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TOEPLITZ OPERATORS ON THE DISK WITH LOCALLY SECTORIAL SYMBOLS

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1. Introduction. Let $\mathbf{D}$ denote the open unit disk in $\mathbf{C}$, and let $A^2(\mathbf{D})$ be the Bergman space of square-integrable holomorphic functions in $\mathbf{D}$. For $a \in L^\infty := L^\infty(\mathbf{D})$, the Toeplitz operator $T(a)$ on $A^2(\mathbf{D})$ is defined by $T(a)\varphi = P(a\varphi)$ $(\varphi \in A^2(\mathbf{D}))$, where P is the orthogonal projection of $L^2(\mathbf{D})$ onto $A^2(\mathbf{D})$. The function a is usually referred to as the symbol of the operator $T(a)$.

The Fredholm properties of Toeplitz operators on $A^2(\mathbf{D})$ were studied by Venugopalkrishna [**15**] and Coburn [**7**] for symbols in $C(\bar{\mathbf{D}})$ (the functions continuous on the closed disk $\bar{\mathbf{D}}$), by McDonald [**10**] (and also in [**8**]) for symbols in $C(\bar{\mathbf{D}})+H^\infty(\mathbf{D})$, and by McDonald and Sundberg [**12**] for symbols in $\operatorname{alg}\mathcal{H}L^\infty(\mathbf{T})$, the smallest closed subalgebra of L^∞ containing the bounded harmonic functions. Note that all these symbol classes are subalgebras of the algebra BC of all bounded continuous functions on $\mathbf{D}$.

Symbols which are not in BC were considered by Luecking [**9**] and McDonald [**11**]. Luecking established an invertibility criterion for $T(a)$ in case $a \geq 0$ a.e. on $\mathbf{D}$. McDonald proved a Fredholm criterion for $T(a)$ in the case where a belongs to $HC(\mathbf{D})$, the set of all functions $a \in L^\infty$ with the following property: for each $\tau \in \mathbf{T} := \partial\mathbf{D}$ there exists a set $U_\tau := \{z \in \mathbf{D} : |z-\tau| < \varepsilon\}$ and a straight line containing τ and dividing U_τ into two subsets U_τ^- and U_τ^+ such that $a|U_\tau^-$ and $a|U_\tau^+$ are uniformly continuous.

We remark that a major part of the aforementioned papers actually deal with Toeplitz operators on the ball $\{z \in \mathbf{C}^n : |z| < 1\}$ or on more general (and even exotic) domains. In addition to the works already cited, we refer in this connection to [**1**] and [**3**].

The present note concentrates on Toeplitz operators whose symbols are locally sectorial in some sense. Assume, for example, λ, μ, ν are

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three distinct points on $\mathbf{T}$ and $a \in L^\infty$ is zero on the triangle Δ made by λ, μ, ν and takes on constant values $\alpha, \beta, \gamma \in \mathbf{C}$ on each of the three segments of $\mathbf{D}\backslash\Delta$. We shall show that $T(a)$ is Fredholm on $A^2(\mathbf{D})$ if only the origin is not located on the periphery of the triangle spanned by α, β, γ. The approach employed here relies heavily on ideas from the circle-case, i.e., for Toeplitz operators on the Hardy space $H^2(\mathbf{D})$, and it is to a large extent originated by the papers [**14**] and [**4**] (also see the book [**6**]).

2. Locally sectorial functions. Recall that L^∞ stands for $L^\infty(\mathbf{D})$, and let L_0^∞ be the set of all $a \in L^\infty$ such that $\operatorname{ess\,sup}\{|a(z)| : r < |z| < 1\} \to 0$ as $r \to 1$. We denote by X the maximal ideal space of L^∞. Since L_0^∞ is a closed ideal of L^∞, there is a closed subset ∂X of X such that $L_0^\infty = \{a \in L^\infty : a|\partial X = 0\}$, and, consequently, L^∞/L_0^∞ may be identified with $C(\partial X)$. In what follows we shall make no distinction between the coset $a^\pi := a + L_0^\infty \in L^\infty/L_0^\infty$ and its Gelfand transform $a^\pi \in C(\partial X)$.

Let $\mathcal{B}$ be any C^*-subalgebra of L^∞ containing the constants (in that case we say that $\mathcal{B}$ is a C^*-algebra between $\mathbf{C}$ and L^∞). Then $\mathcal{B} + L_0^\infty$ is also a C^*-subalgebra of L^∞. The quotient algebra $(\mathcal{B} + L_0^\infty)/L_0^\infty$ is isometrically star-isomorphic to $\mathcal{B}/(\mathcal{B} \cap L_0^\infty)$ and is a C^*-subalgebra of $L^\infty/L_0^\infty = C(\partial X)$. For β in the maximal ideal space $M_\mathcal{B}$ of $(\mathcal{B} + L_0^\infty)/L_0^\infty$, we define the fiber ∂X_β as

$$\partial X_\beta = \{x \in \partial X : b^\pi(x) = b^\pi(\beta) \text{ for all } b \in \mathcal{B}\}.$$

The partition $\partial X = \cup_\beta \partial X_\beta$ is a partition of ∂X into pairwise distinct nonempty compact subsets.

