mathcal{H})$ or $|1+z| > 1$ and $T^*(\mathcal{H}) \subseteq T(\mathcal{H})$; if this is so, then S_z^n is cohyponormal for all $n \geqslant 1$.

(b) S_{-1} is cosubnormal if and only if $T(\mathcal{H}) \subseteq T^*(\mathcal{H})$; if $T(\mathcal{H}) \subsetneq T^*(\mathcal{H})$, then $S_{-1} = N \oplus V^*$, where N is a normal operator and V is an operator which is unitarily equivalent to a unilateral shift of multiplicity $\dim T^*(\mathcal{H}) \ominus T(\mathcal{H})$.

(c) If $z \neq -1$, then S_z is cosubnormal if and only if $T^*(\mathcal{H}) = T(\mathcal{H})$.

Proof of Proposition 19. Part (i) is a direct consequence of Lemma 16.

Part (ii) follows from (i), Lemma 18 and Lemma 16 (e).

(iii) If S_{-1} is subnormal, then it is hyponormal and consequently, by (i), $T^*(\mathcal{H}) \subseteq T(\mathcal{H})$. If $T^*(\mathcal{H}) = T(\mathcal{H})$, then by Lemma 16 the operator S_{-1} is normal. If $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$, then by Lemma 18 the space $\widetilde{\mathcal{R}}$ reduces S_{-1}, $S_{-1}|_{\widetilde{\mathcal{R}}^\perp}$ is normal and $S_{-1}|_{\widetilde{\mathcal{R}}}$ is unitarily equivalent to the bilateral weighted shift W on $l^2_{\widetilde{\Delta}}$ with weights $\{\widetilde{\lambda}_n(-1)\}_{n\in\mathbf{Z}}$. Moreover, the space $\mathcal{K} \stackrel{\mathrm{def}}{=} \{\{h_n\}_{n\in\mathbf{Z}} \in l^2_{\widetilde{\Delta}}\colon h_k = 0 \text{ for all } k \leqslant -1\}$ reduces W, $W|_{\mathcal{K}^\perp} = 0$ and $W|_{\mathcal{K}}$ is a unilateral shift of multiplicity $\dim \widetilde{\Delta}$.

(iv) According to Lemma 16 (e), it suffices to show that if $z \neq -1$ and S_z is subnormal, then $T^*(\mathcal{H}) = T(\mathcal{H})$. Suppose that, contrary to our claim, $T^*(\mathcal{H}) \neq T(\mathcal{H})$. By (i) only two possibilities can occur: either $T(\mathcal{H}) \subsetneq T^*(\mathcal{H})$ or $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$. In the first case, we take a unit vector $f \in U^{*2}(\Delta)$. In view of the proof of Lemma 18 we have $S_z(f) = Uf$ and $S_z^k(f) = (1+z)^{k-2}U^k f$ for all $k \geqslant 2$. Thus

$$(a_0, a_1, a_2, \dots) = (1, 1, 1, |1+z|^2, |1+z|^4, |1+z|^6, \dots), \tag{16}$$

where $a_n = \|S_z^n(f)\|^2$ for $n \geqslant 0$. Since S_z is subnormal, there exists a probability Borel measure μ on the half line $[0, \infty)$ such that $a_n = \int_0^\infty t^n \mathrm{d}\mu(t)$ for all $n \geqslant 0$, cf. [**22**]. By the Schwarz inequality, we have

$$1 = a_1^2 = \left(\int_0^\infty t^0 t \mathrm{d}\mu(t) \right)^2 \leqslant \int_0^\infty t^0 \mathrm{d}\mu(t) \int_0^\infty t^2 \mathrm{d}\mu(t) = a_0 a_2 = 1,$$

which implies that the monomials t^0 and t are linearly dependent in $L^2(\mu)$. Thus, $\operatorname{supp} \mu = \{\theta\}$, where $\theta \in [0, \infty)$, and consequently $a_n = \theta^n$ for all $n \geqslant 0$. This and (16) yields $|1+z| = 1$, a contradiction.

If $T^*(\mathcal{H}) \subsetneq T(\mathcal{H})$, then taking a unit vector $f \in U^{*2}(\widetilde{\Delta})$ leads to

$$(a_0, a_1, a_2, \dots) = (1, |1+z|^2, |1+z|^4, |1+z|^4, |1+z|^4, \dots), \tag{17}$$

where $a_n = \|S_z^n(f)\|^2$ for $n \geqslant 0$. Since $a_1^2 = a_0 a_2$, the reasoning contained in the previous paragraph yields $a_n = \theta^n$ for all $n \geqslant 0$, where $\theta \in [0, \infty)$. This, $z \neq -1$ and (17) gives $|1+z| = 1$, a contradiction. □

Example 21. We show that Theorem 9 is no longer true for real-analytic functions. For this, consider a real Hilbert space $\mathcal{H}_{\mathrm{r}}$, a unitary operator $U \in \mathbf{B}(\mathcal{H}_{\mathrm{r}})$ and an operator $T \in \mathbf{B}(\mathcal{H}_{\mathrm{r}})$ which satisfy the following condition

$$T^*U = T^*T \quad \text{and} \quad TU \neq UT. \tag{18}$$

Notice that this can be done even in a two-dimensional Hilbert space $\mathcal{H}_{\mathrm{r}}$, cf. Lemma 16 and Remark 17. Set $\mathcal{H}_{\mathrm{r}}^{(2)} = \mathcal{H}_{\mathrm{r}} \oplus \mathcal{H}_{\mathrm{r}}$. Then the function $\psi\colon \mathbf{R}^2 \to \mathbf{B}(\mathcal{H}_{\mathrm{r}}^{(2)})$ defined by

$$\psi(x,y) = \begin{bmatrix} U + (x-1)T & -yT \\ yT & U + (x-1)T \end{bmatrix}, \quad x, y \in \mathbf{R},$$

is real-analytic. By (18) and Lemma 16 the operator $\psi(x,y)$ is unitary for all $(x,y) \in \mathbf{R}^2$ such that $x^2+y^2=1$. As a consequence, the unitary-operator-valued function $\varphi\colon \mathbf{R} \to \mathbf{B}(\mathcal{H}_{\mathrm{r}}^{(2)})$ given by $\varphi(t) = \psi(\cos t, \sin t)$ is real-analytic and its range is not commutative because the operators $\varphi(0)$ and $\varphi(\pi/2)$ do not commute.

