ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 4, Fall 1994

ON THE EXISTENCE OF TANGENT HYPERPLANES TO FULL SUBLATTICES OF EUCLIDEAN SPACE

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ABSTRACT. Let L be a full sublattice of Euclidean n-space. We study those points in the boundary of L where L admits a tangent hyperplane. The main result states that this collection of points is dense in the boundary of L. This theorem is a generalization of the well-known fact that monotone increasing real-valued functions are differentiable almost everywhere.

1. Introduction. A standard result in analysis states that monotone increasing real-valued functions are differentiable almost everywhere. In other words, if $f : [0,1] \to \mathbf{R}$ is a monotone (upper semi-continuous) function, and if $L = \{(x,y) \in [0,1]^2 : y \leq f(x)\}$ is the subgraph of f, then the set of points where we can assure the existence of a tangent line to L is dense in the boundary of L. In this note we will extend this result to full sublattices: A sublattice $L \subseteq \mathbf{R}^n$ is called *full*, provided that the interior L° of L is connected and dense in L. Full sublattices of $\mathbf{R}^n$ were first introduced and studied in greater detail in [**2**] and [**3**]. If L is such a full sublattice, then the points p in the boundary of L where L admits a tangent hyperplane is dense in the boundary ∂L of L. Such a point $p \in \partial L$ will be called a $\mathcal{C}_1$-point. The property of being a $\mathcal{C}_1$-point is not an intrinsic property of the point $p \in \partial L$; it rather depends on the particular imbedding of L into $\mathbf{R}^n$. On the other hand, there are certain points $p \in \partial L$ that do not admit a tangent plane under any imbedding of L into $\mathbf{R}^n$.

Another related result is S. Mazur's theorem [**5**] which states that a closed convex set with dense interior in a separable Banach space has a dense set of points of $\mathcal{C}_1$-points in the boundary. From a point of view of order theory, convex sets typically stand at the opposite side of distributivity. So one might hope that there is a generalization of Mazur's result to abstract convex structures along the lines studied by

Received by the editors on December 3, 1992, and in revised form on July 6, 1993.
AMS *Subject Classification.* 26B, 26A, 06D.

M. van de Vel [**6**] or R.E. Jamison-Waldner [**4**] that also would cover our result.

Here is a short guide to our notation:

1. **R** stands for the set of all real numbers.

2. The symbols $q, r, s, t, \varepsilon, \delta$ denote real numbers.

3. As usual, $\mathbf{R}^n$ is the Euclidean n-space, and n is reserved to denote the dimension of $\mathbf{R}^n$.

4. Vectors in $\mathbf{R}^n$ are denoted by x, y, z and lower indices denote the coordinates of those vectors, i.e., $x = (x_1, \dots, x_n)$.

5. A subset $L \subseteq \mathbf{R}^n$ is called a *sublattice* provided that for each pair $x, y \in L$ the elements $x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$ and $x \wedge y = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$ belong to L.

6. The numbers m, i, j, k, and l are integer indices for coordinates, i.e., $m, i, j, k, l \in \{1, \dots, n\}$.

7. As indices for sequences we use the symbols λ, μ, and ν. They are sometimes also written as upper indices.

8. The symbols ρ and π are reserved for (set theoretical) projection maps of various sorts, and the symbols α and β are exclusively used to denotes "seams."

2. Preliminaries. In this section we will summarize some of the results of [**7**] and [**3**].

1. $C_1, \dots, C_n$ is a fixed family of complete chains. The smallest element of C_i is denoted by $\perp$ and the greatest element is denoted by $\top$. Let L be a complete sublattice of $C_1 \times \dots \times C_n$.

2. For every index $i \in \{1, \dots, n\}$ let $\pi_i : C_1 \times \dots \times C_n \to C_i$ denote the i^{th} projection. The restriction of π_i to L will also be denoted by π_i. We will assume in the sequel that $\pi_i : L \to C_i$ is surjective. This is no severe restriction, since we may replace C_i by $\pi_i(L) \subseteq C_i$.

3. Since L is a complete sublattice of $C_1 \times \dots \times C_n$, the map π_i preserves arbitrary infima and arbitrary suprema. Hence $\pi_i : L \to C_i$ has an upper adjoint $\varepsilon_i : C_i \to L$ and a lower adjoint $\delta_i : C_i \to L$.

Explicitly, ε_i and δ_i are given by the equations

$$\varepsilon_j(r) = \sup\{x \in L : \pi_j(x) \leq r\}$$
$$\delta_j(r) = \inf\{x \in L : \pi_j(x) \geq r\}.$$

Then ε_i preserves infima and δ_i preserves suprema.

4. For every pair of indices $1 \leq i,\, j \leq n$, we let

$$\alpha^L_{i,j} = \pi_i \circ \varepsilon_j,$$
$$\beta^L_{i,j} = \pi_i \circ \delta_j.$$

Then $\alpha^L_{i,j}$ is the upper adjoint of $\beta^L_{j,i}$.

Definition 2.1. Let $C_1, \dots, C_n$ be a finite family of complete chains. A family of maps $\alpha_{i,j} : C_j \to C_i$ satisfying

1. $\alpha_{i,j} : C_j \to C_i$ preserves arbitrary infima (i.e., is order preserving and upper semicontinuous);
2. $\alpha_{i,i} = id_{C_i}$;
3. $\alpha_{i,j}(\top) = \top$;
4. $\alpha_{i,j} \circ \alpha_{j,k} \geq \alpha_{i,k}$

is called an *n-dimensional* $\wedge$-seam. The dual notion of $\vee$-seams is defined accordingly.

Proposition 2.2. *Let $i, j, k \in \{1, \dots, n\}$. Then $(\alpha^L_{i,j})_{1 \leq i,j \leq n}$ is an $\wedge$-seam, whereas $(\beta^L_{i,j})_{1 \leq i,j \leq n}$ is a $\vee$-seam.*

The lattice L can be recovered from the maps $\alpha^L_{i,j}$ and $\beta^L_{i,j}$ as follows:

Proposition 2.3.

$$L = \{(c_1, \dots, c_n) \in C_1 \times \cdots \times C_n : (\forall i, j)\ \alpha^L_{i,j}(c_j) \geq c_i\}$$

and

$$L = \{(c_1, \dots, c_n) \in C_1 \times \cdots \times C_n : (\forall i, j)\ \beta^L_{i,j}(c_j) \leq c_i\}.$$

Let us introduce some additional notation: If $f : [0,1] \to [0,1]$ is any function, we let

$$\begin{aligned}\breve{f}(r) &= \sup\{g(r) : g : [0,1] \to [0,1] \text{ is continuous and } g \le f\}\\ &= \sup_{\varepsilon>0} \inf_{|r-s|<\varepsilon} f(s)\end{aligned}$$

and

$$\begin{aligned}\widehat{f}(r) &= \inf\{g(r) : g : [0,1] \to [0,1] \text{ is continuous and } f \le g\}\\ &= \inf_{\varepsilon>0} \sup_{|r-s|<\varepsilon} f(s).\end{aligned}$$

Then $\breve{f}$ is the largest lower semicontinuous function below f, and $\widehat{f}$ is the smallest upper semicontinuous function above f. Moreover, if f is monotone increasing, then

$$\begin{aligned}\breve{f}(r) &= \sup\{f(s) : s < r \text{ or } s = 0\};\\ \widehat{f}(r) &= \inf\{f(s) : s > r \text{ or } s = 1\}.\end{aligned}$$

It follows that for monotone f, the functions $\breve{f}$ and $\widehat{f}$ are also monotone. Moreover, $\breve{f}$ preserves arbitrary suprema and $\widehat{f}$ preserves arbitrary infima.