A function $a \in L^\infty$ is said to be *locally sectorial* over a C^*-algebra $\mathcal{B}$ between $\mathbf{C}$ and L^∞ if for each $\beta \in M_\mathcal{B}$ the origin does not belong to $\operatorname{conv} a^\pi(\partial X_\beta)$, the closed convex hull of the set $a^\pi(\partial X_\beta)$. The set $a^\pi(\partial X)$ coincides with the spectrum of a^π in L^∞/L_0^∞. If $\operatorname{conv} a^\pi(\partial X)$ does not contain the origin, then a is called *globally sectorial*.

Theorem 1. *A function $a \in L^\infty$ is locally sectorial over a C^*-algebra $\mathcal{B}$ between $\mathbf{C}$ and L^∞ if and only if a is of the form $a = bs + d$, where $d \in L_0^\infty$, $s \in L^\infty$ is globally sectorial, and b is a function in $\mathcal{B}$ such that b^π is invertible in $(\mathcal{B} + L_0^\infty)/L_0^\infty$.*

Proof. The "if" part is trivial for, if $\beta \in M_{\mathcal{B}}$, then $a^\pi(\partial X_\beta) = b^\pi(\beta)s^\pi(\partial X_\beta)$. So suppose a is locally sectorial over $\mathcal{B}$. Then $a^\pi(x) \neq 0$ for all $x \in \partial X$, and so there is a function $u \in L^\infty$ satisfying $u^\pi(x) = a^\pi(x)/|a^\pi(x)|$ for all $x \in \partial X$. If $\beta \in M_\beta$, then the set $u^\pi(\partial X_\beta)$ lies on some open half-circle, and hence there is a $c_\beta \in \mathbf{C}$ such that $|u^\pi(x) - c_\beta| < 1$ for all $x \in \partial X_\beta$. Using a theorem by Glicksberg (see [**4**] or [**6**]), we now infer that there exists a function $b \in \mathcal{B}$ with $|u^\pi(x) - b^\pi(x)| < 1$ for all $x \in \partial X$. Since u^π is unimodular on ∂X, we have $|1 - b^\pi(x)/u^\pi(x)| < 1$ for all $x \in \partial X$. Consequently, $\mathrm{Re}\,(b^\pi/u^\pi) > 0$, and thus also $\mathrm{Re}\,(a^\pi/b^\pi) = \mathrm{Re}\,(|a^\pi|u^\pi/b^\pi) > 0$ on ∂X. Let s be any function in L^∞ such that $s^\pi(x) = a^\pi(x)/b^\pi(x)$ for $x \in \partial X$. Then s is globally sectorial and $d := a - bs \in L_0^\infty$. Finally, since $b^\pi \in (\mathcal{B} + L_0^\infty)/L_0^\infty$ is invertible in L^∞/L_0^∞ and $(\mathcal{B} + L_0^\infty)/L_0^\infty$ is a C^*-subalgebra of L^∞/L_0^∞, it follows that b^π is invertible in $(\mathcal{B} + L_0^\infty)/L_0^\infty$. $\square$

For $\tau \in \mathbf{T}$, let $\mathcal{U}_\tau$ be the family of all sets of the form $\{z \in \mathbf{D} : |z - \tau| < \varepsilon\}$. Given $a \in L^\infty$ and $U \in \mathcal{U}_\tau$, we denote by $\mathcal{R}_U(a)$ the essential range of a on U, and we let $\mathcal{R}_\tau(a) = \cap R_U(a)$, the intersection over all $U \in \mathcal{U}_\tau$. The maximal ideal space of $(C(\bar{\mathbf{D}}) + L_0^\infty)/L_0^\infty \cong C(\mathbf{T})$ is homeomorphic to $\mathbf{T}$, and so $\partial X = \cup\{\partial X_\tau : \tau \in M_{C(\bar{\mathbf{D}})}\}$ is a fibration over $\mathbf{T}$. We have $a^\pi(\partial X_\tau) = \mathcal{R}_\tau(a)$, and hence a is locally sectorial over $C(\bar{\mathbf{D}})$ if and only if $0 \notin \operatorname{conv} \mathcal{R}_\tau(a)$ for all $\tau \in \mathbf{T}$.

If $a \in HC(\mathbf{D})$, there is a natural definition of the one-sided limits $a(\tau \pm 0)$ at each point $\tau \in \mathbf{T}$, and a is locally sectorial over $C(\bar{\mathbf{D}})$ if and only if $0 \notin [a(\tau - 0), a(\tau + 0)]$ for all $\tau \in \mathbf{T}$.

