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CONVOLUTION OPERATORS ON SCHWARTZ SPACES FOR CHÉBLI-TRIMÈCHE HYPERGROUPS

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ABSTRACT. The convolution associated to the generalized Fourier transformation related to Chébli-Trimèche hypergroups is investigated on the Schwartz type spaces introduced by Bloom and Xu. In particular, the pointwise multipliers for these spaces are described and the convolution is studied in detail on the corresponding dual spaces.

1. Introduction. In a series of papers [**6–8, 23**], J.J. Betancor and Marrero studied the Hankel convolution on spaces of distributions. They developed for the Hankel convolution a theory analogous to the classical one for the usual convolution on the Schwartz distribution spaces. In this paper, we study the convolution operator associated to Chébli-Trimèche hypergroups on Schwartz type distribution spaces introduced by Bloom and Xu [**12**].

Although the notion of hypergroup was introduced in the 1930's, the harmonic analysis on hypergroups was developed in the 1970's by Dunkl [**14**], Jewett [**20**] and Spector [**26**], amongst others.

Here we deal with a special kind of hypergroup known as Chébli-Trimèche hypergroups. This sort of hypergroup has been quite investigated in the last years, see [**12, 22, 31**]. Chébli-Trimèche hypergroups are a class of one-dimensional hypergroups on $[0,\infty)$ associated to a Sturm-Liouville boundary value problem. The characters of the hypergroup are the solutions of the considered problem.

More specifically, we denote by $\triangle$ the differential operator

$$\triangle = -\frac{d^2}{dx^2} - \frac{A'(x)}{A(x)}\,\frac{d}{dx},$$

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where A is a continuous function on $[0,\infty)$ that is twice continuously differentiable on $(0,\infty)$ and that satisfies the following conditions:

(i) $A(0)=0$ and $A(x)>0$, $x\in(0,\infty)$.

(ii) A is increasing and unbounded on $[0,\infty)$.

(iii) There exist an odd function $B\in C^\infty(\mathbf{R})$, $\alpha>-1/2$, and $\delta>0$ such that

$$\frac{A'(x)}{A(x)}=\frac{2\alpha+1}{x}+B(x),\quad x\in(0,\delta).$$

(iv) $A'/A\in C^\infty(0,\infty)$ and it is decreasing on $(0,\infty)$. Hence, the limit $\lim_{x\to\infty}A'(x)/A(x)$ exists. We define $\rho=(1/2)\lim_{x\to\infty}A'(x)/A(x)$.

If A is a function satisfying the properties listed above, A is called a Chébli-Trimèche function [**13, 29**].

Also we will assume that the function A defining the operator $\triangle$ satisfies the following condition: there exist $R,\delta>0$ for which $A'(x)/A(x)=2\rho+e^{-\delta x}D(x)$, if $\rho>0$, or $A'(x)/A(x)=(2\alpha+1)/x+e^{-\delta x}D(x)$, if $\rho=0$, when $x>R$, and $D\in C^\infty(0,\infty)$, $d^k/(dx^k)D$ being bounded on $(0,\infty)$, for every $k\in\mathbf{N}$.

If $f\in C^\infty(\mathbf{R})$ is even, the generalized translation $u(x,y)=(\tau_x f)(y)$, $x,y\in(0,\infty)$, is defined as the solution of the Cauchy problem

$$(\triangle_x-\triangle_y)u(x,y)=0,$$
$$u(x,0)=f(x),\quad u_y(x,0)=0,\quad x\in(0,\infty).$$

This generalized translation can be extended to the Lebesgue space $L_p(m)$, where $1\le p\le\infty$ and $m=Adx$ is the Haar measure associated to the hypergroup. Thus, τ_x is a contraction on $L_p(m)$, for each $x\in(0,\infty)$ and $1\le p\le\infty$.

The #-convolution induced by the translation τ_x, $x\in(0,\infty)$, is given, as usual, through

$$(f\#g)(x)=\int_0^\infty f(y)(\tau_x g)(y)A(y)\,dy,$$

where f and g are nice functions (for instance, $f,g\in L_1(m)$). For this convolution operation a Young inequality holds. Also, the #-convolution can be extended to the space of all the bounded complex measures on $(0,\infty)$ [**22**].

The pair $([0,\infty),\#)$ is a hypergroup on $[0,\infty)$ called a Chébli-Trimèche hypergroup. The characters of the hypergroup $([0,\infty),\#)$ are the functions φ_λ, $\lambda \in \mathbf{C}$, where $\triangle\varphi_\lambda = (\lambda^2+\rho^2)\varphi_\lambda$, $\varphi_\lambda(0)=1$, $\varphi'_\lambda(0)=0$, $\lambda \in \mathbf{C}$.

We have an integral transformation $\mathcal{F}$ associated to the hypergroup $([0,\infty),\#)$. This transformation $\mathcal{F}$, called generalized Fourier transform, is defined on $L_1(m)$ by

$$(\mathcal{F}f)(\lambda) = \int_0^\infty \varphi_\lambda(x) f(x) A(x)\,dx, \quad \lambda \in (0,\infty).$$

The inverse of the $\mathcal{F}$-transformation is given, under adequate conditions, by means of

$$f(x) = \int_0^\infty \mathcal{F}(f)(\lambda)\varphi_\lambda(x)\,\frac{d\lambda}{|c(\lambda)|^2}, \quad x \in (0,\infty),$$

where c is a continuous and zero free function on $[0,\infty)$. The function c is usually known as a Harish-Chandra function, see [**30**].

A Plancherel theorem holds for the generalized Fourier transform [**11**, Theorem 2.2.13]. Also in [**29**, Theorem 2.2] a Paley-Wiener type theorem was established for $\mathcal{F}$-transforms.

Two important special cases of Chébli-Trimèche hypergroups are the following ones. If $A(x) = x^{2\alpha+1}$, $x \in (0,\infty)$, with $\alpha > -1/2$, $([0,\infty),\#)$ reduces to the Bessel-Kingman hypergroup [**21**] and $\#$ is the Hankel convolution [**18, 19, 24**]. The Jacobi hypergroup appears when $A(x) = \sinh^{2\alpha+1} x \cosh^{2\alpha+1} x$, with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$ [**16, 17**]. If G is a noncompact connected real semi-simple Lie group with finite center and rank one, the spherical functions are Jacobi functions and the spherical Fourier transformation reduces to the Jacobi transformation.

Bloom and Xu [**12**] introduced spaces of Schwartz type (see Section 2 for definitions) on Chébli-Trimèche hypergroups. They investigated the generalized Fourier transformation on those spaces. Also they started the study of the $\#$-convolution on the above spaces. Recently the authors in [**4**] have investigated the $\#$-convolution and the generalized Fourier transform $\mathcal{F}$ on other new spaces of distributions that are different from those considered by Bloom and Xu. Our objective in this paper is to continue the analysis of the #-convolution operators on the Schwartz type spaces introduced by Bloom and Xu. This paper is organized as follows. In Section 2 we recall definitions and fundamental properties of some function and distribution spaces that will appear throughout this paper. In Section 3 we study new properties for the spaces of Bloom and Xu. We describe the pointwise multipliers for these spaces. The #-convolution is studied on the corresponding dual spaces in Section 4. We characterize distribution spaces defining convolution operators on Schwartz type distribution spaces of Bloom and Xu. Our results are inspired by the classical investigations of Schwartz [**25**] about the usual convolution and Euclidean Fourier transform and by the studies of J.J. Betancor and Marrero [**6**–**8, 23**] about the distributional convolution for Bessel-Kingman hypergroups (the so-called generalized Hankel convolutions). As it was mentioned above, Bessel-Kingman hypergroups are special cases of Chébli-Trimèche hypergroups. Thus, our results can be seen as an extension of the ones obtained by J.J. Betancor and Marrero. Also, when we consider the special case of Jacobi hypergroups our results seem to be new and they complete the important investigations of Flensted-Jensen and Koornwinder [**16, 17**].

Throughout this paper by C we always denote a positive constant not necessarily the same in each occurrence.

2. Preliminaries.

In this section we present the function and distribution spaces that will be appear throughout the paper. We recall some of their properties that will be useful in the sequel.

2.1 *Spaces* $\mathcal{D}_a$ *and* $\mathcal{D}_a^m$*,* $a > 0$*,* $m \in \mathbf{N}$. For every $a > 0$, as in [**29**], we denote by $\mathcal{D}_a$ the space constituted by all those complex and even functions $\phi \in C^\infty(\mathbf{R})$ having support contained in $[-a, a]$. On $\mathcal{D}_a$ is considered the topology associated to the family $\{\gamma_k\}_{k\in\mathbf{N}}$ of seminorms, where

$$\gamma_k(\phi) = \max_{x\in[-a,a]} \left| \frac{d^k}{dx^k}\,\phi(x)\right|, \quad \phi \in \mathcal{D}_a \quad \text{and} \quad k \in \mathbf{N}.$$

By $\mathcal{D}$ we represent the inductive space $\cup_{a>0}\mathcal{D}_a$.

In [**29**, Theorem 7.2] Trimèche characterized, for every $a > 0$, the generalized Fourier transform $\mathcal{F}(\mathcal{D}_a)$ of $\mathcal{D}_a$ as the space $\mathcal{L}_a$ defined as follows. An entire and even function Φ is in $\mathcal{L}_a$ if and only if, for every $k \in \mathbf{N}$,

$$\rho_k^a(\Phi) = \sup_{\lambda \in \mathbf{C}} (1+|\lambda|^2)^k e^{-a|\operatorname{Im}\lambda|} |\Phi(\lambda)| < \infty.$$

The topology of $\mathcal{L}_a$ is defined by the family $\{\rho_k^a\}_{k\in\mathbf{N}}$ of norms.

Let $a > 0$ and $m \in \mathbf{N}$. By $\mathcal{D}_a^m$ we denote the space of even functions $\phi \in C^{2m}(\mathbf{R})$ having support contained in $[-a, a]$. On $\mathcal{D}_a^m$ we consider the topology associated to the norm α_m defined by

$$\alpha_m(\phi) = \max_{0\le k\le 2m} \gamma_k(\phi), \quad \phi \in \mathcal{D}_a^m.$$

Note that if $b > a$ and $\phi \in \mathcal{D}_a^m$, there exists a sequence $\{\phi_n\}_{n\in\mathbf{N}} \subset \mathcal{D}_b$ such that $\phi_n \to \phi$, as $n \to \infty$, in $\mathcal{D}_b^m$.

Bloom and Xu [**12**] introduced Fréchet function spaces that are isomorphic under the generalized Fourier transformation $\mathcal{F}$.

Let $0 < p \le 2$. The generalized Schwartz space $\mathcal{S}_p$ is defined as follows. An even function $\phi \in C^\infty(\mathbf{R})$ is in $\mathcal{S}_p$ if and only if

$$\mu_{k,l}^p(\phi) = \sup_{x\in[0,\infty)} (1+x)^l \varphi_0(x)^{-2/p} \left| \frac{d^k}{dx^k}\,\phi(x)\right| < \infty$$

for every $k, l \in \mathbf{N}$. We define on $\mathcal{S}_p$ the topology associated to the family $\{\mu_{k,l}^p\}_{k,l\in\mathbf{N}}$ of semi-norms. Thus, $\mathcal{S}_p$ is a Fréchet and Montel space.

The topology of $\mathcal{S}_p$ is also generated by the system $\{\eta_{k,l}^p\}_{k,l\in\mathbf{N}}$ of semi-norms, where

$$\eta_{k,l}^p(\phi) = \sup_{x\in[0,\infty)} (1+x)^l \varphi_0(x)^{-2/p} |\triangle^k \phi(x)|, \quad \phi \in \mathcal{S}_p$$

Indeed, according to [**12**, Lemma 4.18], $\{\mu_{k,l}^p\}_{k,l\in\mathbf{N}}$ defines on $\mathcal{S}_p$ a topology stronger than the one associated to $\{\eta_{k,l}^p\}_{k,l\in\mathbf{N}}$. On the other hand, by analyzing the proof of [**12**, Proposition 4.24] and taking into account [**12**, Theorem 4.27], we can see that the topology defined by $\{\eta_{k,l}^p\}_{k,l\in\mathbf{N}}$ is stronger than the one generated by $\{\mu_{k,l}^p\}_{k,l\in\mathbf{N}}$.

