ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

AN ALGORITHM FOR THE PROJECTIVE CHARACTERS OF FINITE CHEVALLEY GROUPS

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ABSTRACT. An algorithm is obtained for the Brauer characters afforded by the projective indecomposable modules (in the defining characteristic) for the finite universal Chevalley groups. Tables of character degrees for the special linear group $SL(4, 2^m)$, $m = 1, 2, 3$, are provided.

In [**4**] we expressed in the language of directed graphs an iterative procedure for finding the irreducible constituents (with multiplicity) of a product of irreducible Brauer characters (in the defining characteristic) of a finite universal Chevalley group. Roughly speaking, the first iteration produces edges which originate at the given product (viewed as a vertex). Each of these edges terminates at either an irreducible Brauer character or a product of such; in the latter case, a second iteration is required. In this manner, paths (sequences of edges) are constructed which eventually terminate at the desired irreducible constituents, the multiplicities of which are then determined by the paths.

The method described uses Steinberg's tensor product theorem and depends on a knowledge of the composition factors (with multiplicity) of products of irreducible modules, with restricted highest weights, for the including infinite algebraic group. Indeed, the method is just a formalization of how one possessing this knowledge would naturally proceed by hand. (Although the required composition factors are not known, in general, they would be known, in principle at least, should Lusztig's conjecture be proven.)

We will show in this paper that if we apply our iterative procedure to a product of just two irreducible Brauer characters, then any paths which terminate at the Steinberg character will have at most two nontrivial edges (provided the characteristic is large enough). Because of this, it is easy to determine all such paths and, hence, the multiplicity

Received by the editors on July 31, 1989.
Research supported in part by the National Science Foundation.

of the Steinberg character as a constituent of the product (given the information mentioned earlier regarding the modules for the algebraic group). Since this multiplicity is the main ingredient of the recursion formula obtained in [**4**] for the characters of the projective indecomposable modules, we easily obtain, in turn, a more explicit formula for these important characters (see 2.7 as well as 2.9 and 2.10).

If the characteristic is too small, then our proofs are no longer valid. If this is the case, however, it is possible to use an explicit knowledge of the modules for the algebraic group to amend our techniques in order to obtain similar results. In section 3 we demonstrate this for the group $SL(4, 2^m)$. In particular, we compute the degrees of the projective indecomposable characters of this group in the cases $m = 1, 2, 3$. (The results of section 3 appear in the author's University of Illinois (Urbana) thesis. The author gratefully acknowledges the guidance and kind encouragement of his advisor, Professor Michio Suzuki.)

1. Preliminaries. Let p be a prime number, and let $m \in \mathbf{Z}^+\backslash\{0\} \cup \{\infty\}$. In what follows, m will be assumed to be fixed except that definitions and notations involving m will be considered established for all $m \in \mathbf{Z}^+\backslash\{0\} \cup \{\infty\}$. If $m < \infty$, set $q = p^m$ and let $\mathbf{F}_q$ denote a field of order q; if $m = \infty$, set $q = \infty$ and let $\mathbf{F}_\infty = K$ denote an algebraic closure of $\mathbf{F}_p$.

Fix an irreducible root system R of rank l and let $G = G^{(m)}$ denote the universal Chevalley group of type R defined over $\mathbf{F}_q$. For $m < \infty$, $G^{(m)}$ is a finite group which we view as a subgroup of the infinite algebraic group $G^{(\infty)}$.

Choose a system $\{\alpha_i, 1 \leq i \leq l\}$ of simple roots in R and let $\{\lambda_i, 1 \leq i \leq l\}$ be the corresponding fundamental dominant weights. The λ_is form a $\mathbf{Z}$-basis for the weight lattice Λ associated with R. For $n \in \mathbf{Z}^+$, we define $\Lambda_n = \{\Sigma a_i\lambda_i \in \Lambda | 0 \leq a_i < n\}$ and we denote by $\Lambda_\infty = \Lambda^+$ the set $\Sigma \mathbf{Z}^+\lambda_i$ of dominant weights.

By "G-module" we shall mean finite dimensional KG-module if $m < \infty$ and finite dimensional rational G-module if $m = \infty$. For $\lambda \in \Lambda^+$, $M(\lambda)$ denotes a fixed irreducible $G^{(\infty)}$-module with highest weight λ.

Let $\mathcal{A} = \mathcal{A}^{(m)}$ denote the Grothendieck ring of the category of G-modules and let φ_M denote the element of $\mathcal{A}$ associated with the module M. If $m < \infty$, we view φ_M as the Brauer character afforded by M and thus identify $\mathcal{A}$ with the ring of Brauer characters of G. The elements $\varphi_\lambda := \varphi_{M(\lambda)}$, $\lambda \in \Lambda_q$, are called *irreducible*; they form a $\mathbf{Z}$-basis for $\mathcal{A}$, so that for each $\varphi \in \mathcal{A}$, there are uniquely determined integers $[\varphi : \varphi_\lambda]^{(m)}$ such that $\varphi = \sum_{\lambda\in\Lambda_q}[\varphi : \varphi_\lambda]^{(m)}\varphi_\lambda$. If $\varphi = \varphi_M$ for some G-module M, then $[\varphi : \varphi_\lambda]^{(m)}$ is just the multiplicity of $M(\lambda)$ as a composition factor of M.

Given any $G^{(\infty)}$-module M, we denote by $\mathrm{Fr}(M)$ the $G^{(\infty)}$-module which has the same underlying vector space as M but on which $g \in G$ acts according to the new rule $g \cdot x = \mathrm{Fr}(g)x$ $(x \in M)$ where Fr is the Frobenius automorphism of $G^{(\infty)}$ which raises matrix entries to the pth power. The assignment $\varphi_M \mapsto \varphi_{\mathrm{Fr}(M)}$ induces an endomorphism of the ring $\mathcal{A}^{(\infty)}$ which we also denote by Fr.

Let $\Lambda^m = \oplus_{j=0}^{m-1} Y_j$ (weak direct sum if $m = \infty$), where Y_j is a copy of Λ. We view Y_j as a subgroup of Λ^m and denote by $\iota_j : \Lambda \to Y_j \subseteq \Lambda^m$ and $\pi_j : \Lambda^m \to Y_j \subseteq \Lambda^m$ the natural injection and projection, respectively. We view Λ^m as a subset of Λ^∞ in the natural way.

Let $J = \{(i,j) | 1 \le i \le l, 0 \le j < m\}$ and for $(i,j) \in J$, set $\lambda_{ij} = \iota_j(\lambda_i)$. Then $\{\lambda_{ij} | (i,j) \in J\}$ is a $\mathbf{Z}$-basis for Λ^m.

Set $\alpha_{ij} = \iota_j(\alpha_i)$ and $\kappa_{ij} = p\lambda_{ij} - \lambda_{i,j+1}$ (viewing second subscripts in $\mathbf{Z}/m\mathbf{Z}$ if $m < \infty$ so that $\lambda_{i,j+1}$ is always defined). We obtain a partial order $\prec$ on Λ^m by declaring $x' \prec x$ if $x - x' \in \mathcal{P} := \mathcal{V} + \mathcal{H}$, where $\mathcal{V} = \Sigma\mathbf{Z}^+\alpha_{ij}$ and $\mathcal{H} = \Sigma\mathbf{Z}^+\kappa_{ij}$.

The assignment $\lambda_{ij} \mapsto p^j\lambda_i$ defines a homomorphism $\mathrm{wt} : \Lambda^m \to \Lambda$ which induces a bijection of the set $\Lambda_p^m := \sum_{j=0}^{m-1} \iota_j(\Lambda_p)$ onto Λ_q. We define $M(x) := M(\mathrm{wt}(x))$ and $\varphi_x := \varphi_{\mathrm{wt}(x)}(x \in \Lambda_p^m)$.

