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ON THE REPRESENTATION OF MEASURABLE SET VALUED MAPS THROUGH SELECTIONS

ARRIGO CELLINA AND RINALDO M. COLOMBO

1. Introduction. Under reasonable hypotheses, a measurable multi-function F admits measurable selections. Actually [**1**], in this case, one can describe the whole multi-function through a countable family $\mathcal{F}$ of selections, in the sense that

$$F(x) = \mathrm{cl}\,\{f(x) : f \in \mathcal{F}\}.$$

In the present paper we consider an integrably bounded (or $\mathbf{L}^p$-bounded) multi-function with values in $\mathbf{R}^m$ and we show that the countable family $\mathcal{F}$ can be chosen to be (relatively) compact in $\mathbf{L}^p$.

Equivalently, we show the existence of a family $\mathcal{F}$ of selections of F describing F as above, such that $\alpha(\mathcal{F}) = 0$, where α is the Kuratowski index. In [**2**], $\alpha(\mathcal{F})$ was determined for the family $\mathcal{F}$ of all the integrable selections of F.

2. Notation and preliminary results. For a subset A of a set X, $(A)^{\mathbf{C}}$ is the complement of A in X. For X a metric space and A bounded, $\alpha(A)$ is the Kuratowski index [**14**],

$$\alpha(A) = \inf\left\{\varepsilon : A = \bigcup_{i=1}^{n} A_i, \mathrm{diam}\,(A_i) \le \varepsilon\right\},$$

$\mathrm{cl}\,(A)$ is the closure of A. The finite dimensional space $\mathbf{R}^m$ is supplied with the norm $\|x\| = \sup_i |x_i|$. $\|A\|$ is $\sup\{\|a\| : a \in A\}$. The closed ball in $\mathbf{R}^m$ of center y and radius ε is denoted by $B[y,\varepsilon]$. When F is a set-valued map and A is a set, $F^{-1}(A)$ is $\{x : F(x) \cap A \ne \varnothing\}$. We recall that F from a measure space to the subsets of $\mathbf{R}^m$ is called measurable if $F^{-1}(A)$ is measurable for every closed A; this implies that $F^{-1}(B)$ is measurable for every open B. For the properties of measurable multi-functions on measure spaces, we refer to [**3**].

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Note that $(\{x \in X : F(x) \subseteq A\})^{\mathbf{C}} = \{x : F(x) \cap (A)^{\mathbf{C}} \neq \varnothing\} = F^{-1}((A)^{\mathbf{C}})$. In particular, from the above, we have that $\{x : F(x) \subseteq C\}$ is measurable for every closed C.

3. Main result. It is our purpose to prove the following theorem.

Theorem 1. *Let $(X, \mathcal{S}, \mu)$ be a measure space; let F from X into the nonempty closed subsets of $\mathbf{R}^m$ be measurable and such that*

$$\|F(x)\| \leq l(x) \tag{3.1}$$

for some l in $\mathbf{L}^p(X, \mathbf{R})$, $1 \leq p \leq \infty$. Assume that, for $1 \leq p < \infty$, $(X, \mathcal{S}, \mu)$ is σ-finite. Then there exists a countable family $\mathcal{F}$ of $\mathbf{L}^p$-selections of F, such that

i) $\operatorname{cl}\{f(x) : f \in \mathcal{F}\} = F(x)$, *$x$ in X;*

ii) $\alpha(\mathcal{F}) = 0$.

Proof. The proof is divided into several steps. We first assume that l is bounded a.e. by a constant L and obtain the result in $\mathbf{L}^\infty$. The general case is then a straightforward application. We note that (3.1) implies that every measurable selection of F is in $\mathbf{L}^p$.

