ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 3, Summer 1990

PERIODIC GENERALIZED FUNCTIONS

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ABSTRACT. A class of periodic generalized functions, called periodic Boehmians, is studied. Each periodic Boehmian is the sum of its Fourier series. The class of periodic Boehmians is strictly smaller than the class of periodic Mikusiński operators, and strictly larger than the class of periodic distributions.

1. Introduction. In this paper we shall construct the Boehmians on the unit circle. For a general construction of Boehmians see [**6**].

Generalized functions on the unit circle have been classified by their Fourier coefficients. For example, $\{\alpha_n\}_{-\infty}^{\infty}$ is the sequence of Fourier coefficients of a distribution if the α_n's grow no faster than a polynomial in n [**7**]. $\{\alpha_n\}_{-\infty}^{\infty}$ is the sequence of Fourier coefficients of a hyperfunction if $\overline{\lim}_{|n|\to\infty} |\alpha_n|^{1/|n|} \leq 1$ [**4**]. Any sequence of complex numbers is the sequence of Fourier coefficients of a Mikusiński operator [**3**]. We will show that the coefficients of a periodic Boehmian satisfy a growth condition much like that of a hyperfunction.

§2 is concerned with definitions. Most of the material in §3 and §4 can be found in [**6**] and [**2**], respectively, but is presented here for the convenience of the reader. §3 has results on convergence. §4 gives an example of a periodic Boehmian which is not a distribution. In §5 Fourier coefficients are defined and it is shown that the Fourier coefficients of a periodic Boehmian satisfy a growth condition (Theorem 5.14 and Theorem 5.15). It is not known whether the condition in Theorem 5.14 is necessary and sufficient. Indeed there is a significant gap between the condition in Theorem 5.14 and the condition in Theorem 5.15; it is not even known if each sequence which is $o(e^{o(n)})$ is a sequence of Fourier coefficients for a periodic Boehmian.

1980 *Mathematics subject classification* (1985 *Revision*). Primary 44A40, 46F15, 46F99; Secondary 42A16, 42A24.
Received by the editors on October 1, 1986 and, in revised form, on May 15, 1987.

2. Notation and construction of β. The unit circle will be denoted by T. $C(T)$ will denote the collection of all continuous complex valued functions on T. $C^n(T)$ ($C^n_\infty(T)$) will be the collection of sequences of continuous (infinitely differentiable) complex valued functions on T.

The convolution of f and g in $C(T)$ is denoted by juxtaposition. Thus

$$(fg)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\,dt.$$

If, for $n = 1, 2, \dots, f_n, f \in C(T)$, then $\lim_n f_n = f$ will mean f_n converges uniformly to f on T.

A sequence of continuous real valued functions, $\{\delta_j\}_{j=1}^\infty$, will be called an approximate identity or a delta sequence if the following conditions are satisfied:

(i) for each j, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta_j(t)\,dt = 1$;

(ii) for each j and all t, $\delta_j(t) \geq 0$;

(iii) Given a neighborhood V of 1, there exists a positive integer N such that for all $j \geq N$, the support of δ_j is contained in V.

The collection of delta sequences will be denoted by Δ. The next, well-known theorem gives some indication why an element of Δ is called an approximate identity.

THEOREM 2.1. *Let $f \in C(T)$ and $\{\delta_j\}_{j=1}^\infty \in \Delta$, then $\lim_j f\delta_j = f$.*

DEFINITION 2.2. Let $\mathcal{A} \subset C^n(T) \times \Delta$ be defined by

$$\mathcal{A} = \{(\{f_j\}_{j=1}^\infty, \{\delta_j\}_{j=1}^\infty) : \text{ for each } i \text{ and each } j, f_i\delta_j = f_j\delta_i\}.$$

Two elements $(\{f_j\}_{j=1}^\infty, \{\delta_j\}_{j=1}^\infty)$ and $(\{g_j\}_{j=1}^\infty, \{\sigma_j\}_{j=1}^\infty)$ of $\mathcal{A}$ are said to be equivalent, denoted by

$$(\{f_j\}_{j=1}^\infty, \{\delta_j\}_{j=1}^\infty) \sim (\{g_j\}_{j=1}^\infty, \{\sigma_j\}_{j=1}^\infty),$$

if, for all i and j, $f_i\sigma_j = g_j\delta_i$. A straightforward calculation shows that '$\sim$' is an equivalence relation on $\mathcal{A}$. The equivalence classes will be called periodic Boehmians.

