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DERIVATIONS OF A RESTRICTED WEYL TYPE ALGEBRA I

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ABSTRACT. Several authors find all the derivations of an algebra [**1, 3, 6**]. For two (non)associative algebras A_1 and A_2; if the additive group $\mathrm{Der}\,(A_1)$ of all the derivations of A_1 and the additive group $\mathrm{Der}\,(A_2)$ of all the derivations of A_2 are not isomorphic, then the two (non)associative algebras are nonisomorphic as algebras [**5**]. A Weyl type nonassociative algebra and its sub-algebra are defined in the papers [**2, 3, 9**]. We find all the derivations of the nonassociative algebra $\overline{WN_{0,0,s_1}}$ in this paper [**4**].

1. Introduction. Generally, there is an infinite dimensional simple algebra with an outer derivation. Thus, it is an interesting problem to find all the derivations of an infinite dimensional (non)associative algebra [**2**]. The Weyl type nonassociative algebras are defined in the papers [**3, 11**]. All the derivations of the restricted Weyl type nonassociative algebras $\overline{WN_{0,0,1_1}}$ and $\overline{WN_{0,0,2_1}}$ are found in the papers [**1, 2**]. In this paper, we find all the derivations of the restricted Weyl type nonassociative algebras $\overline{WN_{0,0,s_1}}$. We show that $\mathrm{Der}\,(\overline{WN_{0,0,s_1}})$ is (s^2+s)-dimensional. The nonassociative algebra $\overline{WN_{0,0,s_1}}$ contains the matrix ring $M_s(\mathbf{F})$, and we show that $\mathrm{Der}\,(M_s(\mathbf{F}))$ is s^2-dimensional [**2, 4**].

2. Preliminaries. Let $\mathbf{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbf{N}$ and $\mathbf{Z}$ will denote the nonnegative integers and the integers, respectively. Let $\mathbf{F}[x_1, \dots , x_{m+s}]$ be the polynomial ring with the variables $x_1, \dots , x_{m+s}$. Let $g_1, \dots , g_n$ be given polynomials in $\mathbf{F}[x_1, \dots , x_{m+s}]$. For $n, m, s \in \mathbf{N}$, let us define the commutative, associative $\mathbf{F}$-algebra $F_{g_n,m,s} = \mathbf{F}[e^{\pm g_1}, \dots , e^{\pm g_n}, x_1^{\pm 1}, \dots , x_m^{\pm 1}, x_{m+1}, \dots , x_{m+s}]$ in the formal power series ring $\mathbf{F}[[x_1, \dots , x_{m+s}]]$ which is called a stable algebra

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in the paper [**7**] with the standard basis

$$\mathbf{B} = \{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \mid a_1, \ldots, a_n, i_1, \ldots, i_m \in \mathbf{Z}, \quad i_{m+1}, \ldots, i_{m+s} \in \mathbf{N}\} \tag{1}$$

and with the obvious addition and the multiplication [**3, 7, 10, 12, 13**] where we take $g_1, \ldots, g_n$ so that $\mathbf{B}$ is the standard basis of $F_{g_n,m,s}$. ∂_w, $1 \le w \le m+s$ denotes the usual partial derivative with respect to x_w on $F_{g_n,m,s}$. For partial derivatives $\partial_u, \ldots, \partial_v$ of $F_{g_n,m,s}$, the composition $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ of them is denoted $\partial_u^{j_u} \cdots \partial_v^{j_v}$ where $j_u, \ldots, j_v \in \mathbf{N}$. Let us define the vector space $WN(g_n, m, s)$ over $\mathbf{F}$ which is spanned by the standard basis

$$\begin{aligned} \{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \mid & \\ e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} &\in \mathbf{B}, \\ j_u, \ldots, j_v \in \mathbf{N},\ 1 \le u, \ldots, v &\le m+s\}. \end{aligned} \tag{2}$$

Thus, we can dene the multiplication $*$ on $WN(g_n, m, s)$ as follows:

$$\begin{aligned} & e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \\ & \quad * e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \\ & \quad = e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \\ & \quad \partial_v^{j_v} (e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}}) \partial_h^{j_h} \cdots \partial_w^{j_w} \end{aligned} \tag{3}$$

for any basis elements $e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ and

$$e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \in WN(g_n, m, s).$$

Thus, we can define the Weyl-type nonassociative algebra $\overline{WN_{g_n,m,s}}$ with the multiplication $*$ in (3) and with the set $WN(g_n, m, s)$ [**9, 10, 11**]. For $r \in \mathbf{N}$, let us define the restricted Weyl type nonassociative subalgebra $\overline{WN_{g_n,m,s_r}}$ of the nonassociative algebra $\overline{WN_{g_n,m,s}}$ spanned by

$$\begin{aligned} \{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \mid & \\ e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} &\in \mathbf{B}, \\ j_u, \ldots, j_v \in \mathbf{N},\ j_u + \cdots + j_v \le r,\ 1 \le u, \ldots, v &\le m+s\}. \end{aligned} \tag{4}$$

For any basis element $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v}$ of $\overline{WN_{gn,m,s}}$, let us define the degree $\deg_r(e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v})$ of x_r as i_r and the total degree $\deg_{tot}(e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v})$ of $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v}$ as $i_1 + \cdots + i_s$. Thus, for any element l of $\overline{WN_{g_n,m,s}}$, we define $\deg_r(l)$ of x_r as the highest degree of x_r in the basis terms of l and $\deg_{\mathrm{tot}}(l)$ as the highest total degree of basis terms of l. It is well known that the nonassociative algebra $\overline{WN_{g_n,m,s}}$ is simple, even though it has the right annihilator [**6, 8**]. For any element of l of $\overline{WN_{g_n,m,s}}$, an element l_1, respectively l_2, is the right, respectively left, identity of l if $l * l_1 = l$, respectively $l_2 * l = l$.

3. Derivations of $\overline{WN_{0,0,s_1}}$.

Lemma 1. *For any derivation D of the nonassociative algebra $\overline{WN_{0,0,3_1}}$, then we have that*

$$\begin{aligned} D(x_1^i \partial_1) = {} & (1-i) a_{1,0,0} x_1^i \partial_1 - i b_{1,0,0} x_1^{i-1} x_2 \partial_1 \\ & - i c_{1,0,0} x_1^{i-1} x_3 \partial_1 + i d_{1,0,0} x_1^{i-1} \partial_1 \\ & + a_{2,0,0} x_1^i \partial_2 + a_{3,0,0} x_1^i \partial_3 \end{aligned}$$

$$\begin{aligned} D(x_2^j \partial_2) = {} & (1-j) b_{2,0,0} x_2^j \partial_2 - j c_{2,0,0} x_2^{j-1} x_3 \partial_2 \\ & - j a_{2,0,0} x_1 x_2^{j-1} \partial_2 + j h_{2,0,0} x_2^{j-1} \partial_2 \\ & + b_{1,0,0} x_2^j \partial_1 + b_{3,0,0} x_2^j \partial_3 \end{aligned}$$

$$\begin{aligned} D(x_3^k \partial_3) = {} & (1-k) c_{3,0,0} x_3^k \partial_3 - k a_{3,0,0} x_1 x_3^{k-1} \partial_3 \\ & - k b_{3,0,0} x_2 x_3^{k-1} \partial_3 + k g_{3,0,0} x_3^{k-1} \partial_3 \\ & + c_{1,0,0} x_3^k \partial_1 + c_{2,0,0} x_3^k \partial_2 \end{aligned}$$

where $a_{1,0,0}, \ldots, h_{2,0,0} \in \mathbf{F}$.

