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REPRESENTATIONS OF ARCHIMEDEAN RIESZ SPACES—A SURVEY

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0. Introduction. Representation theorems play an important role in many mathematical disciplines. They provide us with more concrete descriptions of abstract mathematical objects like functionals, operators, spaces equipped with algebraical, topological or order structure. From the philosophical point of view, these descriptions help us to better understand the objects they deal with, and from the methodological point of view, they are a useful tool for getting new and reproving old results. The goal of this paper is to give a survey of representation theorems in the theory of Archimedean Riesz spaces (or vector lattices). For representations of non-Archimedean partially ordered vector spaces and Riesz spaces, see e.g., [**217, 6**, Chapter 2; **88, 174, 141, 178, 126, 66**]. Earlier surveys, only partially covering the material presented here, were given by Goullet de Rugy [**73**], Hackenbroch [**77**] and Wickstead [**211**].

The material is organized as follows: In Section 1 we collect important properties of Stonian spaces, since those spaces play an outstanding role in representation problems. Sections 2 and 3 are concerned with representations of "abstract" Riesz spaces, i.e., Riesz spaces which are not necessarily equipped with a compatible topology. In Section 4 we deal with Banach lattices, and finally, in Section 5, with Orlicz lattices.

In most cases, we will give only the main ideas of the proofs. Hence the word "proof" in this paper has to be read "sketch of proof."

We assume the knowledge of the fundamentals of Riesz space theory, including its terminology, as presented in Sections 1–3 of [**5**].

Throughout the paper, L denotes an Archimedean Riesz space.

For later reference, we want to fix some notation.

For $u \in L$, we denote by L_u the ideal of L generated by u.

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L is called laterally complete if for each disjoint system from L^+ the supremum exists in L.

L is called universally complete if it is Dedekind complete and laterally complete. A universally complete Riesz space M is called a universal completion of L if L can be embedded order densely into M, i.e. there exists an injective Riesz homomorphism from L onto an order dense Riesz subspace of M. (Recall that order dense embeddings are useful because they preserve arbitrary suprema and infima.)

Following [**127**], we denote by $\Gamma(L)$ the extended order continuous dual of L; thus if we put

$$\Phi := \{I : I \text{ order dense ideal of } L\},$$

then

$$\Gamma(L) = \bigcup_{I \in \Phi} I_n^\sim,$$

where $\xi \in I_n^\sim$ and $\eta \in J_n^\sim$ are identified if they coincide on their common domain $I \cap J \in \Phi$, and where algebraic and order structure are defined using representatives. According to [**127**; 1.5], $\Gamma(L)$ is a universally complete Riesz space, which contains $L_n^\sim$ as an order dense ideal provided $L_n^\sim$ separates L [**127**; 1.3]. This result was already obtained by Ermolin [**49**].

For functions $f, g \in \overline{\mathbf{R}}^X$ we set

$$\{f < g\} := \{x \in X : f(x) < g(x)\}$$

and use other abbreviations of the same type.

The characteristic function of a set A is denoted by 1_A, or by 1_A^X if it is necessary to specify the underlying space X.

For a Hausdorff space X, we write

$$\begin{aligned}
C(X) &:= \{f \in \mathbf{R}^X : f \text{ is continuous}\},\\
C_b(X) &:= \{f \in C(X) : f \text{ is bounded}\},\\
C_c(X) &:= \{f \in C(X) : \operatorname{supp} f \text{ is compact}\},\\
C(X, \overline{\mathbf{R}}) &:= \{f \in \overline{\mathbf{R}}^X : f \text{ is continuous}\},\\
C_\infty(X) &:= \{f \in C(X, \overline{\mathbf{R}}) : \{|f| \neq \infty\} \text{ is dense}\}.
\end{aligned}$$

$C(X)$, $C_b(X)$ and $C_c(X)$ are Riesz spaces under their natural (i.e. pointwise) ordering.

We write $\mathfrak{K}(X)$ for the family of compact subsets of X and $\mathfrak{U}(X)$ for the family of open subsets of X, and we denote by $\mathfrak{B}(X)$ the σ-algebra of Borel subsets of X, i.e. the σ-algebra generated by $\mathfrak{U}(X)$; further $\mathfrak{B}_c(X)$ stands for the δ-ring generated by $\mathfrak{K}(X)$, i.e. the family of relatively compact Borel subsets of X.

For a ring $\mathfrak{R}$ of subsets of a set X, we denote by $\mathcal{M}(\mathfrak{R})$ the Dedekind complete Riesz space of real-valued measures on $\mathfrak{R}$, i.e. the set of countably additive maps from $\mathfrak{R}$ to $\mathbf{R}$ with locally finite variation (cf. [**129**; 25.3]).

Given $\mu \in \mathcal{M}(\mathfrak{R})^+$, there exist a σ-algebra $\mathfrak{A}$ of subsets of X (namely the set $\mathfrak{A}$ of locally $\mathfrak{R}_\delta$-measurable subsets of X, i.e. $A \in \mathfrak{A}$ if and only if $A \cap B \in \mathfrak{R}_\delta$ for all $B \in \mathfrak{R}_\delta$) and a unique $\mathfrak{R}_\delta$-regular measure $\mu' : \mathfrak{A} \to [0, \infty]$ extending μ (here $\mathfrak{R}_\delta$ denotes the δ-ring generated by $\mathfrak{R}$); see [**112**; Theorem 4] or [**34**; 5.4.15, 5.4.6, 5.4.17]. Therefore we can assume, when needed, that the measure μ is defined on the whole of $\mathfrak{A}$.

We want to stress that our notion of integral is the one using the concept of "locally negligible sets" (see [**34**]), as it is used in Bourbaki's construction of the "essential integral" on locally compact Hausdorff spaces [**22**], or in [**84**].

We write $L^0(\mu)$ for the Riesz space of equivalence classes modulo μ-null sets of the real-valued μ-measurable functions (with μ extended as above). The measure μ is called localizable if $L^0(\mu)$ is Dedekind complete [**84**; Chapter I, Section 8]; in this case, $L^0(\mu)$ is automatically universally complete [**5**; 23.24].

A positive regular measure on a Hausdorff space X is a measure μ defined on a ring of sets $\mathfrak{R} \supset \mathfrak{K}(X)$ such that

$$\mu(A) = \sup\{\mu(K) : K \in \mathfrak{K}(X), K \subset A\}$$

for all $A \in \mathfrak{R}$. In this situation, μ' is $\mathfrak{K}(X)$-regular [**34**; Exercise 5.2.16(δ)]. Each positive regular measure on a locally compact Hausdorff space is localizable (even strictly localizable); see [**84**; Chapter I, Sections 8,9].

1. Stonian spaces. Let X be a Hausdorff space. Denoting by $\mathfrak{V}(X)$

the neighborhood filter of $x \in X$, we set, for $f \in \overline{\mathbf{R}}^X$:

$$f_{\sup}(x) := \limsup_{y \to x} f(y) := \inf_{V \in \mathfrak{V}(x)} \sup_{y \in V} f(y)$$
$$f_{\inf}(x) := \liminf_{y \to x} f(y) := \sup_{V \in \mathfrak{V}(x)} \inf_{y \in V} f(y)$$

$f_{\sup}$ is upper semicontinuous, $f_{\inf}$ is lower semicontinuous.

Proposition 1.1. *Let X be completely regular such that $C_b(X)$ is Dedekind complete. If $f : X \to \overline{\mathbf{R}}$ is lower semicontinuous, then $f_{\sup}$ is continuous, and $\{f_{\sup} \neq f\}$ is meager.*

Proof. Applying the arctan-function we may assume f to be bounded. Observe that f is the pointwise supremum of $\mathcal{G} := \{g \in C_b(X) : g \leq f\}$. Denoting by f' the supremum of $\mathcal{G}$ in $C_b(X)$ and setting $A_n := \{f' - f \geq 1/n\}$, we find $f' = f$ on $X \setminus \cup A_n$ and $f' = f_{\sup}$. □

A completely regular Hausdorff space X is called Stonian (or extremally disconnected) if $\bar{U}$ is open for all $U \in \mathfrak{U}(X)$, or equivalently, if $\overset{\circ}{F}$ is closed for all closed subsets F of X.

Stonian spaces were first considered by M.H. Stone [**166, 167**] and H. Nakano [**145**]. One should not confuse Stonian spaces with Stonian representation spaces of Boolean algebras; such representation space is Stonian if and only if the Boolean algebra is Dedekind complete [**129**; 47.5].

The Riesz space description of Stonian spaces is as follows:

Theorem 1.2. *The following are equivalent whenever X is a completely regular Hausdorff space.*

(a) *X is Stonian*;

(b) *$C(X)$ is Dedekind complete*;

(c) *$C_b(X)$ is Dedekind complete.*

Proof. (a) $\Rightarrow$ (b). Let $f_\iota \uparrow\leq g$ in $C(X)$. Setting $f(x) := \sup f_\iota(x)$ for all $x \in X$, $f_{\sup}$ is continuous (since $\{f_{\sup} > \alpha\} = \cup\overline{\{f > \alpha + 1/n\}}$ is open for all $\alpha \in \mathbf{R}$), and thus $f_\iota \uparrow f_{\sup}$ in $C(X)$.

(c) $\Rightarrow$ (a). Proposition 1.1 for $f = 1_U$. $\square$

We conclude that the Stone–Čech compactification of a Stonian space is Stonian.

An important extension property is contained in

Theorem 1.3. *If U is a dense open subset of the Stonian space X and if $f \in C_\infty(U)$, then there is a unique $g \in C_\infty(X)$ with $g|_U = f$.*

Proof. Apply Proposition 1.1 to f_1, where

$$f_1(x) := \begin{cases} f(x) & \text{if } x \in U \\ -\infty & \text{otherwise.} \end{cases} \qquad \square$$

In arbitrary Hausdorff spaces, it is in general impossible to define an addition in $C_\infty(X)$, although scalar multiplication and order structure are naturally given:

$$(\alpha f)(x) := \alpha\, f(x)$$
$$f \le g :\Longleftrightarrow f(x) \le g(x) \quad \text{for all } x \in X.$$

But

Theorem 1.4. *Let X be Stonian. Then $C_\infty(X)$ is a universally complete Riesz space, where the addition is defined as follows: For $f, g \in C_\infty(X)$ there is exactly one $h \in C_\infty(X)$ such that $f(x) + g(x) = h(x)$ for all $x \in \{|f| < \infty\} \cap \{|g| < \infty\}$; we put $f + g := h$.*

Proof. All assertions follow easily from Theorem 1.3, except for the Dedekind completeness; for this, let $0 \le f_\iota \uparrow\le f$ in $C_\infty(X)$, and set $U := \{f < \infty\}$. Since $C_b(U)$ and $C_b(X)$ are Riesz isomorphic by Theorem 1.3, U is Stonian, and therefore there exists $g := \sup f_\iota|_U$ in $C(U)$. Now apply Theorem 1.3 again. $\square$

Algebraic and order properties of the set $C_\infty(X)$, for X a compact Hausdorff space, were studied by Veksler, Zaharov and Koldunov

[**190**]. For Stonian X, the spaces $C_\infty(X)$–which are highly important in representation theory–were studied, e.g., by Dikanova, Koldunov, Lozanovskiĭ, Pinsker, Veksler, Vulikh, and the author (see, e.g., [**38, 39, 40, 99, 102, 184, 52, 53**] and others). For instance, a result of Pinsker states that under the assumption of (CH), if $C_\infty(X)$ has the diagonal property [**129**; p. 83] (where X is a compact Stonian space), then X is a Souslin space [**94**]. Vladimirov proved that this assertion cannot be derived in ZFC [**192**]; see also [**19**].

We now turn to measure theory in Stonian spaces. This theory was developed by Dixmier [**41**].

The measures playing the decisive role in Stonian spaces are the so–called normal measures. Let $\mu \in \mathcal{M}(\mathfrak{R})$ be a regular measure on a ring of sets $\mathfrak{R} \supset \mathfrak{K}(X)$, where X is a Hausdorff space. The measure μ is called normal if the map

$$C_c(X) \to \mathbf{R}, \qquad f \mapsto \int f \, d|\mu|$$

is order continuous.

Proposition 1.5. *For a positive regular measure μ on a locally compact Hausdorff space X, the following are equivalent:*

(a) *μ is normal;*

(b) *$\mu(K) = 0$ for all $K \in \mathfrak{K}(X)$ with $\overset{\circ}{K} = \varnothing$;*

(c) *$\mu(A) = \sup\{\mu(U) : U \subset A,\ U$ open$\}$ for all $A \in \mathfrak{R}$.*

Consequently, each meager set is a μ-null set provided μ is normal.

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{F} := \{f \in C_c(X) : f \geq 1_K\}$. Then $\mathcal{F} \downarrow 0$ in $C_c(X)$. Hence $\int f \, d\mu \downarrow 0$, where f runs through $\mathcal{F}$, and thus $\mu(K) = 0$.

(b) $\Rightarrow$ (a). Let $f_\iota \downarrow 0$ in $C_c(X)$, and set $f(x) := \inf f_\iota(x)$ for all $x \in X$. For $\alpha > 0$, $K := \{f \geq \alpha\} \in \mathfrak{K}(X)$, and $\overset{\circ}{K} = \varnothing$. Hence $f = 0$ μ-a.e. and thus, using Dini's Theorem, $\int f_\iota \, d\mu \to 0$. $\square$

For the rest of this section, let X be a locally compact Stonian space. Then

$$\mathfrak{R}(X) := \{K \triangle N : K \text{ open–compact},\ N \text{ meager}\}$$

is a δ-ring containing $\mathfrak{K}(X)$ (since $K = \overset{\circ}{K}\triangle(K\backslash\overset{\circ}{K})$) and thus containing also the δ-ring generated by $\mathfrak{K}(X)$, namely $\mathfrak{B}_c(X)$ (cf. [**93**, Chapter X, Section 2]).

In view of Proposition 1.5, $\mathfrak{N}(X)$ is a suitable domain for the normal measures. We set

$$\mathcal{M}(X) := \{\mu : \mu \text{ is a normal regular measure on } \mathfrak{N}(X)\}.$$

Proposition 1.6. *Let $\mu \in \mathcal{M}(X)$, let $f \in \overline{\mathbf{R}}^X$ be μ-measurable, and let $A \subset X$ be μ-measurable. Then*

(a) $f = f_{\sup} = f_{\inf} = (f_{\sup})_{\inf} = (f_{\inf})_{\sup}$ *μ-a.e.* (*observe that $(f_{\sup})_{\inf}$ and $(f_{\inf})_{\sup}$ are continuous*);

(b) $A = \bar{A} = \overset{\circ}{A} = \overset{\circ}{\bar{A}} = \bar{\overset{\circ}{A}}$ *μ-a.e.*

Proof. (a) Let $K \in \mathfrak{K}(X)$. By normality and Lusin's Theorem, there exist open-compact mutually disjoint sets K_n such that $K\backslash \cup K_n$ is μ-null and $f|_{\cup K_n}$ is continuous. Thus $f = f_{\sup} = f_{\inf} = (f_{\sup})_{\inf} = (f_{\inf})_{\sup}$ on $\cup K_n$, hence μ-a.e. on K. Since K was arbitrary, all is proved. □

Corollary 1.7. *If $\mu \in \mathcal{M}(X)$ and* $\operatorname{supp}\mu = X$, *then $L^0(\mu)$ and $C_\infty(X)$ are canonically Riesz isomorphic.* □

Corollary 1.8. *For $\mu, \nu \in \mathcal{M}(X)^+$, the following hold*:

(a) $\mu(A) = \sup\{\mu(K) : K$ *open-compact,* $K \subset A\}$ *for all μ-integrable A*;

(b) $\operatorname{supp}\mu$ *is open–closed*;

(c) $\operatorname{supp}\inf(\mu, \nu) = \operatorname{supp}\mu \cap \operatorname{supp}\nu$;

(d) $\mu \ll \nu \Longleftrightarrow \operatorname{supp}\mu \subset \operatorname{supp}\nu$.

Proof. (b) $\overline{X\backslash \operatorname{supp}\mu}$ is an open μ-null set.

(c) For $\mu \perp \nu$ and $\operatorname{supp}\mu$, $\operatorname{supp}\nu$ compact, the claim follows from the Hahn decomposition theorem; then the general case can be derived from this. □

X is called hyperstonian if $\cup_{\mu\in\mathcal{M}(X)} \operatorname{supp}\mu$ is dense in X, or equivalently, if $(C_c(X))_n^\sim$ separates $C_c(X)$. Not every Stonian space is hyperstonian (see the example following Proposition 2.7).

The following will be useful for uniqueness considerations.

Theorem 1.9. *Let Y be another locally compact Stonian space. In each of the following cases, the Stone-Čech compactifications βX and βY are homeomorphic*:

(a) *There exist an order dense Riesz subspace L of $C_\infty(X)$ and an order dense Riesz subspace M of $C_\infty(Y)$ which are Riesz isomorphic.*

(b) *There exist $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$ with $\operatorname{supp}\mu = X$, $\operatorname{supp}\nu = Y$ and order dense Riesz subspaces L of $L^0(\mu)$ and M of $L^0(\nu)$ which are Riesz isomorphic.*

(c) *X and Y are hyperstonian, and $\mathcal{M}(X)$ is Riesz isomorphic to $\mathcal{M}(Y)$.*

Proof. (a) The isomorphism extends to an isomorphism of the generated ideals L_1 and M_1. Then observe that each $f \in C_\infty(X)^+$ can be written as the supremum of a disjoint family of elements of L_1^+ [**5**; 23.15] to obtain that $C_\infty(X)$ and $C_\infty(Y)$ are Riesz isomorphic. Hence $C_\infty(\beta X)$ and $C_\infty(\beta Y)$ are Riesz isomorphic by Theorem 1.3, and since multiplication by an invertible element of $C_\infty(\beta X)^+$ is an isomorphism, one can assume $1_{\beta X} \mapsto 1_{\beta Y}$. Since the Boolean algebra of all bands of $C_\infty(\beta X)$ [**129**; 22.7] and the Boolean algebra of all open-compact subsets of βX are isomorphic, it follows that βX and βY are homeomorphic.

(b) follows from Corollary 1.7 and (a).

(c) In order to apply (a), it is enough to show that $(\mathcal{M}(X))_n^\sim$ is isomorphic to $C_\infty(X) \cap (\cap_{\mu\in\mathcal{M}(X)} \mathcal{L}^1(\mu))$, where the isomorphism is given by the formula $f \mapsto \xi$, with

$$\xi(\mu) = \textstyle\int f\, d\mu \quad \text{for all } \mu \in \mathcal{M}(X)$$

(see [**33**; 1.6.1]; his proof is reproduced below).

Let $\xi \in (\mathcal{M}(X))_n^\sim$, $\xi \geq 0$, and fix a maximal disjoint system (μ_ι) of $\mathcal{M}(X)^+$. For each ι, there exists, by the Riesz-Kakutani representation theorem, $\nu_\iota \in \mathcal{M}(X)^+$ such that $\operatorname{supp}\nu_\iota \subset \operatorname{supp}\mu_\iota$ and

$$\xi(h \cdot \mu_\iota) = \int h \, d\nu_\iota \quad \text{for all } h \in C_c(X).$$

By the Radon-Nikodým theorem and Corollaries 1.8(d), 1.7, there is $f_\iota \in C_\infty(X) \cap \mathcal{L}^1(\mu_\iota)$ with $\operatorname{supp} f_\iota \subset \operatorname{supp}\mu_\iota$ and

$$\xi(h \cdot \mu_\iota) = \int h f_\iota \, d\mu_\iota \quad \text{for all } h \in C_c(X).$$

Let $0 \le \mu \ll \mu_\iota$. Then $\mu = g \cdot \mu_\iota$, with $g \in C_\infty(X)$, and since $C_c(X)$ is order dense in $C_\infty(X)$, we get

$$\xi(\mu) = \int g f_\iota \, d\mu_\iota = \int f_\iota \, d\mu.$$

Now let f be the unique element of $C_\infty(X)$ with $f = f_\iota$ on $\operatorname{supp}\mu_\iota$ for all ι (Theorem 1.3). Then $f \in \cap_{\mu \in \mathcal{M}(X)} \mathcal{L}^1(\mu)$ and

$$\xi(\mu) = \int f \, d\mu \quad \text{for all } \mu \in \mathcal{M}(X). \qquad \square$$

It is well known that two compact Hausdorff spaces X, Y are homeomorphic provided that $C(X)$ and $C(Y)$ are Riesz isomorphic [**162**; Chapter II, Corollary 2, p. 104]. Also (c) is known to hold for compact hyperstonian spaces X, Y [**162**; Chapter II, 9.2].

For detailed information on normal measures, see e.g., [**62, 182, 64, 114, 115**]. Confer also [**188, 121, 33** and **61**].