DEFINITION 2.3. The space of periodic Boehmians, denoted by β, is defined as

$$\beta = \left\{ \left[\frac{\{f_j\}_{j=1}^{\infty}}{\{\delta_j\}_{j=1}^{\infty}} \right] : \Big(\{f_j\}_{j=1}^{\infty}, \{\delta_j\}_{j=1}^{\infty} \Big) \in \mathcal{A} \right\}.$$

For convenience a typical element of β will be written as $[f_j/\delta_j]$.

It follows from Theorem 2.1 that if $f, g \in C(T), \{\delta_j\}_{j=1}^{\infty} \in \Delta$ and for each $j, f\delta_j = g\delta_j$, then $f = g$. Thus if $[f\delta_j/\delta_j] = [g\delta_j/\delta_j]$ then $f = g$. So $C(T)$ can be viewed as a subset of β by identifying f with $[f\delta_j/\delta_j]$, where $\{\delta_j\}_{j=1}^{\infty}$ is some fixed delta sequence. Similarly, let $\{\delta_j\}_{j=1}^{\infty}$ be a fixed element of $C_{\infty}^{n}(T) \cap \Delta$. Then $D'(T)$, the class of distributions on the unit circle, can be viewed as a subset of β by identifying u with $[u^*\delta_j/\delta_j]$, where $u^*\delta_j$ denotes the convolution of u and δ_j as distributions (see [**7**]).

By defining a natural addition, multiplication and scalar multiplication, β becomes an algebra.

DEFINITION 2.4. (i) $[f_j/\delta_j] + [g_j/\sigma_j] = [(f_j\sigma_j + g_j\delta_j)/\delta_j\sigma_j]$.

(ii) $[f_j/\delta_j][g_j/\sigma_j] = [f_j g_j/\delta_j\sigma_j]$.

(iii) $\alpha[f_j/\delta_j] = [\alpha f_j/\delta_j]$, where α is a complex number.

Note. It is not difficult to show that if $\{\delta_j\}_{j=1}^{\infty}, \{\sigma_j\}_{j=1}^{\infty} \in \Delta$, then $\{\delta_j\sigma_j\}_{j=1}^{\infty} \in \Delta$.

3. Convergence in β**.** Let $a_n, a \in \beta$ for $n = 1, 2, \ldots,$ as in [**6**]. We say that a_n is δ-convergent to a if there exists a delta sequence $\{\delta_j\}_{j=1}^{\infty}$ such that, for each n and j, $a\delta_j, a_n\delta_j \in C(T)$, and, for each j, $\lim_n a_n\delta_j = a\delta_j$. This will be denoted by $\delta - \lim_n a_n = a$.

We state, without proofs, several lemmas from [**6**].

LEMMA 3.1. *Let* $a_n \in \beta, f_n \in C(T)$ *for* $n = 1, 2, \ldots.$ *If there exists* $\{\delta_n\}_{n=1}^{\infty} \in \Delta$ *such that, for each* n *and* $j, a_n\delta_j \in C(T)$ *and for each* $j, \lim_n a_n\delta_j = f_j$, *then* $\delta - \lim_n a_n = [f_j/\delta_j]$.

LEMMA 3.2. (UNIQUE LIMITS). *Let* $a_n, a, b \in \beta$ *for* $n = 1, 2, \ldots$. *If* $\delta - \lim_n a_n = a$ *and* $\delta - \lim_n a_n = b$, *then* $a = b$.

LEMMA 3.3. *Let* $a \in \beta$ *and* $\{\delta_j\}_{j=1}^{\infty} \in \Delta$. *If, for each* j, $a\delta_j \in C(T)$, *then* $a = [a\delta_j/\delta_j]$.