Proposition 2.4. *Let $L \subseteq [0,1]^n$ be a sublattice of $\mathbf{R}^n$ with $\wedge$-seams $(\alpha_{i,j})_{i,j}$ and $\vee$-seams $(\beta_{j,i})_{j,i}$. Then for $0 < r \le 1$ and $0 \le s < 1$ we have*

$$\begin{aligned}\breve{\alpha}_{i,j}(r) \le s &\Leftrightarrow r \le \widehat{\beta}_{j,i}(s)\\ s < \breve{\alpha}_{i,j}(r) &\Leftrightarrow \widehat{\beta}_{j,i}(s) < r.\end{aligned}$$

Let us consider a sublattice $L \subseteq [0,1]^n \subseteq \mathbf{R}^n$. The interior L° can be described in terms of the seams of L as follows.

Proposition 2.5. *Let $L \subseteq [0,1]^n$, and assume that the $\alpha_{i,j}$ are the seams of L. Then*

$$\begin{aligned}L^\circ &= \bigcap_{1\le i<j\le n} \pi_{i,j}^{-1}(L_{i,j}^\circ)\\ &= \{x \in [0,1]^n : \alpha_{i,j}(x_j) > x_i \textit{ for all } i,j\}.\end{aligned}$$

Theorem 2.6. *Let $L \subseteq \mathbf{R}^n$ be a closed subdirect product of $[0,1]^n$, and let the $\wedge$-seams of L be given by the $\alpha_{i,j}$'s. Then the following conditions are equivalent*:

1. *L is a full sublattice of $\mathbf{R}^n$;*

2. *For all triples $i,j,k \in \{1,\dots,n\}$, $L_{i,j,k} = \pi_{i,j,k}(L)$ is a full sublattice of $\mathbf{R}^3$;*

3. *For all triples $i,j,k \in \{1,\dots,n\}$ we have*

(a) $\breve{\alpha}_{i,k}(r) \le \breve{\alpha}_{i,j} \circ \breve{\alpha}_{j,k}(r)$ *whenever* $0 \le r \le 1$;

(b) $r \le \breve{\alpha}_{i,j} \circ \breve{\alpha}_{j,i}(r)$ *for all* $0 \le r \le 1$ *and* $r < \breve{\alpha}_{i,j} \circ \breve{\alpha}_{j,i}(r)$ *for all* $0 < r < 1$.

Corollary 2.7. *Let $L \subseteq \mathbf{R}^n$ be a full sublattice that is a subdirect product of $[0,1]^n$. If the $\alpha_{i,j}$'s are the $\wedge$-seams of L, then each $\alpha_{i,j}$ is continuous at* 1.

3. Tangent lines to full sublattices of Euclidean 2-space.

Theorem 2.6 suggests that we should first study full sublattices of $\mathbf{R}^2$, and we will do this in this section. However, some of the propositions of this section will be stated and proved in a more general context. Throughout this section let $L \subseteq [0,1]^n$ be a full sublattice of $\mathbf{R}^n$ such that $(0,\dots,0),(1,\dots,1) \in L$. Let $(\alpha_{i,j})_{i,j\in\{1,\dots,n\}}$ be the $\wedge$-seams of L,and let $(\beta_{ij})_{i,j\in\{1,\dots,n\}}$ be the $\vee$-seams of L.

Lemma 3.1. *If $x \in \partial L$ belongs to the boundary of L, then there are indices $i,j \in \{1,\dots,n\}$ such that $\widehat{\alpha}_{i,j}(x_j) \le x_i \le \alpha_{i,j}(x_j)$.*

Proof. This follows immediately from Propositions 2.3 and 2.5. □

We make the following definition:

Definition 3.2. Let L be a full sublattice of $\mathbf{R}^n$. We say that a point $x \in \partial L$ is a $\mathcal{C}_1$-*point*, provided that there is a neighborhood U of

x and a continuous map $\phi : U \to \mathbf{R}$ such that

1. $\phi(y) = 0$ if and only if $y \in \partial L \cap U$,
2. ϕ is differentiable at x, and $\Delta\phi(x) \neq (0, \dots, 0)$.

Obviously, if x is a $\mathcal{C}_1$-point of ∂L, then there is a uniquely determined tangent plane to L passing through x. We will show that the set of $\mathcal{C}_1$-points of ∂L is dense in ∂L. Let us start with a lemma

Lemma 3.3. *Let $f : [a, b] \to \mathbf{R}$ be a function, and let $a < r_0 < b$ be given. If f is differentiable at r_0, then $\overset{\smile}{f}$ and $\overset{\frown}{f}$ are differentiable at r_0, and*

$$f'(r_0) = \overset{\smile}{f}{}'(r_0) = \overset{\frown}{f}{}'(r_0)$$

Proof. Without loss of generality, we may assume that $r_0 = f(r_0) = 0$, and, after replacing $f(r)$ by $f(r) - f'(0) \cdot r$, if necessary, that $f'(0) = 0$. We then would like to show that $\overset{\smile}{f}{}'(0) = 0$. Since f is differentiable at 0 and since $f'(0) = 0$ for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that $-\varepsilon|r| \leq f(r) \leq \varepsilon|r|$ whenever $|r| < \delta$. By the definition of $\overset{\smile}{f}$ this implies that $-\varepsilon|r| \leq \overset{\smile}{f}(r) \leq \varepsilon|r|$ whenever $|r| < \delta$. Hence it follows that $\overset{\smile}{f}{}'(0) = 0$. □

Lemma 3.4. *Let $f : [a, b] \to [c, d]$ be a continuous monotone function with upper adjoint g and lower adjoint d. Then $d(s) = \overset{\smile}{g}(s)$ and $g(s) = \overset{\frown}{d}(s)$ whenever $f(a) < s < f(b)$.*

Proof. Clearly, we are allowed to restrict our attention to the case where $f(a) = c$ and $f(b) = d$. In this case f is surjective and

$$g(s) = \sup\{r \in [a, b] : f(r) = s\}$$
$$d(s) = \inf\{r \in [a, b] : f(r) = s\}.$$

We conclude that $d \leq g$, hence $d \leq \overset{\smile}{g}$, since d is lower semicontinuous. Moreover, if $s' < s$, then $g(s') < d(s)$, since otherwise $s = f(d(s)) \leq$

$f(g(s')) = s'$. therefore for $f(a) < s < f(b)$ we have

$$\breve{g}(s) = \sup\{g(s') : s' < s\}$$
$$\leq d(s).$$

It follows that $d(s) = \breve{g}(s)$ whenever $f(a) < s < f(b)$. □

Lemma 3.5. *Let $f : [a, b] \to [c, d]$ be a monotone continuous function, and let $g(r) = \sup\{s : f(s) \leq r\}$ be the upper adjoint of f. Assume that g is differentiable at a point r_0 with $f(a) < r_0 < f(b)$ and that $g'(r_0) \neq 0$. Then f is differentiable at $g(r_0)$ and $f'(g(r_0)) = 1/g'(r_0)$.*