Let $\varepsilon \ge 0$. A function Φ defined in the region $\Omega_\varepsilon = \{\lambda \in \mathbf{C} : |\operatorname{Im}\lambda| \le \varepsilon\}$ is in S_ε if, and only if, the following two conditions hold

(a) Φ is holomorphic and even in $\Omega_\varepsilon^0 = \{\lambda \in \mathbf{C} : |\mathrm{Im}\,\lambda| < \varepsilon\}$ and, for every $k \in \mathbf{N}, d^k/(d\lambda^k)\Phi$ can be continuously extended to Ω_ε, and

(b) $\tau_{k,l;\varepsilon}(\Phi) = \sup_{\lambda\in\Omega_\varepsilon}(1 + |\lambda|)^l |d^k/(d\lambda^k)\Phi(\lambda)| < \infty$, for every $l, k \in \mathbf{N}$.

The space S_ε is endowed with the topology generated by the family $\{\tau_{k,l;\varepsilon}\}_{k,l\in\mathbf{N}}$ of semi-norms. Thus, S_ε is a Fréchet space.

Bloom and Xu established that the generalized Fourier transformation $\mathcal{F}$ is an isomorphism from $\mathcal{S}_p$ onto $S_{(2/p-1)\rho}$ [**12**, Theorem 4.27]. The generalized Fourier transformation can be defined on the dual space $\mathcal{S}_p'$ of $\mathcal{S}_p$ by transposition. That is to say, if $T \in \mathcal{S}_p'$, the generalized Fourier transform $\mathcal{F}T$ of T is the element of $S'_{(2/p-1)\rho}$, the dual space of $S_{(2/p-1)\rho}$, defined by

$$\langle \mathcal{F}T, \mathcal{F}\phi\rangle = \langle T, \phi\rangle, \quad \phi \in \mathcal{S}_p. \tag{2.1}$$

In [**4**] we introduced for every $m \in \mathbf{Z}$, $m < 0$, the spaces A_m and $\mathcal{A}_m$, that will be very useful in our study about Chébli-Trimèche convolution operators, as follows.

Let $m \in \mathbf{Z}$, $m < 0$. The space A_m consists of all those even functions ϕ in $C^\infty(\mathbf{R})$ such that, for every $k \in \mathbf{N}$,

$$\alpha_m^k(\phi) = \sup_{x\in[0,\infty)} (1+x)^m |\Delta^k \phi(x)| < \infty.$$

A_m is endowed with the topology associated to the family $\{\alpha_m^k\}_{k\in\mathbf{N}}$ of semi-norms. Thus, A_m is a Fréchet space. It is clear that Δ is a continuous operator from A_m into itself. We proved in [**4**, Proposition 2.1] that the system $\{\beta_m^k\}_{k\in\mathbf{N}}$ of semi-norms, where

$$\beta_m^k(\phi) = \sup_{x\in[0,\infty)} (1+x)^m \left|\frac{d^k}{dx^k}\,\phi(x)\right|, \quad \phi \in A_m \quad \text{and} \quad k \in \mathbf{N},$$

generates on A_m the same topology as the one defined by $\{\alpha_m^k\}_{k\in\mathbf{N}}$.

The space S_p is contained in A_m but S_p is not dense in A_m, for every $0 < p \le 2$. Indeed, let ϕ be in the closure of $\mathcal{S}_p$ in A_m. There exists a sequence $\{\phi_j\}_{j\in\mathbf{N}} \subset \mathcal{S}_p$ such that $\phi_j \to \phi$, as $j \to \infty$, in A_m. Since $(1+x)^m\phi_j(x) \to 0$, as $x \to \infty$, for every $j \in \mathbf{N}$, also $(1+x)^m\phi(x) \to 0$, as $x \to \infty$. Consider now an even function $\psi \in C^\infty(\mathbf{R})$ such that $\psi(x) = 0$, $x \in (0,1)$ and $\psi(x) = (1+x)^{-m}$, $x \in (2,\infty)$. Then, taking into account that $d^k/(dx^k)\Big(A'(x)/A(x)\Big)$ is bounded in $(1,\infty)$, for every $k \in \mathbf{N}$ [**12**, p. 93], we deduce that $\alpha_m^k(\psi) < \infty$, for any $k \in \mathbf{N}$. On the other hand, $(1+x)^m\psi(x) \not\to 0$, as $x \to \infty$. Hence, for each $0 < p \le 2$, $\psi \in A_m$ and ψ is not in the closure of $\mathcal{S}_p$ in A_m.

We define the space $\mathcal{A}_{m,p}$ as the closure of $\mathcal{S}_p$ in A_m. Actually, the space $\mathcal{A}_{m,p}$ is not dependent on p, as was proved in [**4**, Proposition 2.2]. Since $\mathcal{A}_{m,p}$ does not depend on $0 < p \le 2$, in the sequel we will write $\mathcal{A}_m$ instead of $\mathcal{A}_{m,p}$ for each $m \in \mathbf{Z}$, $m < 0$. By $\mathcal{A}'_m$ we denote the dual space of $\mathcal{A}_m$, where $m \in \mathbf{Z}$.

It is obvious that $\mathcal{A}_{m+1}$ is continuously contained in $\mathcal{A}_m$, where $m \in \mathbf{Z}$ and $0 < p \le 2$. We equip the union space $\mathcal{A} = \cup_{m\in\mathbf{Z}}\mathcal{A}_m$ with the inductive topology. The operator $\triangle$ defines a continuous linear mapping from the space $\mathcal{A}$ into itself.

3. Generalized Fourier transformable Fréchet function spaces.

3.1 *Function spaces* $\mathcal{S}_p$, $0 < p \le 2$. We denote by $\mathcal{S}_*$ the subspace of the Schwartz space $\mathcal{S}$ constituted by all those even functions in $\mathcal{S}$. If we consider in $\mathcal{S}_*$ the topology induced by $\mathcal{S}$, then by virtue of [**12**, Lemma 3.6, (ii)], the mapping $\phi \to \varphi_0^{2/p}\phi$ is an isomorphism from $\mathcal{S}_*$ onto $\mathcal{S}_p$. Hence the space of pointwise multiplier of $\mathcal{S}_p$ coincides with the space of pointwise multipliers of $\mathcal{S}_*$. Moreover, by [**15**, Corollary 4.8, remark ff.], the space $\mathcal{S}_*$ is constituted by all those even functions $\phi \in C^\infty(\mathbf{R})$ such that

$$\gamma_{k,l}(\phi) = \sup_{x\in[0,\infty)} (1+x)^l \left|\left(\frac{1}{x}\frac{d}{dx}\right)^k \phi(x)\right| < \infty,$$

for every $l, k \in \mathbf{N}$. Therefore, according to [**5**, Theorem 2.3], a function f defined on $\mathbf{R}$ is a pointwise multiplier of $\mathcal{S}_p$, that is, the mapping $\phi \to f\phi$ is continuous from $\mathcal{S}_p$ into itself, if and only if, $f \in C^\infty(\mathbf{R})$ is even and, for every $k \in \mathbf{N}$, there exists $n = n_k$ such that $\sup_{x\in[0,\infty)}(1+x^2)^{-n_k}|(1/xd/dx)^k f(x)| < \infty$. Also it is not hard to see that f is a pointwise multiplier of $\mathcal{S}_p$ when, and only when, f

is even and it is a pointwise multiplier of $\mathcal{S}$. The space of pointwise multipliers of $\mathcal{S}_p$ will be denoted by Θ. Note that Θ is not depending on p. Furthermore, $\mathcal{S}_p$ is contained in Θ.

We will represent by $\mathcal{L}(\mathcal{S}_p)$ the space of the continuous linear mappings from $\mathcal{S}_p$ into itself. $\mathcal{L}_s(\mathcal{S}_p)$, respectively $\mathcal{L}_b(\mathcal{S}_p)$, denotes the space $\mathcal{L}(\mathcal{S}_p)$ endowed with the topology of the pointwise convergence, respectively uniform convergence on the bounded sets of $\mathcal{S}_p$. The space Θ can be seen as a subspace of $\mathcal{L}(\mathcal{S}_p)$. As in [**9**, Proposition 1] we can see that $\mathcal{L}_s(\mathcal{S}_p)$ and $\mathcal{L}_b(\mathcal{S}_p)$ induces the same topology on Θ.

Let $n, k \in \mathbf{N}$. The space $\Theta_{n,k}$ consists of all those even functions $f \in C^n(\mathbf{R})$ such that

$$\alpha_{n,k}(f) = \sup_{\substack{x\in[0,\infty)\\ 0\le j\le n}} (1+x^2)^{-k}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^j f(x)\right| < \infty.$$

$\Theta_{n,k}$ is equipped with the topology defined by the norm $\alpha_{n,k}$. Thus $\Theta_{n,k}$ is a Banach space. It is clear that $\Theta = \cap_{n\in\mathbf{N}} \cup_{k\in\mathbf{N}} \Theta_{n,k}$, where the equality is understood algebraically.

Proposition 3.1. *The topology that $\mathcal{L}_s(\mathcal{S}_p)$, equivalently $\mathcal{L}_b(\mathcal{S}_p)$, defines on Θ coincides with the projective-inductive topology of $\cap_{n\in\mathbf{N}}\cup_{k\in\mathbf{N}} \Theta_{n,k}$.*

Proof. Firstly we prove that the topology that $\mathcal{L}_s(\mathcal{S}_p)$ induces on Θ is stronger than the projective-inductive topology of $\cap_{n\in\mathbf{N}} \cup_{k\in\mathbf{N}} \Theta_{n,k}$.

Suppose that A is a bounded set of Θ when we consider on Θ the topology of the pointwise convergence. Then, for every $n \in \mathbf{N}$, there exist $k \in \mathbf{N}$ and $C > 0$ in such a way that

$$\sup_{x\in[0,\infty)} (1+x^2)^{-k}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n f(x)\right| \le C, \quad f \in A.$$

Indeed, assume that $n \in \mathbf{N}$ is such that, for every $k \in \mathbf{N}$, there exist $x_k \in (0,\infty)$ and $f_k \in A$ such that

$$(1+x_k^2)^{-k}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n f_k(x)_{|x=x_k}\right| \ge k,$$

and, for every $j \in \mathbf{N}$, $0 \le j < n$, we can find $k \in \mathbf{N}$ and $C > 0$ for which

$$\sup_{x\in[0,\infty)} (1+x^2)^{-k} \left| \left(\frac{1}{x}\,\frac{d}{dx}\right)^j f(x) \right| \le C, \quad f \in A.$$

Suppose that we can choose the sequence $\{x_k\}_{k\in\mathbf{N}}$ such that $x_0 > 1/4$ and $x_k \le x_{k+1} - 1$, $k \in \mathbf{N}$. We consider an even function $\phi \in C^\infty(\mathbf{R})$ having its support contained in $[-1/4, 1/4]$ and verifying $\phi(0) = 1$. We also define a function ψ by

$$\psi(x) = \sum_{k=0}^{\infty} \varphi_0(x_k)^{2/p} \frac{\phi(x-x_k)}{(1+x_k^2)^k}, \quad x \in [0,\infty),$$

and $\psi(x) = \psi(-x)$, $x \in (-\infty, 0)$. Thus, $\psi \in C^\infty(\mathbf{R})$ and ψ is even. Moreover, by [**12**, Lemma 3.4], for every $l, m \in \mathbf{N}$, we can write

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \varphi_0(x)^{-2/p}(1+x)^l \left| \frac{d^m}{dx^m}\,\psi(x) \right| \\
&\qquad \le C \sup_{z\in[-1/4,1/4]} \left| \frac{d^m}{dz^m}\,\phi(z) \right| \sum_{k=0}^{\infty} \frac{(1+(x_k+1/4)^2)^{l+2/p}}{(1+x_k^2)^k} < \infty,
\end{aligned}$$

because $x_k \to \infty$, as $k \to \infty$. Hence, $\psi \in \mathcal{S}_p$.

On the other hand, Leibniz's rule leads to

$$\begin{aligned}
&\varphi_0(x_k)^{-2/p} \left| \left(\frac{1}{x}\,\frac{d}{dx}\right)^n (f_k\psi)(x)_{|x=x_k} \right| \\
&\quad \ge \varphi_0(x_k)^{-2/p} \left| \psi(x_k) \left(\frac{1}{x}\,\frac{d}{dx}\right)^n f_k(x)_{|x=x_k} \right| \\
&\qquad - \varphi_0(x_k)^{-2/p} \sum_{j=0}^{n-1} \binom{n}{j} \left| \left(\frac{1}{x}\,\frac{d}{dx}\right)^j f_k(x)_{|x=x_k} \left(\frac{1}{x}\,\frac{d}{dx}\right)^{n-j} \psi(x)_{|x=x_k} \right| \\
&\quad \ge (1+x_k^2)^{-k} \left| \left(\frac{1}{x}\,\frac{d}{dx}\right)^n f_k(x)_{|x=x_k} \right| - C \ge k - C.
\end{aligned}$$

This contradicts that A is a bounded set in Θ when it is endowed with the topology induced by $\mathcal{L}_s(\mathcal{S}_p)$.