A more natural way of looking at δ-convergence is

LEMMA 3.4. *Let* $a_n, a \in \beta$ *for* $n = 1, 2, \ldots$. *Then* $\delta - \lim_n a_n = a$ *if and only if there exist representations* $a_n = [f_{j,n}/\delta_j]$ *and* $a = [f_j/\delta_j]$ *where for each* $j, \lim_n f_{j,n} = f_j$.

LEMMA 3.5. *Let* $a_n, b_n, a, b \in \beta$ *for* $n = 1, 2, \ldots$. *If* $\delta - \lim_n a_n = a$, *and* $\delta - \lim_n b_n = b$, *then* $\delta - \lim_n (a_n + b_n) = a + b$.

LEMMA 3.6. *Let* $a_n, a, b \in \beta$ *for* $n = 1, 2, \ldots$. *If* $\delta - \lim_n a_n = a$, *then* $\delta - \lim_n a_n b = ab$.

4. Quasi-analytic classes.

DEFINITION 4.1. Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $M_0 = 1$. Let $\mathbf{I}$ be a closed interval of $\mathbf{R}$. Then

$$C_{\mathbf{I}}\{M_n\} = \{\varphi \in C^{\infty}(\mathbf{R}) : \exists \alpha_\varphi > 0, B_\varphi > 0 \text{ with } \max_{x \in I} |\varphi^{(n)}(x)| \le \alpha_\varphi B_\varphi^n M_n \text{ for } n = 0, 1, 2, \ldots\}.$$

DEFINITION 4.2. A sequence of real numbers $\{M_n\}_{n=0}^{\infty}$ is called logarithmically convex if, for each $n, M_n^2 \le M_{n-1}M_{n+1}$.

DEFINITION 4.3. $C_{\mathbf{I}}\{M_n\}$ is called quasi-analytic if $\varphi \in C_{\mathbf{I}}\{M_n\}, x_0 \in \mathbf{I}$ and, for each n, $\varphi^{(n)}(x_0) = 0$ implies that, for each $x \in \mathbf{I}, \varphi(x) = 0$.

THEOREM 4.4. *If* $\{M_n\}_{n=0}^{\infty}$ *is a logarithmically convex sequence then* $C_{\mathbf{I}}\{M_n\}$ *is not quasi-analytic if and only if*

$$\sum_{n=0}^{\infty} \frac{M_n}{M_{n+1}} < \infty.$$

PROOF. See [**5**].

Proofs of the following two theorems can be found in [**2**].

THEOREM 4.5. *Suppose* $C_{\mathbf{I}}\{M_n\}$ *is not quasi-analytic. Then there exists a logarithmically convex sequence* $\{\tilde{M}_n\}_{n=0}^{\infty}$ *such that* $C_{\mathbf{I}}\{\tilde{M}_n\} \subset C_{\mathbf{I}}\{M_n\}, C_{\mathbf{I}}\{\tilde{M}_n\}$ *is not quasi-analytic, and, for every* $B > 0$,

$$\sum_{n=0}^{\infty} \frac{B^n \tilde{M}_n}{M_n} < \infty.$$

THEOREM 4.6. *If* $C_{\mathbf{I}}\{M_n\}$ *is not quasi-analytic and* $\mathbf{I}' \subset \mathbf{I}$ *then there is a nontrivial nonnegative function* $\varphi \in C_{\mathbf{I}}\{M_n\}$ *with support in* $\mathbf{I}'$.

Let $\{\delta_j\}_{j=1}^{\infty} \in C_{\infty}^{n}(T) \cap \Delta$, and, for each m, define $s^m = [\delta_j^{(m)}/\delta_j]$.

THEOREM 4.7. *If* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence of complex numbers such that* $C_{\mathbf{I}}\{1/|\alpha_n|\}$ *is not quasi-analytic, then* $\delta - \lim_n \sum_{k=0}^{n} \alpha_k s^k$ *exists.*

PROOF. Suppose $C_{\mathbf{I}}\{1/|\alpha_n|\}$ is not quasi-analytic. Without loss of generality assume $\mathbf{I} = [-1, 1]$.