2. Representations of Archimedean Riesz spaces as spaces of continuous functions. Since the spaces $C_\infty(X)$ are Archimedean, only Archimedean Riesz spaces can be embedded Riesz isomorphically into such spaces.

We start with the fundamental Maeda-Ogasawara-Vulikh representation theorem which we shall refer to in the sequel as "MOV." Its existence part is due to Maeda and Ogasawara [**133**] and, independently, to Vulikh ([**195**]; see also [**198, 199**]). The uniqueness assertion is contained in [**94**].

The proof presented here is taken from [**129**] and follows the original proof given in [**133**]. Other proofs can be found in [**163**; Section 26.2] or in [**66**]; Semadeni's proof is based on the Kakutani–Kreĭns Theorem 4.5, while Fleischer's proof uses Carathéodory's "place functions" (cf. also Remark 3 following Proposition 2.7).

Theorem 2.1 "MOV." *Given an Archimedean Riesz space L, there exist a compact Stonian space X and a Riesz isomorphism $T : L \to C_\infty(X)$ onto an order dense Riesz subspace of $C_\infty(X)$; if $S : L \to C_\infty(X')$ is another isomorphism of this kind, then there exist a homeomorphism $\tau : X \to X'$ and $f \in C_\infty(X)^+$ with* $\operatorname{supp} f = X$ *such that $(Su) \circ \tau = f \cdot (Tu)$ for all $u \in L$.*

Each pair (X, T) with the above properties will be called an MOV–representation of L. Furthermore:

(a) *The element u of L is a discrete element* (*or atom*) *of L if and only if* $\operatorname{supp} Tu$ *is a singleton*;

(b) *L is Dedekind complete if and only if TL is an ideal of $C_\infty(X)$*;

(c) *L is universally complete if and only if $TL = C_\infty(X)$*;

(d) *if w is a weak unit of L, one can arrange that $Tw = 1_X$; if w is a strong unit and $Tw = 1_X$, then $TL \subset C(X)$.*

Proof. Let $\mathfrak{B}$ be the Boolean algebra of all bands of L (with operations: $\inf(I, J) = I \cap J$, $\sup(I, J) = (I \cup J)^{dd}$) [**129**; 22.7], and let X be the Stone representation space of $\mathfrak{B}$. Since $\mathfrak{B}$ is Dedekind complete, X is Stonian [**129**; 47.5].

Now let (u_ι) be a maximal disjoint system of L^+ (its existence is assured by the Kuratowski-Zorn lemma), and put

$$X_\iota := \{x \in X : \{u_\iota\}^{dd} \notin x\}.$$

The open sets X_ι are mutually disjoint, and $\cup X_\iota$ is dense in X.

For $v \in L$ and $x \in X_\iota$ set

$$\hat{v}_\iota(x) := \sup\{\alpha \in \mathbf{R} : \{(\alpha u_\iota - v)^+\}^{dd} \in x\} \in \overline{\mathbf{R}}.$$

Then $\hat{v}_\iota \in C_\infty(X_\iota)$. Finally, let Tv be the extension of $\sum \hat{v}_\iota$ to X (Theorem 1.3).

Let X' be another compact Stonian space such that L is embedded order densely into $C_\infty(X')$. It follows from Theorem 1.9 that X and

X' are homeomorphic via a map τ. Now if (u_ι) is as above, set $f_\iota := (Su_\iota)\circ\tau/(Tu_\iota)$ on $\operatorname{supp}(Tu_\iota)$, and extend the so-defined function $\sum f_\iota$ to an element $f \in C_\infty(X)$.

Since multiplication with a $g \in C_\infty(X)^+$ whose support equals X is an isomorphism from $C_\infty(X)$ onto $C_\infty(X)$, one can arrange that a weak unit $w \in L$ is mapped onto 1_X. □

Remarks 1. Buskes and van Rooij show [**26**; 4.2] that MOV for spaces with a weak unit can be proved avoiding the axiom of choice, but using instead of it the Boolean prime ideal theorem, which asserts that every Boolean algebra contains a proper prime ideal.

2. In some representation problems it is convenient to use locally compact spaces instead of compact ones; observe that $C_\infty(X)$ is Riesz isomorphic to $C_\infty(U)$ provided U is a dense open subset of X, by Theorem 1.3. For instance, L is discrete [**5**; p. 17] if and only if L can be embedded order densely into some $\mathbf{R}^U$: If (u_ι) is a maximal disjoint system of discrete elements of L, then via MOV each u_ι corresponds to an f_ι with support $\{x_\iota\}$, and U, the collection of all these isolated points x_ι, is dense in $X = \beta U$.

3. In general, TL does not separate the points of X, even if L is Dedekind complete and has a strong unit. Indeed, take for L the ideal of $C_\infty(\beta\mathbf{N})$ generated by the function f, where $f(n) := 1/n$. But if L has the projection property and possesses a weak unit which is mapped onto 1_X, then TL does separate the points of X; this follows easily from Theorem 2.2 (a) $\Leftrightarrow$ (d).

Identifying the points of X which cannot be separated by elements of TL, one gets a "minimal" representation space that was considered e.g. by Vulikh [**199**].

4. It follows from MOV that any Archimedean Riesz space L possesses a unique universal completion, and if M is an order dense Riesz subspace of L, then their universal completions coincide (cf. [**5**; 23.20]).

5. Another consequence of MOV is that each Archimedean Riesz space L is Riesz isomorphic to a quotient space M/I, where M is a Riesz space of real-valued functions on some set X and I is a σ-ideal of M [**24**; Theorem 3]. Namely represent L as an order dense Riesz

subspace of $C_\infty(X)$, let M be the set of real-valued functions f on X for which there exists $\tilde{f} \in L$ such that $\{f \neq \tilde{f}\}$ is meager (hence $\tilde{f}$ is unique), and let I be the set of all $f \in M$ for which $\{f \neq 0\}$ is meager.

According to Fleischer [**66**], any Archimedean Riesz space can be represented also as a quotient of a space of measurable functions. The connections of this kind of representation to MOV were investigated by Traynor [**169**].

Even a more general result is true: Every Riesz space L (not necessarily Archimedean) can be embedded Riesz isomorphically into a quotient space M/I, where M is a Riesz space of real-valued functions on some set X and I is an ideal of M (see [**6**, Section 10] or [**81**; Section 6]).

6. Bernau [**15**] gave generalizations of MOV to Archimedean lattice groups and Archimedean lattice rings; he also proved a criterion that all representing functions be finite-valued on some common dense subset of the compact Stonian space X [**15**; Theorem 7]. Vulikh, too, obtained representations of lattice rings. See also [**17**].

Let us now describe the MOV-representations of Riesz spaces with the projection property.

For $\mathcal{F} \subset C_\infty(X)$, call $f \in C_\infty(X)$ $\mathcal{F}$-local if for each $x \in X$ there are an open $U_x \subset X$ and $f_x \in \mathcal{F}$ such that $x \in U_x$ and $f_x = f$ on U_x. Call $\mathcal{F}$ local if $\mathcal{F}$ contains all $\mathcal{F}$-local functions. An Archimedean Riesz space L is called local if for some MOV-representation (X, T) of L (and therefore for all MOV-representations of L) the set TL is local.

Let (X, T) be an MOV-representation of the Archimedean Riesz space L, let $u_0 \in L^+$, and take distinct $x_1, x_2 \in \operatorname{supp} Tu_0$. Call $[x_1, x_2, u_0]$ an absolute bundle of L with respect to (X, T), if $(Tu/Tu_0)(x_1) = (Tu/Tu_0)(x_2)$ for every $u \in L$. Given another MOV-representation (X', S) of L, note that $[x_1, x_2, u_0]$ is an absolute bundle of L with respect to (X, T) if and only if $[\tau x_1, \tau x_2, u_0]$ is an absolute bundle of L with respect to (X', S); here τ denotes the homeomorphism described in Theorem 2.1.

The following result is due to Veksler ([**171**; Theorem 1] and [**177**; Theorem 1]):

Theorem 2.2. *For an Archimedean Riesz space L, the following are* *equivalent*:

(a) *L possesses the projection property*;

(b) *for every MOV-representation (X,T) of L, for every $f \in TL$ and for every open-compact $U \subset X$, we have $f1_U \in TL$*;

(c) *L is local*;

(d) *there does not exist an absolute bundle of L.*

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Let (X,T) be an MOV-representation of L, and let $f \in C_\infty(X)^+$ be TL-local. Then there exist $x_1,\dots,x_n \in X$, open-compact neighborhoods U_i of x_i and functions $g_i \in TL^+$ such that $X = \cup_{i=1}^n U_i$ and $f = g_i$ on U_i. By (b), $f1_{U_i} \in TL$ for every i, and hence $f = \sup f1_{U_i} \in TL$.

(c) $\Rightarrow$ (d). Let $[x_1, x_2, u_0]$ be an absolute bundle of L, for some MOV-representation (X,T) of L. We may suppose that $Tu_0 = 1_U$ for some open-compact $U \subset X$. Decompose U into two open-compact sets U_1, U_2 with $x_i \in U_i$. Then $f := 1_{U_1} \notin TL$, but f is TL-local.

(d) $\Rightarrow$ (a). Let J be a band of L, let $u_0 \in L$, $u_0 > 0$, and set $J_0 := \{u_0\}^{dd}$. To show that the component of u_0 on J exists, we can assume $J \subset J_0$ (otherwise consider first $J \cap J_0$ instead of J) and $J \neq J_0$.

Let (X,T) be an MOV-representation of L with $Tu_0 = 1_U$, for some open-compact $U \subset X$. In the sequel we identify L with TL. Set $V := \overline{\cup_{v\in J} \operatorname{supp} v}$; then V is a proper subset of U.

Let $x \in V$. Using (d), we find, for each $t \in U \backslash V$, an element $v_t \in L^+$ with $v_t(t) = 1$ and $v_t(x) \neq 1$. From this, the existence of $z_t \in J_0^+$ with $z_t(t) = 0$ and $z_t(x) = 1$ is derived. Choose $\alpha > 0$, and set $w_t := ((1+\alpha)z_t - \alpha u_0)^+$ and $U_t := \operatorname{int}\{w_t = 0\}$. Then $X \backslash U \subset U_t$, $t \in U_t$, $w_t(x) = 1$. There exist $t_1,\dots,t_n \in U \backslash V$ with $U \backslash V \subset \cup_{i=1}^n U_{t_i}$. Set $f_x := 2 \inf w_{t_i}$. Then $f_x = 0$ on $X \backslash V$, whence $f_x \in J$, and $f_x(x) = 2$. Set $V_x := \{f_x > u_0\}$.

There exist $x_1,\dots,x_m \in V$ such that $V \subset \cup_{k=1}^m V_{x_k}$. Then $u := \sup f_{x_k} \in J$, and $\inf(u, u_0)$ is the component of u_0 on J. □

From (a) $\Leftrightarrow$ (b), it can be derived that a uniformly complete Riesz space [**129**; 42.1] with the projection property is Dedekind complete, a result which is due to Veksler [**175**], as well as the corresponding statement that a uniformly complete Riesz space with the principal projection property is σ-Dedekind complete; this latter result was reproved in [**7**], using representations on quasi-Stonian spaces.

Veksler obtained also characterizations of the Riesz spaces with the principal projection property and of the σ-Dedekind complete Riesz spaces, which we present without proof. (Note that (a) $\Leftrightarrow$ (b) in Theorem 2.3 is evident.)

Theorem 2.3 [**171**; Theorem 2]**.** *For an Archimedean Riesz space L, the following are equivalent*:

(a) *L possesses the principal projection property*;

(b) *for every MOV-representation (X,T) of L and for every $f,g \in TL$, we have $f1_{\operatorname{supp} g} \in TL$*;

(c) *given an MOV-representation (X,T) of L and an absolute bundle $[x_1,x_2,u_0]$ of L with respect to (X,T), if $u \in L$ satisfies $Tu \equiv 0$ in a neighborhood of x_1, then $Tu \equiv 0$ in a neighborhood of x_2.*

If U_1,U_2 are disjoint open-compact subsets of X, then $B_i(U_1,U_2)$ denotes the set of all points of U_i which form an absolute bundle with some point of U_j $(j \neq i)$. A closed subset F of X is called P-set if $\operatorname{int}(\cap_{n=1}^{\infty} V_n) \supset F$ for each sequence (V_n) of open-compact supersets of F.

Theorem 2.4 [**181**; Theorem 8]**.** *An order dense Riesz subspace L of $C_\infty(X)$, with X compact and Stonian, is σ-Dedekind complete if and only if it satisfies the following conditions*:

(1) *condition* (c) *of Theorem* 2.3 (*for $T = id$*);

(2) *each $f \in C_\infty(X)^+$, majorized by some element of L and satisfying $(f/u_0)(x_1) = (f/u_0)(x_2)$ for all absolute bundles $[x_1,x_2,u_0]$ of L, belongs to L*;

(3) *for each pair (U_1,U_2) of disjoint open-compact subsets of X, the sets $\overline{B_i(U_1,U_2)}$ are P-sets.*

The proof of part (a) in the following result is based on the fact that each laterally complete Archimedean Riesz space possesses the projection property [**5**; 23.4(ii)] and on Theorem 2.2 (a) $\Leftrightarrow$ (b); the essential ingredients in the proof of (b) are [**129**, 24.9(iv); **5**, 23.4(ii)], Theorem 2.2 (a) $\Leftrightarrow$ (b) and Theorem 2.3 (a) $\Leftrightarrow$ (b). Note that the countable analogon of the condition given in (a) does *not* characterize σ-lateral completeness. For details, see [**60**].

Theorem 2.5. *For every Archimedean Riesz space L and every MOV-representation (X, T) of L, we have*:

(a) *L is laterally complete if and only if TL contains all functions $f \in C_\infty(X)$ for which there are a disjoint family (U_ι) of open-compact subsets of X and a family (f_ι) in TL such that $\cup U_\iota$ is dense in X and $f 1_{U_\iota} = f_\iota 1_{U_\iota}$ for every ι.*

(b) *L is σ-laterally complete and possesses the projection property if and only if TL contains all functions $f \in C_\infty(X)$ for which there are a countable disjoint family (U_ι) of open-compact subsets of X and a family (f_ι) in TL such that $\cup U_\iota$ is dense in X and $f 1_{U_\iota} = f_\iota 1_{U_\iota}$ for every ι.*

The boundedly lateral completion of an Archimedean Riesz space was described in terms of its MOV-representation by Veksler and Geiler [**187**; Theorem 11].

Another related result is due to Veksler [**180**] and Koldunov [**101**]. Take an MOV-representation (X, T) of $L^0(\lambda)$, where λ denotes Lebesgue measure. Obviously $T(L^0(\lambda)) = C_\infty(X)$. Then, assuming (CH), there is a dense subset Y of X such that for every $f \in C_\infty(X)$ and for every $y \in Y$ there exist a function $g \in T(L^1(\lambda))$ and a neighborhood U of y such that $f = g$ on U.

The problem of characterizing those Archimedean Riesz spaces L which possess an MOV-representation (X, T) with $TL \subset C(X)$ was solved, for a very special class of Riesz spaces, by Maeda and Ogasawara [**133**, **152**]; strengthening a result of Papert [**153**; Theorem 7], Bernau proved that there exists a dense open subset Y of X (where X is as above) such that L can be embedded order densely into $C(Y)$ if and only if for each $u \in L$, $u > 0$ there is, for each $v \in L$, a number $n(v) \in \mathbf{N}$

such that

$$\sup_{i\in\{1,\ldots,p\}} \frac{1}{n(v_i)} v_i \not\geq u$$

for any finite subset $\{v_1, \ldots, v_p\}$ of L [**15**; Theorem 7]. Here is the general solution of the problem:

Theorem 2.6 [**2**]. *There exists an MOV-representation (X, T) of L with $TL \subset C(X)$ if and only if there exists a family $(u_{\iota\lambda})_{\iota\in I,\lambda\in\Lambda_\iota}$ in L^+ such that*

(1) $L = \{u_{\iota\lambda} : \iota \in I,\ \lambda \in \Lambda_\iota\}^{dd}$;

(2) *for each ι, the net $(u_{\iota\lambda})_{\lambda\in\Lambda_\iota}$ increases and is order bounded in L*;

(3) *for all $\iota_1, \iota_2 \in I$, $\iota_1 \neq \iota_2$, and for all $\lambda_1 \in \Lambda_{\iota_1}$, $\lambda_2 \in \Lambda_{\iota_2}$ we have* $\inf(u_{\iota_1,\lambda_1},\ u_{\iota_2,\lambda_2}) = 0$;

(4) *for each $v \in L^+$ there exists $n \in \mathbf{N}$ satisfying* $\sup_{\lambda\in\Lambda_\iota} \inf(z,\ nu_{\iota\lambda}) = z$ *for each $0 \leq z \in \{u_{\iota\lambda} : \lambda \in \Lambda_\iota\}^{dd}$, $z \leq v$, and for each $\iota \in I$.*

Proof. Denote by L^δ the Dedekind completion of L.

To prove the sufficiency of the condition, set $u_\iota := \sup_{\lambda\in\Lambda_\iota} u_{\iota\lambda}$ in L^δ, and find an MOV-representation (X, T^δ) of L^δ such that $T^\delta u_\iota = 1_{X_\iota}$ for suitable open-compact subsets X_ι of X. Then $(X, T^\delta|_L)$ is as required.

Conversely, let L be an order dense Riesz subspace of $C(X)$, with X compact and Stonian. Then the same holds for L^δ [**129**; 50.8(iii)]. By the Kuratowski-Zorn Lemma, one can find a disjoint family (X_ι) of open-compact subsets of X such that $1_{X_\iota} \in L^\delta$ for each ι and $\cup X_\iota$ is dense in X. For each ι, there is a net $(u_{\iota\lambda})_{\lambda\in\Lambda_\iota}$ in L^+ with $0 \leq u_{\iota\lambda} \uparrow_\lambda 1_{X_\iota}$. Then $(u_{\iota\lambda})_{\iota\in I,\lambda\in\Lambda_\iota}$ satisfies (1)–(4). □

The proof of the following assertion [**53**; Theorem 3] is based on the fact that $\Gamma(L)$ separates L if and only if there is an order dense ideal I of L which is separated by $I_n^\sim$ [**127**; 2.5]:

Proposition 2.7. *If X is a locally compact Stonian space such that L is embedded order densely into $C_\infty(X)$, then X is hyperstonian if and only if $\Gamma(L)$ separates L.*

If $L = C([0,1])$, and if (X,T) is an MOV-representation of L with $T1_{[0,1]} = 1_X$, then $\mathcal{M}(X) = \{0\}$: For $\mu \in \mathcal{M}(X)^+$, the map $L \to \mathbf{R}$, $u \mapsto \int (Tu) d\mu$ is an order continuous linear form; but since $L_n^\sim = \{0\}$ (this follows, e.g., from the Riesz-Kakutani representation theorem and Proposition 1.5), we get $\mu = 0$.

There exist several variants of MOV which we will shortly mention; for a detailed account, see chapter 7 of [**129**].

1. The Nakano representation theorem ([**147**] for L σ-Dedekind complete) is very similar to MOV. L is now assumed to have the principal projection property. The Boolean algebra of all bands of L is replaced by the Boolean ring $\mathfrak{B}_p$ of all principal bands of L, and X is the Stone representation space of $\mathfrak{B}_p$. The space X is Hausdorff, totally disconnected and locally compact, and TL separates the points of X.

2. In the Yosida representation theorem [**217**], the space X consists of the union of the sets $\mathfrak{P}_\iota$, where $\mathfrak{P}_\iota$ denotes the set of all prime ideals of L maximal with respect to the property of not containing u_ι, and (u_ι) is a maximal disjoint system of L^+. Then X, equipped with the hull-kernel topology, is Hausdorff and locally compact, and TL separates the points of X. If L has a weak unit, then X is compact; moreover, X is unique in a natural sense (see [**78**; 2.11]). If L has a strong unit, then $TL \subset C(X)$.

The Johnson-Kist representation theorem [**89**; 6.7] generalizes the Yosida theorem insofar as sets of proper prime ideals with certain properties are used.

3. Sometimes it is convenient to consider quasi-Stonian spaces, i.e., completely regular Hausdorff spaces in which the closure of each open F_σ-set is open (see, e.g., [**203**; Chapter V] or [**129**; Section 43]). For instance, by a result of Ogasawara, the Stone representation space of a Boolean algebra $\mathcal{R}$ is quasi-Stonian if and only if $\mathcal{R}$ is σ-Dedekind complete [**203**; Theorem II.9.2]. By results of Nakano and Gillman-Jerison, a normal Hausdorff space X is quasi-Stonian if and only if $C_b(X)$ is σ-Dedekind complete, and for X completely regular and Hausdorff, $C_b(X)$ is σ-Dedekind complete if and only if $\overline{\{f \neq 0\}}$ is open for every $f \in C_b(X)$ (see, e.g., [**129**; 43.8, 43.9]). Analogously to Theorem 1.4, one can show that $C_\infty(X)$ is a σ-universally complete Riesz space provided X is quasi-Stonian.