Proof. Let d be the lower adjoint of f. By the previous two lemmas we have

$$d(r_0) = \breve{g}(r_0)$$

and

$$d'(r_0) = g'(r_0).$$

Let $s_0 = g(r_0)$. Since g is differentiable at r_0, g is continuous at r_0 and therefore

$$\begin{aligned} d(r_0) &= \breve{g}(r_0) \\ &= \sup\{g(r) : r < r_0\} \\ &= g(\sup\{r : r < r_0\}) \\ &= g(r_0) \\ &= s_0. \end{aligned}$$

Since f is the lower adjoint of g, it follows that $f(s_0) = f(g(r_0)) \leq r_0$. If we had $f(s_0) < r_0$, then for all $f(s_0) < r < r_0$ we had $g(r_0) = g(f(g(r_0))) = g(f(s_0)) \leq g(r) \leq g(r_0)$, and therefore g is constant on the interval $[f(s_0), r_0]$. Hence it would follows that $g'(r_0) = 0$, contradicting our assumptions. Therefore, $f(s_0) = r_0$. Now let s_λ be any sequence converging to s_0, and assume that $s_\lambda \neq s_0$ for all λ. Let $r_\lambda = f(s_\lambda)$. Since f is continuous, it follows that $\lim_{\lambda\to\infty} r_\lambda = r_0$. Since g and d are the upper and lower adjoints of f, we conclude that $d(r_\lambda) - d(r_0) \leq s_\lambda - s_0 \leq g(r_\lambda) - g(r_0)$ which implies

$$\frac{r_\lambda - r_0}{g(r_\lambda) - g(r_0)} \leq \frac{f(s_\lambda) - f(s_0)}{s_\lambda - s_0} \leq \frac{r_\lambda - r_0}{d(r_\lambda) - d(r_0)}.$$

Hence

$$\lim_{n\to\infty} \frac{f(s_\lambda) - f(s_0)}{s_\lambda - s_0} = \frac{1}{g'(r_0)} \qquad \square$$

Lemma 3.6. *Let $L \subseteq [0,1]^2$ be a full sublattice such that $(0,0), (1,1) \in L$. Assume that $0 < r_0 < 1$. If $\alpha_{2,1}$ is differentiable at r_0, then $(r_0, \alpha_{2,1}(r_0))$ is a $\mathcal{C}_1$-point of ∂L.*

Proof. Let $s_0 = \alpha_{2,1}(r_0)$. For each $0 < \varepsilon$, let

$$U_\varepsilon = \{(r,s) : |r - r_0|, |s - s_0| < \varepsilon\}.$$

Since $\alpha_{2,1}$ is differentiable at r_0, it is continuous at r_0. Hence, $s_0 = \alpha_{2,1}(r_0) = \breve{\alpha}_{2,1}(r_0)$, and it follows that

$$\breve{\alpha}_{1,2}(s_0) = \breve{\alpha}_{1,2}\breve{\alpha}_{2,1}(r_0) > r_0.$$

Since $\breve{\alpha}_{1,2}$ is lower semicontinuous, there is a number $\varepsilon > 0$ such that $r_0 + \varepsilon < \breve{\alpha}_{1,2}(s_0 - \varepsilon)$. Clearly, $s_0 - \varepsilon < s_0 = \alpha_{2,1}(r_0) = \breve{\alpha}_{2,1}(r_0) \leq \breve{\alpha}_{2,1}(r_0 + \varepsilon)$, and it follows that

$$(r_\varepsilon, s_\varepsilon) = (r_0 + \varepsilon, s_0 - \varepsilon) \in L^\circ.$$

For every number $q < 0$ consider the line

$$r_q(t) = r_\varepsilon + t$$
$$s_q(t) = s_\varepsilon + qt.$$

Let $P_q(t) = (r_q(t), s_q(t))$. Then

$$A_q = \{t \in \mathbf{R} : P_q(t) \in L\}$$

is a closed interval containing 0 in its interior. Indeed, suppose that $t_1, t_2 \in A_q$, and suppose that $t_1 < t_2$. Then $P_q(t_1), P_q(t_2) \in L$. It follows that $P_q(t_1) \wedge P_q(t_2)$, $P_q(t_1) \vee P_q(t_2) \in L$, and, since L is connected, it follows the square Q with vertices $P_q(t_1)$, $P_q(t_2)$, $P_q(t_1) \wedge P_q(t_2)$ and $P_q(t_1) \vee P_q(t_2)$ belongs to L. But for every $t_1 < t < t_2$ we have $(r_q(t), s_q(t)) \in Q \subseteq L$. It follows that A_q is a closed interval. Moreover, $P_q(0) = (r_q(0), s_q(0)) = (r_\varepsilon, s_\varepsilon) \in L^\circ$, hence 0 belongs to the interior of A_q.

Next, we show that the function

$$\sigma : \{q \in \mathbf{R} : q < 0\} \to \mathbf{R}$$
$$q \mapsto \inf A_q$$

is continuous. First of all, this function is well defined, since for every $q < 0$ the negative number $2/q$ does not belong to A_q (note that $L \subseteq [0,1]^2$). Repeating our above argument involving the square Q, it is easy to see that

$$A_q^\circ = \{t : P_q(t) \in L^\circ\}.$$

Also,

$$\sigma(q) = \inf A_q^\circ.$$

Fix $q_0 < 0$ and $\delta > 0$. Then there is a number $t \in A_{q_0}^\circ$ such that $t < \sigma(q_0) + \delta$. Since L° is an open set, we can find a neighborhood V of q_0 such that $P_q(t) \in L^\circ$ for all $q \in V$. It follows that $\sigma(q) \leq t < \sigma(q_0) + \delta$ for all $q \in V$, and therefore σ is upper semicontinuous at q_0. It remains to show that σ is lower semicontinuous at q_0. Suppose not. Then there is a number $\delta > 0$ and a sequence $q_\lambda < 0$ converging to q_0 such that $\sigma(q_\lambda) \leq \sigma(q_0) - \delta$; without loss of generality we may assume that the sequence $(\sigma(q_\lambda))_\lambda$ converges to a number $r_0 \leq \sigma(q_0) - \delta$. Let

$$\begin{aligned}(r_\lambda, s_\lambda) &= (r_{q\lambda}(\sigma(q_\lambda)), s_{q_\lambda}(\sigma(q_\lambda))) \\ &= (r_\varepsilon + \sigma(q_\lambda), s_\varepsilon + q_\lambda \sigma(q_\lambda)).\end{aligned}$$

Then, by definition, $(r_\lambda, s_\lambda) \in L$. The sequence $(r_\lambda, s_\lambda)_\lambda$ converges to $(r_\varepsilon + r_0, s_\varepsilon + q_0 r_0) = (r_{q_0}(r_0), s_{q_0}(r_0))$, hence this element belongs to L, and therefore $r_0 \in A_{q_0}$. It follows that $\sigma(q_0) \leq r_0$, a contradiction.