Let $\mathcal{B}$ be a C^*-algebra between $C(\mathbf{T})$ and $QC(\mathbf{T}) := L^\infty(\mathbf{T}) \cap VMO(\mathbf{T})$. We denote by $\mathcal{H}b$ the harmonic extension of a function $b \in L^\infty(\mathbf{T})$ into $\mathbf{D}$. Then $\mathcal{H}\mathcal{B} + L_0^\infty := \{\mathcal{H}b + d : b \in \mathcal{B}, d \in L_0^\infty\}$ is a C^*-subalgebra of L^∞, and since

$$(\mathcal{H}\mathcal{B} + L_0^\infty)/L_0^\infty \cong \mathcal{H}\mathcal{B}/(\mathcal{H}\mathcal{B} \cap L_0^\infty) = \mathcal{H}\mathcal{B}/\{0\} \cong \mathcal{B},$$

$M_{\mathcal{H}\mathcal{B}+L_0^\infty}$ is homeomorphic to the maximal ideal space $M_{\mathcal{B}}$ of $\mathcal{B}$. By resorting to results of [**13**] and [**5**], one can show that if $a \in BC$ and $\beta \in M_{\mathcal{B}}$, then $a^\pi(\partial X_\beta)$ is equal to the cluster set

$$\begin{aligned}\mathrm{Cl}\,(a,\beta) := \{\lambda \in \mathbf{C} : \forall \varepsilon > 0 \ \ \forall\, b \in \mathcal{B} \ \ \exists\, z \in \mathbf{D} \text{ such that} \\ |(\mathcal{H}b)(z) - b(\beta)| < 1 \text{ and } |a(z) - \lambda| < \varepsilon\}.\end{aligned}$$

Hence, a function $a \in BC$ is locally sectorial over $\mathcal{H}\mathcal{B} + L_0^\infty$ if and only if $0 \notin \operatorname{conv} \mathrm{Cl}\,(a,\beta)$ for all $\beta \in M_{\mathcal{B}}$.

Finally, let $\mathcal{B}$ be any C^*-algebra between $C(\bar{\mathbf{D}})$ and L^∞, and let $a \in L^\infty$. If, for each $\tau \in \mathbf{T}$, there exist a function $b_\tau \in \mathcal{B}$ and a set $U_\tau \in \mathcal{U}_\tau$ such that $0 \notin \mathcal{R}_{U_\tau}(b_\tau)$ and $0 \notin \operatorname{conv} \mathcal{R}_{U_\tau}(b_\tau a)$, then a is clearly locally sectorial over $\mathcal{B}$.

3. Toeplitz operators with locally sectorial symbols. The Hankel operator $H(a) : A^2(\mathbf{D}) \to L^2(\mathbf{D})$ generated by a function $a \in L^\infty$ is the operator acting by the rule $H(a)\varphi = (I - P)(a\varphi)$ ($\varphi \in A^2(\mathbf{D})$). The question on the compactness of Hankel operators is of great relevancy in the Fredholm theory of Toeplitz operators. Axler [**2**] showed that if $a \in H^\infty(\mathbf{D})$, then $H(\bar{a})$ is compact if and only if a is in the "little Bloch" space. Only recently Zhu [**16**] succeeded in describing the set of all $a \in L^\infty$ for which both $H(a)$ and $H(\bar{a})$ are compact. This set will here be denoted by QC (because it may be regarded as a disk analogue of Sarason's algebra $QC(\mathbf{T})$) and it consists of all $a \in L^\infty$ for which

$$\lim_{|z|\to 1} \frac{1}{|S_z|} \int_{S_z} \left| a(w) - \frac{1}{|S_z|} \int_{S_z} a(u)\, dA(u) \right| dA(w) = 0,$$

where $dA = (1/\pi) r\, dr\, d\theta$ is normalized area measure on $\mathbf{D}$ and, for $z \in \mathbf{D}$, $S_z := \{w \in \mathbf{D} : |w| \ge |z|,\ |\arg w - \arg z| \le 1 - |z|\}$ and $|S_z| = (1 + |z|)(1 - |z|)^2$ is the measure of S_z.

For $z \in \mathbf{D}$, the normalized reproducing kernel $k_z \in A^2(\mathbf{D})$ is defined by

$$k_z(w) = (1 - |z|^2)/(1 - \bar{z}w)^2, \qquad w \in \mathbf{D}.$$

If $a \in L^\infty$, then the function

$$\tilde{a}(z) := \int_{\mathbf{D}} a(w) |k_z(w)|^2\, dA(w), \qquad z \in \mathbf{D},$$

belongs to BC. It is called the Berezin symbol of a. Also, for $z \in \mathbf{D}$, let

$$C_z = \{w \in \mathbf{D} : |w| \geq |z|, |\arg w - \arg z| \leq (1 - |z|)/2\}$$

and given $a \in L^\infty$, define $\hat{a} \in BC$ as

$$\hat{a}(z) = \frac{1}{|C_z|} \int_{C_z} a(w)\, dA(w), \qquad z \in \mathbf{D}.$$

Note that the maps $a \mapsto \tilde{a}$ and $a \mapsto \hat{a}$ of L^∞ into BC are linear, contractive, and order-preserving.