We now turn to a complex version of the above example. Let $U \in \mathbf{B}(\mathcal{H})$ be a unitary operator and $T \in \mathbf{B}(\mathcal{H})$ an operator which satisfies (18). Then by Lemma 16 the function $\varphi\colon \mathbf{R} \to \mathbf{B}(\mathcal{H})$ defined by

$$\varphi(t) = U + (\mathrm{e}^{it} - 1)T, \quad t \in \mathbf{R}$$

is real-analytic and $\varphi(\mathbf{R})$ consists of unitary operators, although the operators $\varphi(0)$ and $\varphi(\pi)$ do not commute.

Example 22. Example 21 may be modified so as to show that in Theorem 9 the open set Ω_0 cannot be replaced by a set of uniqueness. Let U and T be as in the previous paragraph. Define the analytic function $\widetilde{\varphi}\colon \mathbf{C} \to \mathbf{B}(\mathcal{H})$ by

$$\widetilde{\varphi}(z) = U + zT, \quad z \in \mathbf{C}.$$

Then $E \stackrel{\text{def}}{=} \{z \in \mathbf{C}\colon |1+z| = 1\}$ is a set of uniqueness in $\mathbf{C}$ and $\widetilde{\varphi}(E)$ consists of unitary operators, although $\widetilde{\varphi}(\mathbf{C})$ is not commutative and $\widetilde{\varphi}(\mathbf{C})$ is not a family of normal operators, cf. Lemma 16. In fact, if we choose U and T so that $T(\mathcal{H}) \nsubseteq T^*(\mathcal{H})$ and $T^*(\mathcal{H}) \nsubseteq T(\mathcal{H})$, then for all $z \in \mathbf{C} \setminus E$, $\widetilde{\varphi}(z)$ is neither hyponormal nor cohyponormal, cf. Proposition 19 and Remarks 17 and 20.

Remark 23. Regarding Example 21, notice that if $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is an analytic unitary-operator-valued (or an analytic self-adjoint-operator-valued) function defined on a nonempty connected open subset of $\mathbf{C}$, then the function φ is constant (mimic the proof of [**16**, Lemma] playing with a Taylor series expansion of φ at a fixed point of Ω, and then apply Lemma 2).

7. Perturbing subnormal partial isometries. Regarding subnormality it is purposeful to study an operator for which norms of its powers, excluding the 0th exponent, on an arbitrarily fixed vector form a constant sequence. Such operators can be modeled on 2 by 2 operator matrices of the form $\left[\begin{smallmatrix} V & X \\ 0 & 0 \end{smallmatrix}\right]$, where V is an isometry and X is a bounded linear operator. Thus they can be thought of as specific perturbations of operators of the shape $V \oplus 0$ with V as above. In view of [**19**, Problem 161] subnormal partial isometries are exactly those operators which can be represented as the aforesaid $V \oplus 0$.

Proposition 24. *If $S \in \mathbf{B}(\mathcal{H})$, then the following conditions are equivalent*

(i) $\|S^n h\| = \|Sh\|$ *for all* $n \geqslant 1$ *and* $h \in \mathcal{H}$,

(ii) $|S^n| = |S|$ *for all* $n \geqslant 1$,

(iii) $|S^2| = |S|$,

(iv) $S^n = W^{n-1}S$ *for all* $n \geqslant 1$, *where* $W \in \mathbf{B}(\mathcal{H})$ *is an isometry,*

(v) $S^2 = WS$, *where* $W \in \mathbf{B}(\mathcal{H})$ *is an isometry,*

(vi) $S|_{\overline{S(\mathcal{H})}}$ *is an isometry,*

(vii) $S = \left[\begin{smallmatrix} V & X \\ 0 & 0 \end{smallmatrix}\right]$ *with respect to a decomposition* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, *where* $V \in \mathbf{B}(\mathcal{H}_1)$ *is an isometry and* $X: \mathcal{H}_2 \to \mathcal{H}_1$ *is a bounded linear operator,*

(viii) S *satisfies* (vii) *and* $\mathcal{H}_1 = \overline{V(\mathcal{H}_1) + X(\mathcal{H}_2)}$; *such* V *and* X *are unique.*

If $S \in \mathbf{B}(\mathcal{H})$ *satisfies* (vii), *then*

(a) S *is hyponormal if and only if* S *is subnormal,*

(b) S *is subnormal if and only if* $\|S\| \leqslant 1$; *the latter is equivalent to* $\|X\| \leqslant 1$ *and* $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$,

(c) S *is quasinormal if and only if* S *is a partial isometry*; *the latter is equivalent to* X *being a partial isometry and* $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$,

(d) *S is an isometry if and only if X is an isometry and $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$,*

(e) *S is cohyponormal if and only if V is unitary and $X = 0$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is easily seen to be true. The proof of the equivalences (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) is by induction on n. The implication (iv) $\Rightarrow$ (ii) is valid because $W^{*k}W^k = I$ for all $k \geqslant 0$.

(iii) $\Rightarrow$ (vi). This is a consequence of the following equalities

$$\|S(Sh)\| = \big\||S^2|h\big\| = \big\||S|h\big\| = \|Sh\|, \quad h \in \mathcal{H}.$$

(vi) $\Rightarrow$ (iv). One can verify that any isometric extension $W \in \mathbf{B}(\mathcal{H})$ of the isometry $S|_{\overline{S(\mathcal{H})}} \in \mathbf{B}(\overline{S(\mathcal{H})})$ meets our requirements.