Since $P_{-1}(t) = (r_0 + \varepsilon + t, s_0 - \varepsilon - t)$, it follows that $\inf\ A_{-1} = -\varepsilon$, i.e.,

$$\sigma(-1) = -\varepsilon.$$

In the next step, we verify that

$$q_1 \leq q_2 < 0 \Longrightarrow \sigma(q_2) \leq \sigma(q_1).$$

Indeed, assume that $\sigma(q_1) < \sigma(q_2)$. Let

$$r_1 = r_\varepsilon + \sigma(q_1), \qquad s_1 = s_\varepsilon + q_1\sigma(q_1)$$
$$r_2 = r_\varepsilon + \sigma(q_2), \qquad s_2 = s_\varepsilon + q_2\sigma(q_2).$$

Then $r_1 < r_2$ and, since $\sigma(q_1) < \sigma(q_2) \leq 0$, $s_2 < s_1$. Therefore, the rectangle with vertices $(r_1, s_1), (r_2, s_2), (r_1, s_2)$ and (r_2, s_1) belongs to L. The ray starting at $(r_\varepsilon, s_\varepsilon)$ and passing through (r_2, s_2) enters that rectangle at (r_2, s_2) and therefore intersects the interior of the rectangle. But then there is a $t < \sigma(q_2)$ so that $(r_{q_2}(t), s_{q_2}(t)) \in L$, contradicting the definition of $\sigma(q_2)$.

It follows that the function

$$\sigma^* : \{q : q > 0\} \to \{t : t < 0\}$$
$$q \mapsto \sigma(-q)$$

is monotone and continuous, and satisfies $\sigma^*(1) = -\varepsilon$. In the following, we will restrict this map to the closed interval $[1-r, 1+r]$, where r may be chosen in such a way that $\sigma^*(1-r) < -\varepsilon = \sigma^*(1) < \sigma^*(1+r)$. Let $I_1 = [1-r, 1+r]$ and $I_2 = [\sigma^*(1-r), \sigma^*(1+r)]$.

For a fixed number $t \in I_2$, let

$$\tau(t) = \sup\{q \in I_1 : q > 0 \text{ and } \sigma^*(q) = t\}.$$

Then

$$\tau(t) = -\frac{\alpha_{2,1}(r_\varepsilon + t) - s_\varepsilon}{(r_\varepsilon + t) - r_\varepsilon}$$
$$= -\frac{\alpha_{2,1}(r_\varepsilon + t) - s_\varepsilon}{t},$$

and τ is the upper adjoint of $q \mapsto \sigma^*(q)$. Moreover, τ is differentiable at $-\varepsilon = r_0 - r_\varepsilon$, and

$$\tau'(-\varepsilon) = -\frac{-\varepsilon\alpha'_{2,1}(r_0) - (\alpha_{2,1}(r_0) - s_\varepsilon)}{\varepsilon^2}$$
$$= -\frac{-\varepsilon\alpha'_{2,1}(r_0) - \varepsilon}{\varepsilon^2}$$
$$= \frac{\alpha'_{2,1}(r, 0) + 1}{\varepsilon}$$
$$> 0.$$

Since $\tau(-\varepsilon) = (\alpha_{2,1}(r_\varepsilon - \varepsilon) - s_\varepsilon)/\varepsilon = (\alpha_{2,1}(r_0) - (s_0 - \varepsilon))/\varepsilon = 1$, it follows from Lemma 3.5 that the function $q \mapsto \sigma(-q)$ is differentiable at $q = 1$, and hence $\sigma(q)$ is differentiable at $q = -1$.

Finally, let

$$\phi(r, s) = (r - r_\varepsilon) - \sigma((s - s_\varepsilon)/(r - r_\varepsilon)).$$

Then $\phi(r, s) = 0$ if and only if $(r, s) \in \partial L$, and ϕ is differentiable at (r_0, s_0). Moreover,

$$\begin{aligned} \Delta\phi(r_0, s_0) &= \left(1 + \sigma'\left(\frac{s_0 - s_\varepsilon}{r_0 - r_\varepsilon}\frac{s_0 - s_\varepsilon}{(r_0 - r_\varepsilon)^2}\right), -\sigma'\left(\frac{s_0 - s_\varepsilon}{r_0 - r_\varepsilon}\right)\frac{1}{r_0 - r_\varepsilon}\right) \\ &= (1 + \sigma'(-1)/\varepsilon, \sigma'(-1)/\varepsilon) \\ &\neq (0, 0). \qquad \square \end{aligned}$$

Theorem 3.7. *Let $L \subseteq \mathbf{R}^2$ be a full sublattice. Then the $\mathcal{C}_1$-points of ∂L are dense in ∂L.*

Proof. Let $x = (x_1, x_2) \in \partial L$. Then, using Lemma 3.1, we may assume that $\breve{\alpha}_{2,1}(x_1) \leq x_2 \leq \alpha_{2,1}(x_1)$. Assume first that $\alpha_{2,1}(x_1) = x_2 = \alpha_{2,1}(x_1)$. Then $\alpha_{2,1}$ is continuous at x_1. Using the monotonicity of $\alpha_{2,1}$, we can find a sequence $0 < r_\lambda < 1$ such that $\alpha_{2,1}$ is differentiable at r_λ and such that $\lim_{\lambda\to\infty} r_\lambda = x_1$. Then $(r_\lambda, \alpha_{2,1}(r_\lambda))$ is a $\mathcal{C}_1$-point of ∂L by Lemma 3.6, and $\lim_{\lambda\to\infty}(r_\lambda, \alpha_{2,1}(r_\lambda)) = (x_1, x_2) = x$.

Hence we may assume that $\alpha_{2,1}(x_1) < \alpha_{2,1}(x_1)$. Then we may pick a sequence s_λ so that $\breve{\alpha}_{2,1}(x_1) < s_\lambda < \alpha_{2,1}(x_1)$ so that $\lim_{\lambda\to\infty} s_\lambda = x_2$. For each such s_λ we have $\beta_{1,2}(s_\lambda) = x_1$, and $\beta_{2,1}$ is constant on a neighborhood of s_λ. Hence it follows that $(x_1, s_\lambda) \in \partial L$ is a $\mathcal{C}_1$-point, and the points of this form converge to $(x_1, x_2) = x$. $\square$

4. Tangent hyperplanes to full sublattices of $\mathbf{R}^n$. Let $L \subseteq [0,1]^n$ be a full sublattice of $\mathbf{R}^n$. For each $1 \leq i \leq n$, let

$$L^i = \{(x_1, \dots, \hat{x}_i \dots x_n) \in [0,1]^{n-1} : x \in L\}$$

and let

$$\pi^i : L \to L^i$$

be the canonical projection. (As usual, $(x_1, \dots, \hat{x}_i, \dots, x_n)$ abbreviates $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$.) Throughout this whole section, let $L \subseteq [0,1]^n$ be a full sublattice of Euclidean n-space. The $\wedge$-seams of L are given by the $\alpha_{i,j}$'s and the $\vee$-seams are given by the $\beta_{i,j}$'s. We plan to show that the $\mathcal{C}_1$-points are dense in the boundary of L. In a first step, we will show that we can restrict ourselves to points that have only a very restricted number of coordinates with 0's and 1's:

Proposition 4.1. *Let $x \in \partial L$. Then x is the limit of points $y \in \partial L$ such that either $0 < y_i < 1$ for all coordinates, or there is exactly one coordinate i_0 such that $y_{i_0} \in \{0, 1\}$.*

Proof. Let $x \in \partial L$ be given. We shall prove this result by induction on the total number of 0's and 1's in x. First, we will renumber the coordinates in such a way that there is a number $m_0 \le m_1 < n$ so that

$$x_i = 0 \iff i < m_0$$
$$x_i = 1 \iff m_0 \le i < m_1.$$

If $m_1 = 0$, then $0 < x_i < 1$ for all indices, and there is nothing to show. Similarly, if $m_1 = 1$, then $x_i \in \{0, 1\}$ if and only if $i = 1$, and the assertion of the proposition follows again trivially. Hence we may assume that $2 \le m_1$. We shall now reduce m_1 by at least 1.