A function $a \in BC$ is said to be *bounded away from zero* if a^π is invertible in L^∞/L_0^∞, or equivalently, if there exists an $r \in (0,1)$ such that $\inf |a(z)| > 0$ on the annulus $r < |z| < 1$. In this case the winding number (index) about the origin of the restriction of a to the circle $|z| = \rho$ is independent of $\rho \in (r,1)$ and it will be denoted by $\operatorname{ind}(a|\partial\mathbf{D})$.

Theorem 2 (Zhu [**16**] and [**17**]). *For a function $b \in L^\infty$ the following are equivalent*:

(i) $b \in QC$;

(ii) $|\tilde{b}|^2 - |b^2|^\sim \in L_0^\infty$;

(iii) $|\hat{b}|^2 - |b^2|^\wedge \in L_0^\infty$;

(iv) $(ba)^\sim - \tilde{b}\tilde{a} \in L_0^\infty$ *for all* $a \in L^\infty$;

(v) $(ba)^\wedge - \hat{b}\hat{a} \in L_0^\infty$ *for all* $a \in L^\infty$.

If $b \in QC$, then $T(b)$ is Fredholm on $A^2(\mathbf{D})$ if and only if $\tilde{b}$ (equivalently, $\hat{b}$) is bounded away from zero; *in this case, the Fredholm index of $T(b)$ is given by*

$$\operatorname{Ind} T(b) = -\operatorname{ind}(\tilde{b}|\partial\mathbf{D}) = -\operatorname{ind}(\hat{b}|\partial\mathbf{D}).$$

Theorem 3. *Let $a \in L^\infty$ be locally sectorial over a C^*-algebra $\mathcal{B}$ between $\mathbf{C}$ and QC. Then $T(a)$ is Fredholm on $A^2(\mathbf{D})$, the functions $\tilde{a}$ and $\hat{a}$ are bounded away from zero, and*

$$\operatorname{Ind} T(a) = -\operatorname{ind}(\tilde{a}|\partial\mathbf{D}) = -\operatorname{ind}(\hat{a}|\partial\mathbf{D}).$$

Proof. It suffices to prove the theorem for $\mathcal{B} = QC$. Theorem 1 provides a representation $a = sb + d$, where $d \in L_0^\infty$, $s \in L^\infty$ is globally sectorial, $b \in QC$ and b^π is invertible in $(QC + L_0^\infty)/L_0^\infty$. From Theorem 2(iv) we deduce that $\tilde{b}$ is bounded away from zero, and since $\tilde{s}$ is obviously so, we obtain, again by Theorem 2(iv), that $\tilde{a}$ is bounded away from zero. We have

$$T(a) = T(s)T(b) + H^*(\bar{s})H(b) + T(d),$$

and since $T(s)$ is Fredholm of index zero and $T(d)$ and $H(b)$ are compact and, once more by Theorem 2, $T(b)$ is Fredholm with index $-\mathrm{ind}\,(\tilde{b}|\partial\mathbf{D})$, it follows that $T(a)$ is Fredholm and that

$$\begin{aligned}\mathrm{Ind}\,T(a) = \mathrm{Ind}\,T(b) &= -\mathrm{ind}\,(\tilde{b}|\partial\mathbf{D})\\ &= -\mathrm{ind}\,(\tilde{b}|\partial\mathbf{D}) - \mathrm{ind}\,(\tilde{s}|\partial\mathbf{D}) = -\mathrm{ind}\,(\tilde{b}\tilde{s}|\partial\mathbf{D})\\ &= -\mathrm{ind}\,(\widetilde{bs}|\partial\mathbf{D}) \quad \text{(Theorem 2(iv))}\\ &= -\mathrm{ind}\,(\widetilde{bs} + \tilde{d}|\partial\mathbf{D}) = -\mathrm{ind}\,(\tilde{a}|\partial\mathbf{D}).\end{aligned}$$

It can be shown analogously that $\hat{a}$ is bounded away from zero and that $\mathrm{Ind}\,T(a) = -\mathrm{ind}\,(\hat{a}|\partial\mathbf{D})$. □

The preceding theorem yields in particular a result which was essentially established by McDonald [**11**]. Namely, if $a \in HC(\mathbf{D})$, then the following are equivalent:

(i) $T(a)$ is Fredholm on $A^2(\mathbf{D})$;

(ii) a is locally sectorial over $C(\bar{\mathbf{D}})$;

(iii) $\hat{a}$ is bounded away from zero;

(iv) $0 \notin a^\# := \cup_{\tau\in\mathbf{T}}[a(\tau-0), a(\tau+0)]$;

and if $T(a)$ is Fredholm, then $\mathrm{Ind}\,T(a)$ equals minus the winding number about the origin of the naturally oriented curve $a^\#$. Indeed, the equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) are (almost) obvious, the implication (ii) $\Rightarrow$ (i) and the index formula follow from the theorem, and, once the index formula is available, the implication (i) $\Rightarrow$ (ii) can be proved by an index perturbation argument.

Given a function $a \in L^\infty(\mathbf{T})$ and an open arc $V \subset \mathbf{T}$, let $\mathcal{R}_V(a)$ denote the essential range of the restriction of a to V. For $\tau \in \mathbf{T}$, define $\mathcal{R}_\tau(a) = \cap \mathcal{R}_V(a)$, the intersection over all open arcs $V \subset \mathbf{T}$ containing τ.

