ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 1, Winter 1993

NILPOTENCE OF PRODUCTS OF NONNEGATIVE MATRICES

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ABSTRACT. Given m nonnegative n-by-n matrices $A_1, A_2, \dots, A_m$, we consider the circumstances in which the product $A_1 A_2 \dots A_m$ is nilpotent and also the stronger condition that $A_1, A_2, \dots, A_m$ are simultaneously permutation similar to strictly upper triangular matrices. These eventualities coincide in the case of a single matrix, but they differ for several matrices and are each characterized in a variety of ways.

1. Introduction and notations. We consider the question of which sequences of m nonnegative n-by-n matrices have a nilpotent product. This problem comes to our attention from the study of the structural properties of discrete time positive periodic linear systems. Given a discrete-time periodic linear system

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{1}$$

in which $A(k)$ and $B(k)$ are nonnegative N-periodic matrices of sizes n-by-n and n-by-p, respectively, we consider the n discrete time invariant linear systems associated with (1)

$$x_s(k+1) = A_s x_s(k) + B_s u_s(k), \qquad s = 0, 1, \dots, N-1. \tag{2}$$

Recently, in [**1**], there appeared a characterization of positive controllability of (1) in terms of positive controllability of the systems (2); more precisely, it was shown that "The positive periodic system (1) is completely positive orthant controllable at s if and only if the positive invariant system (2) corresponding to index s is completely positive orthant controllable, $s = 0, 1, \dots, N-1$." Further, it is known (see

Received by the editors on March 23, 1990, and in revised form on May 15, 1991.
The work of the first author was carried out while visiting The College of William and Mary, the hospitality of which is gratefully acknowledged and was supported by D.G.I.C.Y.T. of Ministerio de Educación y Ciencia, Spain.
The work of the second author was supported in part by National Science Foundation grant DMS 90-00839 and by Office of Naval Research contract N00014-90-J-1739.

[**2**]) that the invariant system (2), for a fixed s, is completely positive orthant controllable if and only if the set of states x_s that are reachable in finite time with nonnegative inputs u_s is R_+^n and the matrix A_s is nilpotent. Moreover, this matrix is the monodromy matrix $\Phi_A(s+N,s)$ at s of (2), which is defined as

$$A_s = \Phi_A(s+N,s) = A(s+N-1)A(s+N-2)\cdots A(s). \tag{3}$$

Thus, it is of interest when A_s is nilpotent, or, in other words, when the product of m nonnegative matrices is nilpotent. We note that this is a purely combinatorial property depending only upon the nonzero patterns of the factors in order. Thus, our results will be described in combinatorial terms.

Let A be an n-by-n matrix. We denote its entries by a_{ij}, $1 \le i, j \le n$, and by $G(A)$ its *directed graph*, that is $G(A) = \{V, E\}$ where $V = \langle n \rangle$ is the set of vertices $\{1, 2, \ldots, n\}$ and $E = \{(i,j) \in V \times V | a_{ij} \neq 0\}$ is the set of edges. We denote an *edge* from vertex i to vertex j by (i,j). A *path* (of length k) from i to j is a sequence of edges $(i,i_1), (i_1,i_2), \ldots, (i_{k-1},j)$ in $G(A)$. A path from i to j is called a *circuit* if $i = j$, and a circuit is a *cycle* if it has no repeated (interior) vertices.