Recall that for each index i we have

$$\delta_i(r) = (\beta_{1,i}(r), \dots, \beta_{n,i}(r))$$
$$\varepsilon_i(r) = (\alpha_{1,i}(r), \dots, \alpha_{n,i}(r)).$$

Since all the maps $\beta_{i,j}$ are continuous at 0 and all the maps $\alpha_{i,j}$ are continuous at 1 by Corollary 2.7, it follows that

$$\lim_{\lambda \to \infty} \delta_i(1/\lambda) = \bot,$$
$$\lim_{\lambda \to \infty} \varepsilon_i(1 - 1/\lambda) = \top.$$

Assume first that $m_0 < m_1$. Let

$$y^\lambda = x \wedge \varepsilon_{m_1 - 1}(1 - 1/\lambda).$$

Then the sequence y^λ converges to x. Moreover, eventually $y_i^\lambda = x_i$ for all $m_1 \le i$, and also for all λ, $0 < y_{m_1-1}^\lambda \le 1-1/\lambda < 1$. Hence for $m_0 \le i$ we have $0 < y_i^\lambda < 1$. Moreover $y_1^\lambda = \min\{x_1, \alpha_{1,m_1-1}(1-1/\lambda)\}$, hence if $m_0 > 1$ then $y_1^\lambda = x_1 = 0$ and therefore $y^\lambda \in \partial L$. On the other hand, if $m_0 = 1$, then $x_1 = 1$ and $y_1^\lambda = \alpha_{1,m_1-1}(1-1/\lambda) = \alpha_{1,m_1-1}(y_{m_1-1}^\lambda)$, and $y_\lambda \in \partial L$ in this case, too.

Now note that the total number of 0's and 1's in all the y^λ is strictly smaller than the total number 0's and 1's in x.

If $m_0 = m_1$, then x has no 1's but only 0's. In our above argument, we replace y_λ by

$$y^\lambda = x \vee \delta_{m_1-1}(1/\lambda).$$

Then note that $y_1^\lambda = \beta_{1,m_1-1}(1/\lambda) = \beta_{1,m_1-1}(y_{m_1-1}^\lambda)$ and therefore $y^\lambda \in \partial L$. As before, $\lim y^\lambda = x$, and the number of 0's of y^λ is decreased by at least 1 since $y_{m_1-1}^\lambda > 0$. This completes the induction step. □

Lemma 4.2. *If $L \subseteq \mathbf{R}^n$ is a full sublattice and if $x \in L\backslash\{\top\}$ such that $0 < x_i$ for all coordinates i, then there is a coordinate j such that $\alpha_{j,i}(x_i) > x_j$ for all $i \neq j$.*

Proof. Assume not. Then for each j there is a number i such that

$$\breve{\alpha}_{j,i}(x_i) \le x_j.$$

Since $x < \top$, there is an index k_0 such that $x_{k_0} < 1$. Since the assertion of the lemma is not true, there has to be $k_1 \neq k_0$ such that $\breve{\alpha}_{k_0,k_1}(x_{k_1}) \le x_{k_0}$. Moreover, it is true that $x_{k-1} < 1$ since otherwise we would have

$$\begin{aligned} 1 = \breve{\alpha}_{k_0,k_1}(1) &= \breve{\alpha}_{k_0,k_1}(x_{k_1}) \\ &\le x_{k_0} < 1. \end{aligned}$$

If the lemma were incorrect, we could continue in this way and find a sequence of indices k_λ such that

$$\begin{aligned} k_\lambda &\neq k_{\lambda+1} \\ x_{k_\lambda} &< 1 \\ \breve{\alpha}_{k_\lambda,k_{\lambda+1}}(x_{k_{\lambda+1}}) &\le x_{k_\lambda}. \end{aligned}$$

Since there is only a finite set of indices, there are numbers $\nu < \mu$ such that $k_\nu = k_\mu$. Since $k_\nu \neq k_{\mu+1}$, we even have $\nu + 1 < \mu$. We obtain

$$\begin{aligned} x_{k_\nu} &\geq \breve{\alpha}_{k_\nu, k_{\nu+1}} \circ \breve{\alpha}_{k_{\nu+1}, k_{\nu+2}} \circ \cdots \circ \breve{\alpha}_{k_{\mu-1}, k_\mu}(x_{k_\mu}) \\ &\geq \breve{\alpha}_{k_\nu, k_{\nu+1}} \circ \breve{\alpha}_{k_{\nu+1}, k_\mu}(x_{k_\mu}) \\ &= \breve{\alpha}_{k_\nu, k_{\nu+1}} \circ \breve{\alpha}_{k_{\nu+1}, k_\nu}(x_{k_\nu}) \\ &\geq x_{k_\nu} \end{aligned}$$

and hence

$$x_{k_\nu} = \breve{\alpha}_{k_\nu, k_{\nu+1}} \circ \breve{\alpha}_{k_{\nu+1}, k_\nu}(x_{k_\nu}).$$

Since L is a full sublattice, it follows from Theorem 2.6 that $x_{k_\nu} \in \{0, 1\}$. However, by construction $0 < x_{k_\nu} < 1$, a contradiction. $\square$

Lemma 4.3. *If $L \subseteq \mathbf{R}^n$ is a full sublattice, if $n \geq 3$, and if $x \in \partial L \backslash \{\top\}$ such that $0 < x_i$ for all coordinates i, then there is a coordinate j such that*

1. *$(x_1, \dots, \hat{x}_j, \dots x_n) \in \partial L^j$, and*

2. *$\breve{\alpha}_{j,i}(x_i) > x_j$ for all $j \neq i$, or, if this is not true, then $x_j < \breve{\beta}_{j,i}(x_i)$ for all $j \neq i$.*

Proof. Since $x < \top$ and $0 < x_i$ for all indices i, it follows from Lemma 4.2 that there is an index j_0 so that

$$\breve{\alpha}_{j_0, i}(x_i) > x_{j_0}, \qquad \forall\, i \neq j_0.$$

Since $x \in \partial L$, we conclude from Proposition 2.5 that there are indices i_0, i_1 so that

$$\breve{\alpha}_{i_0, i_1}(x_{i_1}) \leq x_{i_0}.$$

Obviously, $i_0 \neq j_0$. If we could find indices $i_0, i_1 \neq j_0$ so that $\breve{\alpha}_{i_0, i_1}(x_{i_1}) \leq x_{i_0}$, then Proposition 2.5 implies that $(x_1, \dots, \hat{x}_{j_0}, \dots, x_n) \in \partial L^{j_0}$. Hence we may assume that

$$(*) \qquad \breve{\alpha}_{k,l}(x_l) > x_k, \qquad \forall\, k, l \notin \{j_0\}, k \neq l.$$

Thus, necessarily $i_1 = j_0$.