(vi) $\Rightarrow$ (viii). Set $\mathcal{H}_1 = \overline{S(\mathcal{H})}$, $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$, $V = S|_{\overline{S(\mathcal{H})}}$ and $X = S|_{\mathcal{H}_2}$. Then $S = \left[\begin{smallmatrix} V & X \\ 0 & 0 \end{smallmatrix}\right]$ is the desired matrix representation of S. The uniqueness of V and X satisfying the condition (vii) and the equality $\mathcal{H}_1 = \overline{V(\mathcal{H}_1) + X(\mathcal{H}_2)}$ is evident.

The implication (viii) $\Rightarrow$ (vii) is obvious.

(vii) $\Rightarrow$ (iii). A matrix computation combined with the isometricity of V leads to $S^{*2}S^2 = S^*S$, as desired.

Assume now that the operator $S \in \mathbf{B}(\mathcal{H})$ satisfies (vii).

(a), (b). If S is hyponormal, then by the isometricity of V we have

$$\|V^*h_1\|^2 + \|X^*h_1\|^2 = \|S^*(h_1 \oplus 0)\|^2 \leqslant \|S(h_1 \oplus 0)\|^2 = \|h_1\|^2, \quad h_1 \in \mathcal{H}_1. \tag{19}$$

This implies that $\|X\| \leqslant 1$. Substituting Vh_1 in place of h_1 into (19) and again using the isometricity of V we get

$$\|h_1\|^2 + \|X^*Vh_1\|^2 \leqslant \|h_1\|^2, \quad h_1 \in \mathcal{H}_1,$$

which yields $X^*V = 0$. Hence $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$.

If $\|X\| \leqslant 1$ and $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$, then by the isometricity of V we have

$$\|S(h_1 \oplus h_2)\|^2 = \|Vh_1 + Xh_2\|^2 = \|Vh_1\|^2 + \|Xh_2\|^2 \leqslant \|h_1 \oplus h_2\|^2$$

for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$. This means that $\|S\| \leqslant 1$.

The operator S is subnormal if and only if for every $h \in \mathcal{H}$ the sequence $a_n(h) \stackrel{\mathrm{def}}{=} \|S^n h\|^2$, $n \geqslant 0$, is a Stieltjes moment sequence, cf. [**22, 23**], see also [**30**, Proposition 2.3]. By (vii) $\Leftrightarrow$ (i) we have

$$a_n(h) = \begin{cases} \|h\|^2 & \text{for } n = 0, \\ \|Sh\|^2 & \text{for } n \geqslant 1. \end{cases}$$

If $\|S\| \leqslant 1$, then one can check that $a_n(h) = \int_0^\infty t^n \mathrm{d}\mu_h(t)$ for all $n \geqslant 0$ with

$$\mu_h = (\|h\|^2 - \|Sh\|^2)\delta_0 + \|Sh\|^2\delta_1, \tag{20}$$

where δ_x is the probability Borel measure on $[0, \infty)$ with total mass at x. Thus S is subnormal. It is well known that subnormal operators are always hyponormal, cf. [**19**, Section 160].

(c) Notice first that S is quasinormal if and only if $S|S|^2 = |S|^2 S$ which is equivalent to the conjunction of the following four equalities

$$XX^*V = 0, \tag{21}$$

$$(X^*V)V = 0, \quad (X^*V)X = 0, \tag{22}$$

$$X = V(X^*V)^* + X|X|^2. \tag{23}$$

Suppose that S is quasinormal. Multiplying both sides of (21) on the left by V^*, we get $(X^*V)^*X^*V = 0$. This implies $X^*V = 0$ which is equivalent to $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$. Therefore, by (23), we have $X = X|X|^2$. This in turn is equivalent to X being a partial isometry, cf. [**19**, Problem 98, Corollary 3]. The converse implication holds because the equalities $X^*V = 0$ and $X = X|X|^2$ imply (21), (22) and (23).

Next, S is a partial isometry if and only if $S = SS^*S$ which is equivalent to

$$V + XX^*V = V, \quad VV^*X + XX^*X = X.$$

This in turn is equivalent to the equalities $X^*V = 0$ and $XX^*X = X$, which is the same as saying that $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$ and X is a partial isometry.

(d) Since, as is easily seen, $S^*S = I_{\mathcal{H}}$ if and only if $X^*V = 0$ and $X^*X = I_{\mathcal{H}_2}$, part (d) is established.

(e) Suppose that S is cohyponormal. Then for all $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$

$$\|Vh_1+Xh_2\|^2 = \|S(h_1\oplus h_2)\|^2 \leqslant \|S^*(h_1\oplus h_2)\|^2 = \|V^*h_1\|^2+\|X^*h_1\|^2. \tag{24}$$

Substituting $h_1 = 0$, we deduce that $X = 0$. Thus, by (24), $\|Vh_1\| \leqslant \|V^*h_1\|$ for all $h_1 \in \mathcal{H}_1$, which implies that the operator V is unitary. The reverse implication is manifestly true. The proof is complete. □

It turns out that bounded subnormal operators satisfying condition (i) of Proposition 24 have minimal normal extensions of a special shape. Indeed, let S be such an operator, N its minimal normal extension and F a semi-spectral measure tied up to S via (2). It follows from (20) that

$$F = (I - S^*S)\,\delta_0 + S^*S\,\delta_1.$$

This yields $\operatorname{supp} F \subseteq \{0,1\}$. Due to (6), the spectrum of N is contained in $\{0\} \cup \{z \in \mathbf{C}: |z| = 1\}$, which forces N to take the form $U \oplus 0$, where U is a unitary operator. In the following theorem we give an explicit matrix construction of a normal extension of S.

