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ASYMPTOTIC AND OSCILLATORY BEHAVIOR OF A CLASS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor G. J. Butler

ABSTRACT. The asymptotic and oscillatory behavior of solutions of functional differential equations of the form

$$(a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[g(t)]) = 0, \quad \left(^{\cdot} = \frac{d}{dt}\right),$$

is discussed. Here q is allowed to change sign on $[t_0, \infty)$.

1. Introduction. Consider the functional differential equations of the type

$$(1.1) \quad (a(t)\psi(x(t))\dot{x}(t))^{\cdot} + p(t)\dot{x}(t) + q(t)f(x[g(t)]) = 0, \quad \left(^{\cdot} = \frac{d}{dt}\right),$$

where $q, g, p, q : [t_0, \infty) \to \mathbf{R}$, $\psi, f : \mathbf{R} \to \mathbf{R}$ are continuous, $a(t) > 0$, $q(t) \geq 0$, and q is not identically zero on any subinterval of $[t_0, \infty)$. Moreover, $g(t) \to \infty$ as $t \to \infty$, $\psi(x) > 0$ for all x and $xf(x) > 0$ for $x \neq 0$, and functions q, g, a and ψ are continuously differentiable.

In what follows, we consider only such solutions which are defined for all $t \geq t_0 \geq 0$. The oscillatory character is considered in the usual sense; i.e., a continuous real-valued function x defined on $[t_x, \infty)$, for some $t_x \geq 0$, is called oscillatory if its set of zeros is unbounded above, otherwise it is called nonoscillatory.

In recent years there has been an increasing interest in the study of the qualitative behavior of solutions of equations of type (1.1). Kulenovic and Grammatikopoulos [**13**] obtained some results on the behavior of the retarded strongly superlinear equation

$$(*) \qquad (a(t)\dot{x}(t)) + q(t)f(x[q(t)]) = 0.$$

Our main purpose is to study the asymptotic and oscillatory behavior of Equation (1.1), where conditions on a, p and ψ are different from those which appeared in [**5–8**]. In §2, we present some criteria which guarantee that every solution x of Equation (1.1) is either oscillatory or else $x(t) \to 0$ monotonically as $t \to \infty$. Oscillation criteria for Equation (1.1) when it is strongly superlinear, i.e., when $\int_{\pm} \psi(u)/f(u) < \infty$, with retarded or advanced arguments are established. We also present some criteria which are applicable to linear equations as well as equations of type (1.1) where $f'(x)/\psi(x) \geq k > 0$ for $x \neq 0$. We include the case when q is of arbitrary sign on $[t_0, \infty)$ and establish results concerning oscillation of the derivative of any solution of this equation. Examples are inserted in the text to illustrate the relevance of the results. Some of the results in this paper overlap with some which appeared in our paper [**5**].

2. Main results. In our first result we allow the argument $g(t)$ to be retarded, advanced, or of advanced type.

THEOREM 2.1. *Assume that* $\dot{g}(t) \geq 0$, $p(t) \geq 0$ *for* $t \geq t_0$ *and*

$$f'(x) \geq 0 \text{ for } x \neq 0, \quad \left(\dot{} = \frac{d}{dx} \right), \tag{2.1}$$

and let there exist $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that*

$$\dot{\rho}(t) \leq 0, \ (p(t)\rho(t))^{\cdot} \leq 0 \text{ and } (a(t)\dot{\rho}(t))^{\cdot} \geq 0 \text{ for } t \geq t_0. \tag{2.2}$$

If

$$\int^{\infty} \rho(s)q(s)\, ds = \infty \tag{2.3}$$

and

$$\int^{\infty} \frac{1}{a(s)\rho(s)} \int_{t_0}^{s} \rho(\tau)q(\tau)\, d\tau\, ds = \infty, \tag{2.4}$$

then every solution x *of Equation* (1.1) *is either oscillatory or* $x(t) \to 0$ *monotonically as* $t \to \infty$.