Suppose now that we cannot find a sequence $\{x_k\}_{k\in\mathbf{N}}$ as above. Then there exist $l \in \mathbf{N}$, $\alpha > 0$, $C > 0$, such that

$$(1+x^2)^{-l}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n f(x)\right| \le C, \quad x \ge \alpha, \quad f \in A.$$

Hence $x_k \in (0,\alpha)$, for k large enough. We choose an even function $\psi \in C^\infty(\mathbf{R})$ such that $\psi(x) = 1$, $x \in (0,\alpha)$, and $\psi(x) = 0$, $x \ge \alpha + 1$. It is clear that $\psi \in \mathcal{S}_p$. Moreover, for k large enough,

$$\begin{aligned}
&\varphi_0(x_k)^{-2/p}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n (f_k\psi)(x)_{|x=x_k}\right| \\
&\quad \ge \varphi_0(x_k)^{-2/p}\left|\psi(x_k)\left(\frac{1}{x}\frac{d}{dx}\right)^n f_k(x)_{|x=x_k}\right| \\
&\qquad - \varphi_0(x_k)^{-2/p}\sum_{j=0}^{n-1}\binom{n}{j}\left|\left(\frac{1}{x}\frac{d}{dx}\right)^j f_k(x)_{|x=x_k}\left(\frac{1}{x}\frac{d}{dx}\right)^{n-j}\psi(x)_{|x=x_k}\right| \\
&\quad \ge \varphi_0(x_k)^{-2/p} k(1+x_k^2)^k - C.
\end{aligned}$$

Again, according to [**12**, Lemma 3.4] we obtain a contradiction because A is bounded in Θ equipped with the topology of $\mathcal{L}_s(\mathcal{S}_p)$.

Hence, A is bounded in the inductive space $\cup_{k\in\mathbf{N}}\Theta_{n,k}$, for every $n \in \mathbf{N}$. Since Θ endowed of the pointwise convergence topology is bornological, see [**9**, Proposition 2], we can conclude that the inclusion $i : \Theta \to \cup_{k\in\mathbf{N}}\Theta_{n,k}$ is continuous, for every $n \in \mathbf{N}$. Therefore, the topology induced by $\mathcal{L}_s(\mathcal{S}_p)$ on Θ is stronger than the projective-inductive topology of $\cap_{n\in\mathbf{N}}\cup_{k\in\mathbf{N}}\Theta_{n,k}$, because the projective topology is the initial topology associated to the inclusions $i : \Theta \to \cup_{k\in\mathbf{N}}\Theta_{n,k}$, $n \in \mathbf{N}$.

To see that the projective-inductive topology of $\cap_{n\in\mathbf{N}}\cup_{k\in\mathbf{N}}\Theta_{n,k}$ is stronger than that of the $\mathcal{L}_s(\mathcal{S}_p)$ topology on Θ, we can proceed as in [**27**, Proposition 2.20]. $\square$

Remark 3.1. We emphasize that the proof of Proposition 3.1 is different than the one presented in [**27**, Corollary 3.37] for the space of multipliers on the Schwartz space $\mathcal{S}$.

By $\mathcal{S}_p'$ we will denote the dual space of $\mathcal{S}_p$. We now obtain a characterization of the elements of $\mathcal{S}_p'$ that will be very useful in the sequel.

Lemma 3.1. *Let T be a functional on $\mathcal{S}_p$. Then, the following properties are equivalent*:

(i) $T \in \mathcal{S}_p'$.

(ii) *There exist $r \in \mathbf{N}$ and essentially bounded, with respect to the Lebesgue measure on $(0,\infty)$, functions f_k on $(0,\infty)$, $k = 0, 1, \dots, r$, for which*

$$\langle T, \phi\rangle = \sum_{k=0}^{r} \int_0^\infty (1+x)^r \varphi_0(x)^{-2/p} \frac{d^k}{dx^k}\,\phi(x) f_k(x)dx, \quad \phi \in \mathcal{S}_p. \tag{3.1}$$

Proof. It is sufficient to use the Hahn-Banach theorem and standard duality arguments, [**28**], by taking into account that, for every $j, r \in \mathbf{N}$, there exists $C > 0$ such that

$$\mu_{j,r}^p(\phi) \le C \int_0^\infty (1+t)^{r+2/p} \varphi_0(t)^{-2/p} \left|\frac{d^{j+1}}{dt^{j+1}}\,\phi(t)\right| dt \quad \phi \in \mathcal{S}_p. \qquad \square$$

Let now f be a measurable function on $(0,\infty)$ such that

$$\int_0^\infty |f(x)| e^{-(2\rho/p)x} (1+x)^{-l+2/p} A(x)\, dx < \infty,$$

for some $l \in \mathbf{N}$. Then, by proceeding as in [**4**, p. 272], we can see that the functional T_f defined on $\mathcal{S}_p$ by

$$\langle T_f, \phi\rangle = \int_0^\infty f(x)\phi(x)A(x)\, dx, \quad \phi \in \mathcal{S}_p, \tag{3.2}$$

is in $\mathcal{S}_p'$.

Thus, $\mathcal{S}_p$ can be seen as a subspace of $\mathcal{S}_p'$ by identifying $\phi \in \mathcal{S}_p$ with $T_\phi \in \mathcal{S}_p'$. In a similar way, we can prove that the space Θ of multipliers of $\mathcal{S}_p$ is contained in $\mathcal{S}_p'$, when $\rho = 0$ or $0 < p \le 1$ and $\rho > 0$.

The operator Δ is defined on $\mathcal{S}_p'$ by transposition.

3.2 *Function spaces S_ε, $\varepsilon \ge 0$.* In the following we characterize the pointwise multipliers of S_ε.

Proposition 3.2. *Let F be a function defined on Ω_ε. Then, the next assertions are equivalent*:

(i) *F is holomorphic and even in $\Omega_\varepsilon^0 = \{\lambda \in \mathbf{C} : |\mathrm{Im}\,\lambda| < \varepsilon\}$ and, for every $k \in \mathbf{N}, (d^k/d\lambda^k)F$ can be continuously extended to Ω_ε, and, for every $k \in \mathbf{N}$, there exists $l \in \mathbf{N}$ such that*

$$\sup_{\lambda\in\Omega_\varepsilon} (1+|\lambda|)^{-l}\left|\frac{d^k}{d\lambda^k}\, F(\lambda)\right| < \infty.$$

(ii) *$F\Phi \in S_\varepsilon$, for every $\Phi \in S_\varepsilon$.*

(iii) *The mapping $\Phi \to F\Phi$ is continuous from S_ε into itself.*

Proof. (i) $\Rightarrow$ (ii). It is sufficient to note that if $k, m \in \mathbf{N}$, then there exist $l \in \mathbf{N}$ and $C > 0$ such that

$$\tau_{k,m;\varepsilon}(F\Phi) = \sup_{\lambda\in\Omega_\varepsilon} (1+|\lambda|)^{m}\left|\frac{d^k}{d\lambda^k}\,(F\Phi)(\lambda)\right| \le C\sum_{j=0}^{k} \tau_{j,m+l;\varepsilon}(\Phi),$$

for all $\Phi \in S_\varepsilon$.

(ii) $\Rightarrow$ (iii). It is a straightforward consequence of the closed graph theorem and the fact that convergence in S_ε implies pointwise convergence.

(iii) $\Rightarrow$ (i). The function $\Phi(\lambda) = e^{-\lambda^2}$, $\lambda \in \Omega_\varepsilon$,is in S_ε. Then, $F\Phi = \Psi$ is also in S_ε. Hence, $F(\lambda) = Psi(\lambda)e^{\lambda^2}$, $\lambda \in \Omega_\varepsilon$, where $\Psi \in S_\varepsilon$. It is immediately deduced that F is holomorphic in Ω_ε^0 and $(d^k/d\lambda^k)F$ can be continuously extended to Ω_ε, for every $k \in \mathbf{N}$.

Assume now that (i) does not hold. Then, we can find $k \in \mathbf{N}$ such that

$$\sup_{\lambda\in\Omega_\varepsilon} (1+|\lambda|)^{-n}\left|\frac{d^k}{d\lambda^k}\, F(\lambda)\right| = \infty,$$

for every $n \in \mathbf{N}$, and there exists $l \in \mathbf{N}$ for which

$$\sup_{\lambda\in\Omega_\varepsilon} (1+|\lambda|)^{-l}\left|\frac{d^j}{d\lambda^j}\, F(\lambda)\right| < \infty, \quad j = 0, 1, 2, \dots, k-1.$$

Hence, for every $n \in \mathbf{N}$, we can choose $\lambda_n \in \Omega_\varepsilon$ satisfying

$$(1+|\lambda_n|)^{-n}\left|\frac{d^k}{d\lambda^k}\,F(\lambda)_{|\lambda=\lambda_n}\right| \geq n.$$

Moreover, without loss of generality, we can suppose that $|\operatorname{Re}\lambda_n| \leq |\operatorname{Re}\lambda_{n+1}| - 1$, $n \in \mathbf{N}$.

Let us define, for every $n \in \mathbf{N}$, the function

$$\Phi_n(\lambda) = \frac{\Phi(\lambda-\lambda_n)+\Phi(\lambda+\lambda_n)}{(1+|\lambda_n|)^n}, \quad \lambda \in \Omega_\varepsilon,$$

where, as above, $\Phi(\lambda) = \exp(-\lambda^2)$. Note that, for every $m, \beta \in \mathbf{N}$,

$$\begin{aligned}
\tau_{m,\beta;\varepsilon}(\Phi_n) &= \sup_{\lambda\in\Omega_\varepsilon} \frac{(1+|\lambda|)^\beta}{(1+|\lambda_n|)^n}\left|\frac{d^m}{d\lambda^m}\,\Phi(\lambda-\lambda_n)\right| \\
&\quad + \sup_{\lambda\in\Omega_\varepsilon} \frac{(1+|\lambda|)^\beta}{(1+|\lambda_n|)^n}\left|\frac{d^m}{d\lambda^m}\,\Phi(\lambda+\lambda_n)\right| \\
&\leq 2 \sup_{\lambda\in\Omega_{2\varepsilon}} \frac{(1+|\lambda|+|\lambda_n|)^\beta}{(1+|\lambda_n|)^n}\left|\frac{d^m}{d\lambda^m}\,\Phi(\lambda)\right| \\
&\leq C(1+|\lambda_n|)^{\beta-n} \sup_{\lambda\in\Omega_{2\varepsilon}} (1+|\lambda|)^\beta\left|\frac{d^m}{d\lambda^m}\,\Phi(\lambda)\right|.
\end{aligned}$$

Here C is not dependent on $n \in \mathbf{N}$. Hence $\Phi_n \to 0$, as $n \to \infty$, in S_ε.

However, Leibniz's rule leads to

$$\begin{aligned}
&\sup_{\lambda\in\Omega_\varepsilon}\left|\frac{d^k}{d\lambda^k}\,\big(F(\lambda)\Phi_n(\lambda)\big)\right| \\
&\quad \geq \left|\frac{d^k}{d\lambda^k}\big(F(\lambda)\Phi_n(\lambda)\big)_{|\lambda=\lambda_n}\right| \\
&\quad \geq \left|\frac{d^k}{d\lambda^k}\,F(\lambda)_{|\lambda=\lambda_n}\right|\Big|\Phi_n(\lambda_n)\Big| \\
&\qquad - \sum_{j=0}^{k-1}\binom{k}{j}\left|\frac{d^j}{d\lambda^j}\,F(\lambda)_{|\lambda=\lambda_n}\right|\left|\frac{d^{k-j}}{d\lambda^{k-j}}\,\Phi(\lambda)_{|\lambda=\lambda_n}\right| \\
&\quad \geq (1+|\lambda_n|)^{-n}\left|\frac{d^k}{d\lambda^k}F(\lambda)_{|\lambda=\lambda_n}\right|\left|1+e^{-4\lambda_n^2}\right| - C
\end{aligned}$$

$$\geq (1+|\lambda_n|) - n\left|\frac{d^k}{d\lambda^k}\, F(\lambda)_{|\lambda=\lambda_n}\right|\left(1-e^{-4((\operatorname{Re}\lambda_n^2)^2-(\operatorname{Im}\lambda_n)^2)}\right) - C$$
$$\geq \frac{1}{2}\, n - C, \quad n \in \mathbf{N}.$$

Therefore, $F\Phi_n \not\to 0$, as $n \to \infty$, in S_ε.

Thus we conclude that (iii) does not hold. The proof is now complete. $\square$

We present in the following assertion some important multipliers of S_ε.