Theorem 25. *Let $S \in \mathbf{B}(\mathcal{H})$. Then the following conditions are equivalent*

(a) *S is subnormal and it satisfies condition* (i) *of Proposition* 24*,*

(b) *S has a normal extension of the form $U \oplus 0$, where U is a unitary operator,*

(c) *S is subnormal and its minimal normal extension is of the form $U \oplus 0$, where U is a unitary operator.*

If S is a subnormal operator which satisfies condition (vii) *of Proposition* 24*, then the operator*

$$N \stackrel{\text{def}}{=} \begin{bmatrix} V & X & Q & P & 0 & 0 \\ 0 & 0 & 0 & 0 & |X|^2 & |X|D \\ 0 & 0 & 0 & 0 & D^*|X| & D^*D \\ 0 & 0 & 0 & V^* & 0 & 0 \\ 0 & 0 & 0 & X^* & 0 & 0 \\ 0 & 0 & 0 & Q^* & 0 & 0 \end{bmatrix} \tag{25}$$

defined on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ *is a normal extension of* S, *where*[8]

$\mathcal{H}_3$ *is the closure of the range of* $|X|\sqrt{I-|X|^2}$,

$D \stackrel{\text{def}}{=} \sqrt{I-|X|^2}\,J$, $J\colon \mathcal{H}_3 \to \mathcal{H}_2$ *is the identity embedding,*

$Q \stackrel{\text{def}}{=} WD$, W *is the partial isometry in the polar decomposition* $X = W|X|$,

P *is the orthogonal projection of* $\mathcal{H}_1$ *onto* $\mathcal{H}_1 \ominus \overline{V(\mathcal{H}_1) + X(\mathcal{H}_2)}$.

The operator N *is of the form* $U \oplus 0$, *where* U *is a unitary operator. Moreover, if* S *is an isometry, then* N *is a unitary operator.*

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious.

The implication (a) $\Rightarrow$ (c) has been justified in the paragraph preceding Theorem 25. Yet another way of proving this (without recourse to the spectral theorem) is to follow the diagrams: (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). When dealing with the implication (a) $\Rightarrow$ (b) we prove all the desired properties of the matrix (25),

(a) $\Rightarrow$ (b) The operator N given by (25) is evidently an extension of S (identify $\mathcal{H}$ with $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \{0\} \oplus \cdots \oplus \{0\}$). We now show that N is a normal operator. Define the operators $C, Z \in \mathbf{B}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3)$ by

$$C = \begin{bmatrix} V & X & Q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} P & 0 & 0 \\ 0 & |X|^2 & |X|D \\ 0 & D^*|X| & D^*D \end{bmatrix}. \tag{26}$$

Then the operator N can be rewritten as

$$N = \begin{bmatrix} C & Z \\ 0 & C^* \end{bmatrix}. \tag{27}$$

Observe that

$$V^*X = 0, \tag{28}$$
$$V^*Q = 0, \tag{29}$$
$$Q^*Q = D^*D, \tag{30}$$
$$X^*Q = |X|D. \tag{31}$$

Indeed, the equality (28) is a consequence of part (b) of Proposition 24. Since

$$Q(\mathcal{H}_3) \subseteq W(\mathcal{H}_2) = \overline{X(\mathcal{H}_2)} \perp V(\mathcal{H}_1), \tag{32}$$

we get (29). The product W^*W is the orthogonal projection of $\mathcal{H}_2$ onto $\overline{|X|(\mathcal{H}_2)}$. Noticing that the range of D is contained in the range of W^*W, we obtain $Q^*Q = D^*W^*WD = D^*D$, which gives (30). The equality (31) can be justified as follows

$$X^*Q = X^*WD = |X|W^*WD = |X|D.$$

Since V is an isometry, the equalities (28), (29), (30) and (31) lead to

$$C^*C = \begin{bmatrix} V^*V & V^*X & V^*Q \\ X^*V & X^*X & X^*Q \\ Q^*V & Q^*X & Q^*Q \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & |X|^2 & |X|D \\ 0 & D^*|X| & D^*D \end{bmatrix}. \tag{33}$$

The standard calculation which is left to the reader yields

$$(C^*C)^2 = \begin{bmatrix} I & 0 & 0 \\ 0 & |X|E|X| & |X|ED \\ 0 & D^*E|X| & D^*ED \end{bmatrix}, \tag{34}$$

where $E \overset{\text{def}}{=} |X|^2 + DD^*$. As $\mathcal{H}_3$ reduces $|X|$, the operator JJ^*, which is the orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{H}_3$, commutes with $\sqrt{I-|X|^2}$. Thus, by the definition of D we have

$$DD^* = \sqrt{I-|X|^2}JJ^*\sqrt{I-|X|^2} = JJ^*(I-|X|^2).$$

This and $(I-|X|^2)\big(\overline{|X|(\mathcal{H}_2)}\big) \subseteq \mathcal{H}_3$ implies

$$Eh = |X|^2h + JJ^*(I-|X|^2)h = |X|^2h + (I-|X|^2)h = h, \quad h \in \overline{|X|(\mathcal{H}_2)}. \tag{35}$$

Since the range of D is contained in $\overline{|X|(\mathcal{H}_2)}$, we get $E|X| = |X|$ and $ED = D$, which together with (33) and (34) leads to

$$(C^*C)^2 = C^*C. \tag{36}$$

This means that C is a partial isometry, cf. [**19**, Problem 98]. The above analysis shows in particular that the operator $\left[\begin{smallmatrix} |X|^2 & |X|D \\ D^*|X| & D^*D \end{smallmatrix}\right]$ defined on

$\mathcal{H}_2 \oplus \mathcal{H}_3$ is an orthogonal projection. As a consequence, the operator Z defined in (26) is an orthogonal projection as well.