Denote by $\mathcal{X} = \mathcal{X}^{(m)}$ the free abelian monoid on the set $B = \cup_{j=0}^{m-1}\iota_j(\Lambda_p)\backslash\{0\}$. We view each $i_j(\Lambda_p)$ as a subset of $\mathcal{X}$ (identifying $0 \in \iota_j(\Lambda_p)$ with $1 \in \mathcal{X}$) and in turn identify Λ_p^m with its image in $\mathcal{X}$ under the map $\Sigma\iota_j(\mu_j) \mapsto \prod \iota_j(\mu_j)$ $(\mu_j \in \Lambda_p)$. For $x = x_1 \dots x_s \in \mathcal{X}$ $(x_i \in B)$ we set $\varphi_x = \prod \varphi_{x_i}$.

The directed graph Υ which was described in the introduction is defined as follows (cf. [**4**]). Its set of vertices is $\mathcal{X}$ and its set of edges is $\{(\zeta_0, \dots, \zeta_{m-1}) | \zeta_j \in A_j\}$ where $A_j = \{(a,b) \in \pi_j(\mathcal{X}) \times \Lambda_p^\infty | [\varphi_{\mathbf{a}} : \varphi_{\mathbf{b}}]^{(\infty)} \neq 0\}$. (Here $\pi_j : \mathcal{X} \to \mathcal{X}$ fixes $\iota_j(\lambda)$ and sends $\iota_k(\lambda)$ to 1 for

$k \neq j$.) If $e = (\zeta_j) = ((a_j, b_j))$ is an edge, it originates at $o(e) := \prod a_j$ and terminates at $t(e) := \prod \operatorname{res}(b_j)$ where $\operatorname{res} : \mathcal{X}^{(\infty)} \to \mathcal{X}$ is defined by $\iota_j(\lambda) \mapsto \iota_{\bar{j}}(\lambda)$ ($j \mapsto \bar{j}$ is reduction modulo m if $m < \infty$ and the identity map if $m = \infty$).

Let $x, x' \in \mathcal{X}$. A *path* c of length s from x to x' with vertices x_i is a sequence $e_1, \dots, e_s$ of edges such that $o(e_1) = x = x_0$, $t(e_s) = x' = x_s$ and $t(e_i) = o(e_{i+1}) = x_i$ $(1 \leq i < s)$. $C_s(x, x')$ denotes the set of all paths from x to x' of length s. The *essential length of the path* c (written e.l.(c)) is the number of edges for which $o(e_i) \neq t(e_i)$; we set $\text{e.l.}(x, x') = \text{lub}\,\{\text{e.l.}(c) | c \in \cup_s C_s(x, x')\}$.

For $\zeta = (a, b) \in A_{j_0}$ $(0 \leq j_0 < m)$ we define

$$v(\zeta) = \iota_{j_0}(p^{-j_0}(\text{wt}(\bar{a}) - \text{wt}(b))) \in \mathcal{V},$$
$$h(\zeta) = \sum_{i=1}^{l} \left(\sum_{j=j_0}^{\infty} \left(\sum_{k=j+1}^{\infty} b_{ik} p^{k-j-1} \right) \kappa_{ij} \right) \in \mathcal{H}$$

where $b = \sum b_{ij} \lambda_{ij}$, and $\text{mult}(\zeta) = [\varphi_a : \varphi_b]^{\infty}$, where the bar indicates the map which takes $x = x_1 \dots x_s \in \mathcal{X}$ $(x_i \in B)$ to $\bar{x} = \Sigma x_i \in \Lambda^m$. We extend these definitions first to an edge $e = (\zeta_j)$ in Υ by setting $v(e) = \Sigma v(\zeta_j)$, $h(e) = \Sigma h(\zeta_j)$ and $\text{mult}(e) = \prod \text{mult}(\zeta_j)$ and then to a path $c = e_1, \dots, e_s$ in Υ by setting $v(c) = \Sigma v(e_i)$, $h(c) = \Sigma h(e_i)$ and $\text{mult}(c) = \prod \text{mult}(e_i)$.

Theorem 1.1. [**4**, 2.6.1]. *If* $x \in \mathcal{X}$ *and* $x' \in \Lambda_p^m$, *then* $\text{e.l.}(x, x') < \infty$ *and for each positive integer* $s \geq \text{e.l.}(x, x')$, *we have*

$$[\varphi_x : \varphi_{x'}]^{(m)} = \sum_{c \in C_s(x, x')} \text{mult}(c).$$

Theorem 1.2. [**4**, 2.5.4, 2.6.2]. *If* c *is a path in* Υ *from* x *to* x' $(x, x' \in \mathcal{X})$, *then* $\bar{x} - \bar{x}' = h(c) + v(c)$. *In particular, if* $x \in \mathcal{X}$, $x' \in \Lambda_p^m$ *and* $[\varphi_x : \varphi_{x'}]^{(m)} \neq 0$, *then* $x' \prec \bar{x}$.

For the remainder of the paper, we assume $m < \infty$. If $x \in \Lambda_p^m$, we denote by Φ_x the Brauer character afforded by the projective indecomposable G-module $P(x)$ that has unique irreducible quotient $M(x)$. If we set $\gamma = \Sigma(p-1)\lambda_{ij}$, then $\Gamma := \Phi_\gamma = \varphi_\gamma$ is the Steinberg character.

Theorem 1.3. [**4**, 3.1.2]. *If* $x \in \Lambda_p^m$, *then*

$$\Phi_x = \Gamma\bar{\varphi}_{\gamma-x} - \sum_{\substack{x \prec y \in \Lambda_p^m \\ y \neq x}} [\varphi_{\gamma-x}\varphi_y : \Gamma]^{(m)}\Phi_y.$$

(*Here* $\bar{\varphi}$ *denotes the complex conjugate of* φ.)

2. The algorithm. In this section, we assume that $p > \langle \rho, \alpha_0^\vee \rangle + 1$ ($= h_R$, the Coxeter number of R) where $\langle\ ,\ \rangle$ is the inner product in the definition of R, $\rho = \Sigma\lambda_i$ and $\alpha_0^\vee = 2\alpha_0/\langle \alpha_0, \alpha_0\rangle$ is the co-root of the short dominant root α_0.

Lemma 2.1. *If* $\Sigma t_{ij}\lambda_{ij} = \Sigma a_{ij}\alpha_{ij} + \Sigma b_{ij}\kappa_{ij} \in \mathcal{P}$ ($t_{ij} \in \mathbf{Z}$ *and* $a_{ij}, b_{ij} \in \mathbf{Z}^+$), *then* $\langle \sum_i b_{ij}\lambda_i, \alpha_0^\vee\rangle \leq D/(p-1)$ *for each* j, *where* $D = \max_k \langle \Sigma t_{ik}\lambda_i, \alpha_0^\vee\rangle$.

Proof. Since $\kappa_{ij} = p\lambda_{ij} - \lambda_{i,j+1}$, we have $\Sigma_i(t_{ij} + b_{i,j-1} - pb_{ij})\lambda_i = \Sigma_i a_{ij}\alpha_i$ for each j (second subscripts in $\mathbf{Z}/m\mathbf{Z}$). If j_0 is chosen with $\langle \Sigma_i b_{ij_0}\lambda_i, \alpha_0^\vee\rangle$ maximal, then for each j we have

$$\begin{aligned} D &- (p-1)\langle \sum_i b_{ij}\lambda_i, \alpha_0^\vee\rangle \\ &\geq \langle \sum_i t_{ij_0}\lambda_i, \alpha_0^\vee\rangle - \left\langle \sum_i pb_{ij_0}\lambda_i, \alpha_0^\vee \right\rangle + \left\langle \sum_i b_{i,j_0-1}\lambda_i, \alpha_0^\vee\right\rangle \\ &\geq 0 \end{aligned}$$

where we have used the fact that $\langle \alpha_i, \alpha_0^\vee\rangle \geq 0$ for each i. □

The following notation will be used throughout the rest of the paper.