Nakano proved that every σ-Dedekind complete Riesz space L can be embedded as an ideal into $C_\infty(X)$, for some locally compact quasi-Stonian space X ([**149**; Theorem 8.10, **150**; Theorem 16.7].

Using valuations on L (see Section 4), Hackenbroch has proved [**74**; Satz 2] that for a σ-Dedekind complete Riesz space L there exists a locally compact quasi-Stonian space X such that $C_c(X) \subset TL \subset C_\infty(X)$. If, moreover, L has a weak unit, then TL can be embedded into $C_\infty(X)$ as an ideal containing $C(X)$, for some compact quasi-Stonian space X [**203**; Theorem V.4.1], and if L is σ-universally complete with a weak unit, then $TL = C_\infty(X)$ for some compact quasi-Stonian space X [**203**; Theorem V.5.1]; see also [**42**]. An example of Rotkovich shows that in general a σ-Dedekind complete Riesz space cannot be embedded as an ideal into $C_\infty(X)$, for a quasi-Stonian compact space X [**160**].

Recently, Fleischer has shown [**65**] that Carathéodory's system of "place functions" on a Boolean σ-ring $\mathcal{R}$ [**28**] is Riesz isomorphic to $C_\infty(X)$, where X is the quasi-Stonian Stone representation space of $\mathcal{R}$. In a subsequent paper [**66**], he has proved that an arbitrary Riesz space L can be represented as a space of place functions on the Boolean algebra of all orthogonal complements in L; in the Archimedean case this result just yields MOV, in view of the isomorphism of $C_\infty(X)$ with the space of place functions.

4. There exist also similar representation theorems for Archimedean lattice groups; see, e.g., [**153, 17, 9** and **11**].

Generalizations in another direction were given by Veksler, Zaharov and Koldunov [**190**]; they deal with so-called semivector lattices, i.e., objects which differ from the usual Riesz spaces in that there is given only a partial addition, obeying the vector lattice laws if defined (a typical example is $C_\infty(X)$, with X not necessarily Stonian).

5. Huber [**81**; Section 3] shows that one can get some of the representation theorems (MOV, Nakano, Hackenbroch, Bernau, Papert) by a unified approach, starting with an arbitrary distributive lattice with a smallest element (replacing, e.g., the Boolean algebra of all bands in the proof of MOV).

Concerning uniqueness of representation spaces, we have the following result [**75**]:

Theorem 2.8. *Let X be a locally compact Hausdorff space, and let $C_c(X)$ be Riesz isomorphic to an order dense ideal of the Dedekind complete Riesz space L (respectively, to a super order dense ideal of the σ-Dedekind complete Riesz space L). Then, if X' has the same properties, βX is homeomorphic to $\beta X'$.*

Proof. It is enough to show that $C(\beta X)$ is Riesz isomorphic to $C(\beta X')$ (see [**162**; Chapter II, Corollary 2, p. 104]).

Let Z_L be the order ideal of $\mathcal{L}_b(L, L)$ generated by the identity map (the so-called centre of L). Z_L is a Riesz space [**218**; 140.4]. Then $Z_{C_c(X)}$ can be identified with $C_b(X)$ via

$$C_b(X) \ni f \mapsto \hat{f}, \quad \text{where } \hat{f}(g) := fg \quad \text{for all } g \in C_c(X).$$

To prove that this map is surjective, let $T \in Z_{C_c(X)}$, $T \geq 0$. For each $x \in X$, the map $g \mapsto (Tg)(x)$ defines a positive Radon measure μ_x on X with $\operatorname{supp} \mu_x = \{x\}$, hence there is $f(x) \in \mathbf{R}$ with $\mu_x = f(x)\delta_x$. Then $f \in C_b(X)$ and $\hat{f} = T$.

Since $C_c(X)$ is an order dense (respectively, a super order dense) ideal of L and all elements of Z_L are order continuous, the map

$$Z_L \to Z_{C_c(X)}, \, T \mapsto T|_{C_c(X)}$$

is a Riesz isomorphism (the surjectivity being a consequence of the denseness assumptions). Thus

$$C(\beta X) = C_b(X) = C_b(X') = C(\beta X'). \qquad \square$$

The following result [**76**; Lemma 1] should also be seen in connection with the Kakutani-Kreĭns Theorem 4.5.

Theorem 2.9. *Let L be a norm dense Riesz subspace of $C(X)$, where X is a compact Hausdorff space and $C(X)$ is equipped with the supremum norm. If $1_X \in L$ and L has the principal projection property, then X is totally disconnected.*

Proof. Since the set $\{\{f > 0\} : f \in C(X)\}$ is a base for $\mathfrak{U}(X)$, the same holds for $\mathfrak{U} := \{\{f > 0\} : f \in L^+\}$.

Let $U \in \mathfrak{U}$, $U = \{f > 0\}$ for some $f \in L^+$. Denoting by P_f the projection of L on the band of L generated by f and setting $u_f := P_f 1_X$, we have $u_f = 1_A$ for some open-closed A. Then $\bar{U} = A$ is open, and $\{\bar{U} : U \in \mathfrak{U}\}$ is a base for $\mathfrak{U}(X)$. □

Recall that the Stone-Weierstraß theorem [**162**; Chapter II, 7.3] asserts that if X is a compact Hausdorff space and L is a Riesz subspace of $C(X)$ containing 1_X and separating the points of X, then L is norm dense in $C(X)$.

The next theorem is due to M. Weber.

Theorem 2.10 [**206**; Theorem 2]**.** *Assume there exists an increasing sequence $0 \leq w_n$ in L such that for each $u \in L$ there exist $n \in \mathbf{N}$ and $c > 0$ with $|u| \leq cw_n$, but that L possesses no strong unit. Then the following are equivalent*:

(a) *L can be endowed with a Riesz norm* (*see Section* 4);

(b) *L can be embedded into a Riesz space M with a strong unit*;

(c) *there exist a locally compact, σ-compact Stonian space X and a Riesz isomorphism T from L onto a Riesz subspace of*

$$C_0(X) := \{f \in C(X) : f \textit{ vanishes at infinity}\}$$

such that for each $x \in X$ there is $f \in TL$ with $f(x) > 0$.

Proof. (a) ⇒ (b). Let p be a Riesz norm on L. The norm completion $\bar{L}$ of L is a Banach lattice [**218**; 100.10], and $L \subset M := \bar{L}_u$, with

$$u := \sum_{n \in \mathbf{N}} \frac{w_n}{2^n\, p(w_n)} \in \bar{L}.$$

(b) ⇒ (c). By MOV, M (hence also L) can be embedded into some $C(Y)$, with Y compact and Stonian. Then $W_n := \{w_n = 0\} \neq \varnothing$ since w_n is not a strong unit, hence also $W := \cap_{n \in \mathbf{N}} W_n \neq \varnothing$. Thus $L \subset C_0(X)$, with $X := Y \backslash W$. In particular $\{w_n \geq 1/m\}$ is compact in X, which shows that $X = \cup_{n,m \in \mathbf{N}} \{w_n \geq 1/m\}$ is σ-compact.

(c) ⇒ (a). Consider the restriction of the supremum norm to L. □

The original representation space in the preceding theorem was a subspace of the space $\mathfrak{M}(L)$ of all proper maximal ideals of L, endowed with the hull-kernel topology. Note that the elements of $\mathfrak{M}(L)$ are precisely the kernels of the real-valued Riesz homomorphisms on L. This relation can be used to define the representing functions on $\mathfrak{M}(L)$. In the papers of Makarow and Weber [**134**–**137**, **205**–**208**], detailed investigations of the spaces $\mathfrak{M}(L)$ are made, and several representation theorems are obtained for Riesz spaces which can be represented as spaces of continuous real-valued functions on (subspaces of) $\mathfrak{M}(L)$. We shall mention without proof another typical result. Here the abstract analogon of a continuous function with compact support plays an important role: Makarow and Weber call an element u_0 of the Riesz space L finite if there exists a $v \in L$ such that for every $u \in L$ there is an $\alpha_u \in \mathbf{R}^+$ satisfying

$$|u| \wedge C|u_0| \;\le\; \alpha_u v \qquad \forall\, C > 0;$$

if v can be chosen to be finite itself, then u_0 is called totally finite.

Theorem 2.11 [**137**; Theorem 2]**.** *Let L be an Archimedean Riesz space containing an order dense ideal I which consists of totally finite elements of L and which is contained in a Riesz space with a strong unit.*

Then L can be embedded into $C(X)$, for an appropriate Hausdorff space X, such that $I \subset C_c(X)$ and such that for every distinct $x_1, x_2 \in X$ there is an $f \in I$ with $f(x_1) = 0$ and $f(x_2) = 1$.

Moreover:

(1) *$I = C_c(X)$ if and only if I is uniformly complete.*

(2) *$I = L$ implies X is locally compact and homeomorphic to $\mathcal{M}(L)$.*

We shall consider now hyper-Archimedean Riesz spaces (investigated first by Amemiya [**6**]), i.e., Riesz spaces L such that for all $u, v \in L^+$ there exists $n \in \mathbf{N}$ with

$$\inf((n+1)u,\, v) = \inf(nu, v).$$

Several equivalent formulations of this property are known; e.g., L is hyper-Archimedean if and only if L/I is Archimedean for each ideal I of L (see [**128**, 7.1, 7.2; **129**, 37.6, 61.1, 61.2; **82**; Theorem 3]).

Clearly each Riesz space consisting of real-valued step functions (i.e., functions with finite range) is hyper-Archimedean.

Theorem 2.12 [**58**; Corollary 8]**.** *Let L be a hyper-Archimedean Riesz space, and let (w_ι) be a maximal disjoint system of L^+. Then there exists a locally compact Stonian space X which is the topological direct sum of compact spaces X_ι such that*

(i) *L is Riesz isomorphic to an order dense Riesz subspace $\hat{L}$ of $C_\infty(X)$, and the isomorphism maps w_ι onto 1_{X_ι}, for all ι;*

(ii) *for each ι, $\{f|_{X_\iota} : f \in \hat{L}\}$ is the set of all step functions with respect to an algebra of open-compact subsets of X_ι.*

Proof. First assume that L has a weak unit w. Let (X, T) be an MOV-representation of L with $Tw = 1_X$, and set $\hat{L} := TL$.

Let $f \in \hat{L}^+$. Since $\hat{L}$ is hyper-Archimedean, there exists $n \in \mathbf{N}$ such that $f \leq n1_X$ and $1_X \leq nf$ on $\operatorname{supp} f$. If the range of f were infinite, it would possess an accumulation point $\alpha \in [1/n, n]$, and $|f - \alpha 1_X|$ would take arbitrarily small values on $\operatorname{supp}(f - \alpha 1_X)$, contradicting the hyper-Archimedeanness of $\hat{L}$.

Since $\hat{L}$ has the Stone property (i.e. $\inf(f, 1_X) \in \hat{L}$ for all $f \in \hat{L}^+$), $\hat{L}$ consists of all step functions with respect to a ring (hence an algebra) of open-compact subsets of X (see [**34**; 2.3.10]).

In the general case, represent L as order dense Riesz subspace of $C_\infty(Y)$, with Y compact and Stonian, such that w_ι is mapped onto 1_{X_ι}, and set $X := \cup X_\iota$. $\square$

It is not difficult to prove [**58**; Proposition 7] that for $f, g \in \hat{L}^+$, the function f/g takes only finitely many values on $\operatorname{supp} g$.

Luxemburg and Zaanen proved [**129**; 37.7] that a hyper-Archimedean Riesz space with a weak unit can be represented as the space of all step functions with respect to an algebra of open-compact subsets of the set $\mathfrak{P}$ of proper prime ideals of L, endowed with the hull-kernel topology (observe however, that $\mathfrak{P}$ is in general not Stonian). This result goes back to Amemiya [**6**], but his proof was incomplete.

Theorem 2.13 [**58**; Theorem 10]**.** *For a Riesz space L, the following* *are equivalent.*

(a) *There exists a compact Stonian space X such that L is Riesz isomorphic to an order dense Riesz subspace $\hat{L}$ of $C_\infty(X)$ consisting of all step functions with respect to a ring of open-compact subsets of X;*

(b) *there exists a hyper-Archimedean Riesz space M with a weak unit that contains L as an order dense ideal;*

(c) *L is hyper-Archimedean, and there exists a maximal disjoint system $(w_\iota)_{\iota\in I}$ of L^+ such that for each $u \in L^+$, $\sup_{\iota\in I}(\inf(u, w_\iota))$ exists in L and the set $\{L_{u,\alpha} : \alpha > 0\}$ is finite, where*

$$L_{u,\alpha} := \cap_{\iota\in I}\{w_\iota - \inf(w_\iota, \alpha u)\}^d;$$

(d) *L is Archimedean, and there exists a weak unit w of the universal completion L^u of L such that each $u \in L$ can be written as a finite linear combination of components of w in L.*

Proof. (a) $\Leftrightarrow$ (d) is not difficult to see.

(a) $\Rightarrow$ (b). If M denotes the Riesz subspace of $C_\infty(X)$ generated by 1_X and $\hat{L}$, then

$$M = \{\alpha 1_X + f : \alpha \in \mathbf{R}, f \in \hat{L}\}$$

which implies the assertion.

(b) $\Rightarrow$ (a) is an application of Theorem 2.12.

(a) $\Rightarrow$ (c). The assumptions of (a) imply the existence of a maximal disjoint family (X_ι) of open-compact subsets of X such that $1_{X_\iota} \in \hat{L}$ for all ι. Define w_ι as the element of L which corresponds to 1_{X_ι} under the isomorphism $L \to \hat{L}$.

(c) $\Rightarrow$ (a). Represent L as an order dense Riesz subspace $\tilde{L}$ of $C_\infty(\cup X_\iota)$, with the properties described in Theorem 2.12. Then $f \in \tilde{L}^+$ can be written in the form $f = \sum \alpha_\lambda 1_{U_\lambda}$, where the U_λ's are mutually disjoint non-empty open-compact sets, each one of them contained in some X_ι.

Assume f is not a step function. Then we can extract a strictly monotone (say decreasing) sequence (α_n) from the set of all α_λ's. Hence the sequence $(\tilde{L}_{f,\alpha_n^{-1}})$ is increasing. For each $n \in \mathbf{N}$ there exists $u_n \in \tilde{L}$ such that

$$0 < u_n \le \frac{\alpha_n - \alpha_{n+1}}{\alpha_n} 1_{U_{n+1}};$$

hence, $u_n \in \tilde{L}_{f,\alpha_{n+1}^{-1}} \setminus \tilde{L}_{f,\alpha_n^{-1}}$, which gives a contradiction since n was arbitrary.

$\sup_{\iota \in I}(\inf(u, w_\iota) \in L$ for all $u \in L^+$ implies $\inf(f, 1_X) \in \tilde{L}$ for all $f \in \tilde{L}^+$, and thus $\tilde{L}$ consists of all step functions with respect to a ring of sets [**34**; 2.3.10].

Finally set $X := \beta(\cup X_\iota)$. □

Bernau has shown [**16**] that each countable-dimensional hyper-Archimedean Riesz space can be represented as a Riesz space of real-valued step functions on some set X. A characterization of the Riesz spaces which can be represented in this way was given by Chuang and Nakano [**29**; Theorem 4.4].

We still mention the following result due to Luxemburg and Moore [**128**; 7.5].

Theorem 2.14. *For a Riesz space L, the following are equivalent.*

(a) *L is Riesz isomorphic to the Riesz space of all real-valued functions f on some set X such that $\{f \neq 0\}$ is finite*;

(b) *every principal ideal of L is finite-dimensional and Archimedean* (*hence Riesz isomorphic to some $\mathbf{R}^n$* [**129**; 26.11]).

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a). Take a maximal disjoint system X of atoms of L. (b) implies that each $u \in L$ is a finite linear combination of atoms of L, hence of elements of X. □

More equivalent conditions can be found in [**128**, 7.5; **129**, 61.4; **83**, Theorem 2; and **25**, 5.2].

Representations of hyper-Archimedean lattice groups and f-rings were studied by Conrad [**32**] and Bigard-Keimel-Wolfenstein [**17**]. In particular, Conrad proved that a hyper-Archimedean lattice group can be represented as a lattice group G of real-valued functions on some set X such that for $f, g \in G^+$ there exists $n \in \mathbf{N}$ with $f(x) \le ng(x)$ for all $x \in \{g > 0\}$ [**32**; 1.1 or **17**; 14.1.2].

The next part of this section is devoted to results of Buskes and van Rooij on "small" Riesz spaces. The crucial point in what follows is that neither the axiom of choice nor the countable axiom of choice are used in the proofs.

According to [**27**], a function $p : L \to \,]-\infty, \infty]$ is called sublinear if for $u, v \in L$ and $\alpha \in \mathbf{R}^+$:

$$p(u+v) \le p(u) + p(v),$$
$$p(\alpha u) = \alpha p(u).$$

A sublinear function $p : L \to [0, \infty]$ is called extended M-seminorm if

$$p(\sup(u,v)) = \sup(p(u), p(v)) \quad \text{for all } u, v \in L^+,$$
$$p(|u|) = p(u) \quad \text{for all } u \in L.$$

An extended M-seminorm not taking the value ∞ is called M-seminorm.

For $a \in L$, the extended M-seminorm $\|.\|_a$ on L is defined by

$$\|u\|_a := \inf\{\alpha \in \mathbf{R}^+ : |u| \le \alpha |a|\}.$$

For $\varepsilon > 0$ and $u \in L$, set

$$B_{a,\varepsilon}(u) := \{v \in L : \|v-u\|_a < \varepsilon\}.$$

A subset U of L is called a-open if for each $u \in U$ there exists $\varepsilon > 0$ with $B_{a,\varepsilon}(u) \subset U$.

For $A \subset L$, a set $U \subset L$ is called A-open if it is a-open for all $a \in A$. The topology formed by the A-open sets is called the A-topology. Then the meaning of expressions like "a-closed," "A-closure", etc., should be clear.

L is called uniformly complete if, given $a \in L$ and a filter $\mathfrak{F}$ of subsets of L such that for every $\varepsilon > 0$ there is $u \in L$ with $B_{a,\varepsilon}(u) \in \mathfrak{F}$, there exists $u \in L$ with $B_{a,\varepsilon}(u) \in \mathfrak{F}$ for all $\varepsilon > 0$.

A subset of L is called a $\mathbf{Q}$-linear sublattice of L if it is a sublattice and a $\mathbf{Q}$-vector subspace of L. If A is a $\mathbf{Q}$-linear sublattice of L, then the A-closure $\bar{A}$ of A in L is a Riesz subspace of L, contained in the ideal of L generated by A. This is proved by applying several times the following assertion: If $\Phi : L \to L$ and $\alpha > 0$ are such that $|\Phi(u) - \Phi(v)| \leq \alpha|u - v|$ for all $u, v \in L$, and if $\Phi(A) \subset \bar{A}$, then $\Phi(\bar{A}) \subset \bar{A}$; indeed, Φ is a-continuous for all $a \in L$.

An Archimedean Riesz space L is called slender if it contains a countable $\mathbf{Q}$-linear sublattice A such that L is the A-closure of A (e.g., the Riesz space c of all convergent real sequences is slender).

Proposition 2.15. *Let L be a slender Riesz space.*

(a) *There exists a sequence (a_n) in L^+ such that for each $u \in L$ there is $n \in \mathbf{N}$ with $|u| \leq a_n$.*

(b) *If $u \in L$, $u > 0$, then there exists an M-seminorm p on L with $p(u) > 0$.*

(c) *If $u \in L$, $u > 0$, and if p is an M-seminorm on L with $p(u) > 0$, then there exists a Riesz homomorphism $\phi : L \to \mathbf{R}$ with $\phi(u) > 0$ and $\phi \leq p$.*

Proof. (a) follows from the remark above.