We denote the *transitive closure* of $G(A)$ by $\overline{G(A)}$; this is just the graph that has an edge from i to j if there is a path in $G(A)$ from i to j. If $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are two directed graphs, by $G_1 \cup G_2$ we just mean the directed graph $\{V_1 \cup V_2, E_1 \cup E_2\}$; thus, if the two vertex sets are the same (as will be the case in this note), then the union of two graphs is just the superposition of their edges. If A is a nonnegative matrix note that $\overline{G(A)}$ is just the same as $G(A + A^2 + \cdots + A^n)$ or $G(A) \cup G(A^2) \cup \cdots \cup G(A^n)$. Given several n-by-n nonnegative matrices $A_1, A_2, \ldots, A_m$, we shall wish to consider the union of their directed graphs, but at the same time remember from which matrices each edge came. Thus, we define the *joint graph* $G(A_1, A_2, \ldots, A_m)$ as $G(A_1) \cup G(A_2) \cup \cdots \cup G(A_m)$, in which, additionally, each edge is labelled ("colored") with a subset of $\{1, 2, \ldots, m\}$ depending upon which of the graphs $G(A_1), G(A_2), \ldots, G(A_m)$ includes that edge. It is clear that $G(A_1 + A_2) = G(A_1) \cup G(A_2)$ for n-by-n nonnegative matrices A_1 and A_2. Thus, there is an edge in $G(A_1 + A_2)$ if and only if there is an edge in $G(A_1, A_2)$ labelled 1 or 2.

The graph of the product of two nonnegative n-by-n matrices A_1 and A_2 is the graph $G(A_1A_2)$, and it is readily observed that (i,j) is an edge of $G(A_1A_2)$ if and only if there is a vertex k, $1 \leq k \leq n$, such that $(i,k) \in G(A_1)$ and $(k,j) \in G(A_2)$. Equivalently, there is a path in $G(A_1, A_2)$ of length 2 colored 1, 2 in order.

By a *word* in a set of factors $\{A_1, A_2, \dots, A_m\}$, we mean an arbitrary (with respect to order, length and repetition) product built from these factors. For example, $A_1^2A_2A_1^3A_3A_2^5$ and $A_1A_2^2A_1$ are words in $\{A_1, A_2, A_3\}$. The *length or degree* of a word is just the total number of factors. For example, $A_1^2A_2A_1^3A_3A_2^5$ has degree 12.

As usual, we use $\rho(A)$ to denote the *spectral radius* (maximum of the absolute values of the eigenvalues) of a square matrix A. If A is nonnegative, $\rho(A)$ is itself an eigenvalue of A by the Perron-Frobenius theorem.

The remainder of this note is organized as follows. We first collect several useful characterizations of nilpotence for a single nonnegative matrix in the next section and then observe in Section 3 a variety of facts to be used later. This allows us to give a simple characterization, via the joint graph, of when a product of several nonnegative matrices (in a given order) is nilpotent in Section 4. This characterization parallels one of the characterizations of nilpotence for a single matrix in terms of the usual graph.

If n-by-n nonnegative matrices $A_1, A_2, \dots, A_m$ are strictly upper triangular, then *any* product (including repetitions) from among them is nilpotent. The links between these two phenomena are studied in Section 5.

2. Characterization of nilpotence for a single nonnegative matrix. For convenience, we record here, with proof, several useful characterizations of nilpotence for an individual nonnegative matrix. Most of these are widely known.

Theorem 0. *For an n-by-n nonnegative matrix A, the following statements are equivalent*:

i) *there exists a permutation matrix P such that P^TAP is strictly upper triangular*;

ii) $\rho(A) = 0$;

iii) $A^n = 0$ (*A is nilpotent*);

iv) $G(A)$ *has no loops*;

v) A^k *has all diagonal entries equal to zero, for all* $k = 1, 2, \dots$;

vi) $G(A)$ *is acyclic*;

vii) $G(A)$ *has no circuits*; *and*

viii) *the vertices of* $G(A)$ *may be labelled* $1, 2, \dots, n$, *so that every edge* (i, j) *is such that* $i < j$ (*ordered labelling*).