PROOF. Let $x(t)$ be a nonoscillatory solution of Equation (1.1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $x(t)$ and $x[g(t)]$ are positive for $t \geq t_0$, since the substitution $u = -x$ transforms Equation (1.1) into an equation of the same form subject to the assumptions of the theorem. Now, we consider the following three cases for the behavior of $\dot{x}$:

CASE 1. $\dot{x}$ is oscillatory. If $\dot{x}(t_1) = 0$, then

$$(a(t)\psi(x(t))\dot{x}(t))^{\cdot}|_{t=t_1} = -q(t_1)f(x[g(t_1)]) \leq 0,$$

from which we can prove that $\dot{x}(t)$ cannot have another zero after it vanishes once. Thus, $\dot{x}(t)$ has a fixed sign for all sufficiently large t.

CASE 2. $\dot{x} > 0$ on $[t_1, \infty)$ from some $t_1 \geq t_0$. We define

$$w(t) = \frac{a(t)\psi(x(t))\dot{x}(t)}{f(x[g(t)])}\rho(t) \text{ for } t \geq t_1.$$

Then, for every $t \geq t_1$,

$$\begin{aligned} \dot{w}(t) = &- \rho(t)q(t) - p(t)\rho(t)\frac{\dot{x}(t)}{f(x[g(t)])} + a(t)\dot{\rho}(t)\frac{\psi(x(t))\dot{x}(t)}{f(x[g(t)])} \\ &- a(t)\rho(t)\dot{g}(t)\frac{\psi(x(t))f'(x[g(t)])\dot{x}[g(t)]\dot{x}(t)}{f^2(x[g(t)])}. \end{aligned} \tag{2.5}$$

Using conditions (2.1) and (2.2), we get

$$\dot{w}(t) \leq -\rho(t)q(t) \text{ for } t \geq t_1.$$

Integrating the above inequality from t_1 to t obtains

$$\int_{t_1}^{t} \rho(s)q(s)\,ds \leq w(t_1) - w(t) \leq w(t_1) < \infty.$$

This contradicts condition (2.3).

CASE 3. $\dot{x} < 0$ on $[t_1, \infty)$ for $t_1 \geq t_0$. Suppose that $\lim_{t\to\infty} x(t) = b$, $b \geq 0$. We claim that $b = 0$. To prove it, assume that $b > 0$ and define

$$u(t) = a(t)\psi(x(t))\dot{x}(t)\rho(t),\ t \geq t_1.$$

Then, for $t \geq t_1$,

$$\dot{u}(t) = -\rho(t)q(t)f(x[g(t)]) - p(t)\rho(t)\dot{x}(t) + a(t)\dot{\rho}(t)\psi(x(t))\dot{x}(t). \tag{2.6}$$

Hence, for all $t \geq t_1$, we have

$$\begin{aligned} u(t) = u(t_1) - f(x[g(t)]) \int_{t_1}^{t} \rho(s)q(s)\,ds \\ + \int_{t_1}^{t} f'(x[g(s)])\dot{x}[g(s)]\dot{g}(s) \int_{t_1}^{s} \rho(\tau)q(\tau)\,d\tau\,ds \\ - \int_{t_1}^{t} (q(s)\rho(s))\dot{x}(s)\,ds + \int_{t_1}^{t} (a(s)\dot{\rho}(s))\psi(x(s))\dot{x}(s)\,ds. \end{aligned}$$

By the Bonnet theorem, for any $t \geq t_1$, there exist ξ_1, $\xi_2 \in [t_1, t]$ so that

$$-\int_{t_1}^{t} (p(s)\rho(s))\dot{x}(s)\,ds = -p(t_1)\rho(t_1)[x(\xi_1) - x(t_1)] \leq p(t_1)\rho(t_1)x(t_1)$$

and

$$\begin{aligned} &\int_{t_1}^{t} (a(s)\dot{\rho}(s))(\psi(x(s))\dot{x}(s))\,ds \\ &\qquad = a(t_1)\dot{\rho}(t_1)\left[\int_0^{x(\xi_2)} \psi(v)\,dv - \int_0^{x(t_1)} \psi(v)\,dv\right] \\ &\qquad \leq -a(t_1)\dot{\rho}(t_1)\left(\int_0^{x(t_1)} \psi(v)\,dv\right). \end{aligned}$$

So, for every $t \geq t_1$,

$$u(t) \leq M - f(b) \int_{t_1}^{t} \rho(s)q(s)\,ds,$$

where $M = u(t_1) + p(t_1)\rho(t_1)x(t_1) - a(t_1)\dot{\rho}(t_1)\left(\int_0^{x(t_1)} \psi(v)\,dv\right)$.