A function $a \in L^\infty(\mathbf{T})$ is said to be *locally normal* (over $C(\mathbf{T})$) if, for each $\tau \in \mathbf{T}$, the set $\mathcal{R}_\tau(a)$ lies on some straight line (possibly depending on τ). Note that real-valued as well as piecewise continuous functions are locally normal. A generalization of two main results of McDonald and Sundberg [**12**] is as follows. If $a \in L^\infty(\mathbf{T})$ is locally normal, then the following are equivalent:

(i) $T(\mathcal{H}a)$ is Fredholm on $A^2(\mathbf{D})$;

(ii) $\mathcal{H}a$ is locally sectorial over $C(\bar{\mathbf{D}})$;

(iii) $\mathcal{H}a$ is bounded away from zero;

(iv) the Toeplitz operator $T(a)$ is Fredholm on the Hardy space $H^2(\mathbf{T})$;

if $T(\mathcal{H}a)$ is Fredholm, then

$$\operatorname{Ind} T(\mathcal{H}a) = -\operatorname{ind}(\mathcal{H}a|\partial\mathbf{D}) = \operatorname{Ind} T(a).$$

Note that Theorems 1 and 3 give the implications (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i), and the first index formula. For the implications (iv) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv), and the second index formula, see [**14**] and [**5**] (or [**6**]). Finally, the implication (i) $\Rightarrow$ (ii) can be again established by an index perturbation argument. We remark that the equivalence (i) $\Rightarrow$ (iv) in conjunction with the Widom-Douglas theorem on the connectedness of the essential spectrum of Hardy space Toeplitz operators implies that the essential spectrum of $T(\mathcal{H}a)$ is connected whenever $a \in L^\infty(\mathbf{T})$ is locally normal. Notice that the same conclusion may also be drawn from the equivalence (i) $\Rightarrow$ (iii) along with the fact that the set

$$\{\lambda \in \mathbf{C} : (\mathcal{H}a - \lambda)^\pi \text{ is not invertible in } L^\infty/L_0^\infty\} = \bigcap_{r\in(0,1)} (\mathcal{H}a)(\{z \in \mathbf{D} : r < |z| < 1\})$$

is connected for every $a \in L^\infty(\mathbf{T})$.

4. Locally subsectorial symbols. Let G be any measurable subset of $\mathbf{D}$. We denote by χ_G the characteristic function of G and define $\tilde{\chi}_G$

and $\hat{\chi}_G$ as in Section 3. Note that $\tilde{\chi}_G(z)$ is just the measure of the set $\Phi_z(G)$, where

$$\Phi_z : \mathbf{D} \to \mathbf{D}, \qquad \Phi_z(w) = (z-w)/(1-\bar{z}w).$$

Recall the definition of $\mathcal{U}_\tau$ (Section 2) and put $\mathcal{U} = \cup_{\tau\in\mathbf{T}}\mathcal{U}_\tau$.

Theorem 4 (Luecking [**9**]). *For a measurable set $G \subset \mathbf{D}$ the following are equivalent*:

(i) *$T(\chi_G)$ is invertible on $A^2(\mathbf{D})$*;

(ii) *there exists a constant $C > 0$ such that*

$$\int_{\mathbf{D}} |f|^2\, dA \le C \int_G |f|^2\, dA \quad \textit{for all } f \in A^2(\mathbf{D});$$

(iii) $\inf\{\tilde{\chi}_G(z) : z \in \mathbf{D}\} > 0$;

(iv) $\inf\{\hat{\chi}_G(z) : z \in \mathbf{D}\} > 0$;

(v) $\inf\{|G\cap U|/|U| : U \in \mathcal{U}\} > 0$.

We call a set G satisfying one (and thus all) of these conditions a *Luecking set*. We emphasize that the property of being a Luecking set is a condition on the behavior of the set near the boundary of $\mathbf{D}$. In particular, (iii) is equivalent to the requirement that $\tilde{\chi}_G^\pi$ be invertible in L^∞/L_0^∞ and (v) may be replaced by the condition that, for any $\varepsilon > 0$,

$$\inf\{|G\cap U|/|U| : U \in \mathcal{U}, |U| < \varepsilon\} > 0.$$

As an example, we remark that if $\Delta \subset \mathbf{D}$ is any convex polygon (whose vertices are allowed to be located on $\mathbf{T}$), then $\mathbf{D}\backslash\Delta$ is a Luecking set.

A function $a \in L^\infty$ is said to be *globally subsectorial* if there are $\lambda \in \mathbf{T}$, $\varepsilon > 0$, and a Luecking set $G \subset \mathbf{D}$ such that $\mathrm{Re}\,(\lambda a) \ge 0$ almost everywhere on $\mathbf{D}$ and $\mathrm{Re}\,(\lambda a) \ge \varepsilon$ almost everywhere on G.