Proof. i) $\rightarrow$ ii). The eigenvalues of a triangular matrix appear on the diagonal.

ii) $\rightarrow$ iii). Use the Jordan form of A.

iii) $\rightarrow$ iv). If $\overline{G(A)}$ had a loop (say the i^{th}), then there would exist a positive integer k, $1 \leq k \leq n$, such that $a_{ii}^{(k)} > 0$. Here, $A^k = (a_{ij}^{(k)})$. Then $a_{ij}^{(nk)}$ would be positive as well, contradicting $A^n = 0$.

iv) $\rightarrow$ v). Obvious.

v) $\rightarrow$ vi). Suppose $G(A)$ has a cycle of $k \leq n$ edges containing vertex i. Then a calculation shows that $a_{ii}^{(k)}$ would be positive.

vi) $\rightarrow$ vii). If $G(A)$ had a circuit $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$, either this circuit is a cycle or we may pick the first subscript j such that $i_j = i_h$, with $h < j$. Then $(i_h, i_{h+1}), \dots, (i_{j-1}, i_j)$ would be a cycle in $G(A)$.

vii) $\rightarrow$ viii). There must be a vertex such that no edge emanates from it; for, if not, a circuit could be constructed. Choose such a vertex; label it n and delete it (and any incoming edges) from $G(A)$. The resulting graph is still acyclic. Choose a vertex in it, with the same property, to label $(n-1)$. Continuing in this manner produces an order labelling.

viii) $\rightarrow$ i). This implication is clear by using the permutation matrix associated with the assumed relabelling. $\square$

We note that statements ii) and iii) are equivalent to nilpotence for general matrices. The remaining statements are all *sufficient* for nilpotence for general matrices, but rely upon the nonnegativity of A

for equivalence to nilpotence (i.e., they are entirely graph theoretic).

Given a nonnegative n-by-n matrix A, we define the *transitive closure* of A as $\overline{A} = A + A^2 + \cdots + A^n$. It follows from Theorem 0 that

Corollary 1. *A nonnegative n-by-n matrix A is nilpotent if and only if each diagonal entry of its transitive closure $\overline{A}$ is 0.*

3. General facts. Consider m nonnegative n-by-n matrices $A_1, A_2, \ldots, A_m$. The following facts are basic to what follows and, though simple, may be of independent interest.

Lemma 2. *There is an edge from i to j in $G(A_1 A_2 \cdots A_m)$ if and only if there is a path of length m from i to j in the joint graph $G(A_1, A_2, \ldots, A_m)$ colored with $1, 2, \ldots, m$, in that order.*

Proof. Since each of $A_1, A_2, \ldots, A_m$ is nonnegative, there is an edge (i, j) in $G(A_1 A_2 \cdots A_m)$ if and only if there are the following edges $(i, k_1) \in G(A_1), (k_1, k_2) \in G(A_2), \ldots, (k_{m-1}, j) \in G(A_m)$. These edges are colored $1, 2, \ldots, m$, in order in $G(A_1, A_2, \ldots, A_m)$. $\square$

Corollary 3. *There is a path in $G(A_1 A_2 \cdots A_m)$ of length k if and only if there is a path of length km in $G(A_1, A_2, \ldots, A_m)$ colored $1, 2, \ldots, m, 1, 2, \ldots, m, \ldots, 1, 2, \ldots, m$, in order.*

Corollary 4. *There is a circuit of length km in $G(A_1, A_2, \ldots, A_m)$ colored $1, 2, \ldots, m, 1, 2, \ldots, m, \ldots, 1, 2, \ldots, m$, in order if and only if there is a circuit in $G(A_1 A_2 \cdots A_m)$ of length k.*

Lemma 5. *There is a circuit in the joint graph $G(A_1, A_2, \ldots, A_m)$ in some colored order if and only if there is a circuit in $\overline{G(A_1)} \cup \overline{G(A_2)} \cup \cdots \cup \overline{G(A_m)}$ in which consecutive edges occur in different $\overline{G(A_i)}$.*

Proof. Suppose that the circuit in $G(A_1, A_2, \ldots, A_m)$ is colored $c_1, c_2, \ldots, c_l, \ldots, c_k$, in order. Corresponding to each color there is an edge in $G(A_{c_l})$. If two or more, say p_l, consecutive colors are equal, then there is an edge in $G(A_{c_l}^{p_l})$ and hence that edge is in $\overline{G(A_{c_l})}$. Thus, there is a circuit in $\overline{G(A_1)} \cup \overline{G(A_2)} \cup \cdots \cup \overline{G(A_m)}$ in which consecutive edges occur in different $\overline{G(A_{c_l})}$.