(b) Take (a_n) as in (a). Set $b_1 := 0$ and, for $n \geq 2$,

$$k_n := \min\left\{k \in \mathbf{N} : \left(u - b_{n-1} - \frac{1}{k}(a_1 + \cdots + a_n)\right)^+ > 0\right\}$$
$$b_n := b_{n-1} + \frac{1}{k_n}(a_1 + \cdots + a_n).$$

Since $\|.\|_{b_n} \geq \|.\|_{b_{n+1}}$, we can define

$$p(v) := \lim_{n\to\infty} \|v\|_{b_n} \quad \text{for all } v \in L.$$

(c) Choose a countable $\mathbf{Q}$-linear sublattice $A = \{a_1 = u, a_2, a_3, \dots\}$ of L such that L is the A-closure of A in L. Define

$$p_0(v) := p(v^+) \quad \text{for all } v \in L$$

and, by recursion,

$$p_n(v) := \lim_{\alpha\to\infty} p_{n-1}(v + \alpha a_n) - \alpha p_{n-1}(a_n) \quad \text{for all } v \in L.$$

For each $n \in \mathbf{N}$, we have:

$$(*)\quad \begin{aligned} &p_n \text{ is sublinear,} \\ &p_n(\sup(v,z)) = \sup(p_n(v), p_n(z)) \quad \text{for all } v, z \in L, \\ &-p_{n-1}(-v) \le -p_n(-v) \le p_n(v) \le p_{n-1}(v) \quad \text{for all } v \in L, \\ &-p_n(-a_n) = p_n(a_n) = p_{n-1}(a_n), \end{aligned}$$

hence also

$$-p_n(-a_k) = p_n(a_k) = p_{n-1}(a_k) \quad \text{for } n \ge k.$$

Thus $\phi(v) := \lim_{n\to\infty} p_n(v)$ exists for all $v \in L$, and ϕ is sublinear and satisfies $(*)$; moreover $\phi(u) = p(u^+) > 0$.

If $|v - z| \le \varepsilon|a|$, then $|\phi(v) - \phi(z)| \le \varepsilon\phi(|a|)$; thus ϕ is a-continuous for each $a \in L$, and therefore the set

$$S := \{v \in L : \phi(v) = -\phi(-v)\}$$

is A-closed in L. Then $A \subset S$ implies $S = L$; thus ϕ is a Riesz homomorphism. □

Theorem 2.16 [**27**; 2.4]**.** *Let L be a slender Riesz space with a weak unit w. Let X be the set of all real Riesz homomorphisms ϕ on L with $\phi(w) = 1$. For $u \in L$ set $\hat{u} : X \to \mathbf{R}$, $\phi \mapsto \phi(u)$, and set $\hat{L} := \{\hat{u} : u \in L\}$. Endow X with the weakest topology that makes each $\hat{u}$ continuous. Then*

(a) *X is metrizable, and the map*

$$T : L \to \hat{L},\ u \mapsto \hat{u},$$

is a Riesz isomorphism onto an order dense Riesz subspace of $C(X)$;

(b) *if w is a strong unit, then X is compact and $\hat{L}$ is uniformly dense in $C(X)$; if, moreover, L is uniformly complete, then $\hat{L} = C(X)$.*

Proof. (a) That T is an injective Riesz homomorphism, follows from Proposition 2.15.

Let $A = \{a_1, a_2, \dots\}$ be a countable $\mathbf{Q}$-linear sublattice of L such that L is the A-closure of A. Since $\{u \in L : \phi(u) = \psi(u)\}$ is A-closed in L, the formula

$$d(\phi, \psi) := \sum_{n \in \mathbf{N}} \inf(2^{-n}, |\phi(a_n) - \psi(a_n)|)$$

defines a metric d on X. As $A \subset B := \{u \in L : \hat{u} \text{ is } d\text{-continuous}\}$ and B is A-closed, we have $B = L$. Since the d-topology is obviously weaker than the given topology on X, this implies that both topologies coincide.

Now let $f \in C_\infty(X)^+$, $\phi \in X$ and $\alpha \in \mathbf{R}$ with $0 < \alpha < f(\phi)$. Since the family of all sets

$$X_a := \{\phi \in X : \phi(a) > 0\},$$

where a runs through L^+, is a base for the topology of X, one can find $a \in L^+$ with $\phi \in X_a \subset \{\psi \in X : f(\psi) > \alpha\}$ and $\phi(a) = \alpha$. Then for $u := \inf(a, \alpha w)$, we have $\hat{u} \le f$ and $\hat{u}(\phi) = \alpha$.

(b) Let $B \subset L^+$ such that $X \subset \cup_{b \in B} X_b$, let I denote the ideal of L generated by B, and define an M-seminorm p on L by

$$p(u) := \inf\{\|u - b\|_w : b \in I\}.$$

If $p(w) > 0$, then Proposition 2.15(c) yields a $\phi \in X$ with $\phi|_I = 0$, which implies $\phi \notin X_b$ for all $b \in B$, a contradiction. Therefore there exists $b \in I$ with $\|w - b\|_w \le 1/2$, and thus $b \ge w/2$. Hence there are $b_1, \dots, b_n \in B$ with $w \le 2(b_1 + \dots + b_n)$, and this implies $X \subset X_{b_1} \cup \dots \cup X_{b_n}$.

To show that $\hat{L}$ is uniformly dense in $C(X)$, apply Dini's Theorem, observing the final part of the proof of (a).

That $\hat{L} = C(X)$ provided L is uniformly complete, is not difficult to see. □

Part (b) of the preceding theorem should be compared with the Kakutani–Kreĭns Theorem 4.5.

We now turn to representations of Archimedean Riesz spaces in which an additional algebraic structure, namely a (partial) multiplication, is given. Investigations in this direction were made, e.g., by Steen [**165**], Vulikh [**193, 194, 196, 197, 200, 201, 202**], Kantorovich and Pinsker [**155**], [**95**], Nakano [**150**], Birkhoff and Pierce [**18**], Domracheva [**43, 44**], Henriksen and Johnson [**80, 87**], Kist [**98**], Veksler [**172, 173, 176**] and Rotkovich [**189**], Bernau [**15**], Lozanovskiĭ [**116**], Rice [**159**], Ivanova [**85**], Hager and Robertson [**78**]; see also [**17**].

Many interesting results are contained in Veksler's paper [**176**] which largely develops and completes Vulikh's investigations. We will present some of them.

A partial multiplication in a Riesz space L is a partial binary relation on L, assigning to certain $u, v \in L$ a product uv, with the following properties:

(1) If uv exists, then so does $vu = uv$;

(2) if uv, vw and $(uv)w$ exist, then so does $u(vw) = (uv)w$;

(3) if uv and uw exist, then so does $u(v+w) = uv + uw$;

(4) if $u, v \geq 0$ and uv exists, then $uv \geq 0$;

(5) if $\alpha \in \mathbf{R}$ and uv exists, then so does $(\alpha u)v = \alpha(uv)$;

(6) uv exists and equals 0 if and only if $u \perp v$.

A partial multiplication is called complete if the product is defined for all $u, v \in L$. It is easy to see that the partial multiplication of L is complete if and only if L is a commutative f-algebra [**218**; 140.8] with no non-zero nilpotent elements.

We will also encounter the following properties that a partial multiplication may or may not have:

(α) If uv exists and $|u_1| \leq |u|$, $|v_1| \leq |v|$, then $u_1 v_1$ exists;

(β) if uv exists, then so does $|u||v| = |uv|$;

(γ) if uv exists, then $\{(uv)^+\}^{dd} \subset \{\sup(\inf(u^+, v^+), \inf(u^-, v^-))\}^{dd}$.

It is not difficult to see that (α) implies (β) and (γ). Vulikh studied exclusively partial multiplications with (α).

In the sequel, let L be endowed with a partial multiplication.

A multiplicative unit of L is an element e of L such that for all $u \in L$ there exists $ue = u$. An element e of L is called partial multiplicative unit if it is a multiplicative unit for $\{e\}^{dd}$. A universal element of L is an element $w \neq 0$ such that uw exists for all $u \in L$. It can be verified without difficulty that $w^2 = (w^+)^2 + (w^-)^2 > 0$ and $\{w^2\}^{dd} = \{w\}^{dd}$ for a universal element w of L.

Now let (X, T) be an MOV-representation of the Archimedean Riesz space L, and set $\hat{u} := Tu$ for all $u \in L$, and $\hat{L} := TL$. If uv exists, we say that uv is represented (represented at $x \in X$, respectively) if $\widehat{uv} = \hat{u}\hat{v}$ ($\widehat{uv}(x) = (\hat{u}\hat{v})(x)$, respectively); here of course the product $\hat{u}\hat{v}$ is the canonical product of $\hat{u}$ and $\hat{v}$ in $C_\infty(X)$.

By multiplication with a suitable fixed weak unit f of $C_\infty(X)$ if necessary (cf. MOV), one can always find, for given weak units u, v, w of L, an MOV-representation of L such that $\hat{u}\hat{v} = \hat{w}$.

A point $x \in X$ will be called ordinary for the products uv, wz if $0 < |\hat{u}\hat{v}\hat{w}\hat{z}|(x) < \infty$ and $|\widehat{uv}\widehat{wz}|(x) < \infty$.

Proposition 2.17. *Let x be an ordinary point for the products uv and uw.*

(a) *If (γ) is satisfied and uv is represented at x, then uw is represented at x as well.*

(b) *If (β) is satisfied, then the assertion of* (a) *remains true for $u \geq 0$.*

Proof. (a) Assume the contrary. Then, with $\lambda := 1/\hat{u}(x)$, one finds $\mu, \nu \in \mathbf{R}$ satisfying

$$\begin{aligned} &\mu\hat{v}(x) + \nu\hat{w}(x) = -1, \\ &\lambda\mu\hat{u}(x)\hat{v}(x) + \lambda\nu(\widehat{uw})(x) = 1, \end{aligned}$$

and this contradicts (γ) for the elements λu and $(\mu v + \nu w)$.

(b) It can be verified that (β) implies (γ) for positive u, and thus (b) follows from (a). □

The partial multiplication in L is called representable if there exists an MOV-representation of L such that for all $u, v \in L$:

$$uv \text{ exists and satisfies } \widehat{uv} = \hat{u}\hat{v} \quad \text{if and only if} \quad \hat{u}\hat{v} \in \hat{L}.$$

If the set of all products uv is complete in L ($D \subset L$ is called complete in L if $D^{dd} = L$), then there exists at most one MOV-representation of

L which represents the multiplication, i.e., the weak unit f of $C_\infty(X)$ occurring in MOV is uniquely determined as $f = 1_X$.

For a representable partial multiplication the properties (β) and (γ) are satisfied, and one has:

If $0 \leq u_\iota \uparrow u$ and $0 \leq v_\iota \uparrow v$ in L, if all $u_\iota v_\iota$ exist and $u_\iota v_\iota \uparrow w$ in L, then $uv = w$ exists.

A partial multiplication on L is called maximal with respect to a property p if it has the property p and cannot be extended with preservation of p.

Proposition 2.18. *Let q be a property which, when possessed by some partial multiplication on L, is also possessed by all its extensions, and such that q implies that the set of all products is complete in L. Let each representable partial multiplication on L satisfy a property p, and assume further that each partial multiplication on L satisfying p and q can be extended to a representable partial multiplication. Then a partial multiplication on L with property q is representable if and only if it is maximal with respect to p.*

Proof. Obviously the condition is sufficient. To show that it is necessary, take a representable partial multiplication on L with property q. By the Kuratowski-Zorn lemma, it can be extended to a partial multiplication on L satisfying q and being maximal with respect to p. But then, by the "sufficiency"-part, this extension is representable. On the other hand, a representable partial multiplication for which the set of all products is complete in L cannot be extended to another representable partial multiplication, and thus already the partial multiplication we were starting with, was maximal with respect to p. $\square$

Theorem 2.19 [**176**; Theorem 1]**.** *Let, for a given partial multiplication on the Archimedean Riesz space L, the set of all universal elements be complete in L. Then the following are equivalent.*

(a) *The multiplication is representable*;

(b) *the multiplication is maximal with respect to* (β);

(c) *the multiplication is maximal with respect to* (γ).

Proof. Let q be the property: "The partial multiplication possesses a complete system of universal elements."

To prove (a) $\Leftrightarrow$ (b) ((a) $\Leftrightarrow$ (c), respectively), we have to show, using Proposition 2.18, that every partial multiplication on L with q and (β) (and (γ), respectively) can be extended to a representable one.

By an earlier remark, we can assume that there exists a complete set D of positive universal elements for such a multiplication. For each $w \in D$, choose an MOV–representation (X_w, T_w) of $\{w\}^{dd}$ such that $\hat{w}\hat{w} = \widehat{w^2}$.

If we can show that each uv is represented on each X_w, then these representations coincide on sets of the form $X_{w_1} \cap X_{w_2}$, and thus each uv is represented on $X := \beta(\cup_{w \in D} X_w)$, which will yield the assertion. Observing condition (6), this has to be verified only for ordinary points for uv and w^2.

(a) $\Leftrightarrow$ (c). Since w^2 is represented on X_w, the assertion follows by applying twice Proposition 2.17(a).

(a) $\Leftrightarrow$ (b). Suppose uv is not represented at the ordinary point x for uv and w^2. One can assume $\hat{u}(x) = 1 = \hat{v}(x)$, $\widehat{uv}(x) = 0$ (otherwise considering λu, $\mu v + \nu w$ for appropriate λ, μ, ν instead of u and v), and moreover, by (β), $u > 0$ and $v > 0$. But then, applying Proposition 2.17(b) twice, uv is represented at x, a contradiction. $\square$

From Theorem 2.19 (a) $\Leftrightarrow$ (b), we get

Corollary 2.20 [**176**; Theorem 2]**.** *A complete multiplication in an Archimedean Riesz space L is representable.*

This result was first presented in [**172**]. It was also proved independently by Bernau [**15**; Theorem 13], who extended the result to the case of Archimedean lattice rings with no nonzero nilpotent elements. For Dedekind complete L, the result was already obtained by Nakano [**150**]. A similar theorem, under the same axiomatic assumptions as in Remark 1 following Theorem 2.1, was proved by Buskes and van Rooij [**26**; 4.3]. Johnson [**87**] and Kist [**98**] showed that representations of Archimedean Riesz spaces with a complete multiplication are possible on different topological spaces. Also in [**6**], a variant of Corollary 2.20

(with X locally compact and quasi-Stonian) is (implicitly) contained.

Corollary 2.21 [**176**; Theorem 3]**.** *If L is Dedekind complete, then a partial multiplication with multiplicative unit is representable if and only if it is maximal with respect to (α).*

Proof. Observe that $(\alpha) \Rightarrow (\beta)$, and that a representable partial multiplication of a Dedekind complete L satisfies (α). Now apply Theorem 2.19. $\square$

Vulikh even proved [**196**; 9.26] that every partial multiplication with multiplicative unit and with (α) is representable provided L is Dedekind complete.

Since in a Riesz space with the principal projection property every partial multiplicative unit is a universal element, we get

Corollary 2.22 [**176**; Theorem 4]**.** *Let, for a given partial multiplication on the Riesz space L with the principal projection property, the set of partial multiplicative units be complete in L. Then the following are equivalent*:

(a) *The multiplication is representable*;

(b) *the multiplication is maximal with respect to* (β);

(c) *the multiplication is maximal with respect to* (γ).

More results of the same type, in particular more equivalent conditions in Theorem 2.19 and Corollary 2.22, can be found in Veksler's paper.

A problem, first treated by Kantorovich and directly connected with representations, is that of functions defined on an Archimedean Riesz space L. The main results are due to Lozanovskiĭ [**120**].

Let X be a compact Stonian space. Then, according to Lozanovskiĭ, if F is a Baire function on a set $B \subset \mathbf{R}^n$, the expression $F(f_1, \dots, f_n)$ is defined for $f_1, \dots, f_n \in C_\infty(X)$; this means that there is a function $g \in C_\infty(X)$ as well as a dense open subset U of X such that $(f_1(x), \dots, f_n(x)) \in B$ and $F(f_1(x), \dots, f_n(x)) = g(x)$ for every $x \in U$.

Now if (X, T) is an MOV-representation of the Archimedean Riesz space L, then one can define a function $\widetilde{F}$ on (a subset of) L^n with values in the universal completion of L by putting $\widetilde{F}(u_1, \ldots, u_n) := F(Tu_1, \ldots, Tu_n)$. Lozanovskiĭ proved that if F is a positively homogeneous continuous function on $\mathbf{R}^n$, then the definition of $\widetilde{F}$ does not depend on the choice of the representation, and if L is uniformly complete, then $\widetilde{F}(u_1, \ldots, u_n)$ is an element of L. Using these results, Lozanovskiĭ constructed new classes of Banach lattices (see, e.g., [**122, 123**]).

Let us continue by mentioning some results that deal with a functional description of the Dedekind completion and of the universal completion of the space $C_b(X)$. Since these results are no proper representation theorems (it is rather described what the Dedekind completion of $C_b(X)$ really *is*), we omit proofs.

A real function f on a Baire space X (i.e., $\cap U_n$ is dense for each sequence (U_n) of dense open subsets of X) is said to have the Baire property if there is a sequence (U_n) of dense open subsets of X such that $f|_{\cap U_n}$ is continuous. Let $Ba(X)$ denote the Riesz space of all functions with the Baire property, where f and g are identified if they coincide on $\cap U_n$ for some sequence (U_n) of dense open subsets of X. Let further $Ba_b(X)$ denote the Riesz subspace of bounded elements of $Ba(X)$.

Theorem 2.23. *If X is a Baire space, then $Ba_b(X)$ is the Dedekind completion of $C_b(X)$, and $Ba(X)$ is the universal completion of $C_b(X)$.*

The first part of this theorem was proved, for a compact Hausdorff space X, by K. Nakano and Shimogaki [**151**; Theorem 1]; the general assertion is due to Zaharov [**219**; 2.5 and **220**; Theorem 3].

A real function f on a completely regular space X is called quasinormal if for each $n \in \mathbf{N}$ there is a dense open subset U_n of X such that for each $x \in U_n$ there is a neighborhood V_x of x satisfying

$$|f(x) - f(y)| < \frac{1}{n} \quad \text{for all } y \in V_x.$$

Let $Q(X)$ denote the Riesz space of all quasinormal functions on X, where f and g are identified if there is a sequence (U_n) of dense open subsets of X such that

$$|f(x) - g(x)| \leq \frac{1}{n} \quad \text{for all } x \in U_n.$$

Let further $Q_b(X)$ denote the Riesz subspace of bounded elements of $Q(X)$.

The following result is due to Zaharov [**219**; 3.6 and **220**; Theorem 3 and Theorem 3'].

Theorem 2.24. *If X is completely regular, then $Q_b(X)$ is the Dedekind completion of $C_b(X)$, and $Q(X)$ is the universal completion of $C_b(X)$.*

Refinements of these results, concerning the structure of the sets on which the functions are defined, were given by Veksler [**178, 179, 183**]. For instance, if X is Stonian, then $C_b(X)$ is its own Dedekind completion, while according to the preceding result, large classes of functions are needed to describe the elements of the Dedekind completion; in Veksler's results however, the corresponding classes contain exactly one element, namely the continuous function in $C_b(X)$ associated to the class.

Several descriptions of the Dedekind completion of $C(X)$, for a compact Hausdorff space X, are discussed in [**96**; Chapter 17].

Descriptions of a sequential completion, the so-called Cantor completion of $C_b(X)$, were obtained by Koldunov [**100, 102**], Dashiell-Hager-Henriksen [**36**] and Zaharov [**221**].

It is possible to represent an Archimedean Riesz space with a weak unit as a Riesz space of continuous real-valued functions on a locale. (Locales are discussed, e.g., in [**90**].) This kind of representation (even in a more general version for Archimedean lattice groups with weak unit) is due to Madden (cf., [**130, 131, 132**]); for further information, see also [**10**]. Following [**130**], we sketch the ideas in a "locale-free setting."

Let L be an Archimedean Riesz space with a weak unit w. For any $u \in L$, denote by I_u the relatively uniformly closed ideal of L generated by u, and let $\mathfrak{I}$ be the lattice of all relatively uniformly closed ideals of

L contained in I_w. Let $C(\mathfrak{I})$ denote the set of all maps g from the set of all open sets of $\mathbf{R}$ to $\mathfrak{I}$ such that

$$g(\varnothing) = \{0\}, \qquad g(\mathbf{R}) = I_w,$$
$$g(U_1 \cap U_2) = g(U_1) \cap g(U_2),$$
$$g\Big(\bigcup_{\iota \in I} U_\iota\Big) = \sup_{\iota \in I} g(U_\iota).$$

On $C(\mathfrak{I})$ one can introduce the structure of a Riesz space.

Given $u \in L$ and $\alpha, \beta \in \mathbf{R}, \alpha < \beta$, put

$$T_0 u\big(]\alpha, \beta[\big) := I_{\inf((u-\alpha w)^+,\,(\beta w - u)^+)} \cap I_w.$$

The map $T_0 u$ from the set of all open intervals of $\mathbf{R}$ to $\mathfrak{I}$ extends uniquely to a map $Tu \in C(\mathfrak{I})$, and one obtains

Theorem 2.25 [**130**]. *The map*

$$T: \ L \to C(\mathfrak{I}), \qquad u \mapsto Tu$$

is an injective Riesz homomorphism.