Proof. Let D be the derivation in the lemma. Since ∂_1 is in the annihilator of itself, we have that

$$D(\partial_1) = \sum_{i,j \geq 0} a_{1,i,j} x_2^i x_3^j \partial_1 + \sum_{i,j \geq 0} a_{2,i,j} x_2^i x_3^j \partial_2 + \sum_{i,j \geq 0} a_{3,i,j} x_2^i x_3^j \partial_3 \tag{5}$$

where $a_{1,i,j}, a_{2,i,j}, a_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. Since ∂_2 is in the left annihilator of ∂_1, by (5), we have that,

$$\begin{aligned}\partial_2 * D(\partial_1) = &\sum_{\substack{i\geq 1\\ j\geq 0}} i a_{1,i,j} x_2^{i-1} x_3^j \partial_1 + \sum_{\substack{i\geq 1\\ j\geq 0}} i a_{2,i,j} x_2^{i-1} x_3^j \partial_2 \\ &+ \sum_{\substack{i\geq 1\\ j\geq 0}} i a_{3,i,j} x_2^{i-1} x_3^j \partial_3 = 0.\end{aligned}$$

This implies $a_{1,i,j} = a_{2,i,j} = a_{3,i,j} = 0$, $i \geq 1$, $j \geq 0$. So we have that

$$D(\partial_1) = \sum_{j\geq 0} a_{1,0,j} x_3^j \partial_1 + \sum_{j\geq 0} a_{2,0,j} x_3^j \partial_2 + \sum_{j\geq 0} a_{3,0,j} x_3^j \partial_3. \tag{6}$$

Since ∂_3 is in the left annihilator of ∂_1, by (6), we have that

$$\begin{aligned}\partial_3 * D(\partial_1) = &\sum_{j\geq 1} j a_{1,0,j} x_3^{j-1} \partial_1 + \sum_{j\geq 1} j a_{2,0,j} x_3^{j-1} \partial_2 \\ &+ \sum_{j\geq 1} j a_{3,0,j} x_3^{j-1} \partial_3 = 0.\end{aligned}$$

This implies $a_{1,0,j} = a_{2,0,j} = a_{3,0,j} = 0$, $j \geq 1$. We have that $D(\partial_1) = \sum_a a_{\alpha,0,0} \partial_a$ where $a_{\alpha,0,0} \in \mathbf{F}$; $1 \leq \alpha \leq 3$. Similarly, we have that $D(\partial_2) = \sum_\beta b_{\beta,0,0} \partial_\beta$ and $D(\partial_3) = \sum_\beta c_{\beta,0,0} \partial_\beta$ where $b_{\beta,0,0}, c_{\gamma,0,0} \in \mathbf{F}$, $1 \leq \beta, \gamma \leq 3$. Since $x_1 \partial_1$ is a right identity of ∂_1, we have that $\partial_1 * D(x_1 \partial_1) = a_{2,0,0} \partial_2 + a_{3,0,0} \partial_3$. This implies

$$\begin{aligned}D(x_1\partial_1) = &\, a_{2,0,0} x_1 \partial_2 + a_{3,0,0} x_1 \partial_3 + \sum_{i,j\geq 0} d_{1,i,j} x_2^i x_3^j \partial_1 \\ &+ \sum_{i,j\geq 0} d_{2,i,j} x_2^i x_3^j \partial_2 + \sum_{i,j\geq 0} d_{3,i,j} x_2^i x_3^j \partial_3\end{aligned} \tag{7}$$

where $d_{1,i,j}, d_{2,i,j}, d_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. Since ∂_2 is in the left annihilator of $x_1\partial_1$, by (7), we have that

$$\begin{aligned}b_{1,0,0}\partial_1 + &\sum_{\substack{i\geq 1\\ j\geq 0}} i d_{1,i,j} x_2^{i-1} x_3^j \partial_1 + \sum_{\substack{i\geq 1\\ j\geq 0}} i d_{2,i,j} x_2^{i-1} x_3^j \partial_2 \\ &+ \sum_{\substack{i\geq 1\\ j\geq 0}} i d_{3,i,j} x_2^{i-1} x_3^j \partial_3 = 0.\end{aligned}$$

This implies that $d_{1,1,0} = -b_{1,0,0}$, $d_{1,1,j} = 0$, $j \geq 1$, $d_{1,i,j} = 0$, $i \geq 2$, $j \geq 0$, and $d_{2,i,j} = d_{3,i,j} = 0$, $i \geq 1$, $j \geq 0$. We have the following identity

(8)
$$\begin{aligned} D(x_1\partial_1) = {} & a_{2,0,0}x_1\partial_2 + a_{3,0,0}x_1\partial_3 - b_{1,0,0}x_2\partial_1 \\ & + \sum_{j\geq 0} d_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} d_{2,0,j}x_3^j\partial_2 + \sum_{j\geq 0} d_{3,0,j}x_3^j\partial_3. \end{aligned}$$

Since ∂_3 is in the left annihilator of $x_1\partial_1$, by (8), we have that

$$c_{1,0,0}\partial_1 + \sum_{j\geq 1} jd_{1,0,j}x_3^{j-1}\partial_1 + \sum_{j\geq 1} jd_{2,0,j}x_3^{j-1}\partial_2 + \sum_{j\geq 1} jd_{3,0,j}x_3^{j-1}\partial_3 = 0.$$

Therefore, $c_{1,0,0} = -d_{1,0,1}$ and $d_{1,0,j} = 0$ hold for $j \geq 2$ and $d_{2,0,j} = d_{3,0,j} = 0$ holds for $j \geq 1$. We have that

$$\begin{aligned} D(x_1\partial_1) = {} & a_{2,0,0}x_1\partial_2 + a_{3,0,0}x_1\partial_3 - b_{1,0,0}x_2\partial_1 \\ & - c_{1,0,0}x_3\partial_1 + d_{1,0,0}\partial_1 + d_{2,0,0}\partial_2 + d_{3,0,0}\partial_3. \end{aligned}$$

Since $x_1\partial_1$ is an idempotent, we have that

$$D(x_1\partial_1) = -b_{1,0,0}x_2\partial_1 - c_{1,0,0}x_3\partial_1 + d_{1,0,0}\partial_1 + a_{2,0,0}x_1\partial_2 + a_{3,0,0}x_1\partial_3.$$

By $D(\partial_1 * x_1^2\partial_1) = 2D(x_1\partial_1)$, we have that $2a_{1,0,0}x_1\partial_1 + \partial_1 * D(x_1^2\partial_1) = 2(-b_{1,0,0}x_2\partial_1 - c_{1,0,0}x_3\partial_1 + d_{1,0,0}\partial_1 + a_{2,0,0}x_1\partial_2 + a_{3,0,0}x_1\partial_3)$. This implies that $\partial_1 * D(x_1^2\partial_1) = 2(-b_{1,0,0}x_2\partial_1 - c_{1,0,0}x_3\partial_1 + d_{1,0,0}\partial_1 - a_{1,0,0}x_1\partial_1 + a_{2,0,0}x_1\partial_2 + a_{3,0,0}x_1\partial_3)$. We have that

$$\begin{aligned} D(x_1^2\partial_1) = {} & -a_{1,0,0}x_1^2\partial_1 + a_{2,0,0}x_1^2\partial_2 + a_{3,0,0}x_1^2\partial_3 \\ & - 2b_{1,0,0}x_1x_2\partial_1 - 2c_{1,0,0}x_1x_3\partial_1 + 2d_{1,0,0}x_1\partial_1 \\ & + \sum_{i,j\geq 0} v_{1,i,j}x_2^ix_3^j\partial_1 + \sum_{i,j\geq 0} v_{2,i,j}x_2^ix_3^j\partial_2 + \sum_{i,j\geq 0} v_{3,i,j}x_2^ix_3^j\partial_3, \end{aligned}$$

where $v_{1,i,j}, v_{2,i,j}, v_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. Since ∂_2 is in the left annihilator of $x_1^2\partial_1$, we have that

$$\sum_{\substack{i\geq 1\\ j\geq 0}} iv_{1,i,j}x_2^{i-1}x_3^j\partial_1 + \sum_{\substack{i\geq 1\\ j\geq 0}} iv_{2,i,j}x_2^{i-1}x_3^j\partial_2 + \sum_{\substack{i\geq 1\\ j\geq 0}} iv_{3,i,j}x_2^{i-1}x_3^j\partial_3 = 0.$$