Conversely, suppose there is a circuit in $\overline{G(A_1)} \cup \overline{G(A_2)} \cup \cdots \cup \overline{G(A_m)}$ in which no two consecutive edges are in the same $\overline{G(A_i)}$. Fixing each edge in the corresponding graph $G(A_{c_l}^{p_l})$, then by Lemma 2 there is a path of length p_l in $G(A_{c_l}, A_{c_l}, \ldots, A_{c_l})$. The concatenation of these paths yields a circuit in $G(A_1, A_2, \ldots, A_m)$ in some colored order. □

In the following, we abbreviate the repetition of the label, c_i, p_i times ($p_i > 1$) as $c_i^{p_i}$.

Lemma 6. *There is a circuit of length $p_1 + p_2 + \cdots + p_k$ in the joint graph $G(A_1, A_2, \ldots, A_m)$ labelled $c_1^{p_1}, c_2^{p_2}, \ldots, c_k^{p_k}$ in order if and only if $A_{c_1}^{p_1} A_{c_2}^{p_2} \cdots A_{c_k}^{p_k}$ has a nonzero diagonal entry.*

Proof. Suppose that the i^{th} diagonal entry of the matrix $A_{c_1}^{p_1} A_{c_2}^{p_2} \cdots A_{c_k}^{p_k}$ is nonzero, so that the edge (i, i) is in $G(A_{c_1}^{p_1} A_{c_2}^{p_2} \cdots A_{c_k}^{p_k})$. By Corollary 4 there is a circuit of length k in $G(A_{c_1}^{p_1}, A_{c_2}^{p_2}, \ldots, A_{c_k}^{p_k})$ labelled $c_1, c_2, \ldots, c_k$ in order, and hence there is a circuit of length $p_1 + p_2 + \cdots + p_k$ in $G(A_1, A_2, \ldots, A_m)$ labelled $c_1^{p_1}, c_2^{p_2}, \ldots, c_k^{p_k}$ in order.

Conversely, suppose there is a circuit of length $p_1 + p_2 + \cdots + p_k$ in $G(A_1, A_2, \ldots, A_m)$ labelled $c_1^{p_1}, c_2^{p_2}, \ldots, c_k^{p_k}$ in order. By Corollary 3 applied to every subpath of length p_l labelled $c_l, \ldots, c_l$ of the circuit, there is a path of length 1 in $G(A_{c_l}^{p_l})$. The concatenation of these edges gives a circuit of length k in $G(A_{c_1}^{p_1}, A_{c_2}^{p_2}, \ldots, A_{c_k}^{p_k})$ labelled $c_1, c_2, \ldots, c_k$ in order. Then by Corollary 4 there is a circuit of length 1 in $G(A_{c_1}^{p_1} A_{c_2}^{p_2} \cdots A_{c_k}^{p_k})$ and hence the word $A_{c_1}^{p_1} A_{c_2}^{p_2} \cdots A_{c_k}^{p_k}$ has a nonzero diagonal entry. □

Remark. The "if" part of each of the last three results remains valid if we replace *circuit* by *cycle*. However, the "only if" part does not remain valid as the following example shows. Suppose that $G(A_1) = \{(1,2), (3,5), (4,5)\}$, $G(A_2) = \{(2,3), (5,2)\}$ and $G(A_3) = \{(2,1), (2,3), (3,4)\}$. Then to the cycle (1,4),(4,3),(3,1) of $G(A_1 A_2 A_3)$

corresponds the circuit (1,2),(2,3),(3,4),(4,5),(5,2),(2,3),(3,5),(5,2),(2,1), colored 1,2,3,1,2,3,1,2,3, which has necessarily interior vertices and colored edges repeated.