Proposition 3.3. *For every $x \in (0,\infty)$ and $k, n \in \mathbf{N}$, we have $(\partial^{k+n}/\partial x^k \partial\lambda^n)\varphi_\lambda(x)$ is a multiplier of S_ε, provided that $0 < \varepsilon < \rho$.*

Proof. Since the mapping $\lambda \to \varphi_\lambda$ is entire, for every $x \in (0,\infty)$, we can write

$$\frac{d^n}{d\lambda^n}\,\varphi_\lambda(x) = \frac{n!}{2\pi i}\int_{C_\lambda} \frac{\varphi_\eta(x)}{(\eta-\lambda)^{n+1}}\, d\eta, \quad n \in \mathbf{N},$$

where C_λ can be parametrized by $\eta = \lambda + r_\lambda e^{it}$, $t \in [0,\infty)$, and $r_\lambda = \rho - |Im\lambda|$.

Hence, proceeding as in the proof of [**12**, Lemma 3.6, (ii)] and of [**1**, Proposition 2.3] and according to [**4**, Propositions 2.2, 2.3], we conclude that, for certain $m \in \mathbf{N}$ and for all $x \in (0,\infty)$ and $n, k \in \mathbf{N}$,

$$\left|\frac{\partial^k}{\partial x^k}\,\frac{\partial^n}{\partial\lambda^n}\,\varphi_\lambda(x)\right| \leq C(1+|\lambda|)^m(\rho - |\operatorname{Im}\lambda|)^{-n}(1+x)e^{-(\rho-|\operatorname{Im}\lambda|)x}.$$

Here $C > 0$ is not dependent on $x \in (0,\infty)$ and $|\operatorname{Im}\lambda| < \rho$. Then,

$$\sup_{\lambda\in\Omega_\varepsilon}(1+|\lambda|)^{-m}\left|\frac{\partial^k}{\partial x^k}\,\frac{\partial^n}{\partial\lambda^n}\varphi_\lambda(x)\right| \leq C(1+x), \quad x \in (0,\infty). \tag{3.3}$$

Thus, the proof is finished. $\square$

By S'_ε we will denote the dual space of S_ε. In the sequel some special elements of S'_ε will be described.

Let F be a measurable function on $(0,\infty)$ such that

$$\int_0^\infty |F(y)|(1+y)^{-l}\frac{dy}{|c(y)|^2}<\infty,$$

for some $l\in \mathbf{N}$. Then, it is not hard to see that [**4**, p. 273] that the functional T_F defined on S_ε by

$$\langle T_F,\Phi\rangle=\int_0^\infty F(y)\Phi(y)\frac{dy}{|c(y)|^2},\quad \Phi\in S_\varepsilon, \tag{3.4}$$

is in S'_ε.

In particular, every $\Psi\in S_\varepsilon$ can be identified with the element $T_\Psi\in S'_\varepsilon$. In fact, since $|c(y)|^{-2}\sim y^{2\alpha+1}$, when y is large [**30**], provided that $\alpha>-1/2$, we have

$$\int_0^\infty |\Psi(y)|\frac{dy}{|c(y)|^2}<\infty.$$

As it was mentioned in Section 2 the generalized Fourier transformation is defined on $\mathcal{S}'_p$ by (2.1). By invoking Fubini's theorem and taking into account the inversion formula for the generalized Fourier transformation [**12**, Theorem 4.27] we can see that

$$\langle T_{\mathcal{F}\psi},\mathcal{F}\phi\rangle=\langle T_\psi,\phi\rangle,\quad \phi\in\mathcal{S}_p,$$

provided that $\psi\in\mathcal{S}_p$. Here $T_{\mathcal{F}\psi}$ and T_ψ are given by (3.4) and (3.2), respectively. Thus, the generalized Fourier transformation on $\mathcal{S}_p$ is a particular case of the transformation defined by (2.1) on $\mathcal{S}'_p$.

Henceforth, to simplify, we will write $\mathcal{H}_p$ instead of $S_{(2/p-1)\rho}$.

4. The generalized #-convolution on the space $\mathcal{S}'_p$. Bloom and Xu [**12**] have recently started the study of the #-convolution on the space $\mathcal{S}'_p$. Specifically, they defined the #-convolution between a functional in $\mathcal{S}'_p$ and a function in $\mathcal{S}_p$.

Throughout this section we assume that $0<p\le 2$. In [**12**, Lemma 5.2] it was proved that $\phi\#\psi\in\mathcal{S}_p$, provided that $\phi,\psi\in\mathcal{S}_p$, and also it

was established that $\tau_x\phi \in \mathcal{S}_p$, for every $\phi \in \mathcal{S}_p$ and $x \in (0,\infty)$. Then, the convolution $T\#\phi$ of $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$ is defined by

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle, \quad x \in (0,\infty). \tag{4.1}$$

Note if $\psi \in \mathcal{S}_p$, then

$$T_\psi\#\phi = \psi\#\phi, \quad \phi \in \mathcal{S}_p,$$

when T_ψ is given by (2.3).

Bloom and Xu proved in [**12**, Theorem 5.17] that $T\#\phi \in C^1(0,\infty)$, for every $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$. Also, they established that if $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$, then $T\#\phi \in \mathcal{S}_p'$ and

$$\langle T\#\phi, \psi\rangle = \langle T, \phi\#\psi\rangle, \quad \psi \in \mathcal{S}_p. \tag{4.2}$$

From (4.2) it is not hard to see that the interchange formula

$$\mathcal{F}(T\#\phi) = (\mathcal{F}T)(\mathcal{F}\phi) \tag{4.3}$$

holds for every $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$, when the equality is understood in $\mathcal{H}_p'$.

In the following proposition we improve the result obtained in [**12**, Theorem 5.17].

Proposition 4.1. *Let $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$. If $\rho > 0$ and $1 < p \leq 2$, then $T\#\phi \in \Theta$.*

Proof. Assume that $\rho > 0$ and $1 < p \leq 2$. Firstly we prove that $T\#\phi$ is an even and $C^\infty(\mathbf{R})$ function.

Since, for every $x \in (0,\infty)$, the mapping $x \to \varphi_\lambda(x)$ is even, $T\#\phi$ is an even function as well.

By virtue of Lemma 3.1 there exist $r \in \mathbf{N}$ and essentially bounded (with respect to the Lebesgue measure on $(0,\infty)$) functions f_k on $(0,\infty)$, $k = 0,1,\dots,r$, such that

$$\langle T, \psi\rangle = \sum_{k=0}^{r} \int_0^\infty f_k(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k}\psi(y)\,dy, \quad \psi \in \mathcal{S}_p.$$

Then

$$(T\#\phi)(x) = \sum_{k=0}^{r} \int_0^\infty f_k(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} (\tau_x\phi)(y)\, dy,$$

$x \in (0,\infty)$. Hence, we only need to prove that

$$\int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} (\tau_x\phi)(y)\, dy \in C^\infty(0,\infty), \tag{4.4}$$

where f is an essentially bounded function on $(0,\infty)$ and $r,k \in \mathbf{N}$. Then, bearing in mind that $(\tau_x\phi)(y) = \mathcal{F}^{-1}(\varphi_z(x)(\mathcal{F}\phi)(z))(y)$, $x,y \in (0,\infty)$, we have

$$\begin{aligned} &\frac{d^l}{dx^l}\left(\int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} (\tau_x\phi)(y)\, dy\right) \\ &= \int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} \mathcal{F}^{-1}\left(\frac{d^l}{dx^l} \varphi_z(x)\big(\mathcal{F}\phi\big)(z)\right)(y)\, dy, \end{aligned} \tag{4.5}$$

for $l \in \mathbf{N}$ and all $x \in (0,\infty)$. The differentiation under the integral sign is justified because, according to Proposition 3.3, $(d^l/dx^l)\varphi_z(x)$ is a multiplier of $\mathcal{H}_p$, for every $x \in (0,\infty)$ and $l \in \mathbf{N}$.

To finish the proof we will show that

$$\int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} (\tau_x\phi)(y)\, dy \in \Theta.$$

In accordance with Proposition 3.3 the expression $(d^l/dx^l)\varphi_x(z)$ is a multiplier of the space $\mathcal{H}_p$, for every $l \in \mathbf{N}$. Hence, by taking into account again that

$$(\tau_x\phi)(y) = \mathcal{F}^{-1}\Big(\varphi_z(x)\big(\mathcal{F}\phi\big)(z)\Big)(y),$$

for every $x,y \in (0,\infty)$, we find

$$\left|\frac{d^l}{dx^l}\left(\int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k} (\tau_x\phi)(y)\, dy\right)\right|$$

$$\leq C\ nt_0^\infty |f(y)|(1+y)^r \varphi_0(y)^{-2/p} \left| \frac{d^k}{dy^k} \mathcal{F}^{-1}\bigg(\Big(\frac{d^l}{dx^l}\varphi_x(z)\Big)(\mathcal{F}\phi)(z)\bigg)(y)\right| dy$$
$$\leq \text{ess sup}\,|f| \int_0^\infty \frac{dy}{(1+y)^2}\, \mu^p_{k,r+2}\bigg(\mathcal{F}^{-1}\Big(\frac{d^l}{dx^l}\varphi_z(x)\big(\mathcal{F}\phi\big)(z)\Big)\bigg),$$

for every $x \in (0,\infty)$ and $l \in \mathbf{N}$.

From [**12**, Theorem 4.27] and by Proposition 3.3, we get for any $x \in (0,\infty)$ and $l \in \mathbf{N}$

$$(4.6)\quad \left|\frac{d^l}{dx^l}\bigg(\int_0^\infty f(y)(1+y)^r \varphi_0(y)^{-2/p} \frac{d^k}{dy^k}(\tau_x\phi)(y)\,dy\bigg)\right| \leq C(1+x).$$

Consequently, $\int_0^\infty f(y)(1+y)^r\varphi_0(y)^{-2/p}(d^k/dy^k)(\tau_x\phi)(y)\,dy \in \Theta$. □

The next result is similar to the above one. Now the operator $\triangle$ replaces the derivative d/dx.

Proposition 4.2. *Let $T \in \mathcal{S}'_p$. Then there exists $l \in \mathbf{N}$ such that for every $\phi \in \mathcal{S}_p$ and $k \in \mathbf{N}$, we have* $\sup_{x\in(0,\infty)}(1+x)^{-l}\big|\triangle^k(T\#\phi)(x)\big| < \infty$, *provided that* $0 < p \leq 2$ *and* $\rho \geq 0$.

Proof. As in the proof of Proposition 4.1, according to Lemma 3.1, it is sufficient to prove the property when $T \in \mathcal{S}'_p$ is given by

$$\langle T,\psi\rangle = \int_0^\infty f(y)(1+y)^r\varphi_0(y)^{-2/p}\frac{d^k}{dy^k}\psi(y)\,dy, \quad \psi \in \mathcal{S}_p,$$

where f is an essentially bounded, with respect to the Lebesgue measure on $(0,\infty)$, function on $(0,\infty)$, and $r, k \in \mathbf{N}$.

Then, for every $\phi \in \mathcal{S}_p$ and $x \in (0,\infty)$,we obtain

$$(T\#\phi)(x) = \int_0^\infty f(y)(1+y)^r\varphi_0(y)^{-2/p}\frac{d^k}{dy^k}(\tau_x\phi)(y)\,dy.$$

By virtue of [**12**, Lemma 3.11], see also Proposition 3.3, the expression $(d^j/dx^j)\varphi_\lambda(x)$ is a multiplier of the space $\mathcal{H}_p$, for $j = 0, 1, 2$, provided that $1 < p \le 2$ and $\rho > 0$ or $0 < p \le 2$ and $\rho = 0$. Consequently, since $\triangle_x(\tau_x\phi)(y) = \tau_x(\triangle\phi)(y)$, $x, y \in (0,\infty)$, we have

$$\begin{aligned}\triangle_x(T\#\phi)(x) &= -\bigg(\frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\,\frac{d}{dx}\bigg)(T\#\phi)(x)\\ &= \int_0^\infty f(y)(1+y)^r\varphi_0(y)^{-2/p}\,\frac{d^k}{dy^k}\,\tau_x(\triangle\phi)(y)\,dy.\end{aligned}$$

for every $x \in (0,\infty)$.

Since $\triangle\phi \in \mathcal{S}_p$, we conclude that

$$\triangle_x^s(T\#\phi)(x) = \int_0^\infty f(y)(1+y)^r\varphi_0(y)^{-2/p}\,\frac{d^k}{dy^k}\,\tau_x(\triangle^s\phi)(y)\,dy,$$

where $x \in (0,\infty)$ and $s \in \mathbf{N}$.