If, for some $k_0 \neq j_0$ we had $\breve{\alpha}_{k_0,j_0}(x_{j_0}) > x_{k_0}$, then we would be able to conclude that $\breve{\alpha}_{k_0,l}(x_l) > x_{k_0}$ whenever $l \neq k_0$ (since this inequality would be true for $l = j_0$ by the choice of k_0 and would follow from $(*)$ for all other indices). Now $i_0 = k_0$ would lead to the contradiction

$$x_{i_0} = x_{k_0} < \breve{\alpha}_{k_0,j_0}(x_{j_0}) = \breve{\alpha}_{i_0,j_0}(x_{j_0}) = \breve{\alpha}_{i_0,i_1}(x_{i_1}) \leq x_{i_0}.$$

Therefore $i_0 \neq k_0$, i.e., $i_0, j_0 \notin \{k_0\}$. It follows that $\breve{\alpha}_{i_0,j_0}(x_{j_0}) = \breve{\alpha}_{i_0,i_1}(x_{i_1}) \leq x_{i_0}$, hence $(x_1, \ldots, \hat{x}_{k_0}, \ldots, x_n) \in \partial L^{k_0}$, and $j = k_0$ would be the index we were looking for.

We are left with the case where we have

$$\begin{aligned} \breve{\alpha}_{j_0,i}(x_i) &> x_{j_0}, && \forall\, i \neq j_0 \\ \breve{\alpha}_{i,j_0}(x_{j_0}) &\leq x_i, && \forall\, i \neq j_0 \\ \alpha_{k,l}(x_l) &> x_k, && \forall\, k, l \notin \{j_0\}, k \neq l. \end{aligned}$$

Note that this implies that $x_i < 1$ for all indices i. Hence we can repeat the above argument with the $\widehat{\beta}_{i,j}$'s in place of the $\breve{\alpha}_{i,j}$'s, employing the dual of Lemma 4.2. If Lemma 4.3 were not true, we would conclude that there is an index j_1 so that

$$\begin{aligned} \widehat{\beta}_{j_1,i}(x_i) &< x_{j_1}, && \forall\, i \neq j_1 \\ \widehat{\beta}_{i,j_1}(x_{j_1}) &\geq x_i, && \forall\, i \neq j_1 \\ \widehat{\beta}_{k,l}(x_l) &< x_k, && \forall\, k, l \notin \{j_1\}, k \neq l. \end{aligned}$$

Now we use Proposition 2.4 to obtain

$$\breve{\alpha}_{i,j_1}(x_{j_1}) > x_i, \qquad \forall\, i \neq j_1$$

from the first of the last three inequalities. This, and the second inequality for the $\breve{\alpha}_{i,j_0}$'s gives $j_0 \neq j_1$. Pick any index i with $j_0 \neq i \neq j_1$. Then we find the following two inequalities:

$$\begin{aligned} \breve{\alpha}_{i,j_0}(x_{j_0}) &\leq x_i \\ \widehat{\beta}_{j_0,i}(x_i) &< x_{j_0}. \end{aligned}$$

This last pair of inequalities contradicts Proposition 2.4. □

We now start the proof of the main result. The argument uses induction over the dimension n. Note that we took care of dimension 2 in Section 3. The following three propositions provide a base for the induction step.

Proposition 4.4. *Let $x \in \partial L$ be given, and assume that*

1. *For a certain $\varepsilon > 0$ and all indices $i > 1$ we have $\breve{\alpha}_{1,i}(r) = x_1$ whenever $x_i - \varepsilon \leq r \leq x_1$.*
2. $(x_2, \dots, x_n) \in (L^1)^\circ$.

Then x belongs to the closure of the $\mathcal{C}_1$-points of ∂L.

Proof. By assumption, the maps $\breve{\alpha}_{1,i}$ are constant on the interval $[x_i - \varepsilon, x_i]$, and hence for each $y_i \in \,]x_i - \varepsilon, x_i[$ we have

$$\breve{\alpha}_{1,i}(y_i) = \alpha_{1,i}(y_i) = x_1.$$

Moreover, the assumptions of the Proposition imply that $r \leq \breve{\alpha}_{i,1}\breve{\alpha}_{1,i}(r) = \breve{\alpha}_{i,1}(x_1)$ whenever $r \in [x_i - \varepsilon, x_i]$, hence

$$x_i \leq \breve{\alpha}_{i,1}(x_1).$$

For each index $i > 1$ pick an element x_i' such that $x_i - \varepsilon < x_i' < x_i$ such that $(x_2', \dots, x_n') \in (L^1)^\circ$, and let $x' = (x_1, x_2', \dots, x_n')$. Then for $i > 1$ we have $\alpha_{1,i}(x_i') = x_1$ and $\alpha_{i,1}(x_1) \geq x_i > x_i'$, hence $x' \in \partial L$. It suffices to show that points of the form x' are $\mathcal{C}_1$-points of ∂L. Fix such an element x', and let $\delta > 0$ be chosen such that $x_i' + \delta < \breve{\alpha}_{i,1}(x_1 - \delta)$ for all indices $i > 1$; such a δ exists, since the maps $\breve{\alpha}_{i,1}$ are lower semicontinuous. Define an open set

$$V = \{(y_2, \dots, y_n) \in (L^1)^\circ : x_i - \varepsilon < y_i < x_i' + \delta \text{ for all } i > 1\}$$

and let $U = \,]x_1 - \delta, x_1 + \delta[\, \times V$. We define

$$\varphi : U \to \mathbf{R}; \quad (y_1, \dots, y_n) \mapsto y_1 - x_1.$$

Then $\varphi(y) < 0$ implies $y_1 < x_1 = \breve{\alpha}_{1,i}(x_i - \varepsilon) = \breve{\alpha}_{1,i}(y_i)$ and $y_i < x_i' + \delta \leq \breve{\alpha}_{i,1}(x_1 - \delta) \leq \alpha_{i,1}(y_1)$, hence $y \in L^\circ$. Moreover, $\varphi(y) = 0$

implies $y_1 = x_1 = \breve{\alpha}_{1,i}(y_i)$ and $y_i \le x_i' + \delta \le \breve{\alpha}_{i,1}(x_1 - \delta) \le \breve{\alpha}_{i,1}(y_1)$, hence $y \in \partial L$. And $\varphi(y) > 0$ implies $y_1 > x_1 = \breve{\alpha}_{1,i}(y_i) = \alpha_{1,i}(y_i)$, hence $y \notin L$. Hence $\varphi(y) = 0$ if and only if $y \in \partial L \cap U$. Since $\Delta\varphi = (1, 0, \dots, 0)$, the given point x' is a $\mathcal{C}_1$-point. □

Proposition 4.5. *Let $x \in \partial L$ be given, and assume that*

1. $\widehat{\beta}_{1,i}(x_i) < x_1 < \breve{\alpha}_{1,i}(x_i)$ *for all* $i > 1$,
2. $\pi^1(x) = (x_2, \dots, x_n)$ *is a* $\mathcal{C}_1$*-point of* ∂L^1.

Then x is a $\mathcal{C}_1$-point of ∂L.