We have that $v_{1,i,j} = v_{2,i,j} = v_{3,i,j} = 0$, $i \geq 1$, $j \geq 0$ and

(9)
$$\begin{aligned} D(x_1^2\partial_1) = & -a_{1,0,0}x_1^2\partial_1 + a_{2,0,0}x_1^2\partial_2 + a_{3,0,0}x_1^2\partial_3 \\ & - 2b_{1,0,0}x_1x_2\partial_1 - 2c_{1,0,0}x_1x_3\partial_1 + 2d_{1,0,0}x^1\partial_1 \\ & + \sum_{j\geq 0} v_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} v_{2,0,j}x_3^j\partial_2 + \sum_{j\geq 0} v_{3,0,j}x_3^j\partial_3. \end{aligned}$$

Since ∂_3 is in the left annihilator of $x_1^2\partial_1$, by (9) we have that $\sum_{j\geq 1} jv_{1,0,j}x_3^{j-1}\partial_1 + \sum_{j\geq 1} jv_{2,0,j}x_3^{j-1}\partial_2 + \sum_{j\geq 1} jv_{3,0,j}x_3^{j-1}\partial_3 = 0$. This implies that $v_{1,0,j} = v_{2,0,j} = v_{3,0,j} = 0$, $j \geq 1$. We have that

(10)
$$\begin{aligned} D(x_1^2\partial_1) = & -a_{1,0,0}x_1^2\partial_1 - 2b_{1,0,0}x_1x_2\partial_1 - 2c_{1,0,0}x_1x_3\partial_1 + 2d_{1,0,0}x_1\partial_1 \\ & + a_{2,0,0}x_1^2\partial_2 + a_{3,0,0}x_1^2\partial_3 + v_{1,0,0}\partial_1 + v_{2,0,0}\partial_2 + v_{3,0,0}\partial_3. \end{aligned}$$

Since $x_1\partial_1$ is in the right identity of $x_1^2\partial_1$, by (10) we have that $D(x_1^2\partial_1) = -a_{1,0,0}x_1^2\partial_1 - 2b_{1,0,0}x_1x_2\partial_1 - 2c_{1,0,0}x_1x_3\partial_1 + 2d_{1,0,0}x_1\partial_1 + v_{1,0,0}\partial_1 + a_{2,0,0}x_1^2\partial_2 + a_{3,0,0}x_1^2\partial_3$. By $x_1\partial_1 * x_1^2\partial_1 = 2x_1^2\partial_1$, we have that

(11)
$$\begin{aligned} D(x_1^2\partial_1) = & -a_{1,0,0}x_1^2\partial_1 - 2b_{1,0,0}x_1x_2\partial_1 - 2c_{1,0,0}x_1x_3\partial_1 + 2d_{1,0,0}x_1\partial_1 \\ & + a_{2,0,0}x_1^2\partial_2 + a_{3,0,0}x_1^2\partial_3. \end{aligned}$$

By $D(x_1^2\partial_1 * x_1^2\partial_1) = 2D(x_1^3\partial_1)$ and (11), we have that

$$\begin{aligned} D(x_1^3\partial_1) = & -2a_{1,0,0}x_1^3\partial_1 - 3b_{1,0,0}x_1^2x_2\partial_1 - 3c_{1,0,0}x_1^2x_3\partial_1 + 3d_{1,0,0}x_1^2\partial_1 \\ & + a_{2,0,0}x_1^3\partial_2 + a_{3,0,0}x_1^3\partial_3. \end{aligned}$$

For the equation $D(x_1^{i-1}\partial_1 * x_1^2\partial_1) = 2D(x_1^i\partial_1)$, by using induction on i of $x_1^i\partial_1$, we can prove that

$$\begin{aligned} D(x_1^i\partial_1) = & (1-i)a_{1,0,0}x_1^i\partial_1 - ib_{1,0,0}x_1^{i-1}x_2\partial_1 - ic_{1,0,0}x_1^{i-1}x_3\partial_1 \\ & + id_{1,0,0}x_1^{i-1}\partial_1 + a_{2,0,0}x_1^i\partial_2 + a_{3,0,0}x_1^i\partial_3. \end{aligned}$$

We have the similar formulas of $D(x_2^j\partial_2)$ and $D(x_3^k\partial_3)$ in the lemma. This completes the proof of the lemma. □

Lemma 2. *For any derivation D of the nonassociative algebra $\overline{WN_{0,0,3_1}}$,*

$$\begin{aligned} D(x_1\partial_2) = & -a_{1,0,0}x_1\partial_2 + b_{2,0,0}x_1\partial_2 - b_{1,0,0}x_2\partial_2 - c_{1,0,0}x_3\partial_2 \\ & + d_{1,0,0}\partial_2 + b_{1,0,0}x_1\partial_1 + b_{3,0,0}x_1\partial_3 \end{aligned}$$

$$D(x_1\partial_3) = -a_{1,0,0}x_1\partial_3 + c_{3,0,0}x_1\partial_3 - b_{1,0,0}x_2\partial_3 - c_{1,0,0}x_3\partial_3 + d_{1,0,0}\partial_3 + c_{1,0,0}x_1\partial_1 + c_{2,0,0}x_1\partial_2$$

$$D(x_2\partial_1) = a_{1,0,0}x_2\partial_1 - b_{2,0,0}x_2\partial_1 - a_{2,0,0}x_1\partial_1 - c_{2,0,0}x_3\partial_1 + h_{2,0,0}\partial_1 + a_{2,0,0}x_2\partial_2 + a_{3,0,0}x_2\partial_3$$

$$D(x_2\partial_3) = -a_{2,0,0}x_1\partial_3 - c_{2,0,0}x_3\partial_3 - b_{2,0,0}x_2\partial_3 + c_{3,0,0}x_2\partial_3 + h_{2,0,0}\partial_3 + c_{1,0,0}x_2\partial_1 + c_{2,0,0}x_2\partial_2$$

$$D(x_3\partial_1) = -a_{3,0,0}x_1\partial_1 - b_{3,0,0}x_2\partial_1 + a_{1,0,0}x_3\partial_1 - c_{3,0,0}x_3\partial_1 + g_{3,0,0}\partial_1 + a_{2,0,0}x_3\partial_2 + a_{3,0,0}x_3\partial_3$$

$$D(x_3\partial_2) = -a_{3,0,0}x_1\partial_2 + b_{2,0,0}x_3\partial_2 - b_{3,0,0}x_2\partial_2 - c_{3,0,0}x_3\partial_2 + g_{3,0,0}\partial_2 + b_{1,0,0}x_3\partial_1 + b_{3,0,0}x_3\partial_3$$

hold with appropriate scalars.