By taking into account [**12**, Theorem 4.27, Lemma 3.4, (iv)], for every $s \in \mathbf{N}$, we can find $l \in \mathbf{N}$ and $C > 0$ for which

$$\begin{aligned}\big|\triangle_x^s(T\#\phi)(x)\big| &\le \text{ess sup}\,|f|\int_0^\infty \frac{dy}{(1+y)^2}\,\mu^p_{k,r+2}(\tau_x\triangle^s\phi)\\ &\le C(1+x)^l,\end{aligned}$$

for all $x \in (0,\infty)$. Thus the proof is finished. □

If $T \in \mathcal{S}_p'$ and $\phi \in \mathcal{S}_p$ we can not assure, in general, that $T\#\phi \in \mathcal{S}_p$. Indeed, if we consider the functional T defined on $\mathcal{S}_p$ by

$$\langle T, \phi\rangle = \int_0^\infty \varphi_0(y)\phi(y)A(y)\,dy, \quad \phi \in \mathcal{S}_p,$$

and we recall that $A(y) \le A(1)y^\beta e^{2\rho y}$, when x is large enough [**12**, (3.5)], then for certain $\beta > 0$, by resorting to [**12**, Lemma 3.4 (iii)], we get

$$\begin{aligned}&\bigg|\int_0^\infty e^{-\frac{2\rho}{p}\,y}(1+y)^{-l+2/p}\varphi_0(y)A(y)\,dy\bigg|\\ &\qquad\le C\int_0^\infty e^{-(2/p-1)\rho y}(1+y)^{-l+2/p+\beta+1}\,dy < \infty,\end{aligned}$$

provided that $l > 2/p + \beta + 2$, with $C > 0$. Accordingly, $T \in \mathcal{S}_p'$.

On the other hand, we can write, for every $\phi \in \mathcal{S}_p$ and $x \in (0, \infty)$,

$$\begin{aligned}(T\#\phi)(x) &= \int_0^\infty \varphi_0(y)(\tau_x\phi)(y)A(y)\,dy \\ &= \mathcal{F}(\tau_x\phi)(0) = \varphi_0(x)\mathcal{F}(\phi)(0) \\ &= \varphi_0(x)\int_0^\infty \varphi_0(y)\phi(y)A(y)\,dy.\end{aligned}$$

Hence, if $\phi \in \mathcal{S}_p$, $\phi \not\equiv 0$ and $\phi \geq 0$, then

$$\varphi_0(x)^{-2/p}|(T\#\phi)(x)| = \varphi_0(x)^{1-2/p}\int_0^\infty \varphi_0(y)\phi(y)A(y)\,dy \not\longrightarrow 0,$$

as $x \to \infty$. Thus, we have proved that $T\#\phi \notin \mathcal{S}_p$.

Our next objective is to describe some elements of $\mathcal{S}_p'$ that define convolution operators on $\mathcal{S}_p$. We are motivated by the classical results about convolution operators on the Schwartz space $\mathcal{S}$ [**25**] and by the studies about Hankel convolution in distribution spaces presented in [**10, 23**].

As a consequence of [**4**, Proposition 2.2] we can complete the results established in Propositions 4.1 and 4.2.

Proposition 4.3. *Let $T \in \mathcal{S}_p'$ and $\rho > 0$. Then, for every $\phi \in \mathcal{S}_p$, one has that $T\#\phi \in \mathcal{A}_{-2}$.*

Proof. By Proposition 4.1, $T\#\phi \in C^\infty(\mathbf{R})$ and it is an even function, for every $\phi \in \mathcal{S}_p$. Moreover, according to [**4**, Proposition 2.2], the inequality (4.6) implies the desired result. $\square$

As we will immediately see, the elements of $\mathcal{A}'$, the dual space of $\mathcal{A}$, give rise to convolution operators on $\mathcal{S}_p$. With this aim, we previously establish some characterizations of the functionals in $\mathcal{A}'$.

Proposition 4.4. *Let $T \in \mathcal{S}_p'$. We list the following statements concerning T:*

(i) $T \in \mathcal{A}'$.

(ii) $\mathcal{F}T$ *is a pointwise multiplier of* $\mathcal{H}_p$.

(iii) *For every* $m \in \mathbf{N}$, *there exist* $l \in \mathbf{N}$ *and continuous functions* f_j *on* $(0,\infty)$, $j = 0, 1, \dots, l$, *such that*

$$T = \sum_{j=0}^{l} \triangle^j f_j \tag{4.7}$$

and, for every $j \in \mathbf{N}$, *the function* $(1+x)^m \varphi_0(x)^{-2/p} f_j(x)$ *is bounded on* $(0,\infty)$.

Then, (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii), *whenever* $\rho = 0$ *and* $0 < p \le 2$ *or* $\rho > 0$ *and* $1 \le p \le 2$. *Finally, under the assumptions* $\rho = 0$ *and* $0 < p \le 2$ *or* $\rho > 0$ *and* $0 < p \le 1$, *we have* (iii) $\Rightarrow$ (i) *and* (iii) $\Rightarrow$ (ii).

Proof. Suppose that $\rho = 0$ and $0 < p \le 2$ or $\rho > 0$ and $1 \le p \le 2$.

(i) $\Rightarrow$ (ii). Assume that $T \in \mathcal{A}'$. Then $T \in \mathcal{A}'_m$, for every $m \in \mathbf{Z}$. In the sequel, the restriction $m < -3$ is imposed. There exist $r \in \mathbf{N}$ and essentially bounded functions f_k on $(0,\infty)$, $k = 0, 1, \dots, r$, for which

$$\langle T, \phi \rangle = \sum_{k=0}^{r} \int_0^\infty f_k(y)(1+y)^{m+2} \triangle^k \phi(y)\, dy, \quad \phi \in \mathcal{S}_p.$$

Indeed, there exist $C > 0$ and $r \in \mathbf{N}$ for which

$$|\langle T, \phi \rangle| \le C \max_{0 \le k \le r} \sup_{x \in [0,\infty)} (1+x)^m |\triangle^k \phi(x)|, \tag{4.8}$$

where $\phi \in \mathcal{A}_m$.

Let $\phi \in \mathcal{S}_p$. We can write, for every $k \in \mathbf{N}$ and $x \in (0,\infty)$,

$$(1+x)^m \triangle^k \phi(x) = -\int_x^\infty \frac{d}{dt}\big((1+t)^m \triangle^k \phi(t)\big)\, dt, \tag{4.9}$$

and

$$\frac{d}{dx}\phi(x) = -\frac{1}{A(x)} \int_0^x A(t) \triangle \phi(t)\, dt, \quad x \in (0,\infty). \tag{4.10}$$

By combining (4.9) and (4.10), for every $x \in (0,\infty)$ and $k \in \mathbf{N}$, we obtain

$$\begin{aligned}
&|(1+x)^m \triangle^k \phi(x)| \\
&\leq \int_x^\infty \left(m(1+t)^{m-1} |\triangle^k \phi(t)| + (1+t)^m \left| \frac{d}{dt} \triangle^k \phi(t) \right| \right) dt \\
&\leq m \int_0^\infty (1+t)^{m-1} |\triangle^k \phi(t)| \, dt \\
&\quad + \int_x^\infty (1+t)^m \frac{1}{A(t)} \int_0^t A(z) |\triangle^{k+1} \phi(z)| \, dz \, dt \\
&\leq C \bigg(\int_0^\infty (1+t)^{m-1} |\triangle^k \phi(t)| \, dt + \int_0^\infty (1+t)^{m+2} |\triangle^{k+1} \phi(t)| \, dt \bigg).
\end{aligned}$$

Then, from (4.8) it follows

$$|\langle T, \phi \rangle| \leq C \max_{0 \leq k \leq r+1} \int_0^\infty (1+t)^{m+2} |\triangle^k \phi(t)| \, dt.$$

Hahn-Banach theorem, by using duality arguments, allows us to obtain the desired representation for T. Hence, for all $\phi \in \mathcal{S}_p$, Fubini's theorem leads to

$$\begin{aligned}
\langle \mathcal{F}T, \mathcal{F}\phi \rangle &= \sum_{k=0}^r \int_0^\infty f_k(y)(1+y)^{m+2} \triangle^k \mathcal{F}^{-1}(\mathcal{F}\phi)(y) \, dy \\
&= \sum_{k=0}^r \int_0^\infty f_k(y)(1+y)^{m+2} \mathcal{F}^{-1}\Big((\lambda^2+\rho^2)^k \mathcal{F}(\phi)(\lambda) \Big)(y) \, dy \\
&= \sum_{k=0}^r \int_0^\infty (\lambda^2+\rho^2)^k \mathcal{F}(\phi)(\lambda) \\
&\qquad\qquad \times \int_0^\infty f_k(y)(1+y)^{m+2} \varphi_\lambda(y) \, dy \, \frac{d\lambda}{|c(\lambda)|^2}.
\end{aligned}$$

Thus, we get

(4.11)
$$(\mathcal{F}T)(\lambda) = \sum_{k=0}^r (\lambda^2+\rho^2)^k \int_0^\infty f_k(y)(1+y)^{m+2} \varphi_\lambda(y) \, dy, \quad |\mathrm{Im}\, \lambda| \leq \rho.$$

Then $\mathcal{F}T$ is an even function.

Let $l \in \mathbf{N}$. If we make use of the representation (4.11) for $\mathcal{F}T$ associated to $m \in \mathbf{Z}$ satisfying that $m + l < -4$ and bear in mind [**12**, Lemma 3.4 (iv)], we find

$$\begin{aligned}\left|\frac{d^l}{d\lambda^l}(\mathcal{F}T)(\lambda)\right| &\le C\sum_{k=0}^{r}\sum_{j=0}^{l}\left|\frac{d^{l-j}}{d\lambda^{l-j}}(\lambda^2+\rho^2)^k\right| \\ &\qquad\qquad \times \int_0^\infty |f_k(y)|(1+y)^{m+l+3}e^{(|\mathrm{Im}\,\lambda|-\rho)y}\,dy \\ &\le C\sum_{k=0}^{r}\sum_{j=0}^{l}(1+|\lambda|^2)^k \operatorname{ess\,sup}|f_k|\int_0^\infty (1+y)^{m+l+3}\,dy,\end{aligned}$$

for every λ such that $|\mathrm{Im}\,\lambda| \le \rho$. Hence, according to Proposition 3.2, $\mathcal{F}T$ is a pointwise multiplier of $\mathcal{H}_p$.

(ii) $\Rightarrow$ (iii). Let $m \in \mathbf{N}$. Set $F = \mathcal{F}T$, where $T \in \mathcal{S}_p'$, and assume that F is a multiplier of the space $\mathcal{H}_p$. Then, resorting again to Proposition 3.2, for every $k \in \mathbf{N}$ exists $n_k \in \mathbf{N}$ for which

$$\sup_{|\mathrm{Im}\,\lambda|\le\rho((2/p)-1)}(1+|\lambda|)^{-n_k}\left|\frac{d^k}{d\lambda^k}F(\lambda)\right| < \infty.$$

We denote by l a nonnegative integer that will be later specified. Now we consider the function

$$G(\lambda) = \Big((\rho+1)^2+\lambda^2\Big)^{-l}F(\lambda), \quad |\mathrm{Im}\,\lambda| \le \rho\bigg(\frac{2}{p}-1\bigg).$$

Thus, G is holomorphic and even in $|\mathrm{Im}\,\lambda| < \rho((2/p)-1)$. Moreover, $(d^k/d\lambda^k)G(\lambda)$ can be continuously extended to $|\mathrm{Im}\,\lambda| \le \rho((2/p)-1)$, for every $k \in \mathbf{N}$.

According to [**30**, p. 99], $|c(\lambda)|^{-2} \sim |\lambda|^{2\alpha+1}$, for large $|\lambda|$, provided that $\alpha > -1/2$. Hence, if $2l > n_0 + 2\alpha + 2$, then

$$\int_0^\infty |G(\lambda)|\,|\varphi_\lambda(x)|\,|c(\lambda)|^{-2}d\lambda < \infty,$$

$x \in (0, \infty)$. Moreover, by applying Fubini's theorem we are led to

$$\begin{aligned}\int_0^\infty \phi(x)\mathcal{F}^{-1}(G)(x)A(x)\,dx \\ &= \int_0^\infty \phi(x)\int_0^\infty G(\lambda)\varphi_\lambda(x)\,\frac{d\lambda}{|c(\lambda)|^2}\,A(x)\,dx \\ &= \int_0^\infty G(\lambda)\int_0^\infty \phi(x)\varphi_\lambda(x)A(x)\,dx\,\frac{d\lambda}{|c(\lambda)|^2} \\ &= \int_0^\infty G(\lambda)\mathcal{F}(\phi)(\lambda)\frac{d\lambda}{|c(\lambda)|^2},\end{aligned}$$

for all $\phi \in \mathcal{S}_p$. In other words, we have seen that the inverse Fourier transform $\mathcal{F}^{-1}(G)$ of G as an element of the space $\mathcal{H}'_p$ coincides with the classical inverse Fourier transform of G.