Since we consider the $\mathbf{L}^\infty$ case first, assume that the images are contained in a closed bounded ball (actually, due to the chosen metric, a cube). By partitioning this cube and taking counter images of each part, we obtain a smaller map from which we take a selection. By changing, in every possible way, the order in which the parts are taken, we obtain measurable selections that pass essentially everywhere, (hence approximating the graph of F), this "essentially" depending on the size of the partition. By taking a further refinement, and all the selections corresponding to different orderings, we obtain a better approximation of the graph. The iteration of this process, however, would lead to a noncompact family of selections. Hence, we limit the construction to a subset of all the possible permutations of the elements of the partition. This subset has to be large enough to yield measurable selections passing essentially everywhere and small enough to give a compact family. The basic rule for the choice of the permutations is given in (3.7). It may be useful for the reader to think of the representation of integers in the base 2^m, since each refinement divides each previous cube into 2^m parts.

Steps a)–c) deal with the construction of the refinements and with preliminaries concerning permutations. In step d) we check that inverse images of a special choice of permutations give back the whole inverse image of a previous partition. The family of selections is defined in e) and in f) we show its relative compactness.

a) Let I_N be the set of integers $\{0, \dots, N-1\}$. Every l in $I_{2^{nm}}$ can be uniquely represented by its expansion in the base 2^m as

$$l = \sum_{r=0}^{n-1} 2^{rm} c_r \quad \text{with} \quad c_r \text{ in } I_{2^m}. \tag{3.2}$$

In addition, for every $h = 1, \dots, n-1$, we introduce the quotient l'_h and the remainder l''_h of the division of l by $2^{(n-h)m}$, i.e.,

$$l = 2^{(n-h)m} l'_h + l''_h. \tag{3.3}$$

Remark. $l'_h = \sum_{r=0}^{h-1} 2^{rm} c_{r+n-h}$ and $l''_h = \sum_{r=0}^{n-h-1} 2^{rm} c_r$.

b) Set: $X_L = \{x \in X : F(x) \subseteq B[0, L]\}$ (X_L is measurable and $\mu((X_L)^{\mathbf{C}}) = 0$) and $F_L = F|_{X_L}$. Let $y_0^0 = 0$ and $\varepsilon_0 = L$. Then $F_L(x) \subseteq B[y_0^0, \varepsilon_0]$ for every x in X_L. We wish to define a family of successive refinements of $B[y_0^0, \varepsilon_0]$. Let $\varepsilon_n = \varepsilon_0 / 2^n$ and assume that the refinement $\{B[y_{n-1}^l, \varepsilon_{n-1}]\}_{l \in I_{2^{(n-1)m}}}$ of $B[y_0^0, \varepsilon_0]$ is known, with $n \geq 1$. Define the points $\{y_n^l\}_{l \in I_{2^m}}$ so that

$$\bigcup_{l''_{n-1}=0}^{2^m-1} B[y_n^{2^m l'_{n-1} + l''_{n-1}}, \varepsilon_n] = B[y_{n-1}^{l'_{n-1}}, \varepsilon_{n-1}]. \tag{3.4}$$

As a consequence, if $h = 1, \dots, n-1$, then

$$\bigcup_{l''_h=0}^{2^{(n-h)m}-1} B[y_n^{2^{(n-h)m} l'_h + l''_h}, \varepsilon_n] = B[y_h^{l'_h}, \varepsilon_h] \tag{3.5}$$

and

$$\bigcup_{l=0}^{2^{nm}-1} B[y_n^l, \varepsilon_n] = B[y_0^0, \varepsilon_0]. \tag{3.6}$$

c) Let Σ_N be the set of the permutations of I_N. Define S_n to be the set of those σ in $\Sigma_{2^{nm}}$ such that

$$\sigma\bigg(\sum_{r=0}^{n-1} 2^{rm} c_r\bigg) = \sum_{r=0}^{n-1} 2^{rm} \sigma_r(c_r) \tag{3.7}$$

with σ_r in Σ_{2^m}. (Note that $S_1 = \Sigma_{2^{nm}}$.)