By assumptions of the theorem, there exists a $t_2 \geq t_1$ such that

$$u(t) \leq -\frac{f(b)}{2} \int_{t_1}^{t} \rho(s)q(s)\,ds \text{ for } t \geq t_2.$$

Thus,

$$\int_{t_2}^{t} \psi(x(s))\dot{x}(s)\,ds \le -\frac{f(b)}{2}\int_{t_2}^{t} \frac{1}{a(s)\rho(s)}\int_{t_1}^{s} \rho(\tau)q(\tau)\,d\tau\,ds.$$

By condition (2.4),

$$\int_{x(t_2)}^{x(t)} \psi(v)\,dv \to -\infty \text{ as } t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \ge t_0$. Thus $b = 0$ and $x(t) \to 0$ monotonically as $t \to \infty$. □

The following examples are illustrative.

EXAMPLE 1. Consider the differential equations

$$(t^3\dot{x})^{\cdot} + \frac{1}{t}\dot{x} + q(t)f(x[g(t)]) = 0, \tag{2.7}$$

$$\left(t(1+t^2)\frac{1}{1+x^2}\dot{x}\right)^{\cdot} + \frac{1}{t}\dot{x} + q(t)f(x[g(t)]) = 0 \tag{2.8}$$

and

$$\left(\frac{t^5}{1+t^2}(1+x^2)\dot{x}\right)^{\cdot} + \frac{1}{t}\dot{x} + q(t)f(x[g(t)]) = 0, \tag{2.9}$$

where $g(t)$ is a continuous and nondecreasing function for $t \ge t_0 = 1$ and $\lim_{t\to\infty} g(t) = \infty$, $q(t) = g^{\alpha}(t)(1+t^{-3})$ and $f(x) = |x|^{\alpha}\operatorname{sgn} x$, $\alpha > 0$.

We take $\rho(t) = 1$. If

$$\int^{t} g^{\alpha}(s)\,ds = O(t^2),$$

then all the conditions of Theorem 2.1 are satisfied, and, hence, every solution x of equations (2.7)–(2.9) is either oscillatory or $x(t) \to 0$ monotonically as $t \to \infty$. Each of the Equations (2.7), (2.9) admits the nonoscillatory solution $x(t) = 1/t \to 0$ monotonically as $t \to \infty$.

REMARK 1. One is tempted to believe that if we replace condition (2.4) by the stronger condition

$$\int^{\infty} \frac{1}{a(s)\rho(s)}\, ds = \infty, \tag{2.10}$$

then conditions (2.3) and (2.10) may ensure the oscillation of Equation (1.1). In fact, this is not enough, since, if we take $\alpha \geq 1$ and $g(t) = t$ in equations (2.7)–(2.9) and let $\rho(t) = 1/t^2$, the hypotheses of Theorem 2.1 and condition (2.10) are satisfied. Therefore, we need further restrictions on the functions in Equation (1.1).

In the following theorem we study the oscillatory behavior of Equation (1.1) subject to the conditions

$$\psi(x) \geq c > 0 \text{ for all } x \tag{2.11}$$

and

$$\int_{+0} \frac{\psi(u)}{f(u)}\, du < \infty \text{ and } \int_{-0} \frac{\psi(u)}{f(u)}\, du < \infty. \tag{2.12}$$