By Theorem 4.5 there exists a logarithmically convex sequence $\{M_n\}_{n=0}^{\infty}$ such that $C_{\mathbf{I}}\{M_n\} \subset C_{\mathbf{I}}\{1/|\alpha_n|\}, C_{\mathbf{I}}\{M_n\}$ is not quasi-analytic, and, for each positive B,

$$(*) \qquad \sum_{n=0}^{\infty} B^n |\alpha_n| M_n < \infty.$$

Let $\mathbf{I}_j = [-1/j, 1/j]$, for $j = 1, 2, \dots$. Then, by the previous theorem for each j, there exists $\varphi_j \in C_{\mathbf{I}}\{M_n\}$ such that φ_j is nontrivial and nonnegative and $\operatorname{supp}\varphi_j \subset \mathbf{I}_j$. For each j let $\tilde{\varphi}_j$ denote the function of period 2π whose restriction to $[-\pi, \pi]$ is φ_j. For each j let

$$\delta_j = \frac{\tilde{\varphi}_j}{\int_{-\pi}^{\pi} \varphi_j \, dt}.$$

Then $\{\delta_j\}_{j=1}^{\infty}$ is a delta-sequence.

Since, for each j, $\delta_j \in C_{\mathbf{I}}\{M_n\}$, there exist positive constants θ_j and B_j such that, for each j and all $n, \max_{x\in\mathbf{I}} |\alpha_n \delta_j^{(n)}(x)| \leq \theta_j B_j^n |\alpha_n| M_n$. So, by $(*)$ for each $j, \sum_{n=0}^{\infty} \alpha_n \delta_j^{(n)}$ converges uniformly. Thus $\delta - \lim_n \sum_{k=0}^{n} \alpha_k s^k$ exists. □

The Gamma function is defined by $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt$ for $x > 0$.

Since, for each $\alpha > 1, C_{\mathbf{I}}\{\Gamma(\alpha n)\}$ is not quasi-analytic (see [**5**]),

$$\sum_{n=1}^{\infty} \frac{s^n}{i^n \Gamma(\alpha n)} \in \beta.$$

5. Main result. The Fourier coefficients of an $L^1(T)$ function are defined in the usual way. That is, if $f \in L^1(T)$ for $k = 0, \pm 1, \pm 2, \dots$ define

$$C_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt.$$

The next two lemmas follow from definition.

LEMMA 5.1. *Let* $\{\delta_n\}_{n=1}^{\infty}$ *be a delta sequence. Then, for each* $k, \lim_n C_k(\delta_n) = 1$.

LEMMA 5.2. *Let* $a = [f_j/\delta_j] \in \beta$. *If, for some positive integer* j_0 *and some* $k_0, C_{k_0}(\delta_{j_0}) = 0$, *then* $C_{k_0}(f_{j_0}) = 0$.

Using Lemmas 5.1 and 5.2 we define the Fourier coefficients of a Boehmian.

DEFINITION 5.3. Let $a = [f_j/\delta_j] \in \beta$, for $k = 0, \pm 1, \pm 2, \ldots$. Define $C_k(a) = C_k(f_j)/C_k(\delta_j)$, where, for a fixed k, j is the smallest index such that $C_k(\delta_j) \neq 0$.

The preceding definition easily gives the following, which we state as a theorem.

THEOREM 5.4. *Let $a, b \in \beta$, then, for each $k, C_k(ab) = C_k(a)C_k(b)$.*

THEOREM 5.5. *Let $a, a_n \in \beta$, for $n = 1, 2, \ldots$. Suppose $\delta - \lim_n a_n = a$. Then, for each k, $\lim_n C_k(a_n) = C_k(a)$.*

PROOF. Let $a_n \in \beta$,for $n = 1, 2, \ldots$, such that $\delta - \lim_n a_n = 0$. That is, there exists a delta sequence $\{\delta_j\}_{j=1}^{\infty}$ such that, for each n and all $j, a_n\delta_j \in C(T)$ and, for each $j, \lim_n a_n\delta_j = 0$. So, for each k and all j, $\lim_n C_k(a_n\delta_j) = 0$. Thus, for each k and all j,

$$C_k(\delta_j) \lim_n C_k(a_n) = \lim_n [C_k(a_n)C_k(\delta_j)] = \lim_n C_k(a_n\delta_j) = 0.$$

Hence by Lemma 5.1 for each $k, \lim_n C_k(a_n) = 0$. □

DEFINITION 5.6. Let $a \in \beta$, then, the Fourier series of a is $\sum_{k=-\infty}^{\infty} C_k(a)e^{ikt}$.