In a way similar to the one used in (3.3) to decompose an element of $I_{2^{nm}}$, any σ in S_n can be written as

$$\begin{aligned}\sigma\bigg(\sum_{r=0}^{n-1} 2^{rm} c_r\bigg) &= \sum_{r=0}^{n-1} 2^{rm} \sigma_r(c_r) \\ &= 2^{(n-h)m}\bigg[\sum_{r=0}^{h-1} 2^{rm} \sigma_{r+n-h}(c_{r+n-h})\bigg] \\ &\quad + \bigg[\sum_{r=0}^{n-h-1} 2^{rm} \sigma_r(c_r)\bigg]\end{aligned}$$

where the term inside the first square bracket is a permutation σ^h in S_h given by

$$\sigma^h(l_h') = \sum_{r=0}^{h-1} 2^{rm} \sigma_{r+n-h}(c_{r+n-h}). \tag{3.8}$$

Although S_n is a subset of the set of all permutations of $\Sigma_{2^{nm}}$, it has the following property that will be of importance later: for every n and every k in $I_{2^{nm}}$, there exists a σ in S_n such that $\sigma(0) = k$. In fact, write k as $\Sigma 2^{rm} c_r$ and choose for every r a σ_r in S_1 such that $\sigma_r(0) = c_r$. The σ defined in (3.7) and computed at 0 is k.

d) We define a family of finite partitions of X_L into measurable subsets as follows. Fix n; for every l in $I_{2^{nm}}$, set

$$E_n^l = F_L^{-1}(B[y_n^l, \varepsilon_n]).$$

From (3.4), it follows that

$$\bigcup_{l''_{n-1}=0}^{2^m-1} E_n^{2^m l'_{n-1}+l''_{n-1}} = E_{n-1}^{l'_{n-1}} \tag{3.9}$$

and from (3.6), one has

$$\bigcup_{l'_h,l''_h} E_n^{2^m l'_h+l''_h} = X_L.$$

For every σ in S_n, we set $A_n^{\sigma,l}$ to be

$$A_n^{\sigma,0} = E_n^{\sigma(0)} \text{ and } A_n^{\sigma,l} = E_n^{\sigma(l)} \Big\backslash \left(\bigcup_{k=0}^{n-1} E_n^{\sigma(k)} \right).$$

We have that the $A_n^{\sigma,l}$ are measurable, disjoint and such that

$$\bigcup_{l=0}^{2^{nm}-1} A_n^{\sigma,l} = X_L. \tag{3.10}$$

Write each l in $I_{2^{nm}}$ as $l = 2^m l'_{n-1} + l''_{n-1}$. Notice that, in the case $l'_{n-1} = 0$, one has

$$\begin{aligned}\bigcup_{l''_{n-1}=0}^{2^m-1} A_n^{\sigma,l''_{n-1}} &= E_n^{\sigma(0)} \cup \left(E_n^{\sigma(1)} \Big\backslash E_n^{\sigma(0)} \right) \cup \dots \\ &\quad \dots \cup \left[E_n^{\sigma(2^m-1)} \Big\backslash \left(E_n^{\sigma(0)} \cup \dots \cup E_n^{\sigma(2^m-2)} \right) \right] \\ &= \bigcup_{l''_{n-1}=0}^{2^m-1} E_n^{\sigma(l''_{n-1})} \\ &= E_{n-1}^{\sigma^{n-1}(0)}\end{aligned}$$

where $\sigma^{n-1}(\cdot)$ is defined as in (3.8). When $l'_{n-1} > 0$,

$$\bigcup_{l''_{n-1}=0}^{2^m-1} A_n^{\sigma, 2^m l'_{n-1}+l''_{n-1}}$$

$$= \left[E^{\sigma(2^m l'_{n-1})} \Big\backslash \left(\bigcup_{k=0}^{2^m l'_{n-1}-1} E_n^{\sigma(k)} \right) \right]$$

$$\cup \left\{ \bigcup_{l''_{n-1}=1}^{2^m-1} \left[E_n^{\sigma(2^m l'_{n-1}+l''_{n-1})} \Big\backslash \left(\bigcup_{k=0}^{l''_{n-1}-1} E_n^{\sigma(2^m l'_{n-1}+k)} \cup \bigcup_{k=0}^{2^m l'_{n-1}-1} E_n^{\sigma(k)} \right) \right] \right\}.$$