THEOREM 2.2. *Let* $g(t) \leq t$, $\dot{g}(t) \geq 0$ *for* $t \geq t_0$, *conditions* (2.1), (2.11) *and* (2.12) *hold and assume that there exists a function* $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that conditions* (2.2),(2.3) *and* (2.10) *hold. Equation* (1.1) *is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1.1). as in the proof of Theorem 2.1, three cases arise. The proof of Cases 1 and 2 are similar to the corresponding cases of Theorem 2.1. Hence, we consider Case 3. By conditions (2.3) and (2.10) we conclude that $x(t) \to 0$ as $t \to \infty$. Let $x(t) > 0$ and $x[g(t)] > 0$ for $t \geq t_1 \geq t_0$ and consider the function w defined earlier in the proof of Theorem 2.1 (Case 2). Then, for every $t \geq t_1$,

$$\begin{aligned} \dot{w}(t) = &- \rho(t)q(t) - p(t)\rho(t)\frac{\dot{x}(t)}{f(x[g(t)])} + a(t)\dot{\rho}(t)\frac{\psi(x(t))\dot{x}(t)}{f(x[g(t)])} \\ &- w(t)\frac{\frac{d}{dt}f(x[g(t)])}{f(x[g(t)])}. \end{aligned} \tag{2.13}$$

Using conditions (2.1), (2.2) and the fact that $g(t) \le t$ for $t \ge t_1$, we get

$$w(t) \le w(t_1) - \int_{t_1}^{t} \rho(s)q(s)\,ds - \int_{t_1}^{t} \frac{1}{c} p(s)\rho(s) \left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))} \right) ds + \int_{t_1}^{t} (a(s)\dot{\rho}(s)) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds - \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])}.$$

By the Bonnet theorem, for any $t \ge t_1$, there exist ξ_1, $\xi_2 \in [t_1, t]$ such that

$$-\int_{t_1}^{t} \frac{1}{c} p(s)\rho(s) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds = \frac{1}{c} p(t_1)\rho(t_1) \int_{x(t_1)}^{x(\xi_1)} \frac{\psi(u)}{f(u)}\,du \le -\frac{1}{c} p(t_1)\rho(t_1) \int_{x(t_1)}^{\infty} \frac{\psi(u)}{f(u)}\,du = M_1$$

and

$$\int_{t_1}^{t} (a(s)\dot{\rho}(s)) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds = a(t_1)\dot{\rho}(t_1) \int_{x(t_1)}^{x(\xi_2)} \frac{\psi(u)}{f(u)}\,du \le a(t_1)\dot{\rho}(t_1) \int_{x(t_1)}^{\infty} \frac{\psi(u)}{f(u)}\,du = M_2.$$

So, for every $t \ge t_1$,

$$w(t) \le M - \int_{t_1}^{t} \rho(s)q(s)\,ds - \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])},$$

where $M = w(t_1) + M_1 + M_2$; hence, by condition (2.3), we derive

$$-w(t) \ge C + \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])}, \tag{2.14}$$

where C is a positive constant. So, for every $t \ge t_1$,

$$\left(w(t) \frac{df(x[g(t)])}{f(x[g(t)])} \right) \left(C + \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])} \right)^{-1} \ge -\frac{df(x[g(t)])}{f(x[g(t)]},$$

and, hence, by integrating over $[t_1, t]$, we obtain

$$\ln \frac{1}{C}\left[C + \int_{t_1}^{t} w(s)\frac{df(x[g(s)])}{f(x[g(s)])}\right] \geq \ln \frac{f(x[g(t_1)])}{f(x[g(t)])}.$$

Thus,

$$C + \int_{t_1}^{t} w(s)\frac{df(x[g(s)])}{f(x[g(s)])} \geq \frac{Cf(x[g(t_1)])}{f(x[g(t)])} \text{ for all } t \geq t_1,$$

and so (2.14) yields

$$\psi(x(t))\dot{x}(t) \leq -C_1\frac{1}{a(t)\rho(t)} \text{ for every } t \geq t_1,$$

where $C_1 = Cf(x[g(t_1)])$. Consequently,

$$\int_{x(t_1)}^{x(t)} \psi(u)\,du \leq -C_1 \int_{t_1}^{t} \frac{1}{a(s)\rho(s)}\,ds \to -\infty \text{ as } t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \geq t_1$. □

EXAMPLE 2. Consider the differential equations (2.7)–(2.9) with $q(t) = t$ and $0 < \alpha < 1$. We let $\rho(t) = 1/t^2$, $t \geq t_0 > 0$. It is easy to check that each of these equations is oscillatory by Theorem 2.2.