Theorem 5. *If $a \in L^\infty$ is globally subsectorial, then $T(a)$ is invertible on $A^2(\mathbf{D})$.*

Proof. Let $\mathrm{Re}\, a \geq 0$ on $\mathbf{D}$ and $\mathrm{Re}\, a \geq \varepsilon > 0$ on some Luecking set G. Then there is a constant $C > 0$ such that $||f||_2 \leq C||\chi_G f||_2$ for all $f \in A^2(\mathbf{D})$. Put $\delta = \varepsilon/(C^2||a||_\infty^2)$. If $f \in A^2(\mathbf{D})$ and $||f||_2 = 1$, then

$$\begin{aligned}\int_{\mathbf{D}} (\mathrm{Re}\, a)|f|^2\, dA &\geq \int_G (\mathrm{Re}\, a)|f|^2\, dA \\ &\geq \varepsilon \int_G |f|^2\, dA \\ &\geq (\varepsilon/C^2) \int_{\mathbf{D}} |f|^2\, dA = \varepsilon/C^2,\end{aligned}$$

whence

$$\begin{aligned}||(I - \delta T(a))f||_2^2 &\leq \int_{\mathbf{D}} |1 - \delta a|^2 |f|^2\, dA \\ &= \int_{\mathbf{D}} (1 - 2\delta \mathrm{Re}\, a + \delta^2 |a|^2)|f|^2\, dA \\ &\leq 1 - 2\delta \int_{\mathbf{D}} (\mathrm{Re}\, a)|f|^2\, dA + \delta^2 ||a||_\infty^2 \\ &\leq 1 - 2\delta \frac{\varepsilon}{C^2} + \delta^2 ||a||_\infty^2 \\ &= 1 - \frac{\varepsilon^2}{C^4} \frac{1}{||a||_\infty^2} < 1,\end{aligned}$$

which implies that $T(a)$ is invertible. □

A function $a \in L^\infty$ is called *locally subsectorial* if there is a function $d \in L_0^\infty$ such that $b := a - d$ has the following property: for each $\tau \in \mathbf{T}$ there exist $\lambda_\tau \in \mathbf{T}$, $\varepsilon_\tau > 0$, $U_\tau \in \mathcal{U}_\tau$, and a Luecking set G_τ such that $\mathrm{Re}\, (\lambda_\tau b) \geq 0$ almost everywhere on U_τ and $\mathrm{Re}\, (\lambda_\tau b) \geq \varepsilon_\tau$ almost everywhere on $U_\tau \cap G_\tau$.

Theorem 6. *A function $a \in L^\infty$ is locally subsectorial if and only if it can be written in the form $a = cs + d$, where $c \in C(\bar{\mathbf{D}})$ is bounded away from zero, $d \in L_0^\infty$, and $s \in L^\infty$ is globally subsectorial.*

Proof. The "if" part is obvious. So suppose a is locally subsectorial. With each $U_\tau = \{z \in \mathbf{D} : |z - \tau| < \delta_\tau\}$ we associate the arc

$V_\tau := \{t \in \mathbf{T} : |t-\tau| < \delta_\tau\}$. There is a finite number $V_{\tau_1}, \dots, V_{\tau_n}$ of these arcs such that each $t \in \mathbf{T}$ belongs to at least one and to at most two of them. We also may assume that the corresponding numbers $\lambda_{\tau_1}, \dots, \lambda_{\tau_n}$ have the property that $\lambda_{\tau_i} \neq -\lambda_{\tau_j}$ whenever $V_{\tau_i} \cap V_{\tau_j} \neq \varnothing$.

Choose $r \in (0,1)$ so that $U_{\tau_1} \cup \dots \cup U_{\tau_n}$ contains $W := \{z \in \mathbf{D} : r < |z| < 1\}$ and put $W_j = \{z = \rho e^{i\theta} \in W : e^{i\theta} \subset V_{\tau_j}\}$. Then $\operatorname{Re}(\lambda_{\tau_j} b(z)) \geq 0$ for $z \in W_j$ and $\operatorname{Re}(\lambda_{\tau_j} b(z)) \geq \varepsilon := \min\{\varepsilon_{\tau_1}, \dots, \varepsilon_{\tau_n}\}$ for $z \in W_j \cap G_{\tau_j}$. Let $\varphi_1, \dots, \varphi_n$ be a continuous partition of unity on W such that each φ_j is constant along the radii $\{z \in W : \arg z = \text{const}\}$ and

$$\sum \varphi_j = 1 \quad \text{on W}, \qquad 0 \leq \varphi_j \leq 1 \quad \text{on } W_j, \qquad \operatorname{supp} \varphi_j = W_j.$$

Define $\varphi(z) = \sum \lambda_{\tau_j} \varphi_j(z)$ for $z \in W$. Then clearly $\operatorname{Re}(\varphi b) \geq 0$ on W. It is easily seen that $G := \cup_{j=1}^n (W_j \cap G_{\tau_j})$ is a Luecking set and that $\operatorname{Re}(\varphi b) \geq \varepsilon$ on G. We claim that $\varphi(z) \neq 0$ for all $z \in W$. Assume there is a $z \in W$ such that $\varphi(z) = 0$. Then there exist $i \neq j$ such that $z \in W_i \cap W_j$, whence

$$\lambda_{\tau_i} \varphi_i(z) + \lambda_{\tau_j} \varphi_j(z) = 0, \qquad \varphi_i(z) + \varphi_j(z) = 1,$$

which is impossible, since $\varphi_i(z)$ and $\varphi_j(z)$ are real numbers, $|\lambda_{\tau_i}| = |\lambda_{\tau_j}| = 1$, and $\lambda_{\tau_i} \neq -\lambda_{\tau_j}$. Thus, φ does indeed not vanish on W.