Furthermore, for certain $c_{j,l} \in \mathbf{R}$, $j = 0, 1, \dots, l$, we get

$$T = \mathcal{F}^{-1}\Big\{\Big((\rho+1)^2+\lambda^2\Big)^l G(\lambda)\Big\} = \sum_{j=0}^{l} c_{j,l}\triangle^j\mathcal{F}^{-1}(G) = \sum_{j=0}^{l}\triangle^j f_j,$$

where $f_j = c_{j,l}\mathcal{F}^{-1}G$.

We now prove that $(1+x)^m\varphi_0^{-2/p}(x)\mathcal{F}^{-1}(G)(x)$ is bounded on $(0,\infty)$ provided that l is chosen large enough.

Set $H = \mathcal{F}^{-1}(G)$ and define $g = \mathcal{F}_0^{-1}(G)$, where $\mathcal{F}_0$ denotes the Euclidean Fourier transform

$$\mathcal{F}_0(g)(y) = G(y) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ixy}g(x)\,dx,$$

its inverse being supplied by

$$\mathcal{F}_0^{-1}(G)(x) = g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{ixy}G(y)\,dy.$$

Note that if l is large enough, then the function g is absolutely integrable on $\mathbf{R}$. So $H = \mathcal{F}^{-1}(\mathcal{F}_0 g)$.

Now we introduce a sequence $\{f_n\}_{n\in\mathbf{N}}$ of even and smooth functions on $(0,\infty)$ such that $f_n(x) = 1, |x| \le n$ and $f_n(x) = 0, |x| \ge n+1$, for every $n \in \mathbf{N}$. Assume also that, for every $k \in \mathbf{N}$, there exists $C_k > 0$ satisfying that

$$\left| \frac{d^k}{dx^k} f_n(x) \right| \leq C_k,$$

for $n \in \mathbf{N}$ and $x \in [0, \infty)$.

It is not difficult to find a sequence $\{f_n\}_{n \in \mathbf{N}}$ verifying all the above properties.

We consider the decomposition

$$g = f_n g + (1 - f_n) g, \quad n \in \mathbf{N}.$$

We write $g_n = (1 - f_n) g$ and define $G_n = \mathcal{F}_0(g_n)$ and $H_n = \mathcal{F}^{-1}(G_n)$, for each $n \in \mathbf{N}$. Note that $f_n g = 0$ outside $[-n-1, n+1]$. Hence, according to [**29**], $\mathcal{F}^{-1}\mathcal{F}_0(f_n g) = 0$, outside $[-n-1, n+1]$. Then $H_n = H$ outside $[-n-1, n+1]$. We have taken into account the inversion formula for the Fourier transformation.

Let $0 \leq j \leq m$. Since $|c(\lambda)|^{-2} \sim |\lambda|^{2\alpha+1}$, for large $|\lambda|$ and $\alpha > -1/2$, one has

$$\sup_{x \in [0,1]} \varphi_0^{-2/p}(x) x^j \left| \int_0^\infty G(\lambda) \varphi_\lambda(x) \, \frac{d\lambda}{|c(\lambda)|^2} \right| \leq C, \tag{4.12}$$

whenever $2l > n_0 + 2\alpha + 2$.

Moreover, we can write

$$\begin{aligned} \sup_{x \in [n+1, n+2)} \varphi_0^{-2/p}(x) x^j & \left| \int_0^\infty G(\lambda) \varphi_\lambda(x) \, \frac{d\lambda}{|c(\lambda)|^2} \right| \\ & \leq C n^{j+2} e^{((2/p)-1)\rho n} \sup_{\lambda \in (0,\infty)} \left| (1+\lambda^2)^r G_n(\lambda) \right|, \quad n \in \mathbf{N}, \end{aligned} \tag{4.13}$$

where $r > \alpha + 1$.

Suppose that we can choose $l, r \in \mathbf{N}$ such that $l - n_0/2 - \alpha - 1 > r > \alpha + 1$. Then, for every $n \in \mathbf{N}$, well-known operational rules for the Fourier transform $\mathcal{F}_0$ lead to

$$\begin{aligned} (1+\lambda^2)^r G_n(\lambda) &= \frac{1}{2} \int_{-\infty}^{\infty} g_n(t) \left(1 - \frac{d^2}{dt^2} \right)^r e^{-it\lambda} \, dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \left(1 - \frac{d^2}{dt^2} \right)^r g_n(t) e^{-it\lambda} \, dt, \quad \lambda \in (0, \infty). \end{aligned}$$

The last equality can be established by partial integration. Indeed, let $n \in \mathbf{N}$. For every $s \in \mathbf{N}$, $0 \leq s \leq 2r$, since $l > n_0/2 + \alpha + 1 + r$, we can write

$$\frac{d^s}{dt^s} g_n(t) = \frac{d^s}{dt^s} g(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} G(y)(yi)^s e^{tyi}\, dy, \quad t \geq n+1.$$

Then, the Riemann-Lebesgue lemma implies that $\lim_{|t|\to\infty}(d^j/dt^j)g_n(t) = 0$, $j \in \mathbf{N}$, $0 \leq j \leq 2r$. Consequently, for every $n \in \mathbf{N}$, we get

$$\sup_{\lambda\in(0,\infty)} \left|(1+\lambda^2)^r G_n(\lambda)\right| \leq C \sum_{s=0}^{r} \sup_{t\in(0,\infty)} (1+t)^2 \left|\frac{d^{2s}}{dt^{2s}} g_n(t)\right|. \tag{4.14}$$

Here C is not dependent on $n \in \mathbf{N}$.

By the properties of the functions f_n, $n \in \mathbf{N}$, mentioned above, from (4.14), we arrive at

$$\sup_{\lambda\in(0,\infty)} \left|(1+\lambda^2)^r G_n(\lambda)\right| \leq C \sum_{s=0}^{2r} \sup_{t\geq n}(1+t)^2 \left|\frac{d^s}{dt^s} g(t)\right|, \quad n \in \mathbf{N}.$$

In this way we obtain

$$n^{j+2} e^{((2/p)-1)\rho n} \sup_{\lambda\in(0,\infty)} \left|(1+\lambda^2)^r G_n(\lambda)\right| \\ \leq C \sum_{s=0}^{2r} \sup_{t\geq n}(1+t)^{j+3} \left|\frac{d^s}{dt^s} g(t)\right| e^{((2/p)-1)\rho t}.$$

Now, taking into account (4.13), we conclude

$$\sup_{x\in[1,\infty)} \varphi_0^{-2/p}(x) x^j \left|\int_0^\infty G(\lambda)\varphi_\lambda(x) \frac{d\lambda}{|c(\lambda)|^2}\right| \\ \leq C \sum_{s=0}^{2r} \sup_{t\in(0,\infty)} (1+t)^{j+3} \left|\frac{d^s}{dt^s} g(t)\right| e^{((2/p)-1)\rho t}. \tag{4.15}$$

Let $s \in \mathbf{N}$, $0 \leq s \leq 2r$. By using again operational rules for the Euclidean Fourier transformation, it follows

$$(1+t)^{j+3} \frac{d^s}{dt^s} \int_{-\infty}^{\infty} e^{ity} G(y)\, dy \\ = (1+t)^{j+3} \int_{-\infty}^{\infty} e^{ity} (iy)^s G(y) dy \\ = \int_{-\infty}^{\infty} e^{ity} \left(1 - \frac{1}{i}\frac{d}{dy}\right)^{j+3} \Big((iy)^s G(y)\Big)\, dy, \quad t \in [0,\infty),$$

provided that l is large enough and $t \in (0,\infty)$. If, in addition, $\rho > 0$ and $1 \le p \le 2$, by invoking Cauchy integral formula, we obtain

$$\int_{-\infty}^{\infty} e^{ity}\Big(1 - \frac{1}{i}\,\frac{d}{dy}\Big)^{j+3}\Big((iy)^s G(y)\Big)\,dy$$
$$= \int_{-\infty+i((2/p)-1)\rho}^{\infty+i((2/p)-1)\rho} e^{ity}\Big(1 - \frac{1}{i}\,\frac{d}{dy}\Big)^{j+3}\Big((iy)^s G(y)\Big)\,dy, \quad t \in (0,\infty),$$

when l is sufficiently large.

Hence, we conclude that

$$\sup_{t\in(0,\infty)} (1+t)^{j+3} e^{((2/p)-1)\rho t}\left|\frac{d^s}{dt^s}\,g(t)\right| < \infty.$$

By combining the above estimations we deduce that

$$\sup_{x\in(0,\infty)} (1+x)^m \varphi_0^{-2/p}(x)\Big|\mathcal{F}^{-1}(G)(x)\Big| < \infty,$$

when l is chosen conveniently large. Thus, (iii) is established.

Assume now $\rho = 0$ and $0 < p \le 2$ or $\rho > 0$ and $0 < p \le 1$.

(iii) $\Rightarrow$ (i). Let $m \in \mathbf{N}$. Suppose that $T \in \mathcal{S}'_p$ and T admits the representation (4.7), where f_j is continuous on $(0,\infty)$ and $(1+x)^m\varphi_0^{-2/p}(x)f_j(x)$ is bounded on $(0,\infty)$, $j = 0, 1, \dots, l$. Then, we can write

$$\langle T, \phi\rangle = \sum_{j=0}^{l}\int_0^\infty f_j(x)\triangle^j\phi(x)A(x)\,dx, \quad \phi \in \mathcal{S}_p.$$

Note that, for every $\phi \in \mathcal{S}_p$, due to [**12**, (3.5)], we can infer

$$|\langle T,\phi\rangle| \le C\sum_{j=0}^{l}\int_0^\infty |f_j(x)|(1+x)^\beta e^{2\rho x}\big|\triangle^j\phi(x)\big|\,dx$$
$$\le C\sum_{j=0}^{l}\sup_{x\in(0,\infty)}(1+x)^m|f_j(x)|\varphi_0(x)^{-2/p}\sup_{x\in(0,\infty)}(1+x)^{\gamma-m}\big|\triangle^j\phi(x)\big|,$$

for certain $C, \gamma > 0$.

Therefore, for every $k \in \mathbf{Z}$, $k < 0$, by choosing $m \in \mathbf{N}$ for which $k > \gamma - m$, we get, the functional T is continuous on $\mathcal{S}_p$, when we consider on $\mathcal{S}_p$ the topology of $\mathcal{A}_k$. Under these hypotheses, T can be extended to $\mathcal{A}_k$ as an element of $\mathcal{A}'_k$. Moreover, we have

$$\langle T, \phi \rangle = \sum_{j=0}^{l} \int_0^\infty f_j(x) \triangle^j \phi(x) A(x)\, dx, \quad \phi \in \mathcal{A}_{k,p},$$

where f_j, $j = 0, 1, \dots, l$, are associated to $m \in \mathbf{N}$ chosen as above.

Thus, we conclude that $T \in \mathcal{A}'_k$, for every $k \in \mathbf{Z}$, $k < 0$. Definitively, $T \in \mathcal{A}'$.