Notation 2.2. Fix two elements $y = \prod \iota_j(\mu_j)$ and $z = \prod \iota_j(\nu_j)$ ($\mu_j, \nu_j \in \Lambda_p$) of $\Lambda_p^m \subseteq \mathcal{X}$ and assume $c = e_1, \dots, e_s$ is a path in Υ

from yz to γ with vertices $yz = x_0, x_1, \dots, x_s = \gamma$. Each edge e_n is an m-tuple ${}^n\zeta^k$ $(0 \le k < m)$ where ${}^n\zeta^k = ({}^na^k, {}^nb^k) \in A_k$. Write ${}^nb^k = \sum_{i,j} {}^nb^k_{ij}\lambda_{ij}$ and set ${}^nb^k_j = \pi_j({}^nb^k) = \sum_i {}^nb^k_{ij}\lambda_{ij}$.

Lemma 2.3. *${}^nb^k_j = 0$ if $j \notin \{k, k+1\}$ or if $n > 1$ and $j \ne k$.*

Proof. From the definitions it is clear that ${}^nb^k_j = 0$ if $j < k$. Now, if we write $y + z - \gamma = \Sigma t_{ij}\lambda_{ij}$ $(t_{ij} \in \mathbf{Z})$, then $t_{ij} \le p-1$ for each $(i,j) \in J$. Also, if $j \ge k+1$, then $\sum_i {}^nb^k_{ij}p^{j-k-1}\kappa_{ik}$ is a summand of $h(c)$ which is a summand of $y + z - \gamma$ (1.2). Thus, by 2.1,

$$\begin{aligned} p^{j-k-1}\left\langle \sum_i {}^nb^k_{ij}\lambda_i, \alpha_0^\vee \right\rangle &\le \frac{1}{p-1}\left\langle \sum_i (p-1)\lambda_i, \alpha_0^\vee \right\rangle \\ &= \langle \rho, \alpha_0^\vee \rangle < p-1 \end{aligned}$$

if $j \ge k+1$. Therefore, since $\langle \lambda_i, \alpha_0^\vee \rangle > 0$ for each i, we have that ${}^nb^k_j = 0$ if $j \notin \{k, k+1\}$.

Now, write $\bar{x}_1 - \gamma = \Sigma s_{ij}\lambda_{ij}$ $(s_{ij} \in \mathbf{Z})$. By the first paragraph, we have $x_1 = \prod_k \text{res}\ [({}^1b^k_k)({}^1b^k_{k+1})]$ so that for each j,

$$\begin{aligned} \left\langle \sum_i s_{ij}\lambda_i, \alpha_0^\vee \right\rangle &= \left\langle \sum_i {}^1b^j_{ij}\lambda_i + \sum_i {}^1b^{j-1}_{ij}\lambda_i - (p-1)\rho, \alpha_0^\vee \right\rangle \\ &\le \left\langle \sum_i {}^1b^{j-1}_{ij}\lambda_i, \alpha_0^\vee \right\rangle < p-1 \end{aligned}$$

(interpreting the superscript $j-1$ as $m-1$ if $j = 0$). Furthermore, if $n > 1$ and $j \ge k+1$, then $\sum_i {}^nb^k_{ij}p^{j-k-1}\kappa_{ik}$ is a summand of $\bar{x}_1 - \gamma$ (1.2), whence $p^{j-k-1}\langle \sum_i {}^nb^k_{ij}\lambda_i, \alpha_0^\vee \rangle < 1$ (2.1). Therefore, ${}^nb^k_j = 0$ if $n > 1$ and $j \ne k$. □

Corollary 2.4. e.l.$(yz, \gamma) \le 2$.

Proof. If c has length at least 2, then 2.3 implies that $x_2 \in \Lambda_p^m \subset \mathcal{X}$. Now it is clear that the only edge in Υ originating at an element of Λ_p^m is the one which terminates at the same element. Hence, $x_2 = \gamma$ and the statement follows. □

Define $\mathcal{H}_0 = \{\Sigma h_{ij}\kappa_{ij} \in \mathcal{H} | \sum_i h_{ij}\lambda_i \in \mathcal{E}$ for each $j\}$ where $\mathcal{E} = \{\eta \in \Lambda_p | \langle \eta, \alpha_0^\vee \rangle \leq \langle \rho, \alpha_0^\vee \rangle\}$.

Corollary 2.5. *If $h(c) = \Sigma h_{ij}\kappa_{ij}$, then $h_{ij} = {}^1b_{i,j+1}^j$ for each $(i,j) \in J$. In particular, $h(c) \in \mathcal{H}_0$.*

Proof. This is clear from 2.3 and its proof. □

For $\mu, \nu, \eta', \eta \in \Lambda_p$ we define

$$\operatorname{mult}(\mu, \nu, \eta', \eta) = \sum_{\beta \in \Lambda_p} [\varphi_\mu \varphi_\nu : \varphi_\beta \operatorname{Fr}(\varphi_\eta)]^{(\infty)} [\varphi_\beta \varphi_{\eta'} : \varphi_{(p-1)\rho}]^{(\infty)}.$$

Lemma 2.6. *If $h = \Sigma h_{ij}\kappa_{ij} \in \mathcal{H}_0$, then*

$$\sum_{\substack{c \in C_2(yz,\gamma) \\ h(c)=h}} \operatorname{mult}(c) = \prod_{k=0}^{m-1} \operatorname{mult}(\mu_k, \nu_k, \eta_{k-1}, \eta_k)$$

where $\eta_j = \sum_i h_{ij}\lambda_i$.

Proof. Assume that c has length 2 and that $h(c) = h$. Then

$$\operatorname{mult}(c) = \prod_{k=0}^{m-1} [\varphi_{\pi_k(y)} \varphi_{\pi_k(z)} : \varphi_{{}^1b_k^k} \varphi_{{}^1b_{k+1}^k}]^{(\infty)} [\varphi_{{}^1b_k^k} \varphi_{{}^1b_k^{k-1}} : \varphi_{\pi_k(\gamma)}]^{(\infty)}$$

by 2.3. Set $\beta_k = \sum_i {}^1b_{ik}^k \lambda_i \in \Lambda_p$. By 2.5 and the fact that $[\varphi : \varphi']^{(\infty)} = [\operatorname{Fr}(\varphi) : \operatorname{Fr}(\varphi')]^{(\infty)}$ ($\varphi, \varphi' \in \mathcal{A}^{(\infty)}$, φ' irreducible) we get

$$\operatorname{mult}(c) = \prod_{k=0}^{m-1} [\varphi_{\mu_k} \varphi_{\nu_k} : \varphi_{\beta_k} \operatorname{Fr}(\varphi_{\eta_k})]^{(\infty)} [\varphi_{\beta_k} \varphi_{\eta_{k-1}} : \varphi_{(p-1)\rho}]^{(\infty)}.$$