Proof. Pick numbers $r, s > 0$ so that

$$\widehat{\beta}_{1,i}(x_i) < r < x_1 < s < \breve{\alpha}_{1,i}(x_i) \qquad \forall\, i > 1.$$

Since $(x_2, \dots, x_n)$ is a $\mathcal{C}_1$-point of ∂L^1, we can find an open set $U^1 \subseteq \{(y_2, \dots, y_n) : y_2, \dots, y_n \in \mathbf{R}\}$ containing $(x_2, \dots, x_n)$ and a real-valued function $\phi^1 : U^1 \to \mathbf{R}$ such that

1. $\phi^1(y_2, \dots, y_n) = 0$ if and only if $(y_2, \dots, y_n) \in U^1 \cap \partial L^1$, and
2. $\Delta\phi^1(x_2, \dots, x_n)$ exists and $\Delta\phi^1(x_2, \dots, x_n) \ne (0, \dots, 0)$.

By making U^1 smaller if necessary, we may also assume that

3. If $(y_2, \dots, y_n) \in U^1 \cap [0,1]^{n-1}$, then $\widehat{\beta}_{1,i}(y_i) < r < s < \breve{\alpha}_{1,i}(y_i)$ for all $i > 1$.

Now let

$$U = \,]r, s[\, \times U^1,$$

and let

$$\phi : U \to \mathbf{R}$$
$$(y_1, \dots, y_n) \mapsto \phi^1(y_2, \dots, y_n).$$

Then

$$\Delta\phi(x_1, \dots, x_n) = (0, \Delta\phi^1(x_2, \dots, x_n)) \ne (0, 0, \dots, 0)$$

and

$$\begin{aligned}\phi(y_1, \dots, y_n) = 0 &\iff \phi^1(y_2, \dots, y_n) = 0 \\ &\iff (y_2, \dots, y_n) \in U^1 \cap \partial L^1 \\ &\iff (y_1, \dots, y_n) \in \,]r, s[\, \times (U^1 \cap \partial L^1).\end{aligned}$$

Hence it remains to show that

$$U \cap \partial L =]r, s[\times (U^1 \cap \partial L^1).$$

Indeed, $y = (y_1, \dots, y_n) \in U \cap \partial L$ implies that $r < y_1 < s$ and $(y_2, \dots, y_n) \in U^1 \cap L^1$. Assume, if possible, that $(y_2, \dots, y_n) \in (L^1)^\circ$, i.e., $(y_2, \dots, y_n) \in U^1 \cap (L^1)^\circ$. Then $(y_2, \dots, y_n) \in U^1 \cap [0,1]^{n-1}$ and hence (3) implies that $\widehat{\beta}_{1,i}(y_i) < r < y_1 < s < \breve{\alpha}_{1,i}(y_i)$ for all $i > 1$. Now $(y_2, \dots, y_n) \in (L^1)^\circ$ implies $0 < y_i < 1$ for all $i > 1$, and therefore $\widehat{\beta}_{1,i}(y_i) < y_1$ and Proposition 2.4 imply that $y_i < \breve{\alpha}_{i,1}(y_1)$. We conclude that

$$\begin{aligned} y_1 &< \breve{\alpha}_{1,i}(y_i), \qquad \forall\, i > 1 \\ y_i &< \breve{\alpha}_{i,1}(y_1), \qquad \forall\, i > 1. \end{aligned}$$

If both $i, j > 1$ and $i \neq j$, then $\breve{\alpha}_{i,j}(y_j) > y_i$ since $(y_2, \dots, y_n) \in (L^1)^\circ$, and therefore $\breve{\alpha}_{i,j}(y_j) > y_i$ for all indices $i \neq j$. It then follows from Proposition 2.5 that $(y_1, \dots, y_n) \in L^\circ$, contradicting the fact that $(y_1, \dots, y_n) \in \partial L$. Hence it had to be true that $(y_2, \dots, y_n) \in \partial L^1$, and thus $U \cap \partial L \subseteq]r, s[\times (U^1 \cap \partial L)$.

Conversely, if $y = (y_1, \dots, y_n) \in]r, s[\times (U^1 \cap \partial L^1)$, then, by definition, $y \in U$ and $(y_2, \dots, y_n) \in U^1 \cap [0,1]^{n-1}$. Moreover, it follows from (3) that $\beta_{1,i}(y_i) \leq \widehat{\beta}_{1,i}(y_i) < r < y_1 < s < \breve{\alpha}_{1,i}(y_i) \leq \alpha_{1,i}(y_i)$. Since $(y_2, \dots, y_n) \in \partial L^1 \subseteq L^1$, it is also true that $y_i \leq \alpha_{i,j}(y_j)$ whenever $i, j > 1$, and we conclude that $y \in L$. Since $y \in L^\circ$ would lead to the contradiction $(y_2, \dots, y_n) \in (L^1)^\circ$, we have $y \in \partial L$. Thus, $y \in U \cap \partial L$, whence $]r, s[\times (U^1 \cap \partial L^1) \subseteq U \cap \partial L$. □

Proposition 4.6. *Let $x \in \partial L$ be given, and assume that $\widehat{\beta}_{1,i}(x_i) < x_1 < \breve{\alpha}_{1,i}(x_i)$ for all $i > 1$. If the $\mathcal{C}_1$-points are dense in the boundary of L^1, then x belongs to the closure of $\mathcal{C}_1$-points in ∂L.*

Proof. First we show that $\pi^1(x) = (x_2, \dots, x_n) \in \partial L^1$. This is certainly the case if $x_i \in \{0, 1\}$ for some index $i > 1$. Hence we may assume that $0 < x_i < 1$ for all $2 \leq i \leq n$. Hence $\widehat{\beta}_{1,i}(x_i) < x_1$ is equivalent to $x_i < \breve{\alpha}_{i,1}(x_1)$ by Proposition 2.4. If $\pi^1(x)$ would belong to the interior of L^1, then we had $x_i < \breve{\alpha}_{i,j}(x_j)$ whenever $2 \leq i,\, j \leq n$, and hence this inequality would hold for all $1 \leq i,\, j \leq n$. We could conclude that $x \in L^\circ$ by Proposition 2.5, a contradiction.

Now let $\varepsilon > 0$ be given. We have to find a $\mathcal{C}_1$-point $y \in \partial L$ so that $|x_i - y_i| < \varepsilon$ for all indices i. As before, pick numbers r and s so that

$$\widehat{\beta}_{1,i}(x_i) < r < x_1 < s < \breve{\alpha}_{1,i}(x_i), \qquad \forall\, i > 1.$$

Consider the open neighborhood $U^1 = \{(y_2, \dots, y_n) \in L^1 : \widehat{\beta}_{1,i}(y_i) < r < s < \breve{\alpha}_{1,i}(y_i)\}$ of $(x_2, \dots, x_n)$. Then there is a $\mathcal{C}_1$-point $(y_2, \dots, y_n) \in U^1 \cap \partial L^1$ such that $|x_i - y_i| < \varepsilon$ for each $i > 1$. By construction, $y = (x_1, y_2, \dots, y_n)$ belongs to ∂L, and by Proposition 4.5 this element is a $\mathcal{C}_1$-point of ∂L. $\square$