The case when Equation (1.1) is of advanced type is covered in

THEOREM 2.3. *let* $g(t) \geq t$, $\dot{g}(t) > 0$ *for* $t \geq t_0$ *and conditions* (2.1), (2.11) *and* (2.13) *hold. Suppose that there exists a function* $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that*

(2.15)
$$\dot{\rho}(t) \leq 0, \left(\frac{a(g(t))\rho[g(t)]p(t)}{a(t)\dot{g}(t)}\right)^{\cdot} \leq 0 \text{ and } (a(t)\dot{\rho}(t))^{\cdot} \geq 0 \text{ for } t \geq t_0.$$

If

$$\int^{\infty} \rho[g(s)]q(s)\,ds = \infty \tag{2.16}$$

and

$$\int^{\infty} \frac{1}{a(s)\rho[g(s)]}\, ds = \infty, \tag{2.17}$$

then Equation (1.1) *is oscillatory.*

PROOF. (See [**5**]). □

In the following result we take p and q of variable sign and $g(t) = t$ for all $t \geq t_0$. It is convenient to make use of the notations

$$Q(t) = q(t) - \frac{1}{4k}\left(\frac{1}{c} - \frac{1}{c_1}\right)\frac{p^2(t)}{a(t)},$$
$$\gamma(t) = a(t)\dot{\rho}(t) - \frac{1}{c_1}p(t)\rho(t).$$

We shall note that when $\psi(x) \equiv 1$, $Q(t) = q(t)$.

THEOREM 2.4. *Assume that*

$$xf(x) > 0 \text{ and } f'(x) \geq k > 0 \text{ for } x \neq 0, \qquad \left({}' = \frac{d}{dx}\right), \tag{2.18}$$

$$0 < c \leq \psi(x) \leq c_1. \tag{2.19}$$

Suppose that there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ *such that*

$$\int^{\infty} \frac{1}{a(s)\rho(s)}\, ds = \infty. \tag{2.20}$$

Then each of the following conditions ensures the oscillation of continuable solutions of Equation (1.1):

$$\text{(I)} \qquad \int^{\infty} \frac{\gamma^2(s)}{a(s)\rho(s)}\, ds < \infty, \quad \int^{\infty} \rho(s)Q(s)\, ds = \infty;$$

$$\text{(II)} \qquad \gamma(t) \geq 0,\ \dot{\gamma}(t) \leq 0 \text{ for } t \geq t_0, \quad \int^{\infty} \rho(s)Q(s)\, ds = \infty;$$

$$\text{(III)} \qquad \gamma(t) > 0,\ \dot{\gamma}(t) \geq 0, \quad \lim_{t\to\infty} \frac{1}{\gamma(t)} \int^{t} \rho(s)Q(s)\, ds = \infty.$$

PROOF. Let $x(t)$ be a nonoscillatory solution of Equation (1.1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geq t_0$. Furthermore, we suppose that $x(t) > 0$ for $t \geq t_0$, since the substitution $u = -x$ transforms (1.1) into an equation of the same form subject to the assumptions of the theorem. Now, we define

$$w(t) = \rho(t) \frac{a(t)\psi(x(t))\dot{x}(t)}{f(x(t))}, \quad t \geq t_0.$$

Then, for every $t \geq t_0$,

(2.21)

$$\begin{aligned}\dot{w}(t) =& -\rho(t)q(t) - \frac{p(t)}{a(t)} \frac{1}{\psi(x(t))} w(t) + \frac{\dot{\rho}(t)}{\rho(t)} w(t) \\ & - \frac{1}{a(t)\rho(t)} \frac{f'(x(t))}{\psi(x(t))} w^2(t) \\ =& -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)} w(t) - \frac{1}{\psi(x(t))} \left[\frac{f'(x(t))}{a(t)\rho(t)} w^2(t) + \frac{p(t)}{a(t)} w(t) \right] \\ =& -\rho(t)q(t) + \frac{1}{\psi(x(t))} \frac{p^2(t)\rho(t)}{4a(t)f'(x(t))} + \frac{\dot{\rho}(t)}{\rho(t)} w(t) \\ & - \frac{1}{\psi(x(t))} \left[\sqrt{\frac{f'(x(t))}{a(t)\rho(t)}} w(t) + \frac{p(t)\sqrt{\rho(t)}}{2\sqrt{a(t)f'(x(t))}} \right]^2 .\end{aligned}$$