Conversely, if this product is nonzero for some m-tuple (β_k) ($\beta_k \in \Lambda_p$), then the assignments ${}^1b^k = \iota_k(\beta_k)\iota_{k+1}(\eta_k)$ and ${}^2b^k = \iota_k(7)$, determine a path $c \in C_2(yz, \gamma)$ with $h(c) = h$ (see 2.5). Therefore,

$$\sum_{\substack{c \in C_2(yz,\gamma) \\ h(c)=h}} \text{mult}(c)$$
$$= \sum_{\substack{(\beta_k) \\ \beta_k \in \Lambda_p}} \prod_{k=0}^{m-1} [\varphi_{\mu_k}\varphi_{\nu_k} : \varphi_{\beta_k}\text{Fr}(\varphi_{\eta_k})]^{(\infty)}[\varphi_{\beta_k}\varphi_{\eta_{k-1}} : \varphi_{(p-1)\rho}]^{(\infty)}$$

and switching the sum and product on the right gives the desired formula. □

For $\mu \in \Lambda_p$, define

$$T(\mu) = \{(\eta', \eta, \tau) | \eta', \eta \in \mathcal{E}, \tau \in \Lambda, \mu + \tau \in \Lambda_p \text{ and } \tau + \eta' - p\eta \in \Sigma \mathbf{Z}^+ \alpha_i\}$$

and for $x = \Sigma \iota_j(\mu_j) \in \Lambda_p^m$, let

$$U(x) = \{((\eta'_j, \eta_j, \tau_j)) \in \times_{j=0}^{m-1} T(\mu_j) | \eta'_j = \eta_{j-1},\ 0 \leq j \leq m-1\}$$

(interpreting η_{-1} as η_{m-1}). Now, for each $u = ((\eta'_j, \eta_j, \tau_j)) \in U(x)$, set $\tau_u = \Sigma \iota_j(\tau_j)$ and $h_u = \Sigma h_{ij}\kappa_{ij}$, where $\eta_j = \Sigma_i h_{ij}\lambda_i$, and define

$$\pi(u) = \prod_{k=0}^{m-1} \text{mult}((p-1)\rho - \mu_k, \mu_k + \tau_k, \eta'_k, \eta_k).$$

Theorem 2.7. *If $x \in \Lambda_p^m$, then*

$$\Phi_x = \Gamma\bar{\varphi}_{\gamma - x} - \sum_{\substack{u \in U(x) \\ \tau_u \neq 0}} \pi(u)\Phi_{x+\tau_u}.$$

Proof. From 1.3, 1.1, 2.4 and 2.5, we see that

$$\Phi_x = \Gamma\bar{\varphi}_{\gamma-x} - \sum_h \sum_\tau \sum_{\substack{c \\ h(c)=h}} \text{mult}(c)\Phi_{x+\tau} \tag{2.8}$$

where the first sum is over all $h \in \mathcal{H}_0$, the second sum is over all $\tau \in \mathcal{P}\backslash\{0\}$ such that $x + \tau \in \Lambda_p^m$ and the third sum is over all $c \in C_2(\tau) := C_2((\gamma - x)(x+\tau), \gamma)$ such that $h(c) = h$.

Let h and τ be fixed indices for the first two summations, respectively, in 2.8 having the property that $h = h(c)$ for some $c \in C_2(\tau)$. As before, write $h = \Sigma h_{ij}\kappa_{ij}$ ($h_{ij} \in \mathbf{Z}^+$) and set $\eta_j = \sum_i h_{ij}\lambda_i$. Similarly, write $\tau = \Sigma t_{ij}\lambda_{ij}$ ($t_{ij} \in \mathbf{Z}$) and set $\tau_j = \sum_i t_{ij}\lambda_i$. Fix $c \in C_2(\tau)$ with $h(c) = h$ and write $v(c) = \Sigma a_{ij}\alpha_{ij}$ ($a_{ij} \in \mathbf{Z}^+$). 1.2 implies that $\tau = h(c) + v(c)$. Therefore,

$$\begin{aligned}\iota_j(\tau_j + \eta_{j-1} - p\eta_j) &= \iota_j\left[\sum_i (t_{ij} + h_{i,j-1} - ph_{ij})\lambda_i\right] \\ &= \pi_j\left(\sum_{i,k} t_{ik}\lambda_{ik} - \sum_{i,k} h_{ik}\kappa_{ik}\right) \\ &= \pi_j\left(\sum_{i,k} a_{ik}\alpha_{ik}\right) = \iota_j\left(\sum_i a_{ij}\alpha_i\right),\end{aligned}$$

whence $\tau_j + \eta_{j-1} - p\eta_j = \sum_i a_{ij}\alpha_i \in \Sigma\mathbf{Z}^+\alpha_i$.

It follows that $(h,\tau) \mapsto ((\eta_{j-1}, \eta_j, \tau_j))$ defines a bijection from the set of all pairs (h,τ) in 2.8 having the property that $h(c) = h$ for some $c \in C_2(\tau)$ onto the set of all $u \in U(x)$ having the properties that $\tau_u \neq 0$ and $h(c) = h_u$ for some $c \in C_2(\tau_u)$; the inverse of this map is $u \mapsto (h_u, \tau_u)$. Therefore, we have

$$\Phi_x = \Gamma\bar{\varphi}_{\gamma-x} - \sum_{\substack{u \in U(x) \\ \tau_u \neq 0}} \sum_{\substack{c \in C_2(\tau_u) \\ h(c) = h_u}} \mathrm{mult}(c)\Phi_{x+\tau_u}.$$

The desired formula now follows from 2.6. □

Remark 2.9. We comment briefly on how 2.7 can be used to compute the values of the projective indecomposable characters at a p'-element (for instance 1_G) of G. It is easy to write a computer program which will determine the sets $T(\mu)$ and $U(x)$. (For the special case $G = SL(4, 2^m)$, see [**3**].) Aside from these sets, one needs to know only the values of the irreducible characters at the given p'-element, the composition factor multiplicities $[\varphi_\mu\varphi_{\mu'} : \varphi_\lambda]^{(\infty)}$ ($\mu, \mu' \in \Lambda_p, \lambda \in \Lambda_{p^2}$) and a linear ordering of the elements of Λ_p^m which places $x \in \Lambda_p^m$ after each $x + \tau_u$ ($u \in U(x), \tau_u \neq 0$). The linear ordering is easy to arrange. Let $f : \Lambda \to \mathbf{R}$ be any homomorphism satisfying $f(\lambda_i) > 0$ and $f(\alpha_i) > 0$ for each i (e.g., $f = \langle \cdot, \rho \rangle$), and let $\bar{f} : \Lambda^m \to \mathbf{R}$ be the homomorphism induced by $\lambda_{ij} \mapsto f(\lambda_i)$. Since $\tau_u \in \mathcal{P}\backslash\{0\}$ (see the proof of 2.7), it follows that $\bar{f}(\tau_u) > 0$. Therefore, given $x, y \in \Lambda_p^m$, it suffices to put y before x if $\bar{f}(y) > \bar{f}(x)$ (and to order them arbitrarily if $\bar{f}(y) = \bar{f}(x)$).