THEOREM 5.7. *For each $a \in \beta, a = \delta - \lim_n \sum_{k=-n}^{n} C_k(a)e^{ikt}$.*

PROOF. Let $a = [f_j/\delta_j]$. We can assume that $\{f_j\}_{j=1}^{\infty} \in C_{\infty}^{n}(T)$. Let $\{\sigma_j\}_{j=1}^{\infty} \in C_{\infty}^{n}(T) \cap \Delta$; then $a = [f_j\sigma_j/\delta_j\sigma_j]$ and $\{f_j\sigma_j\}_{j=1}^{\infty} \in C_{\infty}^{n}(T)$. For $n = 0, 1, 2, \ldots$, let $p_n(t) = \sum_{k=-n}^{n} C_k(a)e^{ikt}$. Then, for each n and all m,

$$\begin{aligned} p_n\delta_m(t) &= \sum_{k=-n}^{n} C_k(a)C_k(\delta_m)e^{ikt} = \sum_{k=-n}^{n} C_k(a\delta_m)e^{ikt} \\ &= \sum_{k=-n}^{n} C_k(f_m)e^{ikt}. \end{aligned}$$

Hence, for each $m, \lim_n p_n \delta_m = f_m = a\delta_m$. That is, $\delta - \lim_n p_n = a$. $\square$

The next several lemmas are needed to prove the main result, Theorem 5.14.

LEMMA 5.8. *Let $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ be the k^{th} roots of unity. Then*

$$\sum_{j=0}^{k-1} \alpha_j^n = \begin{cases} k & \text{if } n \equiv 0 (\operatorname{mod} k) \\ 0 & \textit{otherwise.} \end{cases}$$

LEMMA 5.9. *Let $z \in \mathbf{C}$, then, for each $k = 1, 2, \dots$*

$$\sum_{j=1}^{\infty} \frac{z^j}{\Gamma(kj)} = \frac{1}{k} \sum_{j=0}^{k-1} \xi_j \exp(\xi_j)$$

where $\xi_0, \xi_1, \dots, \xi_{k-1}$ are the k^{th} roots of z.

PROOF. Let $z \in \mathbf{C}$. Fix k. Let $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ be the k^{th} roots of unity. Then, for $j = 0, 1, 2, \dots, k-1, \xi_j = \alpha_j \xi_0$. So

$$\frac{1}{k} \sum_{j=0}^{k-1} \xi_j \exp(\xi_j) = \frac{1}{k} \sum_{j=0}^{k-1} \alpha_j \xi_0 \sum_{n=0}^{\infty} \frac{\alpha_j^n \xi_0^n}{n!} = \frac{1}{k} \sum_{n=0}^{\infty} \frac{\xi_0^{n+1}}{n!} \sum_{j=0}^{k-1} \alpha_j^{n+1}.$$

And applying the previous lemma to the above, we obtain

$$\frac{1}{k} \sum_{j=0}^{k-1} \xi_j \exp(\xi_j) = \sum_{\substack{n \equiv -1 \\ (\operatorname{mod} k)}}^{\infty} \frac{\xi_0^{n+1}}{n!} = \sum_{j=1}^{\infty} \frac{\xi_0^{jk}}{(jk-1)!} = \sum_{j=1}^{\infty} \frac{z^j}{\Gamma(kj)}. \square$$

LEMMA 5.10. *For each $k = 1, 2, 3, \dots$ there exist positive constants A_k and M_k such that, for each $x \geq M_k$,*

$$\sum_{n=1}^{\infty} x^n / \Gamma(kn) \geq A_k \exp(x^{1/k}).$$

PROOF. We can assume $x \geq 0$. Fix k and let $r = x^{1/k}$ and $\alpha = \exp(2\pi i/k)$. Then

$$S = \sum_{n=1}^{\infty} \frac{x^n}{\Gamma(kn)} = \frac{r}{k} \sum_{j=0}^{k-1} \alpha^j \exp(\alpha^j r).$$