Continuing the proof of the normality of N, we show that

$$CC^* = \begin{bmatrix} I-P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

In the matrix representing the operator CC^* the only nonzero entry appears in its left upper corner and it is equal to $VV^* + XX^* + QQ^*$. Applying the definition of E and the equality $W^*(\mathcal{H}_1) = \overline{|X|(\mathcal{H}_2)}$, we get

$$\begin{aligned} QQ^* &= WDD^*W^* = W(E - |X|^2)W^* \\ &= WEW^* - W|X|(W|X|)^* \overset{(35)}{=} WW^* - XX^*. \end{aligned}$$

Therefore, employing the facts that VV^* is the orthogonal projection of $\mathcal{H}_1$ onto $V(\mathcal{H}_1)$, WW^* is the orthogonal projection of $\mathcal{H}_1$ onto $\overline{X(\mathcal{H}_2)}$ and $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$ (because of part (b) of Proposition 24), we have

$$VV^* + XX^* + QQ^* = VV^* + WW^* = I - P,$$

which justifies the equality (37).

The next step of the proof consists in showing that

$$C^*Z = 0 \quad \text{and} \quad ZC = 0. \tag{38}$$

It is easily seen that $C^*Z = 0$ if and only if $V^*P = 0$, $X^*P = 0$ and $Q^*P = 0$. However, the last three equalities hold because the ranges of V, X and Q are all orthogonal to the range of P, use (32). Since $Z = Z^*$, the other equality in (38) follows from the former.

Combining the equalities (26), (33) and (37) with fact that Z is an orthogonal projection leads to $Z^*Z + CC^* = C^*C$. This, the self-adjointness of Z, (38) and (27) imply

$$\begin{aligned} N^*N &= \begin{bmatrix} C^*C & C^*Z \\ Z^*C & Z^*Z + CC^* \end{bmatrix} = \begin{bmatrix} C^*C & 0 \\ 0 & C^*C \end{bmatrix} \\ &= \begin{bmatrix} CC^* + ZZ^* & ZC \\ C^*Z^* & C^*C \end{bmatrix} = NN^*, \end{aligned} \tag{39}$$

which means that N is a normal operator and, by (36), $|N|$ is an orthogonal projection. Hence, N is a normal partial isometry, cf. [**19**, Problem 98], and consequently by [**19**, Problem 161] $N = U \oplus 0$, where U is a unitary operator.

(b) $\Rightarrow$ (c). We show a little bit more, namely that if $\mathcal{L}$ is a closed linear space which is invariant for $U \oplus 0$ and $M \stackrel{\text{def}}{=} (U \oplus 0)|_{\mathcal{L}}$ is normal, then $M = U_1 \oplus 0$, where U_1 is a unitary operator. Indeed, by the implication (i) $\Rightarrow$ (vii) of Proposition 24, $M = \left[\begin{smallmatrix} V_1 & X_1 \\ 0 & 0 \end{smallmatrix}\right]$ with isometric V_1. Since M is normal, part (e) of Proposition 24 implies that V_1 is unitary and $M = V_1 \oplus 0$, as desired.

If S is an isometry, then the operator X is an isometry as well. This and (33) implies that $\mathcal{H}_3 = \{0\}$ and C^*C is the identity matrix. Hence by (39) the operator N is unitary. This completes the proof. □

Remark 26. In our matrix construction (25) of the operator N, the Hilbert space $\mathcal{H}_3$, which is a component of the space $\mathcal{K}$ on which N acts, depends heavily on the entry X. Fixing the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we can easily avoid the dependence of $\mathcal{K}$ on X simply by replacing $\mathcal{K}$ by a new Hilbert space

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2$$

and the operator J by the orthogonal projection of $\mathcal{H}_2$ onto $\mathcal{H}_3$, where $\mathcal{H}_3$ is as in Theorem 25. The reader can check that

$$\begin{aligned} &\mathcal{H}_3 = \{0\} \iff X \text{ is a partial isometry } \stackrel{(*)}{\iff} S \text{ is a partial isometry,} \\ &\mathcal{H}_3 = \mathcal{H}_2 \iff \ker X = \{0\} \quad \text{and} \quad \ker(I - |X|) = \{0\}, \end{aligned}$$

where the equivalence $\stackrel{(*)}{\iff}$ can be deduced from parts (b) and (c) of Proposition 24.

Let us comment on the proof of Theorem 25. The matrix form (25) of the normal extension N of S is obtained in three steps. First, we find a matrix form of the operator A satisfying condition (27) in [**32**]; the operator A turns out to be an orthogonal projection which dilates S^*S. Fortunately, such an A can be created with the help of [**17**]. In the next step, we find a quasinormal extension C of S, cf. (26), via an abstract procedure appearing in the proof of [**32**, Lemma 5] making essential use of the identification $\overline{\mathcal{H} + A(\mathcal{H})} \cong \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ (recall that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$). Finally, we extend C to N with the help of another matrix construction proposed in the proof of [**32**, Theorem 2]. It is worth noting that $\|S\| = \|N\|$ which can be inferred from part (b) of Proposition 24 and Theorem 25.

Let us mention that a general matrix construction of normal extensions of subnormal operators proposed in [**2**] seems to be ineffective in our case. As with matrix constructions, their minimality appears to be rare. Fortunately, in our case we can give a satisfactory description of minimality of N.