Remark 2.10. The proof of 1.3 relies on the fact that, for $x \in \Lambda_p^m$, the $G^{(m)}$-module $M := M(\gamma) \otimes_K M(\gamma - x)^*$ (* denotes contragredient) is projective and hence a direct sum of various $P(y)$ ($y \in \Lambda_p^m$) with $P(x)$ appearing exactly once. In general, M is much larger than $P(x)$ in the sense that M has many summands $P(y)$ with $y \neq x$. Consequently, to simplify computations, it is reasonable to look for a naturally occurring and well understood summand of M which is a direct sum of $P(x)$ and a fewer number of the other $P(y)$'s. For $p \geq 2h_R - 2$, it is shown in [**5**] that the restriction to $G^{(m)}$ of the injective hull $Q(x)$ of $M(x)$ in the category of "p^m-restricted" $G^{(\infty)}$-modules is such a summand. In fact, Jantzen shows in [**6**, 2.10 Corollary 2] that the multiplicity of $P(y)$ as a summand of $Q(x)$ is

$$\sum_{z \in \Lambda_p^\infty} [\varphi_y \varphi_z : \varphi_x \mathrm{Fr}^m(\varphi_z)]^{(\infty)}.$$

(Jantzen remarks that we actually need only sum over those z for which $\rho - \mathrm{wt}(z) \in \Sigma \mathbf{Q}^+ \alpha_i$.) From 1.2 it now follows that if this multiplicity is nonzero, then $y - x \in \mathcal{P}'$, where

$$\mathcal{P}' = \{0\} \cup \left\{ \sum a_{ij}\alpha_{ij} + \sum b_{ij}\kappa_{ij} \in \mathcal{P} = \mathcal{P}^{(m)} \middle| \sum_i b_{ij} \neq 0 \text{ for each } j \right\}.$$

Therefore, denoting by Ψ_x the Brauer character of the $G^{(m)}$-module $Q(x)$, we get a formula for Φ_x which resembles that in 1.3

$$\Phi_x = \Psi_x - \sum_{\substack{x \prec' y \in \Lambda_p^m \\ y \neq x}} \sum_{z \in \Lambda_p^\infty} [\varphi_y \varphi_z : \varphi_x \mathrm{Fr}^m(\varphi_z)]^{(\infty)} \Phi_y,$$

where $x \prec' y$ if and only if $y - x \in \mathcal{P}'$. As anticipated, the index y in this formula ranges over a smaller set than that in 1.3, and so in this respect computations are simplified. On the other hand, here we need to know the characters Ψ_x which are more complicated, in general, than the irreducible characters required for 1.3. (For some computations of $\dim_K P(x)$, via the modules $Q(x)$, in the case $G = SL(3, p^m)$ (as well as $SU(3, p^{2m})$), see the thesis [**1**] of Jantzen's student, Dordowsky.)

3. An example. $SL(4, 2^m)$. If the characteristic p does not satisfy the assumption of the previous section, in other words, if $p \leq \langle \rho, \alpha_0^\vee \rangle + 1$, it is still possible that a result similar to 2.7 can be obtained by modifying the methods. For instance, this is the case for the group $G = SL(4, 2^m)$ which we use here for an illustration.

The following lemma corresponds to 2.1.

Lemma 3.1. *Assume* $\Sigma t_{ij}\lambda_{ij} = \Sigma a_{ij}\alpha_{ij} + \Sigma b_{ij}\kappa_{ij}$ *with* $t_{ij} \in \{-1, 0, 1\}$ *and* $a_{ij}, b_{ij} \in \mathbf{Z}^+$. *We have the following*:

(i) $\sum_i b_{ij} \leq 3$ *for each* j, *and if equality holds for some* j, *then* $t_{ij} = b_{ij} = 1$ *and* $a_{ij} = 0$ *for each* $(i, j) \in J$.

(ii) *If* $b_{i_0 j_0} \geq 2$ *for some* (i_0, j_0), *then* $a_{1j_0} + a_{3j_0} \leq 1$.

Proof. (i) Since $\alpha_{1j} = 2\lambda_{1j} - \lambda_{2j}$, $\alpha_{2j} = -\lambda_{1j} + 2\lambda_{2j} - \lambda_{3j}$, $\alpha_{3j} = 2\lambda_{3j} - \lambda_{2j}$ and $\kappa_{ij} = 2\lambda_{ij} - \lambda_{i,j+1}$, we obtain, for each j, the equations

$$t_{1j} = 2a_{1j} + 2b_{1j} - a_{2j} - b_{1,j-1}, \tag{3.2}$$
$$t_{2j} = 2a_{2j} + 2b_{2j} - a_{1j} - a_{3j} - b_{2,j-1} \tag{3.3}$$

and

$$t_{3j} = 2a_{3j} + 2b_{3j} - a_{2j} - b_{3,j-1}. \tag{3.4}$$

(We view all second subscripts in $\mathbf{Z}/m\mathbf{Z}$.) Adding these equations gives

$$\sum_i t_{ij} = a_{1j} + a_{3j} + 2\sum_i b_{ij} - \sum_i b_{i,j-1} \tag{3.5}$$

for each j.

Fix j with $\sum_i b_{ij}$ maximal. From 3.5, we obtain

$$(3.6)\quad \sum_i b_{ij} = \frac{1}{2}\bigg(\sum_i t_{ij} + \sum_i b_{i,j-1} - a_{1j} - a_{3j}\bigg) \le \frac{1}{2}\bigg(3 + \sum_i b_{ij}\bigg),$$

so that $\sum_i b_{ij} \le 3$. Assume $\sum_i b_{ij} = 3$. Then 3.6 implies that $\sum_i t_{ij} = 3$, $\sum_i b_{i,j-1} = 3$ and $a_{1j} + a_{3j} = 0$, whence $t_{ij} = 1$ for each i and $a_{1j} = a_{3j} = 0$. By induction, $\sum_i b_{ij} = 3$ and $a_{1j} = a_{3j} = 0$ for each j and $t_{ij} = 1$ for each (i, j). Equations 3.2, 3.3 and 3.4 now imply that each b_{ij} is at least 1 and hence exactly 1. Finally, 3.2 implies that $a_{2j} = 0$ for each j.

(ii) If $b_{i_0j_0} \ge 2$ for some pair (i_0, j_0), then (i) implies that $\sum_i b_{ij} \le 2$ for each j. Equation 3.5 then gives $a_{1j_0} + a_{3j_0} = \sum_i t_{ij_0} - 2\sum_i b_{ij_0} + \sum_i b_{i,j_0-1} \le 3 - 4 + 2 = 1$. □

Let vol : $\Lambda^m \to \mathbf{Z}$ be the homomorphism induced by $\lambda_{ij} \mapsto 1$ and extend this map to $\mathcal{X}$ by setting $\mathrm{vol}(x) = \mathrm{vol}(\bar{x})$ for each $x \in \mathcal{X}$.

Lemma 3.7. *Let $x, x' \in \mathcal{X}$. If $\overline{x'} \prec \bar{x}$, then* $\mathrm{vol}(x') \le \mathrm{vol}(x)$.

Proof. The assumption $\overline{x'} \prec \bar{x}$ implies that $\bar{x} - \overline{x'} = \Sigma a_{ij}\alpha_{ij} + \Sigma b_{ij}\kappa_{ij}$ with $a_{ij}, b_{ij} \in \mathbf{Z}^+$. Now, $\mathrm{vol}(\alpha_{1j}) = \mathrm{vol}(a_{3j}) = 1$ and $\mathrm{vol}(\alpha_{2j}) = 0$ for each j, and $\mathrm{vol}(\kappa_{ij}) = 1$ for each (i, j). Therefore, $\mathrm{vol}(x) - \mathrm{vol}(x') = \mathrm{vol}(\bar{x}) - \mathrm{vol}(\overline{x'}) = \mathrm{vol}(\bar{x} - \overline{x'}) \ge 0$. □

We will simplify notation by writing the number $a_1 + 2a_2 + 4a_3$ in place of the weight $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3$, and by writing φ^σ in place of $\mathrm{Fr}(\varphi)$. The next lemma gives all of the composition factor multiplicities $[\varphi_\mu\varphi_{\mu'} : \varphi_\lambda]^{(\infty)}$ $(\mu, \mu' \in \Lambda_p, \lambda \in \Lambda^+)$.

Lemma 3.8. *In the Grothendieck ring $\mathcal{A}^{(\infty)}$, we have the following formulas.*

(1) $\varphi_1\varphi_1 = \varphi_1^\sigma + 2\varphi_2$.

(2) $\varphi_1\varphi_2 = \varphi_3 + \varphi_4$.