Proof. Let D be the derivation in the lemma. By Lemma 1 and $D(\partial_1 * x_1\partial_2) = D(\partial_2)$, we have that $\partial_1 * D(x_1\partial_2) = -a_{1,0,0}\partial_2 + b_{2,0,0}\partial_2 + b_{1,0,0}\partial_1 + b_{3,0,0}\partial_3$. So

(12)
$$D(x_1\partial_2) = -a_{1,0,0}x_1\partial_2 + b_{2,0,0}x_1\partial_2 + b_{1,0,0}x_1\partial_1 + b_{3,0,0}x_1\partial_3 + \sum_{i,j\geq 0} t_{1,i,j}x_2^i x_3^j\partial_1 + \sum_{i,j\geq 0} t_{2,i,j}x_2^i x_3^j\partial_2 + \sum_{i,j\geq 0} t_{3,i,j}x_2^i x_3^j\partial_3$$

where $t_{1,i,j}, t_{2,i,j}, t_{3,i,j} \in \mathbf{F}$; $i,j \geq 0$. Since ∂_2 is in the left annihilator of $x_1\partial_2$, we have that

(13)
$$D(x_1\partial_2) = -a_{1,0,0}x_1\partial_2 + b_{2,0,0}x_1\partial_2 + b_{1,0,0}x_1\partial_1 + b_{3,0,0}x_1\partial_3 - b_{1,0,0}x_2\partial_2 + \sum_{j\geq 0} t_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} t_{2,0,j}x_3^j\partial_2 + \sum_{j\geq 0} t_{3,0,j}x_3^j\partial_3.$$

Since ∂_3 is in the left annihilator of $x_1\partial_2$, we have that

$$c_{1,0,0}\partial_2 + \sum_{j\geq 1} jt_{1,0,j}x_3^{j-1}\partial_1 + \sum_{j\geq 1} jt_{2,0,j}x_3^{j-1}\partial_2 + \sum_{j\geq 1} jt_{3,0,j}x_3^{j-1}\partial_3 = 0.$$

This implies that $t_{2,0,1} = -c_{1,0,0}$ and $t_{2,0,j} = 0$ hold, $j \geq 2$, and $t_{1,0,j} = t_{3,0,j} = 0$ holds, $j \geq 2$. We have that
(14)
$$\begin{aligned} D(x_1\partial_2) = &-a_{1,0,0}x_1\partial_2 + b_{2,0,0}x_1\partial_2 + b_{1,0,0}x_1\partial_1 + b_{3,0,0}x_1\partial_3 \\ &- b_{1,0,0}x_2\partial_2 - c_{1,0,0}x_3\partial_2 + t_{1,0,0}\partial_1 + t_{2,0,0}\partial_2 + t_{3,0,0}\partial_3. \end{aligned}$$

Since $x_1\partial_1$ is a left identity of $x_1\partial_2$, we have that

$$\begin{aligned} D(x_1\partial_2) = &-a_{1,0,0}x_1\partial_2 + b_{2,0,0}x_1\partial_2 - b_{1,0,0}x_2\partial_2 - c_{1,0,0}x_3\partial_2 \\ &+ d_{1,0,0}\partial_2 + b_{1,0,0}x_1\partial_1 + b_{3,0,0}x_1\partial_3. \end{aligned}$$

We have the similar formulas of $D(x_1\partial_3)$, $D(x_2\partial_1)$, $D(x_2\partial_3)$, $D(x_3\partial_1)$ and $D(x_3\partial_2)$ in the lemma. This completes the proof of the lemma. □

Lemma 3. *For any derivation D of the nonassociative algebra $\overline{WN_{0,0,3_1}}$,*

$$\begin{aligned} D(x_1x_2\partial_1) = &-a_{2,0,0}x_1^2\partial_1 - b_{1,0,0}x_2^2\partial_1 - b_{2,0,0}x_1x_2\partial_1 - c_{1,0,0}x_2x_3\partial_1 \\ &- c_{2,0,0}x_1x_3\partial_1 + d_{1,0,0}x_2\partial_1 + h_{2,0,0}x_1\partial_1 \\ &+ a_{2,0,0}x_1x_2\partial_2 + a_{3,0,0}x_1x_2\partial_3 \end{aligned}$$

(15)
$$\begin{aligned} D(x_1x_3\partial_1) = &-a_{3,0,0}x_1^2\partial_1 - b_{1,0,0}x_2x_3\partial_1 - b_{3,0,0}x_1x_2\partial_1 - c_{1,0,0}x_3^2\partial_1 \\ &- c_{3,0,0}x_1x_3\partial_1 + d_{1,0,0}x_3\partial_1 + g_{3,0,0}x_1\partial_1 \\ &+ a_{2,0,0}x_1x_3\partial_2 + a_{3,0,0}x_1x_3\partial_3 \end{aligned}$$

$$\begin{aligned} D(x_2x_3\partial_1) = &\, a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 \\ &- b_{3,0,0}x_2^2\partial_1 - c_{2,0,0}x_3^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 \\ &+ g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3 \end{aligned}$$

hold with appropriate scalars. We have the similar formulas of $D(x_1x_2\partial_2)$, $D(x_1x_3\partial_2)$, $D(x_2x_3\partial_2)$, $D(x_1x_2\partial_3)$, $D(x_1x_3\partial_3)$ and $D(x_2x_3\partial_3)$ as (15)*.*

Proof. Let D be the derivation in the lemma. By Lemma 1 and $D(\partial_1 * x_1x_2\partial_1) = D(x_2\partial_1)$, we have that $(a_{1,0,0}\partial_1 + a_{2,0,0}\partial_2 + a_{3,0,0}\partial_3) *$

$x_1x_2\partial_1 + \partial_1 * D(x_1x_2\partial_1) = D(x_2\partial_1)$. By Lemma 2, we have that

$$\begin{aligned} &a_{1,0,0}x_2\partial_1 + a_{2,0,0}x_1\partial_1 + \partial_1 * D(x_1x_2\partial_1) \\ &\qquad = a_{1,0,0}x_2\partial_1 - b_{2,0,0}x_2\partial_1 - a_{2,0,0}x_1\partial_1 - c_{2,0,0}x_3\partial_1 \\ &\qquad\quad + h_{2,0,0}\partial_1 + a_{2,0,0}x_2\partial_2 + a_{3,0,0}x_2\partial_3. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} D(x_1x_2\partial_1) &= -b_{2,0,0}x_1x_2\partial_1 - a_{2,0,0}x_1^2\partial_1 - c_{2,0,0}x_1x_3\partial_1 \\ &\quad + h_{2,0,0}x_1\partial_1 + a_{2,0,0}x_1x_2\partial_2 + a_{3,0,0}x_1x_2\partial_3 \\ &\quad + \sum_{i,j\geq 0} r_{1,i,j}x_2^ix_3^j\partial_1 + \sum_{i,j\geq 0} r_{2,i,j}x_2^ix_3^j\partial_2 + \sum_{i,j\geq 0} r_{3,i,j}x_2^ix_3^j\partial_3, \end{aligned} \tag{16}$$

where $r_{1,i,j}, r_{2,i,j}, r_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. By $\partial_2 * x_1x_2\partial_1 = x_1\partial_1$ and (16), we have that

$$\begin{aligned} &\sum_{\substack{i\geq 1\\ j\geq 0}} ir_{1,i,j}x_2^{i-1}x_3^j\partial_1 + \sum_{\substack{i\geq 1\\ j\geq 0}} ir_{2,i,j}x_2^{i-1}x_3^j\partial_2 + \sum_{\substack{i\geq 1\\ j\geq 0}} ir_{3,i,j}x_2^{i-1}x_3^j\partial_3 \\ &\qquad = -2b_{1,0,0}x_2\partial_1 - c_{1,0,0}x_3\partial_1 + d_{1,0,0}\partial_1. \end{aligned}$$