4. Characterization of the nilpotence of a product of nonnegative matrices. Now we may simply give a characterization of the nilpotence of a product, which, for the joint graph, parallels that of one of the characterizations of nilpotence for a single matrix.

Theorem 7. *Given m nonnegative n-by-n matrices $A_1, A_2, \ldots, A_m$, the product $A_1 A_2 \cdots A_m$ is nilpotent if and only if there exists no circuit of length km, $1 \le k \le n$, in $G(A_1, A_2, \ldots, A_m)$ whose edges are colored $1, 2, \ldots, m, 1, 2, \ldots, m, \ldots, 1, 2, \ldots, m$, in order.*

Proof. The matrix $A_1 A_2 \cdots A_m$ is nilpotent by Theorem 0 if and only if its graph $G(A_1 A_2 \cdots A_m)$ has no circuit. By Corollary 4 this happens if and only if there does not exist a circuit of length km in $G(A_1, A_2, \ldots, A_m)$ colored as $1, 2, \ldots, m, 1, 2, \ldots, m, \ldots, 1, 2, \ldots, m$. □

Note that the product of two nilpotent matrices is not in general nilpotent as the following example shows:

$$A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are nilpotent, but

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

is not nilpotent. Further, it should also be noted that products involving non-nilpotent matrices may be nilpotent. For example, the product of the following non-nilpotent matrices

$$C = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is nilpotent, in either order.

Remark. Note that if the product $A_1 A_2 \cdots A_m$ is nilpotent, then any cyclic reordering of it is nilpotent because the spectra of the two matrix products MN and NM are the same, for square M and N. Thus, for $m = 2$ the product of the m given matrices is or is not nilpotent independent of their order. For $m \geq 3$, however, order is very important. If

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then $A_1 A_2 A_3$ is nilpotent while $A_1 A_3 A_2$ is not.

Corollary 8. *Given m nonnegative n-by-n matrices $A_1, A_2, \ldots, A_m$, the product of these matrices in any order is nilpotent if and only if there exists no circuit of length km, $1 \leq k \leq n$, in $G(A_1, A_2, \ldots, A_m)$ whose edges in order are colored $1, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(m)}, 1, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(m)}, \ldots, 1, \mathcal{T}_{(2)}, \ldots, \mathcal{T}_{(m)}$ for a permutation $\mathcal{T}$ of $\{2, 3, \ldots, m\}$.*

5. Conditions for simultaneous strict upper triangular form. Thus far we have considered a product of a given set of nonnegative matrices multiplied in a specified order. It is not difficult to construct examples in which a permutation of the order or the allowance of repeats makes a difference. For example, if $+$ denotes a positive entry and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ + & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & + & 0 \\ 0 & 0 & + \\ 0 & 0 & 0 \end{bmatrix}$$

then $A_1 A_2$ is nilpotent, but the word $A_1 A_2^2$ is not. Next we turn our attention to general products from a given set of factors.

If $A_1, A_2, \ldots, A_m$ are strictly upper triangular, it is clear that any word in $\{A_1, A_2, \ldots, A_m\}$ is nilpotent. In a sense our principal result, which is for products partly parallel to Theorem 0, is a converse to this observation.