(3) $\varphi_1\varphi_3 = \varphi_2\varphi_1^\sigma + 2\varphi_2^\sigma + 2 + 3\varphi_5$.

(4) $\varphi_1\varphi_4 = \varphi_5 + 2$.

(5) $\varphi_1\varphi_5 = \varphi_4\varphi_1^\sigma + 2\varphi_6$.

(6) $\varphi_1\varphi_6 = \varphi_7 + \varphi_4^\sigma + 2\varphi_2$.

(7) $\varphi_1\varphi_7 = \varphi_6\varphi_1^\sigma + 2\varphi_4\varphi_2^\sigma + 3\varphi_1\varphi_4^\sigma + 4\varphi_3$.

(8) $\varphi_2\varphi_2 = \varphi_2^\sigma + 2\varphi_5 + 2$.

(9) $\varphi_2\varphi_3 = \varphi_1\varphi_2^\sigma + \varphi_1 + 2\varphi_4\varphi_1^\sigma + 3\varphi_6$.

(10) $\varphi_2\varphi_4 = \varphi_6 + \varphi_1$.

(11) $\varphi_2\varphi_5 = \varphi_7 + \varphi_1^\sigma + \varphi_4^\sigma + 2\varphi_2$.

(12) $\varphi_2\varphi_6 = \varphi_4\varphi_2^\sigma + \varphi_4 + 2\varphi_1\varphi_4^\sigma + 3\varphi_3$.

(13) $\varphi_2\varphi_7 = \varphi_5\varphi_2^\sigma + 2\varphi_5^\sigma + 3\varphi_2\varphi_1^\sigma + 3\varphi_2\varphi_4^\sigma + 6\varphi_5 + 6\varphi_2^\sigma + 8$.

(14) $\varphi_3\varphi_3 = \varphi_3^\sigma + 2\varphi_7 + 2\varphi_5\varphi_1^\sigma + 2\varphi_2\varphi_2^\sigma + 4\varphi_1^\sigma + 4\varphi_4^\sigma + 6\varphi_2$.

(15) $\varphi_3\varphi_4 = \varphi_7 + \varphi_1^\sigma + 2\varphi_2$.

(16) $\varphi_3\varphi_5 = \varphi_6\varphi_1^\sigma + \varphi_1\varphi_1^\sigma + 2\varphi_4\varphi_2^\sigma + 2\varphi_4 + 3\varphi_1\varphi_4^\sigma + 4\varphi_3$.

(17) $\varphi_3\varphi_6 = \varphi_5\varphi_2^\sigma + 2\varphi_5^\sigma + 3\varphi_2\varphi_1^\sigma + 3\varphi_2\varphi_4^\sigma + 6\varphi_2^\sigma + 7\varphi_5 + 10$.

(18) $\varphi_3\varphi_7 = \varphi_4\varphi_3^\sigma + 2\varphi_3\varphi_4^\sigma + 2\varphi_1\varphi_5^\sigma + 2\varphi_6\varphi_2^\sigma + 3\varphi_3\varphi_1^\sigma + 4\varphi_4\varphi_4^\sigma + 4\varphi_1\varphi_2^\sigma + 6\varphi_4\varphi_1^\sigma + 8\varphi_6 + 8\varphi_1$.

(19) $\varphi_4\varphi_4 = \varphi_4^\sigma + 2\varphi_2$.

(20) $\varphi_4\varphi_5 = \varphi_1\varphi_4^\sigma + 2\varphi_3$.

(21) $\varphi_4\varphi_6 = \varphi_2\varphi_4^\sigma + 2\varphi_2^\sigma + 2 + 3\varphi_5$.

(22) $\varphi_4\varphi_7 = \varphi_3\varphi_4^\sigma + 2\varphi_1\varphi_2^\sigma + 3\varphi_4\varphi_1^\sigma + 4\varphi_6$.

(23) $\varphi_5\varphi_5 = \varphi_5^\sigma + 2\varphi_2\varphi_1^\sigma + 2\varphi_2\varphi_4^\sigma + 4\varphi_5 + 4\varphi_2^\sigma + 6$.

(24) $\varphi_5\varphi_6 = \varphi_3\varphi_4^\sigma + \varphi_4\varphi_4^\sigma + 2\varphi_1\varphi_2^\sigma + 2\varphi_1 + 3\varphi_4\varphi_1^\sigma + 4\varphi_6$.

(25) $\varphi_5\varphi_7 = \varphi_2\varphi_5^\sigma + 2\varphi_7 + 2\varphi_3^\sigma + 2\varphi_6^\sigma + 3\varphi_5\varphi_4^\sigma + 3\varphi_5\varphi_1^\sigma + 4\varphi_2\varphi_2^\sigma + 8\varphi_1^\sigma + 8\varphi_4^\sigma + 10\varphi_2$.

(26) $\varphi_6\varphi_6 = \varphi_6^\sigma + 2\varphi_7 + 2\varphi_5\varphi_4^\sigma + 2\varphi_2\varphi_2^\sigma + 4\varphi_4^\sigma + 4\varphi_1^\sigma + 6\varphi_2$.

(27) $\varphi_6\varphi_7 = \varphi_1\varphi_6^\sigma + 2\varphi_6\varphi_1^\sigma + 2\varphi_4\varphi_5^\sigma + 2\varphi_3\varphi_2^\sigma + 3\varphi_6\varphi_4^\sigma + 4\varphi_1\varphi_1^\sigma + 4\varphi_4\varphi_2^\sigma + 6\varphi_1\varphi_4^\sigma + 8\varphi_3 + 8\varphi_4$.

(28) $\varphi_7\varphi_7 = \varphi_7^\sigma + 2\varphi_7\varphi_1^\sigma + 2\varphi_7\varphi_4^\sigma + 2\varphi_5\varphi_5^\sigma + 2\varphi_2\varphi_3^\sigma + 2\varphi_2\varphi_6^\sigma + 4\varphi_1^{\sigma^2} + 4\varphi_4^{\sigma^2} + 4\varphi_2^{\sigma^2} + 8\varphi_5\varphi_2^\sigma + 14\varphi_2\varphi_1^\sigma + 14\varphi_2\varphi_4^\sigma + 16\varphi_5^\sigma + 20\varphi_5 + 32\varphi_2^\sigma + 40$.

In the preceding lemma, formulas (1), (2), (4), (8), (10), (11) and (19) were computed first using weight space decompositions (and duality) and then the remaining formulas were obtained by using the associative law in $\mathcal{A}^{(\infty)}$. For instance, the equation $\varphi_7 + \varphi_1^\sigma + \varphi_4^\sigma + 4\varphi_2 = (\varphi_4\varphi_1)\varphi_2 = \varphi_4(\varphi_1\varphi_2) = \varphi_3\varphi_4 + \varphi_4^\sigma + 2\varphi_2$ gives formula (15).

We return to the notation set up in 2.2 and further define ${}^n\beta_j^k = \sum_i {}^n b_{ij}^k \lambda_i \in \Lambda_p$.