This implies that $r_{1,2,0} = -b_{1,0,0}$, $r_{1,1,1} = -c_{1,0,0}$ and $r_{1,1,0} = d_{1,0,0}$ hold. We have that $r_{1,1,j} = 0$, $j \geq 2$, $r_{1,2,j} = 0$, $j \geq 1$, $r_{1,i,j} = 0$, $i \geq 2$, $j \geq 0$, and $r_{2,i,j} = r_{3,i,j} = 0$, $i \geq 1$, $j \geq 0$ hold. We have that

$$\begin{aligned} D(x_1x_2\partial_1) &= -b_{1,0,0}x_2^2\partial_1 - b_{2,0,0}x_1x_2\partial_1 - a_{2,0,0}x_1^2\partial_1 - c_{1,0,0}x_2x_3\partial_1 \\ &\quad - c_{2,0,0}x_1x_3\partial_1 + d_{1,0,0}x_2\partial_1 + h_{2,0,0}x_1\partial_1 + a_{2,0,0}x_1x_2\partial_2 \\ &\quad + a_{3,0,0}x_1x_2\partial_3 + \sum_{j\geq 0} r_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} r_{2,0,j}x_3^j\partial_2 \\ &\quad + \sum_{j\geq 0} r_{3,0,j}x_3^j\partial_3. \end{aligned} \tag{17}$$

Since ∂_3 is in the left annihilator of $x_1x_2\partial_1$, by Lemma 1 and (17), we have that

$$\sum_{j\geq 1} jr_{1,0,j}x_3^{j-1}\partial_1 + \sum_{j\geq 1} jr_{2,0,j}x_3^{j-1}\partial_2 + \sum_{j\geq 1} jr_{3,0,j}x_3^{j-1}\partial_3 = 0.$$

Then $r_{1,0,j} = r_{2,0,j} = r_{3,0,j} = 0$, $j \geq 1$. This implies that

(18)
$$\begin{aligned} D(x_1x_2\partial_1) = &-b_{1,0,0}x_2^2\partial_1 - b_{2,0,0}x_1x_2\partial_1 - a_{2,0,0}x_1^2\partial_1 \\ &- c_{1,0,0}x_2x_3\partial_1 - c_{2,0,0}x_1x_3\partial_1 + d_{1,0,0}x_2\partial_1 \\ &+ h_{2,0,0}x_1\partial_1 + a_{2,0,0}x_1x_2\partial_2 + a_{3,0,0}x_1x_2\partial_3 \\ &+ r_{1,0,0}\partial_1 + r_{2,0,0}\partial_2 + r_{3,0,0}\partial_3. \end{aligned}$$

Since $x_1\partial_1$ is a left identity of $x_1x_2\partial_2$, we have that

$$\begin{aligned} D(x_1x_2\partial_1) = &-a_{2,0,0}x_1^2\partial_1 - b_{1,0,0}x_2^2\partial_1 - b_{2,0,0}x_1x_2\partial_1 \\ &- c_{1,0,0}x_2x_3\partial_1 - c_{2,0,0}x_1x_3\partial_1 + d_{1,0,0}x_2\partial_1 \\ &+ h_{2,0,0}x_1\partial_1 + a_{2,0,0}x_1x_2\partial_2 + a_{3,0,0}x_1x_2\partial_3. \end{aligned}$$

Similarly, we also have that

$$\begin{aligned} D(x_1x_3\partial_1) = &-a_{3,0,0}x_1^2\partial_1 - b_{1,0,0}x_2x_3\partial_1 - b_{3,0,0}x_1x_2\partial_1 \\ &- c_{1,0,0}x_3^2\partial_1 - c_{3,0,0}x_1x_3\partial_1 + d_{1,0,0}x_3\partial_1 \\ &+ g_{3,0,0}x_1\partial_1 + a_{2,0,0}x_1x_3\partial_2 + a_{3,0,0}x_1x_3\partial_3. \end{aligned}$$

By Lemma 1, Lemma 2, and $D(\partial_2 * x_2x_3\partial_1) = D(x_3\partial_1)$, we have that

(19)
$$\begin{aligned} D(x_2x_3\partial_1) = &\, a_{1,0,0}x_2x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 \\ &- b_{3,0,0}x_2^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + g_{3,0,0}x_2\partial_1 \\ &+ a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3 + \sum_{i,j\geq 0} u_{1,i,j}x_1^ix_3^j\partial_1 \\ &+ \sum_{i,j\geq 0} u_{2,i,j}x_1^ix_3^j\partial_2 + \sum_{i,j\geq 0} u_{3,i,j}x_1^ix_3^j\partial_3 \end{aligned}$$

where $u_{1,i,j}, u_{2,i,j}, u_{3,i,j} \in \mathbf{F}$, $i, j \geq 0$. Since ∂_1 is in the left annihilator of $x_2x_3\partial_1$, by Lemma 1 and (19), we also have that

$$\begin{aligned} &\sum_{\substack{i\geq 1\\ j\geq 0}} iu_{1,i,j}x_1^{i-1}x_3^j\partial_1 + \sum_{\substack{i\geq 1\\ j\geq 0}} iu_{2,i,j}x_1^{i-1}x_3^j\partial_2 \\ &\qquad\qquad + \sum_{\substack{i\geq 1\\ j\geq 0}} iu_{3,i,j}x_1^{i-1}x_3^j\partial_3 = -a_{2,0,0}x_3\partial_1. \end{aligned}$$

This implies that $u_{1,1,1} = -a_{2,0,0}$ holds. We have that $u_{1,1,j} = 0$, $j \geq 2$, $u_{1,i,j} = 0$, $i \geq 2$, $j \geq 0$, and $u_{2,i,j} = u_{3,i,j} = 0$, $i \geq 1$, $j \geq 0$. We have that

(20)
$$\begin{aligned} D(x_2x_3\partial_1) = {} & a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 \\ & - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 \\ & + g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3 \\ & + \sum_{j\geq 0} u_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} u_{2,0,j}x_3^j\partial_2 + \sum_{j\geq 0} u_{3,0,j}x_3^j\partial_3. \end{aligned}$$

By $D(\partial_3 * x_2x_3\partial_1) = D(x_2\partial_1)$, by Lemma 2 and (20), we can prove that

(21)
$$\begin{aligned} D(x_2x_3\partial_1) = {} & a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 \\ & - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{2,0,0}x_3^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 \\ & + g_{3,0,0}x_2\partial_1 + h_{2,0,0}x_3\partial_1 + a_{2,0,0}x_2x_3\partial_2 \\ & + a_{3,0,0}x_2x_3\partial_3 + u_{1,0,0}\partial_1 + u_{2,0,0}\partial_2 + u_{3,0,0}\partial_3. \end{aligned}$$

Since $x_2\partial_2$ is a left identity of $x_2x_3\partial_1$, we have that

$$\begin{aligned} D(x_2x_3\partial_1) = {} & a_{1,0,0}x_2x_3\partial_1 - a_{2,0,0}x_1x_3\partial_1 - a_{3,0,0}x_1x_2\partial_1 \\ & - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{2,0,0}x_3^2\partial_1 \\ & - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 + g_{3,0,0}x_2\partial_1 \\ & + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3. \end{aligned}$$

We have the similar formulas of $D(x_1x_2\partial_2)$, $D(x_1x_3\partial_2)$, $D(x_2x_3\partial_2)$, $D(x_1x_2\partial_3)$, $D(x_1x_3\partial_3)$ and $D(x_2x_3\partial_3)$ as (15). Therefore, we have proven the lemma. $\square$