Now put $\psi(z) = 0$ for $z \in \mathbf{D} \backslash W$ and $\psi(z) = \varphi(z)$ for $z \in W$, and let $s(z) = \psi(z) b(z)$ for $z \in \mathbf{D}$. Then, by construction, s is globally subsectorial and $s - \psi b$ belongs to L_0^∞. Since ψ is bounded away from zero on W, there is a function $c \in C(\bar{\mathbf{D}})$ such that $c\psi - 1 \in L_0^\infty$. It follows that

$$cs - b = c(s - \psi b) + (c\psi - 1) b \in L_0^\infty,$$

and because $a - b \in L_0^\infty$, we arrive at the desired representation. □

Theorem 7. *If $a \in L^\infty$ is locally subsectorial, then $T(a)$ is Fredholm on $A^2(\mathbf{D})$, the functions $\tilde{a}$ and $\hat{a}$ are bounded away from zero, and*

$$\operatorname{Ind} T(a) = -\operatorname{ind}(\tilde{a}|\partial\mathbf{D}) = -\operatorname{ind}(\hat{a}|\partial\mathbf{D}).$$

Proof. Write $a = sc + d$ as in Theorem 6. So $T(a) = T(s)T(c) +$ a compact operator, and since $T(s)$ is invertible (Theorem 5) and $T(c)$

is Fredholm of index $-\mathrm{ind}\,(c|\partial\mathbf{D})$, it results that $T(a)$ is Fredholm and that $\mathrm{Ind}\,T(a) = -\mathrm{ind}\,(c|\partial\mathbf{D})$. If $\mathrm{Re}\,s \geq 0$ on $\mathbf{D}$ and $\mathrm{Re}\,s \geq \varepsilon > 0$ on some Luecking set G, then

$$(\mathrm{Re}\,\tilde{s})(z) \geq \varepsilon \int_G |k_z(w)|^2\, dA(w) = \varepsilon\tilde{\chi}_G(z)$$

and, hence, by Theorem 4, $\mathrm{Re}\,\tilde{s}$ is bounded away from zero. Because $\widetilde{cs} - \tilde{c}\tilde{s} \in L_0^\infty$ (which can be verified by elementary estimation or can also be deduced from Theorem 2), we conclude that $\tilde{a}$ is bounded away from zero and that $-\mathrm{ind}\,(c|\partial\mathbf{D}) = -\mathrm{ind}\,(\tilde{a}|\partial\mathbf{D})$. In the same way one can show that $\hat{a}$ is bounded away from zero and that $-\mathrm{ind}\,(c|\partial\mathbf{D}) = -\mathrm{ind}\,(\hat{a}|\partial\mathbf{D})$. $\square$

5. Real-valued symbols. Let $a \in L^\infty$ be real-valued. Put $m = \mathrm{ess\,inf}\,a$, $M = \mathrm{ess\,sup}\,a$, and let $\mathrm{sp}_{\mathrm{ess}}\,T(a)$ and $\mathrm{sp}\,T(a)$ refer to the essential spectrum and the spectrum of $T(a)$ on $A^2(\mathbf{D})$, respectively. McDonald and Sundberg [**12**] proved that

$$\mathrm{sp}_{\mathrm{ess}}\,T(a) = \mathrm{sp}\,T(a) = [m, M]$$

in case a is harmonic. All we can say in the general case is that

$$\mathrm{sp}_{\mathrm{ess}}\,T(a) \subset \mathrm{sp}\,T(a) \subset [m, M].$$

The question of whether m or M belong to $\mathrm{sp}\,T(a)$ was answered by Luecking [**9**]: $m \notin \mathrm{sp}\,T(a)$ (respectively, $M \notin \mathrm{sp}\,T(a)$) if and only if there are an $\varepsilon > 0$ and a Luecking set G such that $a \geq m + \varepsilon$ (respectively, $a \leq M - \varepsilon$) on G. The following theorem provides additional information about $\mathrm{sp}_{\mathrm{ess}}\,T(a)$ and $\mathrm{sp}\,T(a)$.