Theorem 9. *If $A_1, A_2, \ldots, A_m$ are n-by-n nonnegative matrices, the following statements are equivalent:*

i) *every word in $\{A_1, A_2, \ldots, A_m\}$ is nilpotent;*

ii) *the joint graph* $G(A_1, A_2, \dots, A_m)$ *is acyclic*;

iii) *the joint graph* $G(A_1, A_2, \dots, A_m)$ *has no circuits*;

iv) *the matrix* $A_1 + A_2 + \cdots + A_m$ *is nilpotent*;

v) $A_1, A_2, \dots, A_m$ *are simultaneously strictly upper triangular*; *i.e., there is a single permutation matrix* P *such that* $P^T A_i P$ *is strictly upper triangular,* $1 \le i \le m$;

vi) *every word of degree* n *in* $\{A_1, A_2, \dots, A_m\}$ *is* 0;

vii) *the joint graph* $G(A_1, A_2, \dots, A_m)$ *has an ordered labelling*; *and*

viii) *no word in* $\{A_1, A_2, \dots, A_m\}$ *has a nonzero diagonal entry.*

Proof. i) $\to$ ii). Applying Theorem 7 to every word we deduce that $G(A_1, A_2, \dots, A_m)$ is acyclic.

ii) $\to$ iii). Similar to the corresponding proof of Theorem 0.

iii) $\to$ iv). $G(A_1 + A_2 + \cdots + A_m)$ has no circuits because it has the same edges as $G(A_1, A_2, \dots, A_m)$ and by Theorem 0 the matrix $A_1 + A_2 + \cdots + A_m$ is nilpotent.

iv) $\to$ v). Since $A_1 + A_2 + \cdots + A_m$ is nilpotent, there is a permutation matrix P such that $P^T(A_1 + A_2 + \cdots + A_m)P$ is strictly upper triangular by Theorem 0. Then $P^T A_i P$ is strictly upper triangular as well, because of the nonnegativity of the A_i, $1 \le i \le m$.

v) $\to$ vi). Any product of n strictly upper triangular matrices of size n is 0.

vi) $\to$ vii). Since every word of degree n is 0, it follows by formal expansion that $(A_1 + A_2 + \cdots + A_m)^n = 0$. By Theorem 0, $A_1 + A_2 + \cdots + A_m$ is nilpotent, and its graph has an ordered labelling. But $G(A_1 + A_2 + \cdots + A_m)$ and $G(A_1, A_2, \dots, A_m)$ have the same edges (ignoring coloring), so that $G(A_1, A_2, \dots, A_m)$ has an ordered labelling.

vii) $\to$ viii). Since $G(A_1, A_2, \dots, A_m)$ has an ordered labelling, it has no circuits. By Lemma 6 no word has a nonzero diagonal entry.

viii) $\to$ i). Suppose that there is a word that is not nilpotent. Then some power of it has a nonzero diagonal entry, by Theorem 0. But this power is also a word, which contradicts viii) and completes the proof. $\square$

It is natural to ask if statement (i) of Theorem 9 is equivalent to the formally weaker condition of Corollary 8 in which nilpotence is required only of all the $m!$ permutations of $A_1A_2\cdots A_m$. We note that this is not the case, as indicated by the example for $m = 2$ at the beginning of this section. By condition (vi) of Theorem 9, words of degree beyond n need not be considered, and this appears to be the maximal weakening that is equivalent to statement (i).

By appealing to transitive closure, simpler characterizations of simultaneous strict upper triangularity may be given.

Theorem 10. *If A and B are n-by-n nonnegative nilpotent matrices, then there is a permutation matrix P such that P^TAP and P^TBP are strictly upper triangular if and only if AB is nilpotent.*

Proof. If P^TAP and P^TBP are strictly upper triangular, then $P^T\overline{A}P$ and $P^T\overline{B}P$ are also; thus, $\overline{A}\,\overline{B}$ is nilpotent.