3. Representations of Archimedean Riesz spaces as spaces of measures or as spaces of measurable functions. Since spaces of measures have strong separation properties ($\mathcal{M}(\mathfrak{R})$ is separated by its order continuous dual), Riesz spaces embeddable order densely into these spaces must have similar properties. We call such Riesz spaces L (i.e., spaces L which are separated by $L_n^\sim$) ocs-spaces (where "ocs" stands for "order continuously separated"). We remark that L is an ocs-space if and only if L admits a Hausdorff locally convex-solid Lebesgue topology (i.e., a Hausdorff topology τ having a 0-neighborhood base of convex and solid sets, and satisfying the condition: $u_\iota \downarrow 0 \Rightarrow u_\iota \xrightarrow{\tau} 0$); this follows by [**5**; 9.1] and by considering the topology $|\sigma|(L, L_n^\sim)$, defined by the seminorms $\rho_\xi(u) := |\xi|(|u|)$, where ξ runs through $L_n^\sim$.

For the corresponding uniqueness assertions in the following, confer Theorem 1.9.

Theorem 3.1. *The Riesz space L is an ocs-space if and only if it can be embedded order densely into a band $\mathcal{M}$ of $\mathcal{M}(\mathfrak{R})$ for some ring* *of sets $\mathfrak{R}$. Moreover, L can be embedded as an ideal if and only if it is Dedekind complete, and $L \cong \mathcal{M}$ if and only if L has the following property: There exist a weak unit ω of $\Gamma(L)$ and a set $R \subset L_n^\sim$ of components of ω such that $\sup u_\iota$ exists in L for each upward-directed family (u_ι) from L^+ satisfying $\sup \xi(u_\iota) < \infty$ for all $\xi \in R$.*

Riesz spaces with the last-mentioned property are studied in [**56, 59**] and [**1**]. Another characterization of these spaces which we call hypercomplete, will be given in Theorem 3.9 below; the pair (ω, R) described above is called an hc-pair of L. Note that the "only if"-part in the last equivalence of Theorem 3.1 follows by considering $\omega : \mu \mapsto \int 1\, d\mu$, and the collection R of all maps $\mu \mapsto \mu(A)$, where $A \in \mathfrak{R}$.

Theorem 3.1 is a corollary of the more concrete

Theorem 3.2 [**55**; Theorem A]**.** *Let L be an ocs-space, and fix a weak unit ω of $\Gamma(L)$ and a set $R \subset L_n^\sim$ of components of ω with $\omega = \sup R$.*

Then there exist a unique locally compact hyperstonian space X, a Riesz isomorphism T from $\Gamma(L)$ onto $C_\infty(X)$ and a Riesz isomorphism π from L onto an order dense Riesz subspace of $\mathcal{M}(X)$ such that

(i) *$T\omega = 1_X$ and $T\xi$ is the characteristic function of an open-compact subset of X for all $\xi \in R$;*

(ii) *$X = \cup_{\xi \in R} \operatorname{supp}(T\xi)$;*

(iii) *$\eta(u) = \int (T\eta)\, d(\pi u)$ for all $\eta \in \Gamma(L)$ and all $u \in L$ for which $\eta(u)$ is defined.*

Proof. First observe that the existence of R and ω is guaranteed by the Kuratowski-Zorn lemma and the lateral completeness of $\Gamma(L)$.

MOV applied to $\Gamma(L)$ yields a couple (X', T') with $T'\omega = 1_{X'}$. Then set

$$X := \bigcup_{\xi \in R} \operatorname{supp}(T'\xi)$$

and

$$T\eta := T'\eta|_X \quad \text{for all } \eta \in \Gamma(L).$$

Applying, for $u \in L$, the Riesz-Kakutani representation theorem to the order continuous linear form

$$C_c(X) \to \mathbf{R},\ f \mapsto (T^{-1}f)(u)$$

(observe that $T^{-1}f \in L_n^\sim$), we find a normal regular measure πu on X with

$$\textstyle\int f\, d(\pi u) = (T^{-1}f)(u) \quad \text{for all } f \in C_c(X).$$

The proof of the uniqueness is omitted. □

Constantinescu proved this theorem in the special case $L = \mathcal{M}(\mathfrak{R})$ [**33**; 2.3.6, 2.3.8] using Kakutani's representation theorem 4.1. For detailed information and related results, see also [**54, 57**].

Theorem 3.2 is related (although more flexible, due to the possibility of choosing ω and R) to a representation theorem involving measurable functions which goes back to Luxemburg and Zaanen (see the introduction of [**67**]) and was presented by Fremlin [**67**; Theorem 6]:

Theorem 3.3. *For each ocs-space L there exist a hyperstonian space X which is the topological direct sum of a family of compact spaces, and a measure $\mu \in \mathcal{M}(X)$ with* $\operatorname{supp}\mu = X$ *such that L is Riesz isomorphic to an order dense Riesz subspace of*

$$L^1_{\text{loc}}(\mu) = \{f \in L^0(\mu) : f1_K \in L^1(\mu) \text{ for all } K \in \mathfrak{K}(X)\}.$$

Moreover, there is an embedding $T : L_n^\sim \to C_\infty(X)$ such that $\xi(u) = \int (T\xi) f_u\, d\mu$ for every $u \in L$ and every $\xi \in L_n^\sim$, where f_u denotes the element of $L^1_{\text{loc}}(\mu)$ corresponding to u.

If $L_n^\sim$ possesses a weak unit, then L can be embedded into $L^1(\mu)$.

Proof. Fix a weak unit ω of $\Gamma(L)$ and find a triple (X', T', π') according to Theorem 3.2. Let $(u_\iota)_{\iota\in I}$ be a maximal disjoint system of L such that all sets $\operatorname{supp}(\pi' u_\iota)$ are compact. For ω and

$$R := \{(T')^{-1}(1_{\operatorname{supp}(\pi' u_\iota)}) : \iota \in I\}$$

find (again by Theorem 3.2) a triple (X, T, π). Then $\mu := \sum \pi\mu_\iota \in \mathcal{M}(X)$, and by Corollary 1.8(d), $\mathcal{M}(X) = \{\mu\}^{dd} \cong L^1_{\text{loc}}(\mu)$.

If $\omega \in L_n^\sim$, then each πu is bounded; hence L is embeddable into $L^1(\mu)$.

The second part of Theorem 3.3 was proved in Vietsch's Ph.D. Thesis [**191**; Chapter 5]. A special case of the theorem was considered in [**209**; Section 2] and [**210**; Section 2].

Here is another related result:

Theorem 3.4. *For an Archimedean Riesz space L, the following are equivalent*:

(a) *L can be embedded order densely into a space $L^0(\mu)$ (where μ can be chosen as in Theorem* 3.3);

(b) *L contains an order dense ideal M which admits a Hausdorff locally convex-solid Lebesgue topology*;

(c) *L contains an order dense Riesz subspace M which admits a Hausdorff locally convex-solid Lebesgue topology*;

(d) *L contains an order dense ocs-Riesz subspace M*;

(e) *L contains an order dense ocs-ideal M*;

(f) *$\Gamma(L)$ separates L.*

Proof. (a) $\Rightarrow$ (b). $M := L^1(\mu) \cap L$, with the usual L^1-topology.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d). $M_n^\sim$ separates M, by [**5**; 9.1].

(d) $\Rightarrow$ (e). Take the ideal of L generated by M (cf. [**68**; 17 Gb]).

(e) $\Rightarrow$ (f) follows from [**127**; 2.5].

(f) $\Rightarrow$ (a). By MOV and Proposition 2.7, $L \subset C_\infty(X')$, with X' hyperstonian. Construct X as in the proof of Theorem 3.3, and apply Corollary 1.7. $\square$

The equivalence (a) $\Leftrightarrow$ (e) is due to the Soviet school (for L Dedekind complete), while (a) $\Leftrightarrow$ (c) is a special case of Theorem 3.7. First results concerning representations of Riesz spaces as spaces of measurable functions go back to Pinsker [**154, 156, 157**].

Corollary 3.5 [**139**; Corollary 3]. *Let L be an Archimedean Riesz space. Then $L \cong L^0(\mu)$ (with μ as in Theorem* 3.3) *if and only if L is universally complete and separated by $\Gamma(L)$.*

If L is an order dense Riesz subspace of the Archimedean Riesz space M, then the set of discrete elements of L is exactly the set of discrete elements of M. Hence, if L is embedded order densely into some $L^0(\mu)$, then the discrete elements of L correspond to the (characteristic functions of the) μ-atoms.

We shall give in Theorem 3.6 a condition which implies the equivalent conditions in Theorem 3.4.

A topology on L is called locally concave if it has a 0-neighborhood base consisting of solid sets B for which $L^+ \backslash B$ is convex (as is the case e.g. in the spaces $L^p(\mu)$, $p < 1$). Such topologies were investigated by Matzka in his doctoral thesis [**140**]. He proved [**140**; 3.21] that in Riesz spaces which have the projection property and possess a strong unit, the existence of a locally concave Lebesgue topology implies the existence of a locally convex–solid Lebesgue topology (the proof of this assertion is rather long, so we omit it). Using this result and working with a maximal disjoint system of L^+, it is easy to show

Theorem 3.6. *If L has the projection property and L admits a Hausdorff locally concave Lebesgue topology, then L possesses an order dense ideal admitting a Hausdorff locally convex-solid Lebesgue topology (and therefore can be represented as in Theorem* 3.4).

To formulate an extension of Theorems 3.3 and 3.4, which is due to Labuda, we need the following terminology:

A mapping μ from a δ-ring of sets $\mathfrak{R}$ into $[0, \infty[$ is called a submeasure if $\mu(\varnothing) = 0$ and μ is monotone and (finitely) subadditive. A submeasure μ is called σ-order continuous if for every decreasing sequence (A_n) from $\mathfrak{R}$ with $\mu(\cap A_n) = 0$ we have $\mu(A_n) \downarrow 0$. The notions "μ-a.e." and "μ-measurable" are defined as for measures, i.e., a set B is a μ-null set if for each $A \in \mathfrak{R}$ there exists $C \in \mathfrak{R}$, $\mu(C) = 0$, with $A \cap B \subset C$, and an extended real-valued function f is μ-measurable if $f = g$ μ-a.e. for some $\mathfrak{R}$-measurable g (recall that g is $\mathfrak{R}$-measurable if $\{g < \alpha\} \cap A \in \mathfrak{R}$ for all $\alpha \in \mathbf{R}$ and all $A \in \mathfrak{R}$).

Theorem 3.7 [**108**; 2.7]. *For an Archimedean Riesz space L, the following are equivalent*:

(a) *L can be embedded order densely into a space $L^0(\mu)$, where μ is a σ-order continuous submeasure*;

(b) *L admits a Hausdorff locally solid Lebesgue topology*;

(c) *L contains an order dense Riesz subspace M which admits a Hausdorff locally solid Lebesgue topology.*

Again μ (in (a)*) can be chosen to be a normal regular submeasure on $\mathfrak{R}(X)$, where X is the topological direct sum of a family (X_ι) of compact Stonian spaces (hence also Stonian), and such that $\mu(A) = \sum \mu(A \cap X_\iota)$ for each $A \in \mathfrak{R}(X)$.*

Moreover, M (in (c)*) can be embedded continuously into $L^0(\mu)$, where $L^0(\mu)$ is equipped with the topology of convergence in μ, given by the 0-neighborhoods*

$$U_{A,\varepsilon} := \{f \in L^0(\mu) : \mu(\{|f| \geq \varepsilon\} \cap A) \leq \varepsilon\}.$$

Proof. As easily seen, (a) implies (b). Moreover, (c) obviously follows from (b).

To prove that (c) implies (a), it is enough to show (a) for Dedekind complete L. Then, by [**5**; 11.10], M can be assumed Dedekind complete, too. It is not difficult to see, by considering a maximal disjoint system (u_λ) of M and the restrictions of the given topology on M to the bands M_λ of M generated by u_λ, that we can confine ourselves to the case where M has a weak unit.

Next observe that it is sufficient to show that M can be embedded order densely into $L^0(\mu)$, since then $L^0(\mu) = C_\infty(X)$ (the proof of Corollary 1.7 also works for submeasures) and thus $L^0(\mu)$ is also the universal completion of L.

Now let $M \subset C_\infty(X')$, according to MOV. Let (p_ι) be a maximal family of continuous (hence order continuous) Riesz pseudonorms on M with disjoint open-compact "carriers"

$$X_\iota := X' \setminus \overline{\bigcup_{\substack{f \in M \\ p_\iota(|f|)=0}} \{f \neq 0\}}$$

[**68**; 22C]. For $A \subset X_\iota$, A open-compact, and N meager in X_ι, observe that, by the assumptions, $1_A \in M$, and set

$$\mu_\iota(A \triangle N) := p_\iota(1_A)\,,$$

and let $\mu := \sum \mu_\iota$ on $X := \cup X_\iota$.

The topology generated by p_ι is finer than the topology of convergence in μ_ι [**45**; 3.6], which implies the final assertion of the theorem. □

A similar result for Boolean algebras was proved by Flachsmeyer. He showed [**63**; Theorem 3] that a Dedekind complete Boolean algebra with a compatible order continuous locally solid Hausdorff topology can be represented as a product of hyperstonian submeasure algebras.

If the ocs-space L is represented according to MOV, then we can give a simple description of $L_n^\sim$, which is due to Lozanovskiĭ [**204**; Theorem 2.2]:

Theorem 3.8. *Let the ocs-space L be embedded order densely into $C_\infty(X)$, where X is a locally compact Stonian space. Then there exist a dense open subset Y of X, which is the topological direct sum of a family of compact spaces, and a measure $\mu \in \mathcal{M}(Y)^+$ with* $\operatorname{supp}\mu = Y$ *such that $L_n^\sim$ can be identified with*

$$\Phi := \left\{\phi \in C_\infty(X) :\ f\phi|_Y \in \mathcal{L}^1(\mu) \text{ for every } f \in L\right\}$$

via the map $L_n^\sim \to \Phi$, $\xi \mapsto \phi_\xi$, with $\xi(f) = \int (f\phi_\xi)|_Y\, d\mu$.

Proof. Since L is an order dense Riesz subspace of $\Gamma(L_n^\sim)$ [**218**; 109.3], we have $\Gamma(L_n^\sim) = C_\infty(X)$. Moreover, X is hyperstonian by Proposition 2.7. By the Kuratowski-Zorn lemma, and since $(L_n^\sim)_n^\sim$ is an order dense ideal of $\Gamma(L_n^\sim)$, there exists a maximal disjoint system $R = (1_{K_\iota})$ consisting of characteristic functions of open-compact subsets of X such that for each ι, $K_\iota = \operatorname{supp}\mu_\iota$ for some $\mu_\iota \in \mathcal{M}(X)^+$ and $1_{K_\iota} \in (L_n^\sim)_n^\sim$. Find a space Y and maps $T : \Gamma(L_n^\sim) \to C_\infty(Y)$, $\pi : L_n^\sim \to \mathcal{M}(Y)$ with properties (i)–(iii) of Theorem 3.2. Inspecting the proof of this theorem, we see that Y can be chosen as a dense open subset of X such that $T\phi = \phi|_Y$ for all $\phi \in \Gamma(L_n^\sim)$. Then $\mu := \sum \mu_\iota \in \mathcal{M}(Y)$, and by Corollary 1.9(d), $\mathcal{M}(Y) = \{\mu\}^{dd}$. By the Radon-Nikodým theorem, each $\pi\xi$ can be written in the form $\pi\xi = \phi_\xi \cdot \mu$ with $\phi_\xi \in L^1_{\text{loc}}(\mu)$, which implies the desired result since for all $f \in L$ we have $\xi(f) = \int_Y (Tf) d(\pi\xi)$ by (iii). □

Using Proposition 2.7 and the fact that $\Gamma(L)$ separates L if and only if L contains an order dense ocs-ideal, we get the following

Corollary 3.9. *For a locally compact hyperstonian space X, the spaces $C_\infty(X)$ and $\Gamma(C_\infty(X))$ are Riesz isomorphic.*

Some more information on the representation of functionals on spaces of continuous functions is contained, e.g., in [**204, 119, 124, 185, 186** and **52**]. We mention without proof the following result of Lozanovskiĭ [**119, 124, 204**].

Theorem 3.10. *Suppose that X is a compact Stonian space and that L is an ideal of $C_\infty(X)$. Let M denote the space of all Radon measures on X. Then $L^\sim$ can be embedded into the universal completion M^u of M.*

If I is a band of $L^\sim$, then the image of I under this embedding is contained in M if and only if there is a strictly positive linear functional on I.

Let L be an order dense ideal of $C_\infty(X)$, where X is a locally compact Stonian space, and let N be a nowhere dense closed subset of X. Set

$$\begin{aligned} \mathfrak{D}(N) &:= \{U \subset X : U \text{ open-compact}, U \cap N = \varnothing\}, \\ L_N &:= \{f \in C_\infty(X) : f1_U \in L \text{ for all } U \in \mathfrak{D}(N)\}. \end{aligned}$$

Then L_N is an order dense ideal of $C_\infty(X)$ containing L as an order dense ideal.

Now we can give Abramovich's characterization of hypercomplete spaces [**1**; 4.2].

Theorem 3.11. *A Dedekind complete Riesz space L is hypercomplete if and only if there exist a locally compact Stonian space Y, a measure*

$\mu \in \mathcal{M}(Y)^+$ with $\operatorname{supp}\mu = Y$*, and a nowhere dense closed subset N of Y such that L can be identified with $L^1(\mu)_N$.*

Proof. To check that $M := L^1(\mu)_N$ is hypercomplete, set $\omega := 1_Y$ and $R := \{1_U : U \in \mathfrak{D}(N)\}$. Let $0 \le f_\iota \uparrow$ in M such that $\sup 1_U(f_\iota) < \infty$ for all $U \in \mathfrak{D}(N)$. By the Beppo–Levi Theorem, $(f_\iota 1_U)$ increases to some $f_U \in L^1(\mu)$, for each $U \in \mathfrak{D}(N)$. Then $f := \sup\limits_{U \in \mathfrak{D}(N)} f_U$ is the supremum of (f_ι) in M.

Suppose conversely that L is hypercomplete, and let (ω, R) be an hc-pair of L. It is easy to see that L is an ocs-space [**56**; 2.3]. Embed L into $C_\infty(X)$ according to MOV, and find Y and $\mu \in \mathcal{M}(Y)^+$ according to Theorem 3.8. Then L is an order dense ideal of $C_\infty(Y)$, and $L_n^\sim = \{\phi \in C_\infty(Y) : |\int f\phi\, d\mu| < \infty \text{ for all } f \in L\}$. By multiplying the elements of $L_n^\sim$ with an appropriate $g \in C_\infty(Y)^+$ (and the elements of L with $1/g$), we can assume that $\omega = 1_Y$ and $R = \{1_A : A \in \mathfrak{R}\}$ for some family $\mathfrak{R}$ of open-closed subsets of Y. Set $N := Y \backslash \cup_{A \in \mathfrak{R}} A$. We claim $L = L^1(\mu)_N$.

If $f \in L^+$, then $f 1_A \in L^1(\mu)$ for all $A \in \mathfrak{R}$ which implies $f 1_U \in L^1(\mu)$ for each $U \in \mathfrak{D}(N)$; hence $f \in L^1(\mu)_N$.

Now let $0 \le f \in L^1(\mu)_N$. If $U \in \mathfrak{D}(N)$, then each $0 \le \phi \in L_n^\sim$ is bounded on U (use hypercompleteness for this assertion), and thus $f 1_U \in (L_n^\sim)_n^\sim$. Since L is perfect [**218**; 110.1], we conclude $f 1_U \in L$. We have $f 1_U \uparrow_U f$ and $\sup\limits_{U \in \mathfrak{D}(N)} \int 1_A f 1_U\, d\mu < \infty$ for each $A \in \mathfrak{R}$, which implies $f \in L$. □

It is possible to introduce the concept of a so-called ω-hyper-completion of an ocs-space L, where ω is a fixed weak unit of $\Gamma(L)$ [**59**]. The ω-hypercompletion of L, unique up to a Riesz isomorphism, is, by definition, a Riesz space M containing L as order dense Riesz subspace such that the set R_ω of all components of ω in $L_n^\sim$ already belongs to $M_n^\sim$, and such that (ω, R_ω) is an hc-pair of M. Then, if L is embedded into $C_\infty(Y)$, where Y is again as in Theorem 3.8, and if $\omega = 1_Y$, M turns out to be the space $L^1(\mu)_N$, with $N := Y \backslash \cup_{1_U \in R_\omega} U$ [**1**; 4.5]. Alternatively, if L is embedded into $\mathcal{M}(X)$ according to Theorem 3.2 (with $R = R_\omega$), then $M = \mathcal{M}(X)$. Moreover, M can be described without using any representation; namely, $M = G_n^\sim$, where G denotes the ideal of $L_n^\sim$ generated by R_ω [**59**; Theorem 3].