Theorem 8. *Let $a \in L^\infty$ be real-valued, and denote by $\mathcal{T}$ the set of all points $\tau \in \mathbf{T}$ such that $a^\pi(\partial X_\tau) = \mathcal{R}_\tau(a)$ is a singleton, $\{a(\tau)\}$. If $\mathcal{T}$ is not empty, then*

$$\big[\inf_{\tau\in\mathcal{T}} a(\tau), \sup_{\tau\in\mathcal{T}} a(\tau)\big] \subset \mathrm{sp}_{\mathrm{ess}}\,T(a).$$

Proof. Put $\alpha = \inf\{a(\tau) : \tau \in \mathcal{T}\}$ and $\beta = \sup\{a(\tau) : \tau \in \mathcal{T}\}$. Assume first that $\alpha = \beta =: \lambda$. Then $a(\tau) = \lambda$, and hence $\mathcal{R}_\tau(a - \lambda) =$

$\{0\}$ for all $\tau \in \mathcal{T}$. Fix any $\tau \in \mathcal{T}$ and choose any sequence $\{\varphi_n\}_{n=1}^{\infty}$ of functions $\varphi_n \in C(\bar{\mathbf{D}})$ such that $||\varphi_n||_\infty \le 1$, $\varphi_n(\tau) = 1$, and $\varphi_n(z) = 0$ for $|z-\tau| > 1/n$. Then $||\varphi_n(a-\lambda)||_\infty \to 0$ as $n \to \infty$. Denote by $\mathcal{K}$ the ideal of compact operators on $A^2(\mathbf{D})$ and define the essential norm of an operator T on $A^2(\mathbf{D})$ by

$$||T||_{\mathrm{ess}} := \inf\{||T+K|| : K \in \mathcal{K}\}.$$

From [**7**] we know that $||T(f)||_{\mathrm{ess}} \le ||f||_\infty$ for every $f \in L^\infty$ and that $||T(f)||_{\mathrm{ess}} = ||f|\mathbf{T}||_\infty$ for $f \in C(\bar{\mathbf{D}})$. It follows that $||T(\varphi_n)||_{\mathrm{ess}} = 1$ and

$$\begin{aligned} ||T(\varphi_n)T(a-\lambda)||_{\mathrm{ess}} &= ||T(\varphi_n(a-\lambda))||_{\mathrm{ess}} \\ &\le ||\varphi_n(a-\lambda)||_\infty = o(1) \quad \text{as } n \to \infty, \end{aligned}$$

which implies that $T(a) - \lambda I = T(a-\lambda)$ is not Fredholm and proves the theorem in case $\alpha = \beta$.

Let now $\alpha < \beta$ and $\lambda \in (\alpha, \beta)$. We must show that $T(a) - \lambda I = T(a-\lambda)$ is not Fredholm. Contrary to what we want, assume that $T(a-\lambda)$ is Fredholm.

We have $\alpha + \delta < \lambda < \beta - \delta$ for some $\delta > 0$. By our hypothesis, there are distinct $\tau_1, \tau_2 \in \mathcal{T}$ such that $a(\tau_1) < \alpha + \delta/3$ and $a(\tau_2) > \beta - \delta/3$, and therefore we can find $U_1 \in \mathcal{U}_{\tau_1}$ and $U_2 \in \mathcal{U}_{\tau_2}$ such that

$$\begin{aligned} &a - \lambda < \alpha + 2\delta/3 - \lambda < -\delta/3 \quad \text{a.e. on } U_1, \\ &a - \lambda > \beta - 2\delta/3 - \lambda > \delta/3 \quad \text{a.e. on } U_2. \end{aligned}$$

For $\varepsilon \in \mathbf{R}\backslash\{0\}$, define $(a-\lambda)_\varepsilon \in L^\infty$ as follows:

$$\begin{aligned} (a-\lambda)_\varepsilon(z) &:= a(z) - \lambda + i\varepsilon \qquad \text{for } \arg\tau_1 < \arg z < \arg\tau_2, \\ (a-\lambda)_\varepsilon(z) &:= a(z) - \lambda - i\varepsilon \qquad \text{for } \arg\tau_2 < \arg z < \arg\tau_1 + 2\pi. \end{aligned}$$

The function $(a-\lambda)_\varepsilon$ is clearly locally sectorial, and so Theorem 7 (or Theorem 3 with $\mathcal{B} = C(\bar{\mathbf{D}})$) entails that

$$|\operatorname{Ind} T((a-\lambda)_\varepsilon) - \operatorname{Ind} T((a-\lambda)_{-\varepsilon})| = 2.$$

However, if ε is small, we must have

$$\operatorname{Ind} T((a-\lambda)_\varepsilon) = \operatorname{Ind} T((a-\lambda)_{-\varepsilon}) = \operatorname{Ind} T(a-\lambda).$$

This contradiction completes the proof of the theorem. $\square$

The following result is an immediate consequence of Theorem 8.

Corollary ("Two Peninsulas Theorem"). *Let G be a measurable subset of $\mathbf{D}$ and suppose there are $U_0, U_1 \in \mathcal{U}$ such that $\chi_G|U_0 = 0$ and $\chi_G|U_1 = 1$. Then*

$$\mathrm{sp}_{\mathrm{ess}}\, T(\chi_G) = \mathrm{sp}\, T(\chi_G) = [0,1].$$

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