After noting that S is real and using

$$\mathrm{Re}\,(\alpha^j \exp(\alpha^j r)) = (\mathrm{Re}\,\alpha^j)\mathrm{Re}\,(\exp(\alpha^j r)) - (\mathrm{Im}\,\alpha^j)(\mathrm{Im}\,(\exp(\alpha^j r))),$$

some computation gives

$$\begin{aligned} S &= \frac{r}{k}\mathrm{Re}\Big(\sum_{j=0}^{k-1} \alpha^j \exp(\alpha^j r)\Big) \\ &= \frac{r}{k}\Big\{e^r + \sum_{j=1}^{k-1} \exp(r\cos(2\pi j/k))\cos((2\pi j/k) + r\sin(2\pi j/k))\Big\} \\ &\geq \frac{r}{k}\Big\{e^r - \sum_{j=1}^{k-1} \exp(r\cos(2\pi j/k))\Big\}. \ \square \end{aligned}$$

In Lemma 5.10, by replacing x with x^p, we obtain

LEMMA 5.11. *Let p be a positive number. Then, for each $k = 1, 2, \ldots$, there exist positive constants A_k and M_k such that, for each $x \geq M_k$,*

$$\sum_{n=1}^{\infty} \frac{(x^p)^n}{\Gamma(kn)} \geq A_k \exp(x^{p/k}).$$

For $m = 0, 1, 2, \ldots$, let $\alpha_m = (2^m + 1)2^{-m}$ and

$$a_m = \sum_{n=1}^{\infty} \frac{s^n}{i^n \Gamma(\alpha_m n)} \in \beta.$$

Then, by Theorem 5.5, for each m and all k,

$$C_k(a_m) = \sum_{n=1}^{\infty} \frac{(ik)^n}{i^n \Gamma(\alpha_m n)} = \sum_{n=1}^{\infty} \frac{k^n}{\Gamma(\alpha_m n)}.$$

LEMMA 5.12. *For each $m = 0, 1, 2, \ldots$, there exist positive constants A_m and M_m such that, for each $k \geq M_m$,*

$$C_k(a_m) \geq A_m \exp(k^{1/\alpha_m}).$$

PROOF. Fix m. By Lemma 5.11 there exist positive constants A and M such that, for each $k \geq M$,

$$\sum_{n=1}^{\infty} \frac{(k^{2^m})^n}{\Gamma((2^m+1)n)} \geq A \exp(k^{1/\alpha_m}).$$

So, for each $k \geq M$,

$$C_k(a_m) = \sum_{n=1}^{\infty} \frac{k^n}{\Gamma(\alpha_m n)} \geq \sum_{n=1}^{\infty} \frac{(k^{2^m})^n}{\Gamma((2^m+1)n)} \geq A \exp(k^{1/\alpha_m}). \ \square$$

LEMMA 5.13. *For $0 \leq \gamma < 1$ there exists a delta sequence $\{\delta_j\}_{j=1}^{\infty}$ such that, for each $j, C_k(\delta_j) = 0(\exp(-|k|^{\gamma}))$ as $|k| \to \infty$.*

PROOF. Let $0 \leq \gamma < 1$. Pick m such that $\gamma < 1/\alpha_m < 1$. Let

$$a = \sum_{n=1}^{\infty} \frac{s^n}{i^n \Gamma(\alpha_m n)} \in \beta.$$

Then, by the previous lemma, there exist positive constants A and M such that, for each $k \geq M$,

$$C_k(a) = \sum_{n=1}^{\infty} \frac{k^n}{\Gamma(\alpha_m n)} \geq A \exp(k^{1/\alpha_m}).$$

Now suppose a has the representation $a = [f_j/\delta_j]$. Then, for each j, there exists a positive constant $\tilde{M}_j$ such that, for each $k \geq \tilde{M}_j$,

$$|C_k(\delta_j)| = \frac{|C_k(f_j)|}{|C_k(a)|} \leq \exp(-k^{1/\alpha_m}) < \exp(-k^{\gamma}).$$

The lemma follows by observing that the δ_j's are real; hence, for each k and all j,

$$|C_k(\delta_j)| = |C_{-k}(\delta_j)|. \square$$

With the aid of the previous lemma we can now prove the following theorem.