Proposition 27. *Let $S \in \mathbf{B}(\mathcal{H})$ be a subnormal operator satisfying condition* (vii) *of Proposition* 24*, and let N be as in Theorem* 25*. Then the following conditions are equivalent*

1° *N is a minimal normal extension of S,*

2° *X is an isometry* (*equivalently, S is an isometry*) *and V is unitarily equivalent to a unilateral shift,*

3° *S is unitarily equivalent to a unilateral shift.*

Proof. Set $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. As in Theorem 25, we identify $\mathcal{H}$ with $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \{0\} \oplus \cdots \oplus \{0\}$. Recall that $\mathfrak{M}(N, \mathcal{H}) \overset{\text{def}}{=} \bigvee\{N^{*m}\mathcal{H} : m \geqslant 0\}$ is the least closed linear subspace of $\mathcal{K}$ reducing N and containing $\mathcal{H}$. Given an isometry $T \in \mathbf{B}(\mathcal{L})$ in a Hilbert space $\mathcal{L}$, we write $\mathcal{H}_{\mathrm{u}}(T)$ for the largest closed linear subspace of $\mathcal{L}$ which reduces T to a unitary operator. It follows from the von Neumann-Wold decomposition of T, cf. [**27**, Theorem 4.7.1], that $T|_{\mathcal{L} \ominus \mathcal{H}_{\mathrm{u}}(T)}$ is unitarily equivalent to a unilateral shift.

$2^\circ \Leftrightarrow 3^\circ$. Since S is assumed to be subnormal, part (b) of Proposition 24 implies that the spaces $V(\mathcal{H}_1)$ and $X(\mathcal{H}_2)$ are orthogonal and, consequently, by part (d) of Proposition 24, S is an isometry if and only if X is an isometry. As $V = S|_{\mathcal{H}_1}$ and $S(\mathcal{H}) \subseteq \mathcal{H}_1$, we see that $\mathcal{H}_{\mathrm{u}}(S) = \mathcal{H}_{\mathrm{u}}(V)$, which yields the desired equivalence.

$1^\circ \Leftrightarrow 2^\circ$. First we show that 1° implies that X is an isometry. According to the proof of Theorem 25, the operator $Y \overset{\text{def}}{=} \left[\begin{smallmatrix} |X|^2 & |X|D \\ D^*|X| & D^*D \end{smallmatrix}\right]$

defined on $\mathcal{H}_2 \oplus \mathcal{H}_3$ is an orthogonal projection. Suppose that, contrary to our claim, X is not an isometry. Then the orthogonal projection Y is not equal to $I_{\mathcal{H}_2 \oplus \mathcal{H}_3}$, which means that the range of Y is a proper closed linear subspace of $\mathcal{H}_2 \oplus \mathcal{H}_3$. Hence, there exists a nonzero vector $(g_2, g_3) \in \mathcal{H}_2 \oplus \mathcal{H}_3$ which is orthogonal to the range of Y. Employing (25), we get

$$\langle N^*(k_1, \ldots, k_6), (0, \ldots, 0, g_2, g_3)\rangle = \langle Y(k_2, k_3), (g_2, g_3)\rangle = 0, \quad (k_1, \ldots, k_6) \in \mathcal{K}.$$

This implies that the nonzero vector $g \stackrel{\text{def}}{=} (0, 0, 0, 0, g_2, g_3) \in \mathcal{K}$ is orthogonal to $N^*(\mathcal{K})$, hence to all $N^{*m}(\mathcal{K})$ with $m \geqslant 1$, and finally to $\mathfrak{M}(N, \mathcal{H})$ (because evidently $g \perp \mathcal{H}$). This contradicts the assumed minimality of N.

By virtue of the preceding paragraph, without loss of generality we can assume that X is an isometry. Our next task is to show that in this case

$$\mathfrak{M}(N, \mathcal{H}) = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus (\mathcal{H}_1 \ominus \mathcal{H}_{\mathrm{u}}(V)) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \tag{40}$$

Due to the proof of $2^\circ \Leftrightarrow 3^\circ$, S is an isometry and $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$. By Theorem 25, the operator N is unitary. This and the von Neumann-Wold decomposition applied to the isometry S enables one to deduce that

$$\mathfrak{M}(N, \mathcal{H}) = \mathcal{H}_{\mathrm{u}}(S) \oplus \bigoplus_{n \in \mathbf{Z}} N^{*n}(\mathcal{H} \ominus S(\mathcal{H})) = \mathcal{H} \oplus \bigoplus_{n \geqslant 1} N^{*n}(\mathcal{H} \ominus S(\mathcal{H})). \tag{41}$$

It is easily checked that

$$\mathcal{H} \ominus S(\mathcal{H}) = \Delta \oplus \mathcal{H}_2 \oplus \{0\} \oplus \cdots \oplus \{0\},$$

where $\Delta \stackrel{\text{def}}{=} \mathcal{H}_1 \ominus \big(V(\mathcal{H}_1) \oplus X(\mathcal{H}_2)\big)$. Using (25) and the equalities

$$\Delta = \big(\mathcal{H}_1 \ominus V(\mathcal{H}_1)\big) \cap \big(\mathcal{H}_1 \ominus X(\mathcal{H}_2)\big) = \ker V^* \cap \ker X^* \quad \text{and} \quad \mathcal{H}_3 = \{0\},$$

we successively compute

$$\begin{aligned} N^*(\mathcal{H} \ominus S(\mathcal{H})) &= \{0\} \oplus \{0\} \oplus \mathcal{H}_3 \oplus \Delta \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \\ N^{*n}(\mathcal{H} \ominus S(\mathcal{H})) &= \{0\} \oplus \{0\} \oplus \mathcal{H}_3 \oplus V^{n-2}\big(V(\Delta) + X(\mathcal{H}_2)\big) \\ &\qquad \oplus \{0\} \oplus \mathcal{H}_3, \quad n \geqslant 2. \end{aligned}$$