We conclude this section with a theorem of Hackenbroch which shows that also Riesz spaces without order separation property can be represented as spaces $L^1(\mu)$; but in this case μ is a vector measure. The necessary definitions follow.

Let L be an Archimedean Riesz space, let $\mathfrak{R}$ be a ring of subsets of a set X, and let $\mu : \mathfrak{R} \to L^+$ be a measure; i.e. μ is finitely additive, and $A_n \downarrow \varnothing$ implies $\mu(A_n) \downarrow 0$.

For $\mathfrak{R}$-step functions, the integral is defined in the usual way.

$f \in (\mathbf{R}^+)^X$ is called μ-integrable on $A \in \mathfrak{R}$ if there exist step functions f_n such that $0 \leq f_n 1_A \uparrow f 1_A$ and there exists $\sup \int f_n 1_A \, d\mu =: \int f 1_A \, d\mu$ in L (this definition does not depend on the choice of the sequence (f_n)).

$f \in (\mathbf{R}^+)^X$ is called μ-integrable if f is μ-integrable on each $A \in \mathfrak{R}$ and there exists $\sup \int f 1_A \, d\mu =: \int f \, d\mu$ in L. We set

$$\mathcal{L}^1(\mu) := \{f - g : f, g \in (\mathbf{R}^+)^X,\ f, g\ \mu\text{-integrable}\},$$
$$\textstyle\int (f-g) d\mu := \int f \, d\mu - \int g \, d\mu \quad \text{for } f - g \in \mathcal{L}^1(\mu),$$

where this last definition is again independent of the representation. Furthermore

$$\mathcal{N}(\mu) := \{f \in \mathcal{L}^1(\mu) : \textstyle\int |f| d\mu = 0\},$$
$$L^1(\mu) := \mathcal{L}^1(\mu)/\mathcal{N}(\mu).$$

$L^1(\mu)$ is a Riesz space provided $\mathcal{L}^1(\mu)$ is a Riesz space.

Theorem 3.12 [**76**; Satz 1]**.** *Let L possess the principal projection property. Then there exist a locally compact Stonian space X, which is the topological direct sum of a family of compact spaces, a Riesz isomorphism T from L onto an order dense Riesz subspace of $C_\infty(X)$ and a measure $\mu : \mathfrak{R} \to TL^+$, where $\mathfrak{R}$ is a ring of sets contained in $\mathfrak{R}(X)$, such that $L^1(\mu) = TL$. We have $\mu(A) = 0$ if and only if A is meager; moreover an open-compact $A \subset X$ is contained in $\mathfrak{R}$ if and only if $1_A \in TL$, and in this case $\mu(A) = 1_A$.*

Proof. Assume first that w is a weak unit of L, and let (X, T) be an MOV-representation of L with $Tw = 1_X$. Set

$$\mathfrak{R} := \{A \triangle N : 1_A \in TL,\ N \text{ meager}\},$$
$$\mu(A \triangle N) := 1_A.$$

Then $\mathcal{L}^1(\mu)$ is a Riesz space, and $L^1(\mu)$ is a Riesz subspace of TL.

To verify that $L^1(\mu) = TL$, consider first v with $0 \le v \le w$. Let $0 < \alpha < 1$. Since $(\alpha w - v)^+$ is a projection element, we have $\overline{\{T(\alpha w - v)^+ > 0\}} \in \mathfrak{R}$; hence also $\{Tv < \alpha\} = \overline{\{T(\alpha w - v)^+ > 0\}} \in \mathfrak{R}$. Thus

$$f_n := \sum_{k=1}^{2^n+1} \frac{k-1}{2^n} 1_{\{(k-1)/2^n \le Tv < k/2^n\}}$$

is an $\mathfrak{R}$-step function. From $f_n \uparrow Tv$ one concludes $Tv \in \mathcal{L}^1(\mu)$ and $\int Tv\, d\mu = v$.

Finally, the Beppo-Levi theorem yields the desired equality.

To obtain the general result, consider a maximal disjoint system (w_ι) of L, and apply the first case. $\square$

4. Representations of Banach lattices. A normed Riesz space is, by definition, a Riesz space endowed with a Riesz norm, i.e., a norm $\|.\|$ satisfying

$$|u| \le |v| \Longrightarrow \|u\| \le \|v\|.$$

A normed Riesz space is called Banach lattice if it is norm complete.

A Riesz norm is called M-norm if $\inf(u, v) = 0$ implies $\|\sup(u, v)\| = \sup(\|u\|, \|v\|)$; p-additive (for $1 \le p < \infty$) if $\inf(u, v) = 0$ implies $\|u + v\|^p = \|u\|^p + \|v\|^p$; L-norm if it is p-additive for $p = 1$.

An abstract M-space (AM-space) is a Banach lattice with an M-norm.

An abstract L^p-space is a Banach lattice with a p-additive norm.

An abstract L-space (AL-space) is a Banach lattice with an L-norm.

We begin with representation theorems for the special Banach lattices just mentioned.

Theorem 4.1. *Let L be an abstract L^p-space. Then there exist a locally compact hyperstonian space X, which is the topological direct sum of compact spaces, and a $\mu \in \mathcal{M}(X)$ with $\operatorname{supp} \mu = X$ such that L is Riesz isomorphic and isometric to $L^p(\mu)$. If L possesses a weak unit, X can be chosen compact.*

X is unique in the sense of Theorem 1.9.

Proof. Assume first that w is a weak unit of L. Since L is Dedekind complete [**5**; 10.10], the set $\mathcal{B}$ of components of w is a Dedekind complete Boolean algebra. Thus its Stone representation space X is Stonian [**129**; 47.5].

Each $u \in \mathcal{B}$ corresponds to a unique open-compact $A_u \subset X$. Then set, for $A_u \triangle N \in \mathfrak{R}(X)$,

$$\mu(A_u \triangle N) := \|u\|^p.$$

By the order continuity of the norm, this defines a normal regular measure on X.

The required norm-preserving Riesz isomorphism T is now constructed by setting $Tu := 1_{A_u}$ for $u \in \mathcal{B}$ and by observing that the Riesz subspace of L formed by all finite sums $\sum \alpha_\iota u_\iota$, where $u_\iota \in \mathcal{B}$ are disjoint and $\alpha_\iota \in \mathbf{R}$, is order dense (and hence norm dense) in L.

In the general case, consider again a maximal disjoint system of L. $\square$

Theorem 4.1 (for $p = 1$) goes back to Kakutani [**91**; Theorem 7], who proved the result under additional assumptions; see also Cunningham [**35**; 9.10] and Bernau [**15**; Theorem 9]. Another proof, based on MOV, was suggested by Lozanovskiĭ in 1966 (see [**163**; Section 26.3]). For $p > 1$, the result was obtained, also under additional hypotheses, by Bohnenblust [**20**; 6.5], Nakano [**146**], Gordon [**70**], Bretagnolle, Dacunha-Castelle and Krivine [**23**; Theorem 3], and Marti [**138**; Theorem 11]; the first proof of the general case can be found in a survey by Lacey and Bernau [**109**; 3.1] and is due to Ando.

For $n = 0, 1, 2, \ldots$ denote by ℓ_n^p the space $\mathbf{R}^n$, equipped with the p-norm, and let λ be Lebesgue measure on $[0,1]$. Suppose now that the abstract L^p-space L is separable, and let L_a denote the band generated by the atoms of L. Then L_a is isometrically Riesz isomorphic to ℓ_n^p (if $\dim L_a = n$) or to ℓ^p (if $\dim L_a = \infty$). If ${L_a}^d \neq \{0\}$, then ${L_a}^d$ is isometrically Riesz isomorphic to $L^p(\lambda)$. Hence L is isometrically Riesz isomorphic to one of the spaces

$$\ell_n^p, \quad \ell^p, \quad L^p(\lambda), \quad \ell_n^p \oplus_p L^p(\lambda), \quad \ell^p \oplus_p L^p(\lambda).$$

(The symbol $\oplus_p$ indicates that the norm of the direct sum is again the corresponding p-norm.) Also this result is due to Bohnenblust [**20**; 7.1].

It follows from Theorem 4.1 that each normed Riesz space with p-additive norm can be embedded norm densely into a space $L^p(\mu)$ (since the norm completion has again p-additive norm).

Kakutani's theorem on AL-spaces can also be formulated in the following form [**35**; 9.6].

Theorem 4.2. *If L is an AL-space, then there exists a unique compact hyperstonian space X such that L is Riesz isomorphic and isometric to $\mathcal{M}(X)$.*

Proof. Apply Theorem 3.2 to $\omega : L \to \mathbf{R}, u \mapsto \|u^+\| - \|u^-\|$ and to $R := \{\omega\}$, and observe that the last condition given in Theorem 3.1 is satisfied since each norm bounded $0 \leq u_\iota \uparrow$ is Cauchy. □

Next we present the Banach lattice version of Theorem 3.3.

Theorem 4.3. *Let L be a Dedekind complete Banach lattice. Then L can be embedded as an order dense ideal into a space $L^0(\mu)$, with μ localizable, if and only if $L_n^\sim$ separates L (in fact, μ can be chosen as in Theorem* 3.3).

Proof. The "if"-part follows from Theorem 3.3. For the "only if"-part, note that $L_n^\sim$ can be identified with the set $\mathcal{G}$ of all $g \in L^0(\mu)$ for which $fg \in \mathcal{L}^1(\mu)$ whenever $f \in L$, and that the isomorphism is the map

$$L_n^\sim \to \mathcal{G}, \qquad \xi \mapsto g_\xi,$$

where

$$\xi(f) = \int g_\xi f \, d\mu \qquad \forall\, f \in L$$

(cf. [**93**; Chapter VI, Section 1, Theorem 1]). □

In this connection, the following result of Lozanovskiĭ [**118**], which we present without proof, is of interest.

Theorem 4.4. *Let the Dedekind complete Banach lattice L be embedded order densely in a space $L^0(\mu)$, with μ localizable. Then the* *following hold.*

(a) *If μ is bounded, then there is a $g \in L^0(\mu)$ such that*

$$L^\infty(\mu) \;\subset\; \{fg \,:\, f \in L\} \;\subset\; L^1(\mu).$$

(b) *For every $h \in L^1(\mu)$,*

$$\int |h|\,d\mu \;=\; \inf\{\|\xi_g\|\,\|f\| \,:\, f \in L,\, g \in L^0(\mu),\, fg = h\}\,,$$

where ξ_g is the element of L' assigning to each $f \in L$ the value $\int fg\,d\mu$.

In this equation, the infimum is a minimum if the following conditions hold in L:

$$\begin{aligned} &0 \le u_\iota \uparrow \quad \text{and} \quad \sup\|u_\iota\| < \infty \qquad \Longrightarrow \qquad \exists\ \sup u_\iota \\ &0 \le u_\iota \uparrow u \quad \Longrightarrow \quad \|u_\iota\| \to \|u\| \end{aligned}$$

A more special situation is considered in Theorem 4.16 below.

An element $e \in L^+$ is called norm unit if it is the greatest element of the closed unit ball of L.

The following result is due to Kakutani (and Bohnenblust) ([**92**; Theorem 2 and **21**]) and independently to M. and S. Kreĭn [**105, 106**].

Theorem 4.5. *If (and only if) L is an AM-space with norm unit e, then there exists a unique compact Hausdorff space X such that L is Riesz isomorphic and isometric to $C(X)$. X can be chosen to be the set of all real Riesz homomorphisms ϕ on L with $\phi(e) = 1$, endowed with (the restriction of) $\sigma(L', L)$.*

Proof. The uniqueness follows from Theorem 2.8.

Let $Y := \{\xi \in (L')^+ : \xi(e) = \|\xi\| = 1\}$. Y is convex and $\sigma(L', L)$-compact. Let X be the set of all real Riesz homomorphisms in Y. By [**5**; 3.13], and since L' is an AL-space [**5**; 10.15], X coincides with the set of all extreme points of Y. Then X is $\sigma(L', L)$-compact and

Hausdorff, and the map $T : L \to C(X)$, defined by $(Tu)(\phi) = \phi(u)$ for $u \in L$ and $\phi \in X$, is a Riesz isomorphism which preserves the norm by the Kreĭn-Milman theorem and which is onto by the Stone-Weierstraß theorem. $\square$

Remarks 1. Buskes and van Rooij have shown [**26**; 4.1] that the Kakutani-Kreĭns theorem can be proved avoiding the axiom of choice, but using the countable axiom of choice and the Boolean prime ideal theorem instead.

2. It follows from Theorem 4.5 that a normed Riesz space L with an M-norm can be embedded into a space $C(X)$, since L'' is an AM-space with norm unit e, given by $e(\xi) := \|\xi^+\| - \|\xi^-\|$ for all $\xi \in L'$.

3. Benyamini has shown that each separable AM-space is Riesz isomorphic (but in general not isometric) to a space $C(X)$, with X compact [**13**], and that this result cannot be extended to the non-separable case [**14**].

4. If L is a Banach lattice and at the same time a Banach algebra with unit, then L is Riesz isomorphic, algebraically isomorphic and homeomorphic to a space $C(X)$, with X compact; this result is due to Lozanovskiĭ [**116**].

5. Let the Riesz space L be equipped with a Hausdorff topology which is generated by a family of M-seminorms (an M-seminorm is a seminorm p satisfying $p(|u|) \leq p(|v|)$ provided $|u| \leq |v|$, and $p(\sup(u, v)) = \sup(p(u), p(v))$ for all $u, v \in L^+$). Endow the set X of continuous real Riesz homomorphisms on L with $\sigma(L', L)$, and consider $C(X)$ with the topology of uniform convergence on the sets $U^0 \cap X$, where U runs through the set of 0-neighborhoods in L. Then the map $u \mapsto \hat{u}$, defined by $\hat{u}(\phi) = \phi(u)$ for all $\phi \in X$, is a Riesz isomorphism and a homeomorphism onto a Riesz subspace of $C(X)$. This result was proved by Jameson [**86**; Theorem 6].

6. In his Ph.D. Thesis, von Siebenthal considered complete Hausdorff locally convex-solid Riesz spaces L possessing a base $\mathfrak{V}$ of closed, convex, solid sets such that for each $V \in \mathfrak{V}$ the associated Riesz seminorm ρ_V, given by $\rho_V(u) := \inf\{\alpha > 0 : u \in \alpha V\}$, is p-additive. He showed that there exists a locally compact hyperstonian space X such that L can be identified with an order dense ideal of $C_\infty(X)$ in such a way that the topology of L is generated by a suitable set of Riesz

seminorms of the form $\rho_\mu(f) := \left(\int |f|^p d\mu\right)^{1/p}$, with $\mu \in \mathcal{M}(X)^+$ [**164**; 10.4]. He also obtained a similar result for M-seminorms.

We mention without proof three results on AM-spaces without norm unit.

Theorem 4.6. *Let L be an AM-space, and set*

$$X := \{\phi \in L' : \phi \textit{ is a Riesz homomorphism}\},$$

endowed with (the restriction of) $\sigma(L', L)$.

For $u \in L$, set $\hat{u} : X \to \mathbf{R}$, $\phi \mapsto \phi(u)$, and let denote, finally, $C_h(X)$ the set of all homogeneous (i.e. $f(\alpha x) = \alpha f(x)$) elements of $C(X)$.

Then, if $C_h(X)$ is equipped with the norm of uniform convergence on the unit ball of X, the map

$$T : L \to C_h(X),\ u \mapsto \hat{u}$$

is an isometric Riesz isomorphism.

Theorem 4.6 is due to Goullet de Rugy [**72**; 1.31], as well as the equivalence (a) $\Leftrightarrow$ (b) in the following result [**72**; 2.31]; (a) $\Leftrightarrow$ (c) in Theorem 4.7 was proved by Effros [**48**; 3.10] and (a) $\Leftrightarrow$ (d) by Nakano [**149**]. More characterizations of the spaces $C_0(X)$ of continuous functions on X vanishing at infinity can be found in the same places.

Theorem 4.7. *For an AM-space L, the following are equivalent.*

(a) *There exists a locally compact Hausdorff space X such that L is isometric and Riesz isomorphic to $C_0(X)$, equipped with the supremum norm*;

(b) *the restriction of the norm of L' to the set of non-zero Riesz homomorphisms in L' is $\sigma(L', L)$-continuous*;

(c) *the set of Riesz homomorphisms in the unit ball of L' is a $\sigma(L', L)$-closed subset of the positive unit ball of L'*;

(d) *for every subset A of L^+ which is bounded above, we have*

$$\sup\{\|u\| : u \in A\} = \inf\{\|v\| : v \geq u \textit{ for all } u \in A\}.$$

A proof of the following similar result can be found in [**111**; 1.b.10]. We recall that the norm in a Banach lattice is said to be order continuous if $u_\iota \downarrow 0$ implies $\|u_\iota\| \to 0$.

Theorem 4.8. *In an AM-space L, the norm is order continuous if and only if there is a discrete space X such that L is isometrically Riesz isomorphic to $C_0(X)$, equipped with the supremum norm.*

We will need the following concept. If L is a normed Riesz space, then $u \in L^+$ is called a quasi-interior point of L if L_u is norm dense in L (or equivalently, if $\inf(v, nu) \to v$ in norm for each $v \in L^+$, or equivalently, if $|\xi|(u) > 0$ for each $\xi \in L' \backslash \{0\}$).

Each quasi-interior point is a weak unit, and in AL-spaces the converse holds.

The following theorem is again due to Goullet de Rugy [**72**; 3.18].

Theorem 4.9. *Let L be an AM-space with quasi-interior point u, let $X := \{\phi \in L' : \phi$ is a Riesz homomorphism$\}$, endowed with (the restriction of) $\sigma(L', L)$, and set $Y := \{\phi \in X : \phi(u) = 1\}$. Set further*

$$\mathcal{D}(Y) := \{f \in C(Y) : \text{ for each } \varepsilon > 0 \text{ there exists a compact } K \text{ such that } |f(\phi)| \le \varepsilon\|\phi\| \text{ for all } \phi \in Y \backslash K\}$$

and endow $\mathcal{D}(Y)$ with the norm p, defined by

$$p(f) := \inf\{\alpha > 0 : |f(\phi)| \le \alpha\|\phi\| \text{ for all } \phi \in Y\}.$$

Then Y is σ-compact, for each $v \in L$ we have $\hat{v}|_Y \in \mathcal{D}(Y)$ (where $\hat{v}(\phi) := \phi(v)$, for all $\phi \in X$), and the map

$$T : L \to \mathcal{D}(Y),\ v \mapsto \hat{v}|_Y$$

is an isometric Riesz isomorphism.

Proof. First observe that $Y = \cup_{n \in \mathbf{N}} Y \cap \{\phi \in L' : \|\phi\| \le n\}$.

Let $v \in L$ and $\varepsilon > 0$. Set $K := \{\phi \in X : \|\phi\| \le 1/\varepsilon,\ \phi(v) = 1\}$ and $\tilde{K} := \{\alpha\phi : \phi \in K, \alpha \ge 0\}$. Then $Y \cap \tilde{K}$ is compact in Y and $|\hat{v}(\phi)| \le \varepsilon\|\phi\|$ for all $\phi \in X \backslash \tilde{K}$ which implies $\hat{v}|_Y \in \mathcal{D}(Y)$.

Now let $f \in \mathcal{D}(Y)$. Then $\tilde{f}$, the homogeneous extension of f to X, is continuous on $X\backslash\{0\}$. Since for each $\varepsilon > 0$ there is a compact $K \subset X\backslash\{0\}$ such that $|f(\phi)| \leq \varepsilon\|\phi\|$ for all $\phi \in X\backslash\tilde{K}$ (where $\tilde{K}$ is defined as above), one can show, using Theorem 4.6, that $\tilde{f}$ is continuous at 0. Now Theorem 4.6 implies the existence of $v \in L$ with $\hat{v} = \tilde{f}$, hence $\hat{v}|_Y = f$. Thus T is surjective, and that it is isometric, follows from Theorem 4.6, too. □

The following result, reflecting the duality of AL- and AM-spaces, is due to Kaplan [**96**; 58.1].

Theorem 4.10. *Let L be an AL-space, and let M be a norm closed Riesz subspace of L' containing the norm unit e of L' and separating the elements of L.*

Then there exist a compact Hausdorff space X and a band I of $C'(X)$ such that

(i) *L is Riesz isomorphic and isometric to I;*

(ii) *L' is Riesz isomorphic and isometric to $J := (I^0)^d \subset C''(X)$;*

(iii) *M is Riesz isomorphic and isometric to the space $C(X)_J$ consisting of all projections f_J on J of the elements f of $C(X) \subset C''(X)$.*

Proof. By Theorem 4.5, there exist a compact Hausdorff space X and an isometric Riesz isomorphism $T_0 : C(X) \to M$. Set $T := \phi \circ T_0$, where $\phi : M \to L'$ is the canonical injection. Since L is a band of L'' [**162**; Chapter II, 8.3(v)], one can show that $I := T'L$ is a band of $C'(X)$; here T' denotes the adjoint of T. Since M separates L, the map $S := T'|_L$ from L to I is injective. It follows that also $S' : J \to L'$ is an isometric Riesz isomorphism, and one has $S'f_J = Tf$ for all $f \in C(X)$, which implies (iii). □

We proceed with Ando's result; its proof can also be found in [**111**; 1.b.8].