Using conditions (2.18) and (2.19), we have

$$\begin{aligned}\dot{w}(t) \leq& -\rho(t)Q(t) + \gamma(t) \frac{\psi(x(t))\dot{x}(t)}{f(x(t))} \\ & - \frac{1}{c_1} a(t)\rho(t)f'(x(t)) \left(\frac{\psi(x(t))\dot{x}(t)}{f(x(t))} \right)^2 .\end{aligned} \tag{2.22}$$

Thus,

$$\begin{aligned}w(t) \leq w(t_0) &- \int_{t_0}^{t} \rho(s)Q(s)\,ds + \int_{t_0}^{t} \gamma(s) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds \\ &- \frac{1}{c_1} \int_{t_0}^{t} a(s)\rho(s)f'(x(s)) \left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))} \right) ds.\end{aligned} \tag{2.23}$$

We consider the following cases:

CASE 1. Let (I) hold. It follows from the Schwarz inequality that

$$\begin{aligned}
&\left|\int_{t_0}^{t} \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\,ds\right| \\
&\qquad \le \left(\int_{t_0}^{t} \frac{\gamma^2(s)}{a(s)\rho(s)}\,ds\right)^{\frac{1}{2}} \left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{\frac{1}{2}} \\
&\qquad \le K\left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

where $K = (\int_{t_0}^{\infty}(\gamma^2(s)/a(s)\rho(s))\,ds)^{1/2}$ is finite. Thus (2.23) gives

$$\begin{aligned}
w(t) \le w(t_0) &- \int_{t_0}^{t} \rho(s)Q(s)\,ds \\
&+ K\left(\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^2 ds\right)^{1/2} \\
&- \frac{k}{c_1}\int_{t_0}^{t} a(s)\rho(s)\left(\frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\right)^{1/2} ds.
\end{aligned}$$

Clearly, the sum of the last two integrals in the right hand of the above inequality remains bounded above as $t \to \infty$. Thus, in view of (I),

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} \frac{a(t)\rho(t)\psi(x(t))\dot{x}(t)}{f(x(t))} = -\infty.$$

Consequently, there exists a $t_1 \ge t_0$ such that

$$\dot{x}(t) < 0 \text{ for } t \ge t_1.$$

This means that there exists a $t_2 \ge t_1$ such that

$$\begin{aligned}
1 + k_1\int_{t_2}^{t} a(s)\rho(s)f'(x(s))\psi(x(s))\left(\frac{\dot{x}(s)}{f(x(s))}\right)^2 ds& \\
\le a(t)\rho(t)\frac{\psi(x(t))(-\dot{x}(t))}{f(x(t))}&,
\end{aligned} \tag{2.24}$$

where $k_1 = kc/c_1$.

Case 2. If (II) holds, then, by the Bonnet theorem, for any $t \geq t_0$, there exists a $\xi \in [t_0, t]$ so that

$$\begin{aligned} \int_{t_0}^{t} \gamma(s) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds &= \gamma(t_0) \int_{t_0}^{\xi} \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds \\ &= \gamma(t_0) \int_{x(t_0)}^{x(\xi)} \frac{\psi(u)}{f(u)}\, du \\ &\leq c_1 \gamma(t_0) \int_{x(t_0)}^{\infty} \frac{du}{f(u)} = M < \infty. \end{aligned}$$

As in Case 1, there exists a $t_2 \geq t_0$ so that (2.24) holds.