Adding the fourth "coordinates" of the righthand sides of the above equalities and making use of the orthogonality $V^{i+1}(\varDelta) \perp V^i(X(\mathcal{H}_2))$, $i \geqslant 0$, (as $\varDelta \oplus X(\mathcal{H}_2) = \mathcal{H}_1 \ominus V(\mathcal{H}_1)$ is a wandering space for the isometry V) we come to

$$\begin{aligned}
&\varDelta \oplus \big(V(\varDelta) + X(\mathcal{H}_2)\big) \oplus V\big(V(\varDelta) + X(\mathcal{H}_2)\big) \oplus \cdots \\
&\quad = \big(\varDelta \oplus X(\mathcal{H}_2)\big) \oplus V\big(\varDelta \oplus X(\mathcal{H}_2)\big) \oplus V^2\big(\varDelta \oplus X(\mathcal{H}_2)\big) \oplus \cdots \\
&\quad = \big(\mathcal{H}_1 \ominus V(\mathcal{H}_1)\big) \oplus V\big(\mathcal{H}_1 \ominus V(\mathcal{H}_1)\big) \oplus V^2\big(\mathcal{H}_1 \ominus V(\mathcal{H}_1)\big) \oplus \cdots \\
&\quad = \mathcal{H}_1 \ominus \mathcal{H}_{\mathrm{u}}(V).
\end{aligned}$$

This and the above description of $N^{*n}(\mathcal{H} \ominus S(\mathcal{H}))$, $n \geqslant 1$, leads to

$$\bigoplus_{n \geqslant 1} N^{*n}(\mathcal{H} \ominus S(\mathcal{H})) = \{0\} \oplus \{0\} \oplus \mathcal{H}_3 \oplus \big(\mathcal{H}_1 \ominus \mathcal{H}_{\mathrm{u}}(V)\big) \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \tag{42}$$

Combining (41) with (42), we get (40). Now the equivalence $1^\circ \Leftrightarrow 2^\circ$ is a direct consequence of (40). This completes the proof. □

Specifying Proposition 27 for S being an isometry we are able to write the matrix (25) in a form which is independent of the matrix representation of S.

Corollary 28. *Let $S \in \mathbf{B}(\mathcal{H})$ be an isometry and R the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H} \ominus S(\mathcal{H})$. Then the operator $N \stackrel{\text{def}}{=} \left[\begin{smallmatrix} S & R \\ 0 & S^* \end{smallmatrix}\right] \in \mathbf{B}(\mathcal{H} \oplus \mathcal{H})$ is a unitary extension of S. Moreover, N is minimal if and only if S is unitarily equivalent to a unilateral shift.*

Proof. Apply Theorem 25 and Proposition 27 to $\mathcal{H}_1 = S(\mathcal{H})$. □

Remark 29. Notice that if a subnormal operator $S \in \mathbf{B}(\mathcal{H})$ satisfies condition (vii) of Proposition 24 and $X = W|X|$ is the polar decomposition of X, then

$$S = \begin{bmatrix} V & W \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}_1} & 0 \\ 0 & |X| \end{bmatrix}$$

is the polar decomposition of S. This can be verified with the help of parts (b) and (c) of Proposition 24.

Example 30. We construct a class of analytic operator-valued functions $\varphi: \mathbf{C} \to \mathbf{B}(\mathcal{H})$ such that

1° $\varphi(z)$ is subnormal but not quasinormal if $0 < |z| < 1$, and $\varphi(0)$ is quasinormal,

2° $\varphi(z)$ is a non-unitary isometry if $|z| = 1$,

3° $\varphi(z)$ is not hyponormal if $|z| > 1$,

4° $\varphi(z)$ is never cohyponormal.

For this, consider an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_1 \neq \{0\}$ and $\mathcal{H}_2 \neq \{0\}$, a nonunitary isometry $V \in \mathbf{B}(\mathcal{H}_1)$ and a linear isometry $X: \mathcal{H}_2 \to \mathcal{H}_1$ such that $V(\mathcal{H}_1) \perp X(\mathcal{H}_2)$. Then in view of Proposition 24 the function $\varphi: \mathbf{C} \to \mathbf{B}(\mathcal{H})$ defined by

$$\varphi(z) = \begin{bmatrix} V & zX \\ 0 & 0 \end{bmatrix}, \quad z \in \mathbf{C},$$

has all the desired properties. Since any two distinct values of φ do not commute, no restriction of φ to a nonempty open subset of $\mathbf{C}$ is jointly subnormal. Recall that in view of Theorem 9 there is no analytic normal-operator-valued function $\Phi : \Omega \to \mathbf{B}(\mathcal{K})$ defined on a nonempty open subset Ω of $\mathbf{C}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $\varphi(z) = \Phi(z)|_{\mathcal{H}}$ for all $z \in \Omega$.

Example 31. Following Example 3.2 of [**7**], we can define a subnormal-operator-valued analytic function φ on $\mathbf{C}$ which is not jointly subnormal:

$$\varphi(z) = A + zB, \quad z \in \mathbf{C},$$

where $A, B \in \mathbf{B}(\mathcal{H})$ are isometries with mutually orthogonal nonzero ranges. Indeed, each operator $\varphi(z)$ being a multiple of an isometry is subnormal, while the operators A and B do not commute and consequently the function φ is not jointly subnormal. The same reasoning shows that the 1-homogeneous subnormal-operator-valued polynomial $\varphi(z_1, z_2) \stackrel{\mathrm{def}}{=} z_1 A + z_2 B$, $z_1, z_2 \in \mathbf{C}$, is not jointly subnormal.

8. Open questions.

Question 1. Let Θ and Ψ be as in Corollary 5. Is Θ a unique continuous algebra homomorphism which extends Ψ?

Regarding the definition of flatness of jointly subnormal operator-valued functions, cf. Section 3, one can ask

Question 2. Is every jointly subnormal operator-valued function flat?