(iii) $\Rightarrow$ (ii). Assume that f is a continuous function on $(0, \infty)$ such that $(1+x)^m \varphi_0(x)^{-2/p} f(x)$ is bounded on $(0, \infty)$, where $m \in \mathbf{N}$ is such that $m > 1 + \beta + (4/p)$, β being given as in [**12**, (3.5)]. Then $f \in L^1(m)$. Indeed, according to [**12**, Lemma 3.4 and (3.5)], we can write

$$\int_0^\infty |f(x)| A(x)\, dx \leq \int_0^\infty (1+x)^{-m+\beta+4/p} e^{2\rho x(1-2/p)}\, dx < \infty.$$

Hence, the generalized Fourier transform $\mathcal{F}f$ of f is bounded on the strip $\{\lambda : |\mathrm{Im}\,\lambda| \leq \rho\}$. Moreover f defines an element T_f of $\mathcal{S}'_p$ by

$$\langle T_f, \phi \rangle = \int_0^\infty f(x) \phi(x) A(x)\, dx, \quad \phi \in \mathcal{S}_p.$$

By invoking the Fubini theorem, we get that

$$\langle \mathcal{F}T_f, \mathcal{F}\phi \rangle = \langle T_f, \phi \rangle, \quad \phi \in \mathcal{S}_p.$$

Let $k \in \mathbf{N}$. Suppose now that T admits the representation (4.7) where, for every $j \in \mathbf{N}$, the function $(1+x)^m \varphi_0(x)^{-2/p} f_j(x)$ is continuous and bounded on $(0, \infty)$, with $m \in \mathbf{N}$ and $m > 2 + \beta + k + (2/p)$. Then, we have

$$\mathcal{F}T = \sum_{j=0}^{l} (\rho^2 + \lambda^2)^j \mathcal{F} f_j.$$

Hence, by differentiation under the integral sign and by [**12**, Lemma 3.4 (ii), (iv)], one has

$$\begin{aligned}\left|\frac{d^k}{d\lambda^k}\,(\mathcal{F}T)(\lambda)\right| &\le C\sum_{s=0}^{k}\sum_{j=0}^{l}\left|\frac{d^{k-s}}{d\lambda^{k-s}}\,(\rho^2+\lambda^2)^j\right|\left|\frac{d^s}{d\lambda^s}\,(\mathcal{F}f_j)(\lambda)\right| \\ &\le C\sum_{s=0}^{k}\sum_{j=0}^{l}\left|\frac{d^{k-s}}{d\lambda^{k-s}}\,(\rho^2+\lambda^2)^j\right| \\ &\qquad\times\int_0^\infty (1+x)^{\beta+s+1-m+2/p}\,dx, \quad |\mathrm{Im}\,\lambda|\le\rho.\end{aligned}$$

Then, Proposition 3.2 implies that $\mathcal{F}T$ is a multiplier of $\mathcal{H}_p$. □

Remark 4.1. It is not hard to see that the hypothesis $(1+x)^m\times\varphi_0(x)^{-2/p}f_j(x)$ is bounded on $(0,\infty)$ could be replaced by $(1+x)^m\times\varphi_0(x)^{-2/p}f_j(x)$ is in the Lebesgue space $L_q(0,\infty)$, for each $j\in\mathbf{N}$ and $1\le q<\infty$.

As a consequence of Proposition 4.4, we can describe some elements of $\mathcal{S}_p'$ that define convolution operators on $\mathcal{S}_p$.

Proposition 4.5. *Let $0<p\le 2$. Suppose that $T\in\mathcal{S}_p'$ and that $\mathcal{F}T$ is a multiplier of $\mathcal{H}_p$. Then, the mapping $\phi\to T\#\phi$ is continuous from $\mathcal{S}_p$ into itself.*

Proof. Let $T\in\mathcal{S}_p'$ such that $\mathcal{F}T$ is a multiplier of $\mathcal{H}_p$. According to (4.3), we can write

$$\mathcal{F}(T\#\phi)=\mathcal{F}(T)\mathcal{F}(\phi),\quad \phi\in\mathcal{S}_p',$$

in the sense of the equality in $\mathcal{H}_p'$. That is, since $\mathcal{F}T$ is a multiplier of $\mathcal{H}_p$, for every $\phi,\psi\in\mathcal{S}_p$, we obtain

$$\begin{aligned}\langle T\#\phi,\psi\rangle &= \langle\mathcal{F}(T\#\phi),\mathcal{F}\psi\rangle \\ &= \langle\mathcal{F}(T)\mathcal{F}(\phi),\mathcal{F}\psi\rangle = \langle\mathcal{F}^{-1}\Big(\mathcal{F}(T)\mathcal{F}(\phi)\Big),\psi\rangle.\end{aligned}$$

Hence, [**12**, Theorem 4.7] implies that, for every $\phi\in\mathcal{S}_p$, $T\#\phi=\mathcal{F}^{-1}(\mathcal{F}(T)\mathcal{F}(\phi))\in\mathcal{S}_p$ and the mapping $\phi\to T\#\phi$ is continuous from the latter space into itself. □

To simplify in the sequel we will write $T \in \mathcal{M}_p$, $0 < p \leq 2$, to say that $T \in \mathcal{S}_p'$ and $\mathcal{F}T$ is a multiplier of $\mathcal{H}_p$. According to Proposition 4.4, if one of the following two conditions

(i) $T \in \mathcal{A}'$, and either $\rho = 0$ and $0 < p \leq 2$ or $\rho > 0$ and $1 \leq p \leq 2$,

(ii) $T \in \mathcal{S}_p'$ satisfies property (iii) in Proposition 4.4 and $\rho = 0$ and $0 < p \leq 2$ or $\rho > 0$ and $0 < p \leq 1$,

holds, then $T \in \mathcal{M}_p$.

Next, suppose that $T \in \mathcal{S}_p'$ and $L \in \mathcal{M}_p$. Then we can define the convolution $T\#L$ of T and L as the functional on $\mathcal{S}_p$ given by

$$\langle T\#L, \phi\rangle = \langle T, L\#\phi\rangle, \quad \phi \in \mathcal{S}_p. \tag{4.16}$$

Note that $T\#L \in \mathcal{S}_p'$ according to Proposition 4.5.

As was proved by Bloom and Xu [**12**, Theorem 5.17, (iii)], for every $T \in \mathcal{S}_p'$ and $\psi \in \mathcal{S}_p$, one has

$$\langle T\#\psi, \phi\rangle = \langle T, \psi\#\phi\rangle, \quad \phi \in \mathcal{S}_p.$$

Inasmuch as $\mathcal{S}_p$ is contained in $\mathcal{A}'$ and in $\mathcal{M}_p$, definition (4.16) can be seen as an extension of Definition (4.1).

Next we present the main algebraic properties of the #-convolution defined by (4.16)

Proposition 4.6. *Let $0 < p \leq 2$. Assume that $T \in \mathcal{S}_p'$ and L_1, $L_2 \in \mathcal{M}_p$. Then*

(i) $\mathcal{F}(T\#L_1) = (\mathcal{F}T)(\mathcal{F}L_1)$.

(ii) *The Dirac functional $\delta \in \mathcal{A}' \cap \mathcal{M}_p$ and $T\#\delta = T$.*

(iii) $\Delta(T\#L_1) = (\Delta T)\#L_1 = T\#(\Delta L_1)$.

(iv) $L_1\#L_2 \in \mathcal{M}_p$ *and* $L_1\#L_2 = L_2\#L_1$.

(v) $T\#(L_1\#L_2) = (T\#L_1)\#L_2$.

Proof. To see the interchange formula (i) it is sufficient to note that from Propositions 4.2, 4.3 and 4.5 we deduce

$$\begin{aligned}\langle \mathcal{F}(T\#L_1), \mathcal{F}\phi\rangle &= \langle T\#L_1, \phi\rangle = \langle T, L_1\#\phi\rangle \\ &= \langle \mathcal{F}(T), \mathcal{F}(L_1\#\phi)\rangle = \langle (\mathcal{F}T)(\mathcal{F}L_1), \mathcal{F}\phi\rangle,\end{aligned}$$

for all $\phi \in \mathcal{S}_p$. On the other hand, for each $\phi \in \mathcal{A}_m$ and $m \in \mathbf{Z}$, we obtain

$$|\langle \delta, \phi \rangle| = |\phi(0)| \leq \sup_{x \in [0,\infty)} (1+x)^m |\phi(x)|$$

Therefore, $\delta \in \mathcal{A}'$. Furthermore, for every $\phi \in \mathcal{S}_p$, we get

$$\begin{aligned}\langle \mathcal{F}\delta, \mathcal{F}\phi \rangle &= \langle \delta, \phi \rangle = \phi(0)\\ &= \int_0^\infty \varphi_\lambda(0)(\mathcal{F}\phi)(\lambda) \frac{d\lambda}{|c(\lambda)|^2} = \langle \varphi_\lambda(0), (\mathcal{F}\phi)(\lambda) \rangle.\end{aligned}$$

Then, $(\mathcal{F}\delta)(\lambda) = \varphi_\lambda(0) = 1$, $|\operatorname{Im} \lambda| \leq \rho$. Hence $\delta \in \mathcal{M}_p$. Property (i) now allows us to derive (ii).

Equality (iii) is obtained by using (i) and taking into account that $\mathcal{F}(\Delta T) = (\lambda^2 + \rho^2)\mathcal{F}(T)$.

Finally, (iv) and (v) can be deduced from (i). $\square$

Our next objective is to prove a converse of Proposition 4.5. Previously we need to show some results.

We firstly obtain a representation for the fundamental solution of the operator $(1+\Delta)^r$, for every $r \in \mathbf{N}$.

Lemma 4.1. *Let $l \in \mathbf{N}$. The function h_l defined by*

$$h_l(x) = \int_0^\infty \varphi_\lambda(x)(1+\rho^2+\lambda^2)^{-l} \frac{d\lambda}{|c(\lambda)|^2}, \quad x \in \mathbf{R},$$

is even, bounded, and continuous on $\mathbf{R}$ and lies in $C^\infty(\mathbf{R} \setminus \{0\}) \cap \mathcal{S}_p'$, provided that $l > \alpha+1$. Also, for every $k \in \mathbf{N}$, there exists $l_k \in \mathbf{N}$ such that $h_l \in C^k(\mathbf{R})$, provided that $l \geq l_k$.

Moreover, if δ denotes the Dirac functional, then

$$\delta = (1+\Delta)^l h_l,$$

in the sense of equality in $\mathcal{S}_p'$, for each $0 < p \leq 2$, when $l > \alpha + 1$.

Proof. Since $|\varphi_\lambda(x)| \leq 1$, $x, \lambda \in \mathbf{R}$, [**12**, Lemma 3.4, (i)], by taking into account that $|c(\lambda)|^{-2} \sim \lambda^{2\alpha+1}$, when λ is large [**30**], it is not hard to see that h_l is an even, bounded and continuous mapping on $\mathbf{R}$, provided that $l > \alpha + 1$. Moreover, if $k \in \mathbf{N}$, according to [**12**, Lemma 3.6, (ii)], we can find $l_k \in \mathbf{N}$ for which $h_l \in C^k(\mathbf{R})$, when $l \geq l_k$.

On the other hand, $\mathcal{F} = \mathcal{F}_0\mathbf{A}$, where $\mathbf{A}$ represents the Abel transformation defined by [**12**, (4.9)], see also [**29**]. The inverse of $\mathbf{A}$ is obtained in [**29**, Theorem 6.3].

Let $x_0 \in (0,\infty)$. We choose an even function $\alpha \in C^\infty(\mathbf{R})$ such that $\alpha(x) = 0$, $x \in (-x_0/4, x_0/4)$, and $\alpha(x) = 1$, $x \in (x_0/2, \infty)$. We can write

$$h_l(x) = \mathbf{A}^{-1}(\alpha\mathcal{F}_0^{-1}((1+\rho^2+\lambda^2)^{-l}))(x), \quad x > x_0.$$

Partial integration allows us to see that the function $\mathcal{F}_0^{-1}((1+\rho^2 + \lambda^2)^{-l})$ is in $C^\infty(\mathbf{R}\setminus\{0\})$ and that the function $\alpha\mathcal{F}_0^{-1}((1+\rho^2+\lambda^2)^{-l})$ is in the Schwartz space on $\mathbf{R}$. Moreover, the function $\alpha\mathcal{F}_0^{-1}((1+\rho^2 + \lambda^2)^{-l})$ is even.

Hence, according to [**31**, Corollary 6.II.4, (ii)], h_l is smooth in $(x_0/2,\infty)$ and by [**12**, Lemma 3.6, (ii)] $h_l(x) \leq Ce^{-\rho x}(1+x)$, for $x \in (0,\infty)$. Thus we show that $h_l \in C^\infty(\mathbf{R}\setminus\{0\})$ and $h_l \in \mathcal{S}_p'$.

Let now $0 < p \leq 2$ and $\phi \in \mathcal{S}_p$. We can write that

$$\begin{aligned}
&\langle (1+\Delta)^m h_l, \phi\rangle \\
&\quad = \langle h_l, (1+\Delta)^m\phi\rangle \\
&\quad = \int_0^\infty h_l(x)(1+\Delta)^m\phi(x)A(x)\,dx \\
&\quad = \int_0^\infty \int_0^\infty \varphi_\lambda(x)(1+\rho^2+\lambda^2)^{-m}\,\frac{d\lambda}{|c(\lambda)|^2}\,(1+\Delta)^m\phi(x)A(x)\,dx \\
&\quad = \int_0^\infty (1+\rho^2+\lambda^2)^{-m}\int_0^\infty \varphi_\lambda(x)(1+\Delta)^m\phi(x)A(x)\,dx\,\frac{d\lambda}{|c(\lambda)|^2} \\
&\quad = \int_0^\infty \varphi_\lambda(0)\int_0^\infty \varphi_\lambda(x)\phi(x)A(x)\,dx\,\frac{d\lambda}{|c(\lambda)|^2} \\
&\quad = \langle \delta, \phi\rangle.
\end{aligned}$$

Thus, the proof is finished. $\square$

In the following proposition we present families of semi-norms in $\mathcal{D}_a$ defining on $\mathcal{D}_a$ the same topology as $\{\gamma_k\}_{k\in\mathbf{N}}$.