Case 3. Let (III) hold. Once again, by Bonnet's theorem, for some $M_1 > 0$, $t \geq t_0$, we have

$$\left| \int_{t_0}^{t} \gamma(s) \frac{\psi(x(s))\dot{x}(s)}{f(x(s))}\, ds \right| \leq M_1 \gamma(t),$$

and, as in Case 2 of Theorem 7 in [**8**], we obtain inequality (2.24).

The rest of the proof is similar to that of Theorem 7 in [**8**] and, hence, is omitted. □

EXAMPLE 3. Consider the differential equation

(2.25)
$$(t\psi(x(t))\dot{x}(t))^{\cdot} + \frac{\sin t}{t}\dot{x}(t) + \left(\frac{1}{t} + \sin t\right)(x(t) + x^3(t)) = 0, \quad t > 0,$$

where $\psi(x) = 1 + e^{-|x|}$ or $2 - \sin x$. The hypotheses of Theorem 2.4(I) are satisfied with $\rho(t) = 1$, and, hence, every solution of (2.25) is oscillatory. We note that some of the oscillation criteria in the literature fail to apply to (2.25).

THEOREM 2.5. *In addition to conditions* (2.18) *and* (2.19), *let*

(2.26)
$$\int_{+0} \frac{du}{f(u)} < \infty \text{ and } \int_{-0} \frac{du}{f(u)} < \infty.$$

Suppose that there exists a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ *such that*

(2.27)
$$\int^{\infty} \rho(s) Q^*(s)\, ds = \infty$$

and

$$\int^{\infty} \frac{1}{a(s)\rho(s)} \int_T^s \rho(\tau) Q^*(\tau)\, d\tau\, ds = \infty, \quad T \geq t_0, \tag{2.28}$$

where

$$Q^*(t) = q(t) - \frac{1}{4k} \left[\frac{p^2(t)}{ca(t)} - 2p(t)\frac{\dot{\rho}(t)}{\rho(t)} + c_1 a(t) \left(\frac{\dot{\rho}(t)}{\rho(t)} \right)^2 \right].$$

Then every solution of (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we suppose that $x(t) > 0$ for $t \geq t_0$. Furthermore, we consider the function w defined in the proof of Theorem 2.4. Then, for every $t \geq t_0$, we obtain

$$\begin{aligned} \dot{w}(t) \leq & - \rho(t) q(t) \\ & - \frac{1}{\psi(x(t))} \left[\frac{k}{a(t)\rho(t)} w^2(t) + \left(\frac{p(t)}{a(t)} - \frac{\dot{\rho}(t)}{\rho(t)} \psi(x(t)) \right) w(t) \right]. \end{aligned}$$

Completing the square and using condition (2.19), we have

$$\dot{w}(t) \leq -\rho(t) Q^*(t), \quad t \geq t_0.$$

Thus,

$$a(t)\rho(t) \frac{\psi(x(t))\dot{x}(t)}{f(x(t))} \leq C - \int_{t_0}^t \rho(s) Q^*(s)\, ds, \tag{2.29}$$

where

$$C = a(t_0)\rho(t_0) \frac{\psi(x(t_0))\dot{x}(t_0)}{f(x(t_0))}.$$

It follows from condition (2.27) that there exists a $t_1 \geq t_0$ so that

$$\int_{t_0}^{t_1} \rho(s) Q^*(s)\, ds = 0 \text{ and } \int_{t_1}^{t} \rho(s) Q^*(s)\, ds \geq 2|C| \text{ for } t \geq t_1.$$

Thus, inequality (2.29) leads to

$$\frac{\dot{x}(t)}{f(x(t))} \leq -\frac{1}{2}\frac{1}{\psi(x(t))}\frac{1}{a(t)\rho(t)}\int_{t-1}^{t} \rho(s)Q^*(s)\,ds$$
$$\leq -\frac{1}{2c_1}\frac{1}{a(t)\rho(t)}\int_{t_1}^{t} \rho(s)Q^*(s)\,ds,$$

which implies that

$$G(x(t)) \leq G(x(t_1)) - \frac{1}{2c_1}\int_{t_1}^{t} \frac{1}{a(s)\rho(s)}\int_{t_1}^{s} \rho(\tau)Q^*(\tau)\,d\tau\,ds,$$

where

$$G(x(t)) = \int_0^{x(t)} \frac{du}{f(u)}.$$

Consequently, $G(x(t)) \to -\infty$ as $t \to \infty$, contradicting the fact that $G(x(t)) \geq 0$. □

REMARKS. 1. The deviating argument $g(t)$ is chosen to be either retarded or advanced, and, hence, our results are applicable to ordinary, retarded, as well as advanced, equations.