The set in braces may be rewritten as

$$\bigcup_{l''_{n-1}=1}^{2^m-1} \left[\left(E_n^{\sigma(2^m l'_{n-1}+l''_{n-1})} \Big\backslash \bigcup_{k=0}^{l''_{n-1}-1} E_n^{\sigma(2^m l'_{n-1}+k)} \right) \Big\backslash \left(\bigcup_{k=0}^{2^m l'_{n-1}-1} E_n^{\sigma(k)} \right) \right]$$

so that

$$\bigcup_{l''_{n-1}=0}^{2^m-1} A_n^{\sigma, 2^m l'_{n-1}+l''_{n-1}}$$

$$= \left[E_n^{\sigma(2^m l'_{n-1})} \cup \bigcup_{l''_{n-1}=1}^{2^m-1} \left(E_n^{\sigma(2^m l'_{n-1}+l''_{n-1})} \Big\backslash \bigcup_{k=0}^{l''_{n-1}-1} E_n^{\sigma(2^m l'_{n-1}+k)} \right) \right]$$

$$\Big\backslash \left(\bigcup_{k=0}^{2^m l'_{n-1}-1} E_n^{\sigma(k)} \right)$$

$$= \left(\bigcup_{l''_{n-1}=0}^{2^m-1} E_n^{\sigma(2^m l'_{n-1}+l''_{n-1})} \right) \Big\backslash \left(\bigcup_{k=0}^{2^m l'_{n-1}-1} E_n^{\sigma(k)} \right).$$

However, by (3.9), one has

$$\bigcup_{l''_{n-1}=0}^{2^m-1} E_n^{\sigma(2^m l'_{n-1}+l''_{n-1})} = E_{n-1}^{\sigma^{n-1}(l'_{n-1})} \quad (\sigma^{n-1} \text{ as in } (3.8))$$

while

$$\bigcup_{h=0}^{2^m l'_{n-1}-1} E_n^{\sigma(h)} = \bigcup_{h=0}^{l'_{n-1}-1} \bigcup_{l=0}^{2^m-1} E_n^{\sigma(2^m h+l)}$$
$$= \bigcup_{h=0}^{l'_{n-1}-1} E_{n-1}^{\sigma^{n-1}(h)}$$

so that

$$\bigcup_{l''_{n-1}=0}^{2^m-1} A_n^{\sigma,2^m l'_{n-1}+l''_{n-1}} = E_{n-1}^{\sigma^{n-1}(l'_{n-1})} \Big\backslash \left(\bigcup_{h=0}^{l'_{n-1}-1} E_{n-1}^{\sigma^{n-1}(h)} \right)$$

and, by the very definition of $A_{n-1}^{\sigma,l'_{n-1}}$, it follows that

$$\bigcup_{l''_{n-1}=0}^{2^m-1} A_n^{\sigma,2^m l'_{n-1}+l''_{n-1}} = A_{n-1}^{\sigma^{n-1},l'_{n-1}}. \tag{3.11}$$

We wish to generalize this last relation as

$$\bigcup_{l''_h=0}^{2^{(n-h)m}-1} A_n^{\sigma,2^{(n-h)m} l'_h+l''_h} = A_h^{\sigma^h,l'_h}. \tag{3.12}$$

In fact,

$$\bigcup_{l''_h=0}^{2^{(n-h)m}-1} A_n^{\sigma,2^{(n-h)m} l'_h+l''_h}$$
$$= \bigcup_{c_{n-h-1}=0}^{2^m-1} \bigcup_{c_{n-h-2}=0}^{2^m-1} \cdots \bigcup_{c_0=0}^{2^m-1} A_n^{\sigma,2^{(n-h)m} l'_h+\sum_{r=0}^{n-h-1} 2^{rm} c_r}$$
$$= \bigcup_{c_{n-h-1}=0}^{2^m-1} \bigcup_{c_{n-h-2}=0}^{2^m-1} \cdots \bigcup_{c_1=0}^{2^m-1} A_{n-1}^{\sigma^{n-1},2^{(n-h-1)m} l'_h+\sum_{r=0}^{n-h-2} 2^{rm} c_{r+1}}$$
$$= \bigcup_{d_{n-h-2}=0}^{2^m-1} \bigcup_{d_{n-h-3}=0}^{2^m-1} \cdots \bigcup_{d_0=0}^{2^m-1} A_{n-1}^{\sigma^{n-1},2^{(n-h-1)m} l'_h+\sum_{r=0}^{n-h-2} 2^{rm} d_r}$$

and applying (3.11) again, in a finite number of steps we obtain (3.12).