Conversely, if $G(A,B)$ has a cycle, then $G(\overline{A},\overline{B})$ must have one with alternating colors. Since $G(\overline{A})$ is transitively closed, there must then be an alternating cycle in $G(\overline{A},\overline{B})$ whose length is a multiple of 2; then $\overline{A}\,\overline{B}$ is not nilpotent by Theorem 7. □

Corollary 11. *If $G(A)$ and $G(B)$ are transitively closed, then AB is nilpotent if and only if A and B are simultaneously strictly upper triangular.*

Note that the length of the cycle in $G(\overline{A},\overline{B})$ used in the proof of Theorem 10 is a multiple of 2 (the number of matrices). This fact plays a fundamental role in the above characterization. Since in the general case such a cycle need not occur, we should not expect to have a natural and simple extension of this result. Indeed, in order to establish an analogous theorem for m nonnegative matrices, $m > 2$, call a map $f : \langle p\rangle \to \langle p\rangle$ *locally distinguishing* if $f(i) \neq f(i+1)$, $1 \leq i \leq p$, (here $p+1$ is identified with 1 and $\langle p\rangle$ means the set $\{1,2,\dots,p\}$).

Theorem 12. *Let $A_1, A_2, \dots, A_m$ be n-by-n nonnegative nilpotent matrices. There is a permutation matrix P such that PA_iP is strictly* *upper triangular,* $1 \le i \le m$, *if and only if* $\overline{A}_{f(1)}\overline{A}_{f(2)}\cdots\overline{A}_{f(k)}$ *is nilpotent for all* $k \le n$ *and all locally distinguishing maps* f.

Proof. If there is a permutation matrix P such that $P^T A_i P$ is strictly upper triangular, $1 \le i \le m$, then $P^T \overline{A}_i P$ is as well, $1 \le i \le m$, and hence $\overline{A}_{f(1)}\overline{A}_{f(2)}\cdots\overline{A}_{f(k)}$ is nilpotent, $1 \le k \le n$, for all locally distinguishing maps f.

Conversely, suppose that $G(A_1, A_2, \ldots, A_m)$ has a cycle labelled in some order. Then by Lemma 5 (and the Remark following) there is a cycle in $\overline{G(A_1)} \cup \overline{G(A_2)} \cup \cdots \cup \overline{G(A_m)}$ in which consecutive edges occur in different graphs; thus this cycle is in $G(\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_m)$. Suppose that the length of this cycle is $k \le n$ (and, without loss of generality, that it involves at most $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_k$) and that the locally distinguishing map $f : \langle k \rangle \to \langle k \rangle$ is defined by its labels (i.e., $f(j)$ is the label of the j^{th} edge). Therefore, the joint graph $G(\overline{A}_{f(1)}, \overline{A}_{f(2)}, \ldots, \overline{A}_{f(k)})$ has a cycle of length k colored $f(1), f(2), \ldots, f(k)$ in order, and $\overline{A}_{f(1)}\overline{A}_{f(2)}\cdots\overline{A}_{f(k)}$ is not nilpotent by Theorem 7. □

Finally, we note a characterization of the uniqueness of the permutation matrix P for which $P^T A P$ is strictly upper triangular.

Theorem 13. *Let* A *be an* n*-by-*n *nilpotent matrix. There is a unique permutation matrix* P *such that* $P^T A P$ *is strictly upper triangular if and only if* $G(A)$ *has a path of length* $n-1$.

Proof. For the reverse implication, note that the assumed path of length $n-1$ must contain all the vertices $1, 2, \ldots, n$ or else A would contain a cycle and not be nilpotent. It is then clear that the only permutation P for which $P^T A P$ is strictly upper triangular is the one that places the edges of the path of length $n-1$ on the super-diagonal.

For the forward implication, assume that $G(A)$ does not have a path of length $n-1$ and that P_1 is a permutation matrix such that $P_1^T A P_1$ is strictly upper triangular. Then $P_1^T A P_1$ must have at least one 0 along the super-diagonal, say in the $i-1$, i position. If Q is the transposition permutation matrix that interchanges $i-1$ and i, then $P_2^T A P_2$ is also strictly upper triangular for $P_2 = P_1 Q$. □

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