Theorem 4.11 [**8**; Theorem 2]**.** *For a Banach lattice L of dimension ≥ 3, the following are equivalent.*

(a) *L is isometrically Riesz isomorphic to either $L^p(\mu)$ for some measure μ and some $1 \le p < \infty$, or to $C_0(X)$, for some discrete space X.*

(b) *Every closed Riesz subspace of L is the range of a positive projection of norm one on L.*

A similar result was obtained by Lindenstrauss and Tzafriri [**110**; Theorem 5]; see also [**170**; Theorem 8] and [**111**; 1.b.12].

Theorem 4.12. *For a σ-Dedekind complete Banach lattice L, the following are equivalent.*

(a) *L is Riesz isomorphic to either $L^p(\mu)$ for some measure μ and some $1 \le p < \infty$, or to $C_0(X)$, for some discrete space X.*

(b) *Every closed Riesz subspace of L is complemented.*

The following terminology will prove useful.

A locally compact Hausdorff space X is called a representation space for the Banach lattice L if the Riesz space $C_c(X)$ can be identified with a dense ideal of L; X is called a strong representation space if it is, moreover, the topological direct sum of a family of compact spaces.

Replacing in the proof of Theorem 2.8 the words "order dense" by "norm dense" and "order continuous" by "norm continuous," we get the following uniqueness assertion on representation spaces.

Theorem 4.13 [**75**]. *If X and X' are representation spaces for the Banach lattice L, then βX is homeomorphic to $\beta X'$.*

A map ϕ from the Riesz space L into $\overline{\mathbf{R}}^+$ is called a valuation on L if it satisfies

$$\phi(u+v) = \phi(u) + \phi(v) \quad \text{and} \quad \phi(\inf(u,v)) = \inf(\phi(u), \phi(v)) \\ \text{for all } u, v \in L^+,$$

$$\phi(\alpha u) = |\alpha| \phi(|u|) \quad \text{for all } \alpha \in \mathbf{R}, u \in L.$$

The set of all real valuations ϕ on L corresponds to the set of all real Riesz homomorphisms ψ on L via $\psi \mapsto \phi^{\psi}$, with $\phi^{\psi}(u) := \psi(|u|)$. Moreover, the set of all non-zero real Riesz homomorphisms on an ideal I of L corresponds to the set of all valuations ϕ on L which are real and non-zero on I (since a non-zero real valuation ψ on I can be extended uniquely to L by setting $\phi(u) := \sup\{\psi(v) : v \in I, 0 \leq v \leq |u|\}$).

For $u \in L^+$, let X_u be the set of all valuations ϕ on L with $\phi(u) = 1$. X_u, being a closed subspace of $\left(\overline{\mathbf{R}}^L\right)^+$, is compact in the topology of pointwise convergence (in $\overline{\mathbf{R}}$) on L.

If L is a Banach lattice, then L_u is complete under the norm $\|v\|_u := \inf\{\lambda : |v| \leq \lambda u\}$, and thus L_u is Riesz isomorphic and isometric (for $\|.\|_u$) to $C(X_u)$, by Theorem 4.5 and the identification above.

The following result (generalizing the Kakutani-Kreĭns theorem and the Kakutani theorem for AL-spaces with weak unit) is due to Davies [**37**; Theorem 10], and independently to Lotz [**113**] and Goullet de Rugy [**71**; 3.27]; it can also be easily derived from a theorem of Vulikh [**199**; Theorem 4], as was pointed out by Abramovich and Veksler [**3**].

Theorem 4.14. *Let L be a Banach lattice with quasi-interior point u. Then there exists a unique compact representation space X of L, namely $X = X_u$, such that L can be embedded as a Riesz space into the lattice $C_\infty(X)$. X is Stonian if and only if L is Dedekind complete, and X is quasi-Stonian if and only if L is σ-Dedekind complete.*

Proof. Realize L_u as $C(X_u)$ as described above, and define the embedding of L into $C_\infty(X_u)$ by $v \mapsto \hat{v}$, with $\hat{v}(\phi) := \phi(v^+) - \phi(v^-)$ for all $\phi \in X_u$. The uniqueness follows from Theorem 4.13.

Now let X be Stonian, and let $0 \leq v_\iota \uparrow\leq v$ in L. v can be assumed quasi-interior, and thus, by the previous, $X = X_v$; hence $L_v = C(X)$. Then L_v is Dedekind complete, and thus $v_\iota \uparrow v_0$ in L. The proof for quasi-Stonian X is similar. □

Take L and $X = X_u$ as before. Then L' can be identified with an ideal of $C(X)'$; hence each $\xi \in L'$ corresponds to a uniquely determined Radon measure μ_ξ on X with $\xi(v) = \int v \, d\mu_\xi$ for every $v \in L$, and for each $\xi \in L_n^{\sim}$, the measure μ_ξ is normal. The following result was proved by Nagel [**143**; 4.5].

Theorem 4.15. *Let L be a Banach lattice with quasi-interior point u, and let $X = X_u$ be its compact representation space. Then the norm of L is order continuous if and only if each nowhere dense subset of X is a μ_ξ-null set for every $\xi \in L'$; in this case X is hyperstonian.*

Proof. First suppose that the norm is order continuous. Then $L' = L_n^{\sim}$ [**5**; 9.1] and L is Dedekind complete [**5**; 10.3]. Hence $C(X)$ is Dedekind complete and $C(X)_n^{\sim}$ separates $C(X)$. Thus X is hyperstonian, and the nowhere dense sets are μ_ξ-null sets for every $\xi \in L' = L_n^{\sim}$.

To prove the converse, let $v_\iota \downarrow 0$ in $L \subset C_\infty(X)$. We may suppose that $v_\iota \le u$. Take $n \in \mathbf{N}$, and put $A_n := \cap\{v_\iota \ge \frac{1}{n}\}$. Then A_n is a μ_ξ-null set for every $\xi \in L'$. Therefore the downward directed family $(u - f)_{f \in \mathcal{F}}$, with $\mathcal{F} := \{f \in L : 0 \le f \le u,\ f|_{A_n} = 0\}$, converges to 0 in norm [**162**; Chapter II, Corollary to 5.9]. It is now easy to construct a $w_n \in L^+$ with $w_n|_{A_n} = \infty$ and $\|w_n\| = 1$. We can find a ι_0 such that $v_{\iota_0} \le (1/n)(w_n + u)$, which implies that $\lim \|v_\iota\| = 0$. □

In [**143**; 4.4], Nagel obtained a similar characterization of σ-order continuous norms. For some related results, see [**144**].

Also the following result was proved, except for condition (iii), by Nagel [**143**; 3.4, 4.2]; see also Lindenstrauss and Tzafriri [**111**; 1.b.14]. Note that in Banach lattices with order continuous norm, quasi-interior points and weak units are the same.

Theorem 4.16. *If the Banach lattice L possesses a quasi-interior point and has order continuous norm, then there exist a unique compact hyperstonian space X, a $\mu \in \mathcal{M}(X)$ with $\operatorname{supp}\mu = X$ and $\|\mu\| = 1$, an ideal I of $L^1(\mu)$, and a Riesz norm $\|.\|$ on I such that:*

(i) *L is Riesz isomorphic and isometric to I;*

(ii) *$L^\infty(\mu) \subset I$ norm densely, and $I \subset L^1(\mu)$ norm densely;*

(iii) *$\|f\|_1 \le \|f\| \le 2\|f\|_\infty$ for all $f \in L^\infty(\mu)$;*

(iv) $I' = \{g \in L^1(\mu) : \sup\{\int fg\,d\mu : f \in I, \|f\| \leq 1\} < \infty\}$
and $\langle f, g\rangle = \int fg\,d\mu$ *for all* $(f,g) \in I \times I'$.

Conversely, if a Banach lattice L *satisfies* (i)–(iv)*, then it possesses a quasi-interior point and has order continuous norm.*

Proof. To begin with the second assertion, let $f_\iota \downarrow 0$ in I. Then, by (iv), $f_\iota \to 0$ weakly, and hence $f_\iota \to 0$ in norm [**5**; 9.8].

To prove the main assertion, observe first that there exists a strictly positive $\xi \in L'$ with $\|\xi\| = 1$ [**218**; 103.12]. Let w be a weak unit of L satisfying $\|w\| = 2$, and take $\eta \in (L')^+$ with $\|\eta\| = 1$ and $\eta(w) = 2$. Set

$$\omega := (\xi + \eta)/\|\xi + \eta\| \quad \text{and} \quad u := w/\omega(w).$$

Then $\|\omega\| = \omega(u) = 1$, $\|u\| \leq 2$.

Let again $X = X_u$ be the representation space of L. By the preceding result, X is hyperstonian. Put $\mu := \mu_\omega$. Then $L \subset L^1(\mu)$, and since L is Dedekind complete [**5**; 10.3], L is a dense ideal of $L^1(\mu)$. Moreover, $C(X) = L^\infty(\mu)$, by Corollary 1.7. For every $f \in C(X)$, we have

$$\|f\|_1 = \int |f|\,d\mu = \omega(|f|) \leq \|\omega\|\,\|f\| = \|f\| \leq \|f\|_\infty \|u\| \leq 2\|f\|_\infty.$$

To prove (iv), take $\xi \in L'$. Then $\mu_\xi \ll \mu$, by Corollary 1.8(d). From the Radon-Nikodým theorem we conclude the existence of a $g \in L^1(\mu)$ with $\mu_\xi = g \cdot \mu$, which implies the assertion. □

A similar result for injective Banach lattices possessing a weak unit and admitting a strictly positive order continuous linear functional was proved by Haydon [**79**; 6A].

By a result of Wolff [**214**; 3.4], a Banach lattice with order continuous norm is separable if and only if it is Riesz isomorphic to a Riesz subspace of $L^1(\lambda)$, where λ denotes Lebesgue measure on $[0,1]$.

It was shown by Wolff, too, that each Banach lattice can be represented as a space of (in an abstract sense) "integrable" functions; the details follow.

Let X be a compact Hausdorff space. A mapping $N : \left(\overline{\mathbf{R}}^X\right)^+ \to \overline{\mathbf{R}}^+$ is called upper norm if it satisfies the following conditions.

(i) $f \le g \implies N(f) \le N(g)$;

(ii) $\alpha \in \overline{\mathbf{R}}^+ \implies N(\alpha f) = \alpha N(f)$;

(iii) $N(f+g) \le N(f) + N(g)$;

(iv) $f_n \uparrow f \implies N(f) = \sup N(f_n)$;

(v) $N(1_X) < \infty$.

$f \in \overline{\mathbf{R}}^X$ is called N-negligible if $N(|f|) = 0$. Set

$$\mathcal{N} := \{f \in \mathbf{R}^X : N(|f|) = 0\}, \qquad \mathcal{F} := \{f \in \mathbf{R}^X : N(|f|) < \infty\}.$$

By usual arguments from integration theory, $\mathcal{F}$ is an ideal of $\mathbf{R}^X$, and $f \mapsto N(|f|)$ is a Riesz seminorm for which $\mathcal{F}$ is complete. Thus $F := \mathcal{F}/\mathcal{N}$ is a Banach lattice. Let $\mathcal{L}_N$ be the closure of $C(X)$ in $\mathcal{F}$, and set $L_N := \mathcal{L}_N/\mathcal{N}$. (If $N(f) = \int f\, d\mu$ for some Radon measure μ, then $L_N = L^1(\mu)$).

Theorem 4.17 [**213**; 2.2]**.** *Let L be a Banach lattice possessing a compact representation space X. Then there is an upper norm N on $\left(\overline{\mathbf{R}}^X\right)^+$ such that the embedding $T : C(X) \to L$ extends to an isometric Riesz isomorphism from L_N onto L.*

Proof. Let $B := \{f \in C(X) : \|Tf\| \le 1\}$, and set

$$N(h) := \sup\left\{\int h\, d\mu : \mu \in B^0, \mu \ge 0\right\}.$$

Since $N(|f|) = \|Tf\|$ for all $f \in C(X)$, the norm completions L_N and L of $C(X)$ coincide. □

In particular, if u is a quasi-interior point of L, then X_u is a compact representation space of L, and thus L can be identified with a space L_N, which gives another version of Theorem 4.14.

Also some other special situations were investigated by Wolff [**212**]. We shall mention one of his results.

For a Banach lattice L, let Φ be the set of all σ-order closed Riesz subspaces of L'' containing L, and set $B(L) := \cap_{M \in \Phi} M$.

L is called quasi-discrete if the set of real Riesz homomorphisms on L separates $B(L)$.

Theorem 4.18 [**212**; 1.7]**.** *For a Banach lattice L with quasi-interior point u, the following are equivalent.*

(a) *L is quasi-discrete*;

(b) *there exist a σ-compact representation space X for L and a Riesz isomorphism T from L onto an ideal of $C(X)$ such that $Tu = 1_X$ (hence $T(L_u) = C_b(X)$).*

Proof. Realizing L as a Riesz subspace of $C_\infty(X_u)$ (cf. Theorem 4.14), one can show that L is quasi-discrete if and only if there exists a maximal Borel set N of X_u such that for each $x \in N$ there is $f \in L$ with $|f|(x) = \infty$.

If L is quasi-discrete, set $X := X_u \backslash N$. Then N is a closed G_δ, and X_u is the Stone-Čech compactification of X so that all follows from Theorem 4.14.

Conversely, assume (b). Then Y, the Stone-Čech compactification of X, is the representation space for L postulated in Theorem 4.14. Since $T^{-1}(C_c(X))$ is norm dense in L, $N := Y \backslash X$ is a maximal Borel set of the kind required in the condition above. □

In order to formulate Schaefer's theorem, which extends Theorem 4.14, we need the following generalization of the concept of quasi-interior point:

A disjoint system (u_ι) of strictly positive elements of the Banach lattice L is called topological orthogonal system of L if the ideal generated by the u_ι's is norm dense in L. Each topological orthogonal system is a maximal disjoint system.

The next two theorems are due to Schaefer [**161**; Propositions 6,7, Theorem 2]; see also [**162**; Chapter III, Section 5]. We remark that (a) $\Rightarrow$ (b) in the following result can also be derived without difficulty from [**202**]; see also [**3**].

Theorem 4.19. *For a Banach lattice L, the following are equivalent.*

(a) *L possesses a topological orthogonal system*;

(b) *there exists a strong representation space for L*;

(c) *there exists a paracompact representation space for L.*

Moreover, the topological orthogonal systems S and the strong representation spaces X of L are in one-to-one correspondence by assigning to S the set X_S of all valuations ϕ on L such that $\phi(u) = 1$ for some $u \in S$, where X_S is equipped with the topology of pointwise convergence on L.

L is Dedekind complete if and only if each strong representation space is Stonian.

Proof. (c) $\Rightarrow$ (a). The paracompact representation space X can be written as topological direct sum of σ-compact spaces X_ι. There exist $f_{\iota n} \in C_c(X_\iota)$ with $X_\iota = \cup\{f_{\iota n} > 0\}$; set $u_\iota := \sum u_{\iota n}/(n^2\|u_{\iota n}\|)$.

The last assertion of the theorem can be proved using the corresponding claim of Theorem 4.14, while (a) $\Rightarrow$ (b) and the "Moreover" part are consequences of the "concrete" Theorem 4.20. $\square$

Theorem 4.20. *Let L be a Banach lattice possessing a topological orthogonal system.*

(a) *There exist a strong representation space X for L and a minimal $\sigma(\mathcal{M}_B(X), C_c(X))$-compact set $\mathcal{M}$ of positive regular Borel measures on X (here $\mathcal{M}_B(X)$ denotes the set of all regular Borel measures on X) such that*:

L is embedded as a Riesz space into $C_\infty(X)$, and via this embedding, L is (isometric to) the norm completion of $(C_c(X), p_{\mathcal{M}})$ under the norm

$$p_{\mathcal{M}}(f) := \sup_{\mu\in\mathcal{M}} \int |f| d\mu\,,$$

and hence equal to the set of all $f \in C_\infty(X)$ which satisfy $p_{\mathcal{M}}(f - f_n) \to 0$ for some $p_{\mathcal{M}}$-Cauchy sequence (f_n) from $C_c(X)$.

(b) *$(X, \mathcal{M})$ is unique in the following sense*:

If $(Y, \mathcal{N})$ satisfies (a), *then there exist a homeomorphism of a dense open subspace X_0 of X onto a dense open subspace Y_0 of Y and a Riesz isomorphism $C_c(X_0) \to C_c(Y_0)$ whose adjoint maps $\mathcal{M}_0 := \mathcal{M}|_{X_0}$ onto $\mathcal{N}_0 := \mathcal{N}|_{Y_0}$, and $(X_0, \mathcal{M}_0)$ satisfies* (a).

Proof. (a) Let (u_ι) be a topological orthogonal system, and let X be the topological direct sum of the spaces X_{u_ι}.

Denoting by I the ideal of L generated by the u_ι's, we have $C_c(X) = \oplus C(X_{u_\iota}) = \oplus L_{u_\iota} = I$. Hence there exists a Riesz isomorphism $T : C_c(X) \to L$ which maps onto I, and which is continuous for the norm topology on L and for the inductive limit-topology on $C_c(X)$ with respect to the spaces $\{f \in C_c(X) : \operatorname{supp} f \subset K\}$, $K \in \mathfrak{K}(X)$. I being dense, the adjoint T' is injective, and $T'(L')$ is $\sigma(\mathcal{M}_B(X), C_c(X))$-dense in $\mathcal{M}_B(X)$.

Let P be the Šilov boundary of L^+, i.e., the minimal $\sigma(L', L)$-closed subset of $U^0 \cap (L')^+$ (where U denotes the unit ball of L) such that each $u \in L \subset L''$ attains its norm on P; the existence of P is assured by Bauer's theorem [**12** or **162**; II 5.7]. T' being weakly continuous, $\mathcal{M} := T'(P)$ is weakly compact.

If $p_{\mathcal{M}}$ is defined as indicated, then T becomes an isometry onto I. Then T can be extended to an isometric Riesz isomorphism $\overline{T}$ which maps the completion of $(C_c(X), p_{\mathcal{M}})$ onto L.

By Theorem 4.14, $C(X_{u_\iota}) \subset \{u_\iota\}^{dd} \subset C_\infty(X_{u_\iota})$, and since $L = \oplus\{u_\iota\}^{dd}$, we have

$$C_c(X) \subset L \subset \oplus C_\infty(X_{u_\iota}) = C_\infty(X).$$

(b) Let Y be the topological direct sum of the compact spaces Y_λ. Then (v_λ) is a topological orthogonal system of L, where v_λ corresponds to 1_{Y_λ}. Now consider the topological orthogonal system formed by all nonzero $\inf(u_\iota, v_\lambda)$, and observe that the corresponding representation space $Z = \cup Z_{\iota\lambda}$ is homeomorphic to open dense subsets X_0 of X and Y_0 of Y, respectively. That N_0 will be mapped onto M_0 follows from the uniqueness in Bauer's theorem. □

Both Kakutani theorems are special cases of Theorem 4.20:

If L is an AM-space with norm unit e, then P is the set of all Dirac measures on X_e.

If L is an AL-space, then P consists of one element, and since $L' = L_n^\sim$, this measure belongs to $\mathcal{M}(X)$.

While Schaefer determined the Banach lattices with strong representation spaces, Feldman and Porter, following similar lines, solved the analogous problem for representation spaces.

A set of positive elements u_ι of the Banach lattice L is called topological order partition of L if

(i) the ideal I generated by the u_ι's is dense in L;

(ii) for each ι there is λ such that for each ρ there exists $\alpha = \alpha(\rho) \in \mathbf{R}$ with

$$\inf(u_\rho, nu_\iota) \leq \alpha u_\lambda \quad \text{for all } n \in \mathbf{N};$$

(iii) denoting by H the set of all nonzero real Riesz homomorphisms on I and endowing H with $\sigma(H, I)$, there exists $F \in C(H)$ such that

$$F(\phi) \geq \phi(u_\iota) \quad \text{for all } \iota,$$
$$F(\alpha\phi) = \alpha F(\phi) \quad \text{for all } \alpha > 0.$$

Each topological orthogonal system is a topological order partition.