Lemma 3.9. *${}^n b_j^k = 0$ if $j \notin \{k, k+1\}$.*

Proof. We proceed by induction on n. First assume that $n = 1$ and suppress the superscript n in the notation. Since $a^k = \iota_k(\mu_k)\iota_k(\nu_k)$, 3.8 implies that $b_j^k = 0$ if $j \notin \{k, k+1, k+2\}$ so we need only show that $b_{k+2}^k = 0$. Assume $b_{k+2}^k \neq 0$ for some k. Then 3.8 implies that $\mu_k = \nu_k = 7$ and $d := (\beta_k^k, \beta_{k+1}^k, \beta_{k+2}^k) = (0,0,1), (0,0,2)$ or (0,0,4). Therefore, we have

$$v(\zeta^k) = \iota_k(\mu_k + \nu_k - \beta_k^k - 2\beta_{k+1}^k - 4\beta_{k+2}^k)$$

and

$$h(\zeta^k) = \sum_i [(b_{i,k+1}^k + 2b_{i,k+2}^k)\kappa_{ik} + b_{i,k+2}^k \kappa_{i,k+1}].$$

If $d = (0,0,1)$, then $v(\zeta^k) = -2\lambda_{1k} + 2\lambda_{2k} + 2\lambda_{3k} = 2\alpha_{2k} + 2\alpha_{3k}$ and $h(\zeta^k) = 2\kappa_{1k} + \kappa_{1,k+1}$. But this contradicts 3.1(ii) since $v(\zeta^k)$ and $h(\zeta^k)$ are summands of $v(c) + h(c) = \overline{yz} - \gamma = y + z - \gamma \in \{\Sigma t_{ij}\lambda_{ij} | t_{ij} \in \{-1,0,1\}\}$ (see 1.2). We obtain a similar contradiction if either $d = (0,0,4)$, in which case $v(\zeta^k) = 2\alpha_{1k} + 2\alpha_{2k}$ and $h(\zeta^k) = 2\kappa_{3k} + \kappa_{3,k+1}$, or $d = (0,0,2)$, in which case $v(\zeta^k) = \alpha_{1k} + \alpha_{3k}$ and $h(\zeta^k) = 2\kappa_{2k} + \kappa_{2,k+1}$. This handles the case $n = 1$.

If $n > 1$, the induction hypothesis gives ${}^n a^k = ({}^{n-1}b_k^k)({}^{n-1}b_k^{k-1}) = \iota_k({}^{n-1}\beta_k^k)\iota_k({}^{n-1}\beta_k^{k-1})$ (interpreting the superscript $k-1$ as $m-1$ if $k = 0$) so the argument given above for the case $n = 1$ applies here to complete the proof. $\square$

Lemma 3.10. e.l.$(yz, \gamma) \leq 3$.

Proof. It is enough to assume that the length of c is at least 3 and prove that $x_i = \gamma$ for some $i \leq 3$.

Suppose $\mathrm{vol}({}^1b^k_{k+1}) = 3$ for some k. Then ${}^1b^k_{i,k+1} = 1$ for each i, so that $h({}^1\zeta^k) = \sum_i \kappa_{ik}$ (3.9). By 3.1(i) we have that $y = z = \gamma$ and $a_{ij} = 0$ for each $(i,j) \in J$ where $v(c) = \Sigma a_{ij}\alpha_{ij}$ (cf. proof of 3.9). In particular, ${}^1a^k = \iota_k(7)\iota_k(7)$ and $v({}^1\zeta^k) = 0$ for each k, so that, by 3.8, ${}^1b^k = \iota_{k+1}(7)$ for each k. Therefore, $x_1 = \prod \mathrm{res}({}^1b^k) = \gamma$.

Now suppose that $\mathrm{vol}({}^1b^k_{k+1}) < 3$ for each k. Then $\mathrm{vol}({}^2a^k) = \mathrm{vol}({}^1b^k_k) + \mathrm{vol}({}^1b^{k-1}_k) \leq 5$, whence $\mathrm{vol}({}^2b^k) \leq 3$ for each k (3.8). Since $\gamma \prec \overline{x_n}$ for each n (1.2), we have from 3.7 that $3m = \mathrm{vol}(\gamma) \leq \mathrm{vol}(x_2) = \Sigma\mathrm{vol}({}^2b^k) \leq \Sigma 3 = 3m$. Thus, $\mathrm{vol}(x_2) = 3m$ and $\mathrm{vol}({}^2b^k) = 3$ for each k.

We now prove that $\mathrm{vol}({}^3a^k) = \mathrm{vol}({}^3b^k) = 3$ for each k. If $\mathrm{vol}({}^3a^k) \neq 3$ for some k, then, since $\Sigma\mathrm{vol}({}^3a^k) = \mathrm{vol}(x_2) = 3m$, we must have $\mathrm{vol}({}^3a^k) > 3$ for some k in which case $\mathrm{vol}({}^3b^k) < \mathrm{vol}({}^3a^k)$ by 3.8. But, in any event, $\mathrm{vol}({}^3b^k) \leq \mathrm{vol}({}^3a^k)$ for each k (3.8), whence $3m = \mathrm{vol}(\gamma) \leq \mathrm{vol}(x_3) = \Sigma\mathrm{vol}({}^3b^k) \leq \Sigma\mathrm{vol}({}^3a^k) = 3m$. We conclude that $\mathrm{vol}({}^3a^k) = \mathrm{vol}({}^3b^k) = 3$ for each k.

Finally, the preceding paragraph and 3.8 show that each ${}^3b^k = \iota_k(7)$, whence $x_3 = \gamma$. □

Corollary 3.11. $\mathrm{vol}({}^2b^k) = \mathrm{vol}({}^3a^k) = \mathrm{vol}({}^3b^k) = 3$ *for each* k.

Proof. This follows from the proof of 3.10. □

Lemma 3.12. *Assume that the length of* c *is* 3.

(i) *If* $x_2 = \gamma$, *then* $h(c) \in \mathcal{H}_0$.

(ii) *If* $x_2 \neq \gamma$, *then* $y = z = \gamma$ *and* $\mathrm{mult}(c) = 2^m$.

Moreover, $C_3(\gamma^2, \gamma)$ *contains exactly two paths for which* $x_2 \neq \gamma$.

Proof. We consider two cases.

Case 1. $\mathrm{vol}({}^2a^k) = 3$ for some k. Fix such a k. 3.8 and 3.11 imply that ${}^2b^k = \iota_k(7)$. Since ${}^3a^{k+1} = ({}^2b^{k+1}_{k+1})({}^2b^k_{k+1})$ (by 3.9) and ${}^2b^k_{k+1} = 1 \in \mathcal{X}$, we have that $\mathrm{vol}({}^2b^{k+1}_{k+1}) = \mathrm{vol}({}^3a^{k+1}) = 3$ (by 3.11), whence, ${}^2b^{k+1} = \iota_{k+1}(7)$ (3.11 again). Continuing this process, we obtain ${}^2b^k = \iota_k(7)$ for each k, so that $x_2 = \prod \operatorname{res}({}^2b^k) = \gamma$. Combining our results with 3.9, we now have that ${}^nb_j^k = 0$ if $j \notin \{k, k+1\}$ or if $n > 1$ and $j \neq k$ (cf. 2.3). Also, if $h(c) = \Sigma h_{ij}\kappa_{ij}$, then $h_{ij} = {}^1b_{i,j+1}^j$ so that $h(c) \in \mathcal{H}_0$ (cf. 2.5).

Case 2. $\operatorname{vol}({}^2a^k) \neq 3$ for each k. If $\operatorname{vol}({}^2a^k) < 3$ for some k, then 3.11 and 3.8 give the contradiction $3 = \operatorname{vol}({}^2b^k) \leq \operatorname{vol}({}^2a^k) < 3$. Therefore, $\operatorname{vol}({}^2a^k) \geq 4$ for each k. Since $\operatorname{vol}({}^1b^k) \leq 4$ for each k (3.8), we obtain $4m \leq \Sigma\operatorname{vol}({}^2a^k) = \Sigma\operatorname{vol}({}^1b^k) \leq 4m$, so that $\operatorname{vol}({}^1b^k) = 4$ for each k. 3.8 now implies that $y = z = \gamma$ and, for each k, $({}^1\beta_k^k, {}^1\beta_{k+1}^k) = (7,1)$, (7,4) or (5,5). Set ${}^nd^k = ({}^n\beta_k^k, {}^n\beta_{k+1}^k)$. We will establish the following statements.