2. If $\psi(x) \equiv 1$ and $p(t) \equiv 0$, then our results are related to Theorems 3 and 4 in [**13**].

3. Due to space limitations we have avoided comparison of our results with those cited in the references.

REFERENCES

1. N. P. Bhatia, *Some oscillation theorems for second order differential equations*, J. Math. Anal. Appl. **15** (1966), 442–446.

2. W. J. Coles, *Oscillation criteria for nonlinear second order equations*, Ann. Mat. Pura Appl. (4), **82** (1969), 123–134.

3. ——, *A nonlinear oscillation theorem*, International Conference on Differential Equations, Academic Press, New York, 1975, 193–202.

4. S. R. Grace and B. S. Lalli, *Oscillation theorems for perturbed nonlinear differential equations*, J. Math. Anal. Appl. **77** (1980), 205–214.

5. —— and ——, *Asymptotic and oscillatory behavior of solutions of a class of second order differential equations with deviating arguments*, J. Math. Anal. Appl. **145** (1990), 112–136.

6. —— and ——, *Oscillations in second order differential equations with alternating coefficients*, Period. Math. Hungar. **19** (1988), 69–78.

7. —— and ——, *Oscillation and convergence to zero of solutions of damped second order nonlinear differential equations*, J. Math. Anal. Appl. **102** (1984), 539–548.

8. S. R. Grace, B. S. Lalli, and C. C. Yeh, *Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term*, SIAM J. Math. Anal. **15** (1984), 1082–1093.

9. J. R. Graef, S. M. Rankin and P. W. Spikes, *Oscillation theorems for perturbed nonlinear differential equations*, J. Math. Anal. Appl. **65** (1978), 375–390.

10. I. V. Kamenev, *Integral criterion for oscillation of linear differential equations of second order*, Mat. Zametki **23** (1978), 249–251 [In Russian].

11. I. V. Kamenev, *Oscillation of solutions of second order nonlinear equations with sign-variable coefficients*, Differencial'nye Uravnenija **6** (1970), 1718–1721 [In Russian].

12. A. G. Kartsatos, *Recent results on oscillation of solutions of forces and perturbed nonlinear differential equations of even order*, in *Stability of dynamic systems: Theory and applications*, Lecture Notes in Pure and Applied Mathematics, Vol. **28**, Dekker, New York, 1977, 17–72.

13. M. R. Kulenovic and M. K. Grammatikopoulos, *Oscillation and asymptotic behavior of nonlinear differential inequalities and equations with deviating arguments*, Ukrain. Mat. Zh. **36** (1984), 309–316 [In Russian].

14. G. G. Legatos and A. G. Kartsatos, *Further results on the oscillation of solutions of second order equations*, Math. Japon. **14** (1968), 67–73.

15. Ch. G. Philos, *A second order superlinear oscillation criterion*, Canad. Math. Bull. **27** (1984), 102–112.

16. ——, *Integral averages and second order superlinear oscillation*, Math. Nachr. **120** (1985), 127–138.

17. C. C. Travis, *A note on second order nonlinear oscillation*, Math. Japon. **18** (1973), 261–264.

18. D. Willett, *Classification of second order linear differential equations with respect to oscillation*, Av. Math. **3** (1969), 594–623.

19. J. S. Wong, *On second order nonlinear oscillation*, Fukcial Ekvac. **11** (1068), 207–234.

20. ——, *Oscillation theorems for second order nonlinear differential equations*, Bull. Inst. Math. Acad. Sinica **3** (1975), 283–309.

21. C. C. Yeh, *Oscillation theorems for nonlinear second order differential equations with damped term*, Proc. Amer. Math. Soc. **84** (1982), 397–402.

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