Lemma 4.2. *Let $a > 0$ and $1 \leq q \leq \infty$. For every $k \in \mathbf{N}$, η_k^q is defined on $\mathcal{D}_a$ by*

$$\eta_k^q(\phi) = \|\Delta^k \phi\|_q, \quad \phi \in \mathcal{D}_a,$$

where $\|.\|_q$ denotes the usual norm in the Lebesgue space $L_q(0,\infty)$.

The system $\{\eta_k^q\}_{k\in\mathbf{N}}$ of semi-norms is equivalent to $\{\gamma_k\}_{k\in\mathbf{N}}$ on $\mathcal{D}_a$.

Proof. By taking into account [**12**, Lemma 4.18], we can see that $\{\gamma_k\}_{k\in\mathbf{N}}$ defines on $\mathcal{D}_a$ a topology stronger than the one induced in it by $\{\eta_k^q\}_{k\in\mathbf{N}}$.

On the other hand, for every $k \in \mathbf{N}$, we find

$$(\lambda^2+\rho^2)^k(\mathcal{F}\phi)(\lambda) = \int_0^a \varphi_\lambda(x)\Delta^k\phi(x)A(x)\,dx, \quad \lambda \in \mathbf{C} \quad \text{and} \quad \phi \in \mathcal{D}_a.$$

Then, according to [**12**, Lemma 3.4], we can write

$$|(\lambda^2+\rho^2)^k(\mathcal{F}\phi)(\lambda)| \leq Ce^{a|\operatorname{Im}\lambda|}\|\Delta^k\phi\|_q, \quad \lambda \in \mathbf{C} \quad \text{and} \quad \phi \in \mathcal{D}_a.$$

Hence,

$$\rho_k^a(\mathcal{F}\phi) \leq C\eta_k^q(\phi), \quad \phi \in \mathcal{D}_a.$$

Thus, we have established that the generalized Fourier transformation is continuous from $\mathcal{D}_a$, when we consider on $\mathcal{D}_a$ the topology associated to $\{\eta_k^q\}_{k\in\mathbf{N}}$, into $\mathcal{L}_a$.

By invoking now [**29**, Theorem 7.2] we can conclude that the topology generated by $\{\eta_k^q\}_{k\in\mathbf{N}}$ on $\mathcal{D}_a$ is stronger than the one induced on it by $\{\gamma_k\}_{k\in\mathbf{N}}$. $\square$

We now establish a converse of Proposition 4.5.

Theorem 4.1. *Let $T \in \mathcal{S}'_p$, where $0 < p \leq 2$. If $T\#\phi \in \mathcal{S}_p$, for every $\phi \in \mathcal{D}$, then, for each $m \in \mathbf{N}$ there exist $l \in \mathbf{N}$ and continuous functions f_j, $j = 0, 1, \ldots, l$, such that*

$$T = \sum_{j=0}^{l} \Delta^j f_j,$$

and, for every $j \in \mathbf{N}$, $j = 0, 1, \dots, l$, $(1+x)^m \varphi_0(x)^{-2/p} f_j(x)$ *is bounded on* $(0, \infty)$.

Proof. Assume that $m \in \mathbf{N}$, $m > 2$. Let $\phi \in \mathcal{D}$. Since $T\#\phi \in \mathcal{S}_p$, one has

$$\sup_{x\in(0,\infty)} (1+x)^m \varphi_0(x)^{-2/p} |(T\#\phi)(x)| \\ = \sup_{x\in(0,\infty)} |\langle (1+x)^m \varphi_0(x)^{-2/p} \tau_x T, \phi\rangle| < \infty.$$

Hence the set $\{(1+x)^m \varphi_0(x)^{-2/p} \tau_x T\}_{x\in(0,\infty)}$ is a weakly bounded subset of $\mathcal{D}'$.

Let $a > 0$. By using the Hanh-Banach theorem and duality arguments we can see that there exist $\theta \in \mathbf{N}$ and $C > 0$ such that, for every $x \in (0, \infty)$, there exist $f_{j,x} \in L_\infty(0, \infty)$, $j = 0, 1, \dots, \theta$, for which

(4.17)
$$\langle (1+x)^m \varphi_0(x)^{-2/p} \tau_x T, \phi\rangle = \sum_{j=0}^{\theta} \int_0^\infty f_{j,x}(t) \Delta^j \phi(t)\, dt, \quad \phi \in \mathcal{D}_a,$$

where $\sum_{j=0}^{\theta} \|f_{j,x}\|_\infty \leq C$. Hence, for every $x \in (0, \infty)$,

$$(1+x)^m \varphi_0(x)^{-2/p} \tau_x T$$

can be continuously extended to $\mathcal{D}_a^\theta$. Such an extension is given by (4.17).

Moreover, if $x \in (0, \infty)$ and $S \in (\mathcal{D}_a^\theta)'$ for which

$$S = (1+x)^m \varphi_0(x)^{-2/p} \tau_x T$$

on $\mathcal{D}_a$, then S is given in the space $\mathcal{D}_b^\theta$ by the right-hand side of (4.17), for every $0 < b < a$.

We now choose $k \in \mathbf{N}$ such that the fundamental solution h_k of the operator $(1+\Delta)^k$, obtained in Lemma 4.1, is in $C^{2\theta}(\mathbf{R})$. Moreover, we pick an even function $\varphi \in C^\infty(\mathbf{R})$ such that $\varphi(x) = 0$, $|x| > (3a)/4$, and $\varphi(x) = 1$, $|x| < a/2$. The Leibniz rule leads to

$$(1+\Delta)^k (h_k \varphi) = \varphi (1+\Delta)^k h_k + \beta,$$

where β is in $\mathcal{D}_a$. Note also that, according to Lemma 4.1, we get

$$\langle \varphi(1+\Delta)^k h_k, \phi\rangle = \langle (1+\Delta^k)h_k, \phi\varphi\rangle = \phi(0), \quad \phi \in \mathcal{S}_p.$$

Therefore, we can write

$$\delta = (1+\Delta)^k (h_k\varphi) - \beta. \tag{4.18}$$

The function $h_k\varphi$ belongs to $\mathcal{D}^\theta_{(3a/4)}$. Consequently, $h_k\varphi \in \mathcal{S}'_p$ and the generalized Fourier transform $F(h_k\varphi)$, given by

$$\mathcal{F}(h_k\varphi)(\lambda) = \int_0^a \varphi_\lambda(x) h_k(x)\varphi(x)A(x)\,dx, \quad |\mathrm{Im}\,\lambda| \le \rho(2/p-1),$$

is a multiplier of $\mathcal{H}_p$, or, in other words, $h_k\varphi \in \mathcal{M}_p$. Indeed, for every $s \in \mathbf{N}$, by [**12**, Lemma 3.4, (iv)], we have

(4.19)
$$\left|\frac{d^s}{d\lambda^s}\,\mathcal{F}(h_k\varphi)(\lambda)\right| \le C\int_0^a |h_k(x)|\,|\varphi(x)|A(x)\,dx, \quad |\mathrm{Im}\,\lambda| \le \rho(2/p-1).$$

Hence, we can define the #-convolution $T\#(h_k\varphi)$ of T and $h_k\varphi$, according to (4.16), as follows:

$$\langle T\#(h_k\varphi), \phi\rangle = \langle T, (h_k\varphi)\#\phi\rangle, \quad \phi \in \mathcal{S}_p.$$

Our next objective is to see that

$$\varphi_0(x)^{-2/p}(1+x)^m (T\#(h_k\varphi))(x) = \sum_{j=0}^{\theta}\int_0^\infty f_{j,x}(t)\Delta^j(h_k\varphi)(t)\,dt.$$

Since $h_k\varphi \in \mathcal{D}^\theta_{(3a/4)}$, there exists a sequence $\{\phi_\nu\}_{\nu\in\mathbf{N}} \subset \mathcal{D}_a$ such that $\phi_\nu \to h_k\varphi$, as $\nu \to \infty$, in $\mathcal{D}^\theta_a$. Hence, because $\sup_{x\in(0,\infty)} \|f_{j,x}\| < \infty$, $j = 0, \dots, \theta$, one infers

$$\varphi_0(x)^{-2/p}(1+x)^m (T\#\phi_\nu)(x) \longrightarrow \sum_{j=0}^{\theta}\int_0^\infty f_{j,x}(t)\Delta^j(h_k\varphi)(t)\,dt, \quad \text{as} \quad \nu \to \infty,$$

uniformly in $x \in (0,\infty)$.

Moreover, for every $\phi \in \mathcal{S}_p$, $\phi_\nu \# \phi \to (h_k\varphi)\#\phi$, as $\nu \to \infty$, in $\mathcal{S}_p$. Indeed, let $\phi \in \mathcal{S}_p$. By taking into account [**12**, Lemma 3.4, (iv)] and that $\phi_\nu \to h_k\varphi$, as $\nu \to \infty$, in $\mathcal{D}_a^l$, we can deduce that

$$(\mathcal{F}(\phi_\nu) - \mathcal{F}(h_k\varphi))\mathcal{F}(\phi) \longrightarrow 0, \quad \text{as} \quad \nu \to \infty,$$

in $\mathcal{H}_p$.

Hence, by the interchange formula and by [**12**, Theorem 4.27], we conclude that

$$\phi_\nu \# \phi \longrightarrow (h_k\varphi)\#\phi, \quad \text{as} \quad \nu \to \infty,$$

in $\mathcal{S}_p$.

Thus, we have, for every $\phi \in \mathcal{S}_p$,

$$\langle T\#\phi_\nu, \phi\rangle = \langle T, \phi_\nu \# \phi\rangle \longrightarrow \langle T, (h_k\varphi)\#\phi\rangle = \langle T\#(h_k\varphi), \phi\rangle, \\ \text{as} \quad \nu \to \infty.$$

Therefore, we obtain

(4.20)

$$\varphi_0(x)^{2/p}(1+x)^{-m} \sum_{j=0}^{\theta} \int_0^\infty f_{j,x}(t)\Delta^j(h_k\varphi)(t)\,dt = \Big(T\#(h_k\varphi)\Big)(x),$$

in the sense of equality in $\mathcal{S}_p'$.

Now, from (4.18) one infers that

$$\begin{aligned} T &= (1+\Delta)^k(T\#(h_k\varphi)) - T\#\beta \\ &= \sum_{s=0}^{k} \binom{k}{s} \Delta^s(T\#(h_k\varphi)) - T\#\beta. \end{aligned}$$

Since $\beta \in \mathcal{D}_a$, we have $T\#\beta \in \mathcal{S}_p$, and, in particular,

$$\sup_{x\in(0,\infty)} \varphi_0(x)^{-2/p}(1+x)^m|(T\#\beta)(x)| < \infty.$$

Moreover, by (4.20) it follows that

$$\sup_{x\in(0,\infty)} \varphi_0(x)^{-2/p}(1+x)^m|(T\#(h_k\varphi))(x)| < \infty.$$

Thus, the proof is complete. □

From Propositions 4.4 and 4.5 and Theorem 4.1, we can deduce the following properties.

Corollary 4.1. *Let $T \in S_p'$, where $0 < p \leq 2$ when $\rho = 0$, and $0 < p \leq 1$ when $\rho > 0$. The following assertions are equivalent.*

(i) *The mapping $\phi \to T\#\phi$ is continuous from S_p into itself.*

(ii) *For every $m \in \mathbf{N}$, there exist $l \in \mathbf{N}$ and continuous functions f_j, $j = 0, 1, \ldots, l$, such that*

$$T = \sum_{j=0}^{l} \Delta^j f_j,$$

and, for every $j = 0, 1, \ldots, l$, $(1+x)^m \psi_0^{-2/p}(x) f_j(x)$ is bounded on $(0, \infty)$.

Corollary 4.2. *Let $T \in S_p'$, where $0 < p \leq 2$ when $\rho = 0$, and let $p = 1$ when $\rho > 0$. The following assertions are equivalent.*

(i) *The mapping $\phi \to T\#\phi$ is continuous from S_p into itself.*

(ii) $T \in \mathcal{A}'$.

(iii) *$\mathcal{F}'T$ is a multiplier of $\mathcal{H}_p$.*

(iv) *For every $m \in \mathbf{N}$, there exist $l \in \mathbf{N}$ and continuous functions f_j, $j = 0, 1, \ldots, l$, such that*

$$T = \sum_{j=0}^{l} \Delta^j f_j,$$

and, for every $j = 0, 1, \ldots, l$, $(1+x)^m \psi_0^{-2/p}(x) f_j(x)$ is bounded on $(0, \infty)$.

Remark 4.2. We do not know at this moment if the results in Corollaries 4.1 and 4.2 are optimal in ρ and p. This is an open question.

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