A partial answer to Question 2 is contained in

Proposition 32. *If a function $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ is jointly subnormal and $\varphi(\Omega)$ is separable with respect to the operator norm topology, then φ is flat.*

Proof. Let $\Phi\colon \Omega \to \mathbf{B}(\mathcal{K})$ be a minimal normal extension of φ, and let Ω_0 be a countable subset of Ω such that $\varphi(\Omega) \subseteq \overline{\varphi(\Omega_0)}$. By Theorem 4 (use Θ), we get

$$\Phi(\Omega) \subseteq \overline{\Phi(\Omega_0)}. \tag{43}$$

Denote by $\mathfrak{D}$ the countable set of all finite products of operators from $\Phi(\Omega_0)^* \cup \{I_{\mathcal{K}}\}$. Take an orthonormal basis $\mathcal{E}$ of $\mathcal{H}$. One can deduce from (43) that

$$\mathcal{K} = \mathfrak{M}(\Phi, \mathcal{H}) = \overline{\mathrm{lin}\mathfrak{D}\mathcal{E}}. \tag{44}$$

Consider now two cases. If $\mathcal{H}$ is finite dimensional, then $\varphi = \Phi$ (because subnormal operators in finite dimensional Hilbert spaces are normal, cf. [**19**, Problem 163]). If $\mathcal{H}$ is infinite dimensional, then one can infer from (44) that[9]

$$\begin{aligned}\dim \mathcal{K} \leqslant \mathrm{card}\,(\mathfrak{D}\mathcal{E}) \leqslant \mathrm{card}\,(\mathfrak{D} \times \mathcal{E}) = \mathrm{card}\,(\mathfrak{D}) \cdot \mathrm{card}\,(\mathcal{E}) \leqslant \aleph_0 \cdot \dim \mathcal{H} \\ = \dim \mathcal{H},\end{aligned}$$

which completes the proof. □

It follows from Proposition 32 that every continuous jointly subnormal operator-valued function defined on an open subset of a separable metric space is flat.

What most interests us is the following.

Question 3. Let $\Omega \subseteq \mathbf{C}$ be a nonempty connected open set and $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ be an analytic subnormal-operator-valued function whose values mutually commute. Does φ have to be jointly subnormal?

The answer to Question 3 seems to be unknown even in the case of a polynomial of degree 1, cf. [**25**, Problem]. If $\Omega = \mathbf{C}$, then by Lemma 1, Question 3 can be reformulated in an equivalent way by assuming the commutativity of Taylor coefficients of φ at 0 in place of the commutativity of $\varphi(\mathbf{C})$. Notice that if the assumption on the connectedness of Ω is dropped, then the answer to Question 3 is in the negative. This can be deduced from the fact that there exist two commuting subnormal operators which are not jointly subnormal, cf. [**1, 24, 26**].

Consider an operator-valued function $\varphi\colon \Omega \to \mathbf{B}(\mathcal{H})$ defined on a nonempty set Ω. Assume that the space $\overline{\mathrm{lin}\varphi(\Omega)}$ is separable, e.g., this is the case if Ω is an open subset of a separable metric space and φ is continuous. Then there exists a sequence $\{\omega_n\}_{n=0}^{\infty} \subseteq \Omega$ such that $\overline{\mathrm{lin}\{\varphi(\omega_n)\colon n \geqslant 0\}} = \overline{\mathrm{lin}\varphi(\Omega)}$. For every $n \geqslant 0$ choose a real number $\varepsilon_n > 0$ such that $\varepsilon_n \|\varphi(\omega_n)\| < 1/n!$, and define the analytic function $\psi\colon \mathbf{C} \to \mathbf{B}(\mathcal{H})$ by

$$\psi(z) = \sum_{n=0}^{\infty} \varepsilon_n \varphi(\omega_n) z^n, \quad z \in \mathbf{C}.$$

Then by Lemma 1 $\overline{\mathrm{lin}\varphi(\Omega)} = \overline{\mathrm{lin}\psi(\mathbf{C})}$, which means that the sets $\varphi(\Omega)$ and $\psi(\mathbf{C})$ are simultaneously commutative, and the functions φ and ψ are simultaneously jointly subnormal (the latter can be deduced from Theorem 4). However, in general the equality $\overline{\mathrm{lin}\varphi(\Omega)} = \overline{\mathrm{lin}\psi(\mathbf{C})}$ does not imply that φ and ψ are simultaneously subnormal-operator-valued functions. Indeed, let $\widetilde{\varphi}$ and E be as in Example 22; then

$$\overline{\mathrm{lin}\widetilde{\varphi}(\mathbf{C})} = \overline{\mathrm{lin}\widetilde{\varphi}(E)} = \mathrm{lin}\{U, T\},$$

$\widetilde{\varphi}(E)$ consists of unitary operators and $\widetilde{\varphi}(\mathbf{C})$ is not a set of subnormal operators.

ENDNOTES

1. The righthand side of (2) is related to the Hamburger operator moment problem which goes back at least as far as [**34**]; for operator-valued integrals the reader may consult [**27**].

2. In our settings the measures under consideration are necessarily regular, e.g., see [**28**, Theorem 2.18]; as a consequence, they admit closed supports.

3. With respect to the operator norm topologies on $\mathfrak{B}$ and $\mathbf{B}(\mathcal{K})$.

4. Because all minimal normal extensions of φ are unitarily equivalent.

5. This means that the series $\sum_{n=0}^{\infty} P_n(x)$ converges to $\varphi(x)$ in operator norm for every $x \in \mathbb{B}_{\mathcal{X}}(r)$.

6. Apply [**8**, Theorem 9.18 (c)] to $\varPhi(zx) = \sum_{n=0}^{\infty} z^n Q_n(x)$, $z \in \mathbb{B}_{\mathbf{C}}(r/\|x\|)$, $x \neq 0$ being fixed.

7. In view of Proposition 11 such $\varPhi$ must be analytic.

8. By part (b) of Proposition 24, X is a contraction and so the definition of D makes sense.

9. Recall that if $\mathcal{F}$ is a subset of a Hilbert space $\mathcal{M}$ and $\overline{\mathrm{lin}\mathcal{F}} = \mathcal{M}$, then $\dim \mathcal{M} \leqslant \operatorname{card} \mathcal{F}$.

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