Lemma 4. *For any derivation D of the nonassociative algebra $\overline{WN_{0,0,3_1}}$ and for any basis element $x_1^n x_2^m x_3^k \partial_u$, $1 \leq u \leq 3$, of $\overline{WN_{0,0,3_1}}$,* *then we have that*

$$
\begin{aligned}
(22)\quad & D(x_1^n x_2^m x_3^k \partial_1) \\
&= (1-n)a_{1,0,0}x_1^n x_2^m x_3^k \partial_1 - mb_{2,0,0}x_1^n x_2^m x_3^k \partial_1 - kc_{3,0,0}x_1^n x_2^m x_3^k \partial_1 \\
&\quad + nd_{1,0,0}x_1^{n-1} x_2^m x_3^k \partial_1 + mh_{2,0,0}x_1^n x_2^{m-1} x_3^k \partial_1 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1} \partial_1 \\
&\quad - ma_{2,0,0}x_1^{n+1} x_2^{m-1} x_3^k \partial_1 - ka_{3,0,0}x_1^{n+1} x_2^m x_3^{k-1} \partial_1 \\
&\quad - nb_{1,0,0}x_1^{n-1} x_2^{m+1} x_3^k \partial_1 - kb_{3,0,0}x_1^n x_2^{m+1} x_3^{k-1} \partial_1 \\
&\quad - nc_{1,0,0}x_1^{n-1} x_2^m x_3^{k+1} \partial_1 - mc_{2,0,0}x_1^n x_2^{m-1} x_3^{k+1} \partial_1 \\
&\quad + a_{2,0,0}x_1^n x_2^m x_3^k \partial_2 + a_{3,0,0}x_1^n x_2^m x_3^k \partial_3,
\end{aligned}
$$

where $a_{i,0,0}, b_{i,0,0}, c_{i,0,0} \in \mathbf{F}$, $1 \le i \le 3$, *and* $d_{1,0,0}, h_{2,0,0}, g_{3,0,0} \in \mathbf{F}$. *We have the similar formulas of* $D(x_1^n x_2^m x_3^k \partial_2)$ *and* $D(x_1^n x_2^m x_3^k \partial_3)$ *as* (22).

Proof. Let D be the derivation in the lemma. By Lemma 1 and $D(\partial_1 * x_1x_2x_3\partial_1) = D(x_2x_3\partial_1)$, we have that $\partial_1 * D(x_1x_2x_3\partial_1) = -2a_{2,0,0}x_1x_3\partial_1 - c_{2,0,0}x_3^2\partial_1 - 2a_{3,0,0}x_1x_2\partial_1 - b_{2,0,0}x_2x_3\partial_1 - b_{3,0,0}x_2^2\partial_1 - c_{3,0,0}x_2x_3\partial_1 + h_{2,0,0}x_3\partial_1 + g_{3,0,0}x_2\partial_1 + a_{2,0,0}x_2x_3\partial_2 + a_{3,0,0}x_2x_3\partial_3$. This implies that

$$
\begin{aligned}
(23)\quad & D(x_1x_2x_3\partial_1) \\
&= -a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 \\
&\quad - c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 \\
&\quad + a_{2,0,0}x_1x_2x_3\partial_2 + a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{i,j\ge 0} s_{1,i,j}x_2^i x_3^j \partial_1 \\
&\quad + \sum_{i,j\ge 0} s_{2,i,j}x_2^i x_3^j \partial_2 + \sum_{i,j\ge 0} s_{3,i,j}x_2^i x_3^j \partial_3,
\end{aligned}
$$

where $s_{1,i,j}$, $s_{2,i,j}$, $s_{3,i,j} \in \mathbf{F}$, $i,j \ge 0$. By $D(\partial_2 * x_1x_2x_3\partial_1) = D(x_1x_3\partial_1)$, we have that

$$
\begin{aligned}
\sum_{\substack{i\ge 1\\ j\ge 0}} is_{1,i,j}x_2^{i-1}x_3^j\partial_1 + \sum_{\substack{i\ge 1\\ j\ge 0}} is_{2,i,j}x_2^{i-1}x_3^j\partial_2 + \sum_{\substack{i\ge 1\\ j\ge 0}} is_{3,i,j}x_2^{i-1}x_3^j\partial_3 \\
= -2b_{1,0,0}x_2x_3\partial_1 - c_{1,0,0}x_3^2\partial_1 + d_{1,0,0}x_3\partial_1.
\end{aligned}
$$

So we have that

(24)
$$\begin{aligned}D(x_1x_2x_3\partial_1) = &-a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1\\ &- b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1\\ &- c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1\\ &+ h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2\\ &+ a_{3,0,0}x_1x_2x_3\partial_3 + \sum_{j\geq 0} s_{1,0,j}x_3^j\partial_1 + \sum_{j\geq 0} s_{2,0,j}x_3^j\partial_2\\ &+ \sum_{j\geq 0} s_{3,0,j}x_3^j\partial_3.\end{aligned}$$

By $D(\partial_3 * x_1x_2x_3\partial_1) = D(x_1x_2\partial_1)$ and (24), we have that

(25)
$$\begin{aligned}D(x_1x_2x_3\partial_1) = &-a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1\\ &- b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1\\ &- c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1\\ &+ h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2\\ &+ a_{3,0,0}x_1x_2x_3\partial_3 + s_{1,0,0}\partial_1 + s_{2,0,0}\partial_2 + s_{3,0,0}\partial_3.\end{aligned}$$

Since $x_1\partial_1$ is a left identity of $x_1x_2x_3\partial_1$, by Lemma 1 and (25), we have that

(26)
$$\begin{aligned}D(x_1x_2x_3\partial_1) = &-a_{2,0,0}x_1^2x_3\partial_1 - a_{3,0,0}x_1^2x_2\partial_1 - b_{1,0,0}x_2^2x_3\partial_1\\ &- b_{2,0,0}x_1x_2x_3\partial_1 - b_{3,0,0}x_1x_2^2\partial_1 - c_{1,0,0}x_2x_3^2\partial_1\\ &- c_{2,0,0}x_1x_3^2\partial_1 - c_{3,0,0}x_1x_2x_3\partial_1 + d_{1,0,0}x_2x_3\partial_1\\ &+ h_{2,0,0}x_1x_3\partial_1 + g_{3,0,0}x_1x_2\partial_1 + a_{2,0,0}x_1x_2x_3\partial_2\\ &+ a_{3,0,0}x_1x_2x_3\partial_3.\end{aligned}$$

By $D(x_1^n\partial_1 * x_1x_2x_3\partial_1) = D(x_1^nx_2x_3\partial_1)$, Lemma 1 and (26), we have that

(27)
$$\begin{aligned}D(x_1^nx_2x_3\partial_1) = &(1-n)a_{1,0,0}x_1^nx_2x_3\partial_1 - a_{2,0,0}x_1^{n+1}x_3\partial_1 - a_{3,0,0}x_1^{n+1}x_2\partial_1\\ &- nb_{1,0,0}x_1^{n-1}x_2^2x_3\partial_1 - b_{2,0,0}x_1^nx_2x_3\partial_1 - b_{3,0,0}x_1^nx_2^2\partial_1\\ &- nc_{1,0,0}x_1^{n-1}x_2x_3^2\partial_1 - c_{2,0,0}x_1^nx_3^2\partial_1 - c_{3,0,0}x_1^nx_2x_3\partial_1\\ &+ nd_{1,0,0}x_1^{n-1}x_2x_3\partial_1 + h_{2,0,0}x_1^nx_3\partial_1 + g_{3,0,0}x_1^nx_2\partial_1\\ &+ a_{2,0,0}x_1^nx_2x_3\partial_2 + a_{3,0,0}x_1^nx_2x_3\partial_3.\end{aligned}$$