e) For every n, set

$$F_n^\sigma(x) = F_L(x) \cap B[y_n^{\sigma(l)}, \varepsilon_n] \quad \text{for } x \text{ in } A_n^{\sigma,l}.$$

From (3.10) in d), F_n^σ is defined on X_L. Furthermore, F_n^σ is a measurable multifunction with nonempty closed values. Set f_n^σ to be a measurable selection from F_n^σ, and let $\mathcal{F}_L$ be

$$\mathcal{F}_L = \{f_n^\sigma : \sigma \text{ in } S_n \text{ and } n = 1, 2, \dots\}.$$

$\mathcal{F}_L$ is countable. Moreover, we claim that

$$F_L(x) = \operatorname{cl}\{f(x) : f \text{ in } \mathcal{F}_L\}, \quad \text{for every } x \text{ in } X_L.$$

Fix x, choose y in $F_L(x)$ and $\varepsilon > 0$. Let n be such that $\varepsilon_n < \varepsilon/2$. From (3.6), y is in $B[y_n^j, \varepsilon_n]$ for some j. From c), there is a σ in S_n such that $\sigma(0) = j$. From the definition of $A_n^{\sigma,0}$, we have that

$$A_n^{\sigma,0} = E_n^{\sigma(0)} = E_n^j = F^{-1}(B[y_n^j, \varepsilon_n])$$

so that $f_n^\sigma(x) \in B[y_n^j, \varepsilon_n]$, hence $\|f_n^\sigma(x) - y\| \le 2\varepsilon_n < \varepsilon$.

f) We prove the relative compactness of $\mathcal{F}_L$ in the case $p = \infty$ by showing the existence of a finite ε-net in $\mathcal{F}_L$. Fix ε. Let n^* be such that $\varepsilon_{n^*} < \varepsilon/2$. We claim that

$$\{f_n^\sigma \in \mathcal{F}_L : \sigma \in S_n, n \le n^*\}$$

is the required ε-net. Choose any $f_{\bar n}^{\bar\sigma}$ in $\mathcal{F}_L$ (clearly we can assume that $n > n^*$). Fix x. For some l, x is in $A_{\bar n}^{\bar\sigma,l}$, i.e.,

$$f_{\bar n}^{\bar\sigma}(x) \in B[y_{\bar n}^{\bar\sigma(l)}, \varepsilon_{\bar n}].$$

As in (3.5), write $l = 2^{(\bar n - n^*)m} l'_{n^*} + l''_{n^*}$ and, as in (3.8), write

$$\sigma^{n^*}(l'_{n^*}) = \sum_{r=0}^{n^*-1} 2^{rm} \sigma_{r+\bar n - n^*}(c_{r+\bar n - n^*}).$$

Writing σ^* for σ^{n^*}, then by (3.12), we have $A_{\bar{n}}^{\bar{\sigma},l} \subset A_{n^*}^{\sigma^*,l'_{n^*}}$ and hence

$$f_{n^*}^{\sigma^*} \in B[y_{n^*}^{\sigma^*(l'_{n^*})}, \varepsilon_{n^*}].$$

Since by (3.5), $B[y_{\bar{n}}^{\bar{\sigma}(l)}, \bar{n}] \subset B[y_{n^*}^{\sigma^*(l'_{n^*})}, \varepsilon_{n^*}]$, we have

$$\|f_{\bar{n}}^{\bar{\sigma}}(x) - f_{n^*}^{\sigma^*}(x)\| \leq 2\varepsilon_{n^*} < \varepsilon. \tag{3.13}$$

g) Let φ be any measurable selection from F, i.e., $\varphi(x) \in F(x)$ for every x in X. The family $\mathcal{F}_L$ of selections on X_L can be considered a family of selections on X by extending each f_n^σ to be $\varphi(x)$ for x in $(X_L)^{\mathbf{C}}$. Then (3.13) holds on X. Consider now the map F_∞ defined as

$$F_\infty(x) = \begin{cases} \{\varphi(x)\}, & x \text{ in } X_L \\ F(x), & x \text{ in } (X_L)^{\mathbf{C}}. \end{cases}$$