THEOREM 5.14. *Let* $\{\xi_n\}_{n=-\infty}^{\infty}$ *be a sequence of complex numbers such that, for some* $0 \leq \gamma < 1$, $\xi_n = 0(\exp(|n|^\gamma))$ *as* $|n| \to \infty$, *then there exists* $a \in \beta$ *such that, for each* $k, C_k(a) = \xi_k$.

PROOF. Pick α such that $\gamma < \alpha < 1$. Let $\{\delta_j\}_{j=1}^{\infty}$ be a delta sequence such that, for each j,

$$C_k(\delta_j) = 0(\exp(-|k|^\alpha)) \text{ as } |k| \to \infty.$$

Now, for $n = 1, 2, \ldots$, let

$$p_n(t) = \sum_{k=-n}^{n} \xi_k e^{ikt}.$$

Then, for each m and all n,

$$p_n \delta_m(t) = \sum_{k=-n}^{n} \xi_k C_k(\delta_m) e^{ikt}.$$

Since $\xi_k = 0(\exp(|k|^\gamma))$ as $|k| \to \infty$, where $\gamma < \alpha < 1$, for each $m, p_n \delta_m$ converges uniformly.

Hence, by Lemma 3.1, there exists an $a \in \beta$ such that

$$a = \delta - \lim_n \sum_{k=-n}^{n} \xi_k e^{ikt}.$$

Therefore, by Theorem 5.5, for each $k, C_k(a) = \xi_k$. $\square$

The Fourier coefficients of a Boehmian can not grow too fast, as the next theorem will show.

THEOREM 5.15. *Let $\varepsilon > 0, A$ and N be positive numbers. Suppose $\{\xi_n\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers such that, for each $n \geq N, |\xi_n| \geq Ae^{\varepsilon n}$, then, for each $a \in \beta$, there exists an integer k_a such that $C_{k_a}(a) \neq \xi_{k_a}$.*

PROOF. For $n = 0, 1, 2, \ldots$, let $p_n(t) = \sum_{k=-n}^{n} \xi_k e^{ikt}$. Let $\{\delta_j\}_{j=1}^{\infty}$ be a delta sequence. Fix j. Since δ_j is not analytic there exists a subsequence $\{k_q\}_{q=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ such that, for each q, $|C_{k_q}(\delta_j)| \geq \exp(-\varepsilon k_q)$ (see [**1**]). Now, for each n,

$$p_n \delta_j(t) = \sum_{k=-n}^{n} \xi_k C_k(\delta_j) e^{ikt}.$$

From the above there exists a positive integer M such that, for each $q \geq M$,

$$|\xi_{k_q} C_{k_q}(\delta_j)| \geq A.$$

Therefore, $p_n \delta_j$ does not converge uniformly as $n \to \infty$.

Since the above is true for each j and $\{\delta_j\}_{j=1}^{\infty}$ was an arbitrary delta sequence the conclusion follows. □

Thus, the set of all distributions on the unit circle is properly contained in β, which is itself properly contained in the set of all Mikusiński operators on the unit circle. If $\{\xi_n\}_{n=-\infty}^{\infty}$ is a sequence of complex numbers which satisfies Theorem 5.14, then $\overline{\lim}_{|n| \to \infty} |\xi_n|^{1/|n|} \leq 1$ and hence the ξ_n's are the Fourier coefficients of some hyperfunction [**4**]. An interesting open problem is: how does β compare to the set of all hyperfunctions?

Recently, Theorems 5.14 and 5.15 have been strengthened and an example of a Hyperfunction that is not a Boehmian has been found by the author (Periodic Boehmians, Internat. J. Math. and Math. Sci. **12**(4), 1989, 685-692).

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