By $D(x_2^m\partial_2 * x_1^n x_2 x_3\partial_1) = D(x_1^n x_2^m x_3\partial_1)$, Lemma 1 and (27), we have that

$$\begin{aligned}
(28)\quad & D(x_1^n x_2^m x_3\partial_1) \\
&= (1-n)a_{1,0,0}x_1^n x_2^m x_3\partial_1 - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3\partial_1 - a_{3,0,0}x_1^{n+1}x_2^m\partial_1 \\
&\quad - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3\partial_1 - mb_{2,0,0}x_1^n x_2^m x_3\partial_1 - b_{3,0,0}x_1^n x_2^{m+1}\partial_1 \\
&\quad - nc_{1,0,0}x_1^{n-1}x_2^m x_3^2\partial_1 - mc_{2,0,0}x_1^n x_2^{m-1}x_3^2\partial_1 - c_{3,0,0}x_1^n x_2^m x_3\partial_1 \\
&\quad + nd_{1,0,0}x_1^{n-1}x_2^m x_3\partial_1 + mh_{2,0,0}x_1^n x_2^{m-1}x_3\partial_1 + g_{3,0,0}x_1^n x_2^m\partial_1 \\
&\quad + a_{2,0,0}x_1^n x_2^m x_3\partial_2 + a_{3,0,0}x_1^n x_2^m x_3\partial_3.
\end{aligned}$$

By $D(x_3^k\partial_3 * x_1^n x_2^m x_3\partial_1) = D(x_1^n x_2^m x_3^k\partial_1)$, Lemma 1 and (28), we have that

$$\begin{aligned}
(29)\quad & D(x_1^n x_2^m x_3^k\partial_1) \\
&= (1-n)a_{1,0,0}x_1^n x_2^m x_3^k\partial_1 - mb_{2,0,0}x_1^n x_2^m x_3^k\partial_1 - kc_{3,0,0}x_1^n x_2^m x_3^k\partial_1 \\
&\quad + nd_{1,0,0}x_1^{n-1}x_2^m x_3^k\partial_1 + mh_{2,0,0}x_1^n x_2^{m-1}x_3^k\partial_1 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1}\partial_1 \\
&\quad - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k\partial_1 - ka_{3,0,0}x_1^{n+1}x_2^m x_3^{k-1}\partial_1 \\
&\quad - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k\partial_1 - kb_{3,0,0}x_1^n x_2^{m+1}x_3^{k-1}\partial_1 \\
&\quad - nc_{1,0,0}x_1^{n-1}x_2^m x_3^{k+1}\partial_1 - mc_{2,0,0}x_1^n x_2^{m-1}x_3^{k+1}\partial_1 \\
&\quad + a_{2,0,0}x_1^n x_2^m x_3^k\partial_2 + a_{3,0,0}x_1^n x_2^m x^k 3\partial_3.
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
D(x_1^n x_2^m x_3^k\partial_2) &= -na_{1,0,0}x_1^n x_2^m x_3^k\partial_2 + (1-m)b_{2,0,0}x_1^n x_2^m x_3^k\partial_2 \\
&\quad - kc_{3,0,0}x_1^n x_2^m x_3^k\partial_2 + nd_{1,0,0}x_1^{n-1}x_2^m x_3^k\partial_2 \\
&\quad + mh_{2,0,0}x_1^n x_2^{m-1}x_3^k\partial_2 + kg_{3,0,0}x_1^n x_2^m x_3^{k-1}\partial_2 \\
&\quad - ma_{2,0,0}x_1^{n+1}x_2^{m-1}x_3^k\partial_2 - ka_{3,0,0}x_1^{n+1}x_2^m x_3^{k-1}\partial_2 \\
&\quad - kb_{3,0,0}x_1^n x_2^{m+1}x_3^{k-1}\partial_2 - nc_{1,0,0}x_1^{n-1}x_2^m x_3^{k+1}\partial_2 \\
&\quad - mc_{2,0,0}x_1^n x_2^{m-1}x_3^{k+1}\partial_2 - nb_{1,0,0}x_1^{n-1}x_2^{m+1}x_3^k\partial_2 \\
&\quad + b_{1,0,0}x_1^n x_2^m x_3^k\partial_1 + b_{3,0,0}x_1^n x_2^m x_3^k\partial_3
\end{aligned}$$

$$\begin{aligned}
D(x_1^n x_2^m x_3^k \partial_3) = & -na_{1,0,0} x_1^n x_2^m x_3^k \partial_3 - mb_{2,0,0} x_1^n x_2^m x_3^k \partial_3 \\
& + (1-k) c_{3,0,0} x_1^n x_2^m x_3^k \partial_3 + nd_{1,0,0} x_1^{n-1} x_2^m x_3^k \partial_3 \\
& + mh_{2,0,0} x_1^n x_2^{m-1} x_3^k \partial_3 + kg_{3,0,0} x_1^n x_2^m x_3^{k-1} \partial_3 \\
& - ma_{2,0,0} x_1^{n+1} x_2^{m-1} x_3^k \partial_3 - ka_{3,0,0} x_1^{n+1} x_2^m x_3^{k-1} \partial_3 \\
& - nb_{1,0,0} x_1^{n-1} x_2^{m+1} x_3^k \partial_3 - kb_{3,0,0} x_1^n x_2^{m+1} x_3^{k-1} \partial_3 \\
& - nc_{1,0,0} x_1^{n-1} x_2^m x_3^{k+1} \partial_3 - mc_{2,0,0} x_1^n x_2^{m-1} x_3^{k+1} \partial_3 \\
& + c_{1,0,0} x_1^n x_2^m x_3^k \partial_1 + c_{2,0,0} x_1^n x_2^m x_3^k \partial_2
\end{aligned}$$

as (29). This completes the proof of the lemma. □

Note 1. For any basis element $x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u$, $1 \le u \le 3$, of $\overline{WN_{0,0,3_1}}$, we define **F**-maps D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, of $\overline{WN_{0,0,3_1}}$ as follows:

$$\begin{aligned}
D_v(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= (\delta_{u,v} - n_v) x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u, \quad \text{for} \quad 1 \le v \le 3, \\
D_{v,w}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= -n_w x_v^{n_v+1} x_w^{n_w-1} x_k^{n_k} \partial_u + \delta_{u,v} x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_w, \\
&\qquad \text{for} \quad 1 \le v \ne w \le 3, \quad k \notin \{v,w\}, \\
D_{9+v}(x_1^{n_1} x_2^{n_2} x_3^{n_3} \partial_u) &= n_v x_v^{n_v-1} x_k^{n_k} x_w^{n_w} \partial_u, \\
&\qquad \text{for} \quad 1 \le v \le 3 \quad \text{and} \quad 1 \le k \ne w \le 3, \quad v \notin \{k,w\},
\end{aligned}$$

where $\delta_{u,v}$ is Kronecker delta. The **F**-maps D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, of $\overline{WN_{0,0,3_1}}$ can be linearly extended to nonassociative algebra derivations of $\overline{WN_{0,0,3_1}}$.

Theorem 1. *The additive group* $\mathrm{Der}(\overline{WN_{0,0,3_1}})$ *of the nonassociative algebra* $\overline{WN_{0,0,3_1}}$ *is spanned by* D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, *and* D_{9+v}, $1 \le v \le 3$, *which are defined in Note* 1 *and which are inner* [**4**].