Let $\mathcal{F}_\infty$ be a countable family of selections from F_∞ such that i) holds for F_∞ (see [**1**, Chapter III, Section 2). Such a family is obviously compact in $\mathbf{L}^\infty$. Then the family

$$\mathcal{F} = \mathcal{F}_L \cup \mathcal{F}_\infty$$

has the required properties.

h) We proceed to prove the relative compactness of $\mathcal{F}$ in the case $1 \leq p < \infty$ (under the same boundedness assumption on l). Write X as $\cup_n X_n$, $\mu(X_n) < \infty$, $X_n \subset X_{n+1}$, and choose N so large that

$$\int_{(X_N)^{\mathbf{C}}} l^p < \varepsilon/(2^{p+1}).$$

Then, since

$$(\|f-g\|_{L^{\mathbf{p}}})^p = \int_{X^N} \|f-g\|^p + \int_{(X_N)^{\mathbf{C}}} \|f-g\|^p,$$

any finite $\varepsilon/(2\mu(X_N))$-net in $\mathbf{L}^\infty$ is a finite $\varepsilon^{1/p}$-net in $\mathbf{L}^p$.

Hence, the theorem is proved for a measurable F defined on any σ-finite measurable space X and bounded by a function l in $\mathbf{L}^p(X) \cap \mathbf{L}^\infty(X)$.

i) Let φ be any $\mathbf{L}^p$ selection from F. Set

$$\begin{aligned} Y_1 &= B[0,1] \\ Y_n &= \mathrm{cl}\,(B[0,n] \setminus B[0,n-1]), \quad n \geq 2 \\ X_n &= F^{-1}(Y_n), \quad n \geq 1 \end{aligned}$$

and define

$$F_n(x) = \begin{cases} F(x) \cap Y_n, & x \text{ in } X_n \\ \{\varphi(x)\}, & x \text{ in } (X_n)^{\mathbf{C}}. \end{cases}$$

Define $\mathcal{F}_0 = \{\varphi\}$ and, for $n \geq 1$, let $\mathcal{F}_n$ be a countable family of selections satisfying i) and ii) for F_n. Set

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n.$$

To prove i), fix x in X and y in $F(x)$. For some n, y is in Y_n. Hence,

$$y \in \mathrm{cl}\,\{f(x) : f \in \mathcal{F}_n\} \subset \mathrm{cl}\,\{f(x) : f \in \mathcal{F}\}.$$

To prove ii), fix $\varepsilon > 0$. Since $X_n \subset \{x \in X : l(x) \geq n-1\}$, from the integrability of l^p, there exists an n^* such that

$$\int_{\cup_{n>n^*} X_n} l^p < \left(\frac{\varepsilon}{2}\right)^p.$$

Consider the finite set of functions consisting of all the elements of the (finite) ε-net of $\mathcal{F}_0, \dots, \mathcal{F}_{n^*}$. We claim that it is a finite ε-net for $\mathcal{F}$. Fix f in $\mathcal{F}$. Then f is in $\mathcal{F}_n$ for a suitable n. If $n \leq n^*$ there is nothing to prove; therefore, we assume $n > n^*$. Since f is a selection from F_n, it follows that

$$f(x) = \varphi(x) \text{ for all } x \text{ in } (X_n)^{\mathbf{C}}.$$

And, finally, we have

$$\begin{aligned} \|f - \varphi\|_{\mathbf{L}^p} &= \left(\int_{X_n} \|f - \varphi\|^p\right)^{1/p} \\ &\leq 2\left(\int_{\cup_{n>n^*} X_n} l^p\right)^{1/p} \\ &\leq \varepsilon. \qquad \square \end{aligned}$$

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SISSA, Strada Costiera 11, 34014 Trieste, Italy