(1) If ${}^1d^k = (7,1)$ for some k, then ${}^1d^k = (7,1)$ and ${}^2d^k = (6,1)$ for each k.

(2) If ${}^1d^k = (7,4)$ for some k, then ${}^1d^k = (7,4)$ and ${}^2d^k = (3,4)$ for each k.

(3) ${}^1d^k \neq (5,5)$ for each k.

Assume that ${}^1d^k = (7,1)$ for some fixed k. If ${}^1d^{k'} = (5,5)$ for some k', we may assume k' is chosen so that ${}^1d^{k'-1} = (7,1)$ or (7,4) (interpreting ${}^1d^{-1}$ as ${}^1d^{m-1}$). But then 3.8 implies that $\operatorname{vol}({}^2b^{k'}) < 3$, contrary to 3.11. Now, if ${}^1d^{k''} = (7,4)$ for some k'', we may assume that $k'' = k-1$. Then 3.8 and 3.11 imply that ${}^2d^k = (3,4)$ and ${}^2d^{k+1} = (6,1)$. So ${}^3a^{k+1} = ({}^2b_{k+1}^{k+1})({}^2b_{k+1}^k) = \iota_{i+1}(6)\iota_{k+1}(4)$ and $\operatorname{vol}({}^3b^{k+1}) < 3$ (by 3.8) contradicting 3.11. Thus, ${}^1d^k = (7,1)$ for each k, and from 3.8 and 3.11 we find that ${}^2d^k = (6,1)$ for each k. This proves (1) and a similar argument proves (2). Finally, if ${}^1d^k = (5,5)$ for some k, then (1) and (2) imply that ${}^1d^k = (5,5)$ for each k. But then 3.8 implies that $\operatorname{vol}({}^2b^k) < 3$ (for each k), contrary to 3.11. This proves (3).

We have shown that, under the assumption $\operatorname{vol}({}^2a^k) \neq 3$ for each k, there are only two possibilities (given by the conditions in (1) and (2), respectively) for the path c and that for either of these possibilities, $x_2 \neq \gamma$, $y = z = \gamma$ and $\operatorname{mult}(c) = \prod_{n=1}^3 \prod_k \operatorname{mult}({}^n\zeta^k) = 2^m1^m1^m = 2^m$(3.8). □

Theorem 3.13. *For each $x \in \Lambda_p^m$, we have*

$$\Phi_x = \Gamma\bar{\varphi}_{\gamma-x} - \sum_{\substack{u\in U(x)\\ \tau_u\neq 0}} \pi(u)\Phi_{x+\tau_u} - 2^{m+1}\delta_{x0}\Gamma$$

($\delta_{x0} =$ *Kronecker delta*).

Proof. Because of the modified lemmas 3.10 and 3.12 the proof of 2.7 carries over here provided we subtract $2^{m+1}\delta_{x0}\Gamma$ from the right hand side of 2.8. □

The following tables give the degrees of the projective indecomposable characters for $G = SL(4, 2^m)$ in the cases $m = 1, 2, 3$; the degrees were computed from 3.13 with the aid of a computer. If $x = \Sigma a_{ij}\lambda_{ij} \in \Lambda_p^m$ $(p = 2)$, then, in the table corresponding to the choice of m, the integer $64^{-m}\Phi_x(1)$ can be found in the (s_1, s_2)-position, where $\Sigma a_{ij}p^{i-1+3j} = 10s_1 + s_2$ $(0 \leq s_2 < 10)$. We remark that the equation $|G| = \dim KG = \sum_{x\in\Lambda_p^m} \Phi_x(1)\varphi_x(1)$ [**2**, p. 146, Lemma 3.8] provides a check for our computations. We have verified that the degrees printed in the tables satisfy this requirement.

TABLE 1. $(m = 1)$.

	0	1	2	3	4	5	6	7	8	9
0	7	3	5	3	3	5	3	1		

TABLE 2. $(m = 2)$.

	0	1	2	3	4	5	6	7	8	9
0	431	188	286	132	188	198	132	36	188	61
1	114	42	119	68	48	12	286	114	113	44
2	114	72	44	12	132	42	44	15	48	24
3	16	4	188	119	114	48	61	68	42	12
4	198	68	72	24	68	35	24	6	132	48
5	44	16	42	24	15	4	36	12	12	4
6	12	6	4	1						

TABLE 3. ($m = 3$).

	0	1	2	3	4	5	6	7	8	9
0	20239	8408	11604	4904	8408	7180	4904	1296	8408	2960
1	4380	1628	4076	2444	1716	432	11604	4304	4606	1680
2	4304	2544	1680	432	4904	1580	1656	564	1728	864
3	576	144	8408	4076	4380	1716	2960	2444	1628	432
4	7180	2492	2584	864	2492	1278	864	216	4904	1728
5	1656	576	1580	864	564	144	1296	432	432	144
6	432	216	144	36	8408	2960	4304	1580	4076	2492
7	1728	432	2960	969	1442	524	1498	832	576	144
8	4380	1442	1608	552	1668	858	576	144	1628	524
9	546	186	576	288	192	48	4076	1498	1668	567
10	1498	848	576	144	2444	832	864	288	848	428
11	288	72	1716	576	576	192	567	288	192	48
12	432	144	144	48	144	72	48	12	11604	4380
13	4606	1656	4380	2584	1656	432	4304	1442	1608	546
14	1668	864	576	144	4606	1608	1601	560	1608	864
15	560	144	1680	552	560	188	576	288	192	48
16	4304	1668	1608	576	1442	864	546	144	2544	858
17	864	288	858	432	288	72	1680	576	560	192
18	552	288	188	48	432	144	144	48	144	72
19	48	12	4904	1628	1680	564	1716	864	576	144
20	1580	524	552	186	567	288	192	48	1656	546
21	560	188	576	288	192	48	564	186	188	63
22	192	96	64	16	1728	576	576	192	576	288
23	192	48	864	288	288	96	288	144	96	24
24	576	192	192	64	192	96	64	16	144	48
25	48	16	48	24	16	4	8408	4076	4304	1728
26	2960	2492	1580	432	4076	1498	1668	576	1498	848
27	567	144	4380	1668	1608	576	1442	858	552	144
28	1716	567	576	192	576	288	192	48	2960	1498
29	1442	576	969	832	524	144	2444	848	864	288

TABLE 3. (Continued)

	0	1	2	3	4	5	6	7	8	9
30	832	428	288	72	1628	576	546	192	524	288
31	186	48	432	144	144	48	144	72	48	12
32	7180	2444	2544	864	2444	1278	864	216	2492	832
33	858	288	848	428	288	72	2584	864	864	288
34	864	432	288	72	864	288	288	96	288	144
35	96	24	2492	848	858	288	832	428	288	72
36	1278	428	432	144	428	215	144	36	864	288
37	288	96	288	144	96	24	216	72	72	24
38	72	36	24	6	4904	1716	1680	576	1628	864
39	564	144	1728	576	576	192	576	288	192	48
40	1656	576	560	192	546	288	188	48	576	192
41	192	64	192	96	64	16	1580	567	552	192
42	524	288	186	48	864	288	288	96	288	144
43	96	24	564	192	188	64	186	96	63	16
44	144	48	48	16	48	24	16	4	1296	432
45	432	144	432	216	144	36	432	144	144	48
46	144	72	48	12	432	144	144	48	144	72
47	48	12	144	48	48	16	48	24	16	4
48	432	144	144	48	144	72	48	12	216	72
49	72	24	72	36	24	6	144	48	48	16
50	48	24	16	4	36	12	12	4	12	6
51	4	1								

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