Proof. By Lemma 4 and Note 1, any derivation D of $\overline{WN_{0,0,3_1}}$ can be written as a linear sum of D_v, $1 \le v \le 3$, $D_{v,w}$, $1 \le v \ne w \le 3$, and D_{9+v}, $1 \le v \le 3$, which are defined in Note 1. This completes the proof of the theorem. □

Note 2. For any $x_1^{n_1} x_2^{n_2} \cdots x_s^{n_s} \partial_u$, $1 \le u \le s$, of $\overline{WN_{0,0,s_1}}$, we define **F**-maps $D_1, \cdots, D_{s^2+s}$ of $\overline{WN_{0,0,s_1}}$ as follows:

$$D_v(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) = (\delta_{u,v}n_v)x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u, \quad \text{for} \quad 1 \le v \le s$$

$$\begin{aligned} D_{v,w}&(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) \\ &= -n_w x_1^{n_1}\cdots x_v^{n_v+1}x_{v+1}^{n_v+1}\cdots x_w^{n_w-1}x_{w+1}^{n_{w+1}}\cdots x_s^{n_s}\partial_u \\ &\quad + \delta_{u,v}x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_w, \quad for 1 \le v \ne w \le s \end{aligned}$$

$$D_{s^2+v}(x_1^{n_1}x_2^{n_2}\cdots x_s^{n_s}\partial_u) = n_v x_1^{n_1}\cdots x_v^{n_v-1}x_{v+1}^{n_v+1}\cdots x_s^{n_s}\partial_u \quad \text{for} \quad 1 \le v \le s. \tag{30}$$

Then the **F**-maps $D_1, \ldots, D_s$, $D_{v,w}$, $1 \le v \ne w \le s$, D_{s^2+v}, $1 \le v \le s$, can be linearly extended to nonassociative algebra derivations of $\overline{WN_{0,0,s_1}}$.

Theorem 2. *The additive group* $\mathrm{Der}(\overline{WN_{0,0,s_1}})$ *of the nonassociative algebra* $\overline{WN_{0,0,s_1}}$ *is spanned by* D_u, $1 \le u \le s$, $D_{v,w}$, $1 \le v \ne w \le s$, *and* D_{s^2+v}, $1 \le v \le s$, *which are defined in Note* 2.

Proof. Let D be any derivation $\mathrm{Der}(\overline{WN_{0,0,s_1}})$. For any element $x_q^{n_q}\partial_v$ of $\overline{WN_{0,0,s_1}}$, it is easy to prove that

$$D(\partial_v) = \sum_{1\le t\le s} c_t\partial_t \tag{31}$$

$$\begin{aligned} D(x_q^{n_q}\partial_v) &= a_{q,v}(\delta_{q,v} - n_q)x_q^{n_q}\partial_v + b_{q,v}n_q x_q^{n_q-1}\partial_v \\ &\quad + \sum_{1\le w\le s}(1-\delta_{w,q})c_{q,w,v}n_q x_q^{n_q-1}x_w\partial_v \\ &\quad + \sum_{1\le t\le s}(1-\delta_{t,v})d_{q,t,v}x_q^{n_q}\partial_t. \end{aligned} \tag{32}$$

Without loss of generality, we may assume that at least one of c_t, $1 \le t \le s$, is nonzero with appropriate scalars. For any basis element

$x_1^{n_1}\cdots x_s^{n_s}\partial_v$ of $\overline{WN_{0,0,s_1}}$, let us put the following formula:
(33)
$$\begin{aligned} D(x_1^{n_1}\cdots x_s^{n_s}\partial_v) = &\sum_{1\le t\le s} a_{t,v}(\delta_{t,v}-n_t)x_1^{n_1}\cdots x_s^{n_s}\partial_v \\ &+\sum_{1\le t\le s} b_{t,v}n_t x_1^{n_1}\cdots x_t^{n_t-1}x_{t+1}^{n_{t+1}}\cdots x_s^{n_s}\partial_v \\ &+\sum_{1\le r\ne t\le s} c_{r,t,v}(n_t x_1^{n_1}\cdots x_r^{n_r+1}x_{r+1}^{n_{r+1}}\cdots \\ &\qquad x_t^{n_t-1}x_{t+1}^{n_{t+1}}\cdots x_s^{n_s}\partial_v+\delta_{v,r}x_1^{n_1}\cdots x_s^{n_s}\partial_t) \end{aligned}$$

with appropriate scalars. By (31) and (32), we know that D holds (31), (32) and (33) for ∂_u, $1\le u\le s$, and $x_q^{n_q}\partial_v$, $1\le q$, $v\le s$. Thanks to Theorem 3.3 in [**1**], Theorem 1 in [**2**], and Theorem 1, for any basis element of $\overline{WN_{0,0,1_1}}$, $\overline{WN_{0,0,2_1}}$, and $\overline{WN_{0,0,3_1}}$, D holds (30) appropriately. Thus, by induction on s of $\overline{WN_{0,0,s_1}}$, we assume that the theorem holds for $\overline{WN_{0,0,s-1_1}}$. This implies that by (31), (32) and (33), we can assume that the theorem holds for any $x_1^{n_1}\cdots x_s^{n_s}\partial_v$ of $\overline{WN_{0,0,s_1}}$ such that $\deg_{\,\mathrm{tot}}(x_1^{n_1}\cdots x_s^{n_s}\partial_v)\in\mathbf{N}$ by induction on the total degree of $x_1^{n_1}\cdots x_s^{n_s}\partial_v$ and a fixed positive integer n_s. We take a basis element $x_1^{n_1}\cdots x_s^{n_s+1}\partial_v$ such that $\deg_{\,s}(x_1^{n_1}\cdots x_s^{n_s+1}\partial_v)=n_s+1$. By $D(\partial_s * x_1^{n_1}\cdots s_s^{n_s+1}\partial_v)=(n_s+1)D(x_1^{n_1}\cdots x_s^{n_s}\partial_v)$, $D(x_1^{n_1}\cdots x_s^{n_s+1}\partial_w * x_w\partial_v)=D(x_1^{n_1}\cdots x_s^{n_s+1}\partial_v)$ and by appropriate inductions, we can prove that $D(x_1^{n_1}\cdots x_s^{n_s+1}\partial_v)$ holds (30) routinely where $1\le w\le s$ and $w<v$. Thus, we can prove that D is the sum of derivations in Note 2 and let us omit the remaining steps of its proof because of routine calculations. □

Corollary 1. *If $s_1\ne s_2$, then the nonassociative algebras $\overline{WN_{0,0,s_11}}$ and $\overline{WN_{0,0,s_21}}$ are not isomorphic to each other as nonassociative algebras.*

Proof. By Theorem 2, the dimension of $\mathrm{Der}\,(\overline{WN_{0,0,s_11}})$ is $s_1^2+s_1$ and the dimension of $\mathrm{Der}\,(\overline{WN_{0,0,s_21}})$ is $s_2^2+s_2$. Thus, there is no isomorphism between them. This completes the proof of the corollary. □

$\overline{WN_{0,0,s_1}}$ is Lie-admissible with respect to its commutator [**1, 4**]. There is an s^2 dimensional simple subalgebra A_{s^2} in $\overline{WN_{0,0,s_1}}$ spanned by $\{x_u\partial_v \mid 1 \leq u, v \leq s\}$ which is isomorphic to the matrix ring $M_s(\mathbf{F})$. By Theorem 1 we know that, for any derivation D of $\overline{WN_{0,0,s_1}}$ the restriction $D\mid_{A_{s^2}}$ of D to A_{s^2} is a derivation of A_{s^2} which is inner, i.e., $\mathrm{Der}\,(M_s(\mathbf{F}))$ is spanned by the restrictions $D_u \mid_{A_{s^2}}$ of D_u, $1 \leq u \leq s$, and $D_{v,w}\mid_{A_{s^2}}$ of $D_{v,w}$, $1 \leq v \neq w \leq s$, of $\overline{WN_{0,0,s_1}}$ which are defined in Note 2 [**4**].

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