Here are the results of Feldman and Porter [**50**; Theorem 2 and **51**]:

Theorem 4.21. *There exists a representation space for the Banach lattice L if and only if L possesses a topological order partition.*

Proof. For the "only if" consider $L = C_c(X)$. For each $K \in \mathfrak{K}(X)$ there is $f_K \in C_c(X), 0 \leq f_K \leq 1$, and $f_K = 1$ on K. Then the collection of all f_K is a topological order partition since each non–zero real Riesz homomorphism ϕ on $C_c(X)$ has the form $\phi = \phi_{\alpha,x}$, where $\phi_{\alpha,x}(f) = \alpha f(x)$, with $x \in X$ and $\alpha > 0$, and thus $F(\phi_{\alpha,x}) := \alpha$ is a possible choice for F.

The "if" part is contained in Theorem 4.22. □

Theorem 4.22. *The same assertions as in Theorem* 4.20 *hold, with "topological orthogonal system" replaced by "topological order partition," and "strong representation space" replaced by "representation space."*

Proof. Let X be the set of all $\phi \in H$ with $F(\phi) = 1$, endowed with $\sigma(H, I)$ (we use the same notations as in the definition above). Then $X = \cup X_\iota$, where $X_\iota = \{\phi \in X : \phi(u_\iota) > 0\}$. The X_ι's being relatively compact, X is locally compact. X can be identified with a set V of valuations on L (see the remarks preceding Theorem 4.14). Now set, for $u \in L$ and $\phi \in V$,

$$\hat{u}(\phi) := \phi(u^+) - \phi(u^-)$$

to obtain the embedding $u \mapsto \hat{u}$ of L into $C_\infty(X)$.

The further properties of (a) follow similarly to the proof of Theorem 4.20, while the proof of (b) is somewhat more complicated. □

Haydon studied representations of injective Banach lattices, i.e., of Banach lattices L with the property that for every Banach lattice M, for every closed Riesz subspace M_1 of M and for every positive linear map $T_1 : M_1 \to L$ there is a positive linear extension $T : M \to L$ with $\|T\| = \|T_1\|$. He showed that such a Banach lattice is isometrically Riesz isomorphic to a space of order continuous operators between spaces of continuous functions [**79**; 5C], and to $L^1(\mu)$ for some vector measure μ [**79**; 6H]. Representations of certain special classes of Banach lattices (including the injective Banach lattices) as sections of bundles of AL-spaces were studied by Giertz [**69**; Section 8] and also by Haydon [**79**; Section 7].

Let us finish this section by remarking that representations of locally convex Riesz spaces were studied e.g. by Kuller [**107**], Kawai [**97**; 6.6], Portenier [**158**; Section 3], and von Siebenthal [**164**]; see also Goullet de Rugy [**73**; 3.2 and 3.3].

5. Representations of Orlicz lattices. We follow essentially Wnuk's paper [**216**].

Let L be a Riesz space. A mapping $\rho : L \to [0, \infty]$ is called a modular on L ([**216**]; see also [**46**]) if

(ρ1) $\rho(u) = 0$ if and only if $u = 0$;

(ρ2) $|u| \le |v|$ implies $\rho(u) \le \rho(v)$;

(ρ3) $\rho(\alpha u) \to 0$ for $\alpha \to 0$;

(ρ4) $0 \le u_n \uparrow u$ implies $\rho(u_n) \uparrow \rho(u)$;

(ρ5) $u \perp v$ implies $\rho(u + v) = \rho(u) + \rho(v)$.

A modular ρ is called convex if

$$\rho(\alpha u+\beta v) \leq \alpha\rho(u)+\beta\rho(v) \quad \text{for all } u, v \in L \text{ and all } \alpha, \beta \geq 0,\ \alpha+\beta=1.$$

If a modular ρ exists on L, then L is Archimedean and has the countable sup property [**216**; 4.1]. Each modular on L can be extended uniquely to a modular on the Dedekind completion of L [**216**; 4.2].

To a modular ρ, there is associated the F-norm

$$\|u\|_\rho = \inf\{\alpha > 0 : \rho(u/\alpha) \leq \alpha\}$$

which has the σ-Fatou property

$$0 \leq u_n \uparrow u \Rightarrow \|u_n\|_\rho \uparrow \|u\|_\rho.$$

An Orlicz lattice L^ρ is, by definition, a Riesz space L together with a modular ρ (topological completeness is not included in the definition).

If ρ is finite, then $\|.\|_\rho$ is order continuous [**216**; p. 27], and hence

$$u_\iota \downarrow u \geq 0 \Rightarrow \rho(u_\iota) \downarrow \rho(u).$$

Two Orlicz lattices L^ρ and K^η are called isomodular if there exists a Riesz isomorphism T from L onto K such that $\rho(u) = \eta(Tu)$ for all $u \in L$.

Let μ be a positive measure on a ring of sets $\mathfrak{R}$ of subsets of a set X. A function

$$\psi : [0, \infty[\times X \to [0, \infty]$$

is called a Musielak-Orlicz function ([**216**]; see also [**142**; Section 2.3]) if

(ψ1) for all x: $\psi(\cdot, x)$ is nondecreasing, left continuous, continuous at 0, and $\psi(t, x) = 0$ if and only if $t = 0$;

(ψ2) for all t: $\psi(t, \cdot)$ is μ-measurable.

Since the mapping $x \mapsto \psi(|f(x)|, x)$ is μ-measurable provided f is μ-measurable (this is obvious for step functions and follows in the general case by approximation with step functions), one can define the associated functional M_ψ on $L^0(\mu)$ by

$$M_\psi(f) := \int_X \psi(|f(x)|, x) d\mu\,.$$

Then, according to Wnuk,

$$L^\psi(\mu) := \{f \in L^0(\mu) : \text{there is } \alpha > 0 \text{ with } M_\psi(\alpha f) < \infty\}$$

is called a Musielak-Orlicz space. $L^\psi(\mu)$ is an ideal of $L^0(\mu)$, and applying [**129**; 23.6] and [**68**; 18D] it is easy to see that $L^\psi(\mu)$ is super Dedekind complete (observe that elements of $L^\psi(\mu)$ have σ-finite support).

$(L^\psi(\mu))^{M_\psi}$ is an Orlicz lattice possessing the σ-Levi property, i.e., each increasing $\|.\|_{M_\psi}$-bounded sequence from L^+ has a supremum in L [**216**; 1.1], and $L^\psi(\mu)$ is $\|.\|_{M_\psi}$-complete (this follows by applying [**5**; 16.1 and 13.2]).

If $\psi(t,x) = \psi(t,y)$ for all $x, y \in X$, then the mapping

$$\phi : [0, \infty[\to [0, \infty],\ t \mapsto \psi(t, x)$$

is called an Orlicz function, and the space $L^\phi(\mu)$ is called an Orlicz space.

Theorem 5.1 [**216**; 5.1]**.** *Let the Orlicz lattice L^ρ be isomodular to a super order dense Riesz subspace K^{M_ψ} of some Musielak-Orlicz space $L^\psi(\mu)$. Then*

(a) *L is Dedekind complete if and only if K is an ideal of $L^\psi(\mu)$;*

(b) *L^ρ has the σ-Levi property if and only if $K = L^\psi(\mu)$;*

(c) *if $L^\rho_f := \{u \in L^\rho : \rho(\alpha u) < \infty \text{ for all } \alpha > 0\}$ is $\|.\|_\rho$-complete, then L^ρ_f is mapped onto $L^\psi_f(\mu) := \{g \in L^\psi(\mu) : M_\psi(\alpha g) < \infty \text{ for all } \alpha > 0\}$; hence $L^\psi_f(\mu) \subset K$.*

Proof. (a) is obvious.

(b) By [**216**; A.13] the condition is necessary, and it is sufficient since every Musielak-Orlicz space has the σ-Levi property.

(c) Let $g \in L^\psi_f(\mu)$, $g \geq 0$. Identifying L with its image in $L^\psi(\mu)$, we find a sequence (u_n) in L^+ with $u_n \uparrow g$; hence, $u_n \in L^\rho_f$. Then $\|g - u_n\|_{M_\psi} \to 0$. Thus (u_n) is Cauchy and converges to $u \in L^\rho_f$. By [**5**; 5.6(iii)], $u_n \uparrow u$ and hence $g = u$. ◻

Theorem 5.2 [**216**; 5.1, 5.2]**.** *For every Orlicz lattice L^ρ there exist a locally compact hyperstonian space X which is the topological direct sum of a family of compact spaces, a $\mu \in \mathcal{M}(X)$ with $\operatorname{supp} \mu = X$, a Musielak-Orlicz function $\psi : [0, \infty[\times X \to [0, \infty]$ and a super order dense Riesz subspace K of $L^\psi(\mu)$ (which is also super order dense in $L^0(\mu)$) such that L^ρ is isomodular to K^{M_ψ}.*

Moreover:

(a) *If w is a weak unit of L^ρ, then one can arrange that w is mapped onto 1_X, and X is compact*;

(b) *if ρ is convex, then ψ can be chosen convex (in the first variable)*;

(c) *if $\rho(u + v) = \rho(u) + \rho(v)$ for all u, v, then ψ can be chosen to satisfy*

$$\psi(s + t, x) = \psi(s, x) + \psi(t, x) \quad \textit{for all } s, t \textit{ and all } x;$$

(d) *if ρ is finite, then ψ can be chosen finite*;

(e) *if ρ satisfies the (Δ_2)-condition*

$$\exists C > 0 \, \forall \, u \in L : \rho(2u) \leq C\rho(u)\,,$$

then ψ can be chosen to satisfy the condition

$$\exists C > 0 \, \forall \, t \geq 0 \, \forall \, x \in X : \psi(2t, x) \leq C\psi(t, x)\,.$$

Proof. Since L is super order dense in its Dedekind completion $\widetilde{L}$ and ρ can be extended to $\widetilde{L}$, it is enough to consider Dedekind complete L.

First assume that L possesses a weak unit w. By (ρ3), there exists $\alpha > 0$ such that $\rho(\alpha w) < \infty$. Let (X, T) be an MOV-representation of L with $T(\alpha w) = 1_X$ (Theorem 2.1). For $A \subset X$, A open-compact, and N meager in X set

$$\mu(A \triangle N) := \rho(T^{-1} 1_A).$$

μ is a positive measure on $\mathfrak{R}(X)$ by (ρ4) and (ρ5). We have $\mu(A_\iota) \downarrow \mu(\cap A_\iota)$ for each downward-directed family (A_ι) of compact sets since finite modulars are order continuous, and hence μ is regular [**34**;

Exercise 2.2.13]; then by Proposition 1.5, $\mu \in \mathcal{M}(X)$. By (ρ1) we have $\operatorname{supp}\mu = X$. Hence $L^0(\mu) = C_\infty(X)$ (Corollary 1.7). Since μ is bounded, $L^0(\mu)$ has the countable sup property, and thus TL is super order dense.

We set

$$M(f) := \rho(T^{-1}f) \quad \text{for all } f \in TL.$$

TL being Dedekind complete, the Drewnowski-Orlicz theorem [**46**; Theorem 2.1] can be applied and yields the existence of a function $\psi : [0,\infty[\, \times X \to [0,\infty]$ such that

(i) $\psi(t,.)$ is μ-measurable for all t;

(ii) $\psi(0,x) = 0$ for a.e. x;

(iii) for all $t > 0$: $\psi(t,x) > 0$ for a.e. x;

(iv) for all $t > s > 0$: $\psi(t,x) \geq \psi(s,x)$ for a.e. x;

(v) $\psi(\cdot,x)$ is left continuous for a.e. x;

(vi) $M(f) = \int_X \psi(|f(x)|,x)d\mu$ for all $f \in TL$.

Then there exists a μ-null set A such that (i)–(v) hold on $X \backslash A$, with "for a.e. x" replaced by "for each x" (observe that after removing the exceptional set in (v), it is sufficient to consider only rational t, s in (iii) and (iv)). Hence redefining ψ on $[0,\infty[\, \times A$ by setting $\psi(t,x) := t$ yields (except for continuity at 0) that ψ is the required Musielak-Orlicz function.

To fill this last gap, fix a sequence (t_n) with $1 > t_n \downarrow 0$. Then $\int \psi(t_n,x)d\mu = \rho(t_n \alpha w) \to 0$. Thus $\psi(t_n,\cdot) \xrightarrow{\mu} 0$, and therefore there exists a subsequence (t_{n_k}) of (t_n) such that $\psi(t_{n_k},x) \to 0$ for all x not belonging to a μ-null set B. Redefining ψ on $[0,\infty[\, \times B$ as above, we find (observing (iv)) that $\psi(\cdot,x)$ is continuous at 0 for all x.

To show that one can arrange that w is mapped onto 1_X, set $\overline{T}u := \alpha(Tu)$ for all u and $\overline{\psi}(t,x) := \psi(\alpha t,x)$ for all (t,x). Then $\overline{T}, \overline{\psi}$ meet the requirements of (a).

In the general case, let (w_ι) be a complete disjoint system of L. Let P_ι denote the projection from L onto $L_\iota := \{w_\iota\}^{dd}$, set $u_\iota := P_\iota u$ for all $u \in L$, and set $\rho_\iota := \rho|_{L_\iota}$. Then $L = \oplus L_\iota$ (cf. [**54** and **5**; 2.15]) and $\rho(u) = \sum \rho_\iota(u_\iota)$ for all $u \in L$, by (ρ4) and (ρ5). Find $X_\iota, \psi_\iota, \mu_\iota$ for $L_\iota^{\rho_\iota}$ according to the first part of the proof; of course the X_ι may be assumed disjoint. Set $X := \cup X_\iota$, $\mu := \sum \mu_\iota$, and $\psi(t,x) := \psi_\iota(t,x)$ for $x \in X_\iota$.

(b) Referring to the construction just described, it is enough to prove the assertion for Dedekind complete L possessing a weak unit w with $Tw = 1_X$. Then for all rational $\alpha \in [0,1]$, all rational $s,t \in [0,\infty[$ and each open-compact $A \subset X$ we have, by convexity of ρ,

$$\int_A \psi(\alpha s + (1-\alpha)t,\, x)d\mu \;\le\; \int_A (\alpha\psi(s,x) + (1-\alpha)\psi(t,x))d\mu\,,$$

hence

$$\psi(\alpha s + (1-\alpha)t,\, x) \;\le\; \alpha\psi(s,x) + (1-\alpha)\psi(t,x) \qquad \mu\text{-a.e.}$$

Since ψ is left continuous in the first variable, this inequality holds for all $\alpha \in [0,1]$, all $s,t \in [0,\infty[$ and all $x \in X\backslash B$, where B is a μ-null set. Now redefine ψ on $[0,\infty[\, \times B$ by setting $\psi(t,x) := t$.

(c), (d), and (e) are proved similarly. □

Special cases of the above theorem were proved in [**215**] and [**104**]. The first result in this direction was obtained by Nakano [**149**] who proved that a Dedekind complete Riesz space with a convex modular is isomodular to a Riesz subspace of some Musielak-Orlicz space.

If an Orlicz lattice L^ρ does not satisfy the $(\triangle_2)$-condition, then there exist, by results of Koldunov [**101**] and Lozanovskiĭ [**125**], nontrivial linear functionals on L^ρ which are Riesz homomorphisms; Koldunov described these functionals in terms of MOV.

Theorem 5.3 [**216**; 6.7]**.** *If L^ρ has the σ-Levi property, and if there exists a finite family D of discrete elements of L such that $\rho|_{D^d}$ is bounded in $\mathbf{R}$, then in Theorem* 5.2 *L^ρ is mapped onto $L^\psi(\mu) = L^0(\mu)$, with compact X, and $\|.\|_{M_\psi}$ generates the topology of convergence in μ on $L^0(\mu)$.*

Conversely, if L^ρ is isomodular to some $L^\psi(\mu)$ with bounded μ such that $L^\psi(\mu) = L^0(\mu)$, and $\|.\|_{M_\psi}$ generates the topology of convergence in μ on $L^0(\mu)$, then L^ρ possesses the properties above.

Proof. To show the "if" part, observe that the assumptions imply that L^ρ is σ-laterally complete, hence laterally complete since every disjoint system of nonzero elements from L^ρ is countable, by the boundedness of $\rho|_{D^d}$. Therefore L^ρ equals its universal completion, namely $L^0(\mu)$.

If $\|f_n\|_{M_\psi} \to 0$, then $f_n \overset{\mu}{\to} 0$ by [**45**; 3.6].

To prove that $\|f_n\|_{M_\psi} \to 0$ holds provided $f_n \overset{\mu}{\to} 0$, it is enough to show (with $S = X\setminus \cup_{u\in D} \operatorname{supp} Tu$): $f_n 1_S \to 0$ μ-a.e. implies $M_\psi(f_n 1_S) \to 0$. By Egorov's theorem [**34**; 5.4.24] there is a decreasing sequence (A_k) with $\mu(\cap A_k) = 0$ such that $f_n 1_{S\setminus A_k} \to 0$ uniformly in n, for each k. Since $\rho|_{D^d}$ is finite, it follows by an indirect argument that $\sup_n M_\psi(f_n 1_{A_k}) \to 0$, from which the claim can be derived.

For the converse implication, observe first that $L^0(\mu)$ has the σ-Levi property with respect to the topology of convergence in measure. Since $\|.\|_{M_\psi}$ generates this topology, there is $\delta > 0$ such that $M_\psi(f) < 1$ for all $f \in L^0(\mu)$ with $\mu(\operatorname{supp} f) < \delta$. By a result of Saks (see [**47**; IV.9.7]), one finds a partition of X into finitely many atoms A_i with $\mu(A_i) \geq \delta$ and finitely many sets B_j with $\mu(B_j) < \delta$, which implies the remaining assertion. □

Our final theorem is concerned with the embedding of Orlicz lattices into Orlicz spaces. We need one more notion:

A modular ρ on L is called u-component invariant if $\rho(u) < \infty$ and $\rho(\alpha v)\rho(u) = \rho(v)\rho(\alpha u)$ for all $\alpha \geq 0$ and for all components v of u.

Theorem 5.4 [**216**; 6.1, 6.5]**.** (a) *If the Orlicz lattice L^ρ has a complete disjoint system (u_ι) with the properties*

(i) *ρ is u_ι-component invariant for all ι,*

(ii) *$\rho(\alpha u_\iota)\rho(u_\lambda) = \rho(u_\iota)\rho(\alpha u_\lambda)$ for all $\alpha \geq 0$ and all ι, λ,*

then in Theorem 5.2 *$\psi(t, x)$ can be replaced by an Orlicz function $\phi(t)$ which is given by $\phi(t) = \rho(tu_\iota)/\rho(u_\iota)$ for some* (*hence, by* (ii), *all*) *ι.*

(b) *If ρ is u-component invariant for all $u \in L$, then in* (a) *$\phi(t) = t^p$ for some $p \in\,]0, \infty[$; hence L^ρ embeds into $L^p(\mu)$. If ρ is moreover convex, then $p \geq 1$. If ρ satisfies $\rho(u + v) = \rho(u) + \rho(v)$ for all u, v, then $p = 1$.*

Proof. (a) We can assume L Dedekind complete and confine ourselves to the case that L has a weak unit w such that ρ is w-component invariant.

It is enough to show (using left continuity) that $\phi(t) = \psi(t,x)$ for all rational $t \geq 0$ and μ-a.e. x. But for each open–compact $A \subset X$ there is a component u of w with $\mu(A) = \rho(u)$, hence (by an easy calculation) $\int_A \phi(t)d\mu = \int_A \psi(t,x)d\mu$ for all t, which implies the assertion.

(b) Let (u_ι) be a complete disjoint system of L. Application of the assumption to the components u_ι, u_λ of $u_\iota + u_\lambda$ shows that (u_ι) satisfies (a)(ii). Using the same argument for the component u_λ of $\alpha u_\iota + u_\lambda$, we find that

$$\rho(\alpha t u_\iota)\rho(u_\lambda) = \rho(\alpha u_\iota)\rho(t u_\lambda)$$

and hence

$$\phi(\alpha t) = \phi(\alpha)\phi(t)$$

for all $\alpha, t > 0$. Since ϕ is left continuous and continuous at 0, the assertion follows from [**4**; Section 2.1.2, Theorem 3].

The remaining claims are consequences of Theorem 5.2 (b), (c). □

Using the fact that elements of $L^\phi(\mu)$ have σ-finite supports, it is not difficult to see that if L^ρ is isomodular to a super order dense Riesz subspace of an Orlicz space $L^\phi(\mu)$ such that the image of L contains all μ-integrable functions, then L has a complete disjoint system satisfying (i) and (ii) of (a).

Also, if L^ρ is isomodular to a Riesz subspace of some $L^p(\mu)$, then all $u \in L$ are u-component invariant.

Theorem 5.4 was first proved, under additional assumptions, by Claas [**30**]; see also [**31**].

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