Unfortunately, it is not true that we always have $\widehat{\beta}_{1,i}(x_i) < x_1 < \breve{\alpha}_{1,i}(x_i)$ for all $i > 1$, or, more generally, that there exists an index j so that $\widehat{\beta}_{j,i}(x_i) < x_j < \breve{\alpha}_{j,i}(x_i)$ for all $i \neq j$. However, since we are only interested in the closure of the $\mathcal{C}_1$-points, we can apply Proposition 4.1. Therefore, if $x \in \partial L$ is given, we may assume that either $0 < x_i$ for all coordinates, or, if this is not possible, that $x_i < 1$ for all coordinates; let us assume that $x_i < 1$ for all indices. If we work with this assumption, we can apply the dual statement of Lemma 4.2 in order to find an index j such that $\widehat{\beta}_{j,i}(x_i) < x_j$ for all $i \neq j$. After renumbering the coordinates, we may assume that $j = 1$. Hence we have

$$\widehat{\beta}_{1,i}(x_i) < x_1 \leq \alpha_{1,i}(x_i), \qquad \forall\, i > 1.$$

Proposition 4.7. *Assume that $x \in \partial L$ is given and that $\widehat{\beta}_{1,i}(x_i) < x_1 \leq \alpha_{1,i}(x_i)$ for all $i > 1$. Furthermore, assume that the $\mathcal{C}_1$-points of ∂L^1 are dense in ∂L^1. Then x belongs to the closure of the $\mathcal{C}_1$-points, or for every given $\varepsilon > 0$ there is an element x_1' such that*

1. $|x_1 - x_1'| < \varepsilon$,
2. $x_1' \notin \cup_{i=1}^n \{\alpha_{1,i}(x_i), \breve{\alpha}_{1,i}(x_i)\}$
3. *all the maps $\beta_{i,1}$ are continuous at x_1'.*

Proof. Clearly, if $\widehat{\beta}_{1,i}(x_i) < x_1' \leq \alpha_{1,i}(x_i)$, then the results of Section 1 show that $(x_1', x_2, \dots, x_n) \in L$. If $(x_2, \dots, x_n) \in \partial L^1$, then $(x_1', x_2, \dots, x_n) \in \partial L$ whenever $\widehat{\beta}_{1,i}(x_i) < x_1' \leq \alpha_{1,i}(x_i)$, and since monotone maps are continuous almost everywhere, we could satisfy conditions (1)–(3) of the proposition. Hence we will from now on assume that

$$(x_2, \dots, x_n) \in (L^2)^\circ.$$

If for some index $i > 1$ we had $\breve{\alpha}_{1,i}(x_i) < x_1$, then we could pick elements $\breve{\alpha}_{1,i}(x_i) < x_1' < x_1$, $|x_1 - x_1'| < \varepsilon$, and we had $(x_1', x_2, \dots, x_n) \in \partial L$. Again it would follow that we could satisfy conditions (1)–(3) of the proposition. Hence we may also assume that $x_1 \leq \breve{\alpha}_{1,i}(x_i)$ for all $i > 1$. Then, if we had $x_1 < \breve{\alpha}_{1,j}(x_j)$ for some index $j > 1$, then $\widehat{\beta}_{1,j}(x_j) < x_1 < \breve{\alpha}_{1,j}(x_j)$ and Lemma 2.4 would imply that $\widehat{\beta}_{j,1}(x_1) < x_j < \breve{\alpha}_{j,1}(x_1)$. Since $(x_2, \dots, x_n) \in (L^1)^\circ$, we could conclude that $\widehat{\beta}_{j,i}(x_i) < x_j < \breve{\alpha}_{j,i}(x_i)$ whenever $i \neq j$. Hence Proposition 4.6 would imply that x belongs to the closure of the $\mathcal{C}_1$-points of ∂L. We now can restrict our attention to the case where

$$x_1 = \breve{\alpha}_{1,i}(x_i), \qquad \forall\, i > 1.$$

Now pick $\delta > 0$ so small that $x_2 - \delta < x_2' \leq x_2$ implies $(x_2', x_3, \dots, x_n) \in (L^1)^\circ$ and $\widehat{\beta}_{1,j}(x_j) < \breve{\alpha}_{1,2}(x_2')$ for all j. The elements $(\breve{\alpha}_{1,2}(x_2'), x_2', x_3, \dots, x_n) \in \partial L$ approximate x for $x_2 - \delta < x_2' \leq x_2$. If we had $\breve{\alpha}_{1,2}(x_2') < \breve{\alpha}_{1,j}(x_j)$ for some index j, then we could again use Proposition 4.6 to show that $(\breve{\alpha}_{1,2}(x_2'), x_2', x_3, \dots, x_n)$ belongs to the closure of the $\mathcal{C}_1$-points of ∂L. These elements approximate x and hence x itself would belong to the closure of the $\mathcal{C}_1$-points. Hence we conclude that $\breve{\alpha}_{1,2}(x_2') = \breve{\alpha}_{1,3}(x_3) = x_1$ whenever $x_2 - \delta < x_2' \leq x_2$. Repeating this argument for the other coordinates, we conclude that there is a $\delta > 0$ such that $x_1 = \breve{\alpha}_{1,i}(x_i')$ whenever $x_i - \delta < x_i' \leq x_i$. Now Proposition 4.4 would imply that x belongs to the closure of the $\mathcal{C}_1$-points. □

Using this last proposition, we may assume without loss of generality that $x_1 \notin \cup_{i=1}^n \{\alpha_{1,i}(x_i), \breve{\alpha}_{1,i}(x_i)\}$ and that all the maps $\beta_{i,1}$ are continuous at x_1. We then can divide the indices into two classes: there are those indices i for which $\breve{\alpha}_{1,i}(x_i)$. Since Proposition 4.6

already completely exhausts the case where there is no index of the first type, we may assume that there is at least one index i for which the first inequality is true. We now renumber the indices in such a way that the indices of the first type come first.

Proposition 4.8. *Assume that for each $n' < n$ and each full sublattice $M \subseteq \mathbf{R}^{n'}$ the $\mathcal{C}_1$-points are dense in the boundary of M. Let $x \in \partial L$ be given and assume that there is an index $1 < m \leq n$ so that*

1. $\widehat{\beta}_{1,i}(x_i) < x_1$ *for all indices* $i > 1$,
2. $\breve{\alpha}_{1,i}(x_i) < x_1 < \alpha_{1,i}(x_i)$ *for all* i *with* $2 \leq i \leq m$
3. $x_1 < \breve{\alpha}_{1,i}(x_i)$ *whenever* $m < i \leq n$
4. $\beta_{i,1}$ *is continuous at* x_1 *for each* $i \leq n$.

Then x belongs to the closure of $\mathcal{C}_1$-points of ∂L.

Proof. The proof will be an induction on m. We start however with a few general remarks.

First, note that the continuity of the $\alpha_{i,j}$'s at 0 and 1 and (2) imply that

$$0 < x_i < 1, \qquad \forall\, 1 \leq i \leq m.$$

For each index i, let

$$b_i = \beta_{i,1}(x_1) = \widehat{\beta}_{i,1}(x_1).$$

Let $i \leq m$. Then, since $\breve{\alpha}_{1,i}(x_i) \leq x_1 \leq \alpha_{1,i}(x_i)$, it follows that $b_i = \beta_{i,1}(x_1) \leq x_i \leq \widehat{\beta}_{i,1}(x_1) = b_i$, hence

$$b_i = x_i, \qquad \forall\, i \leq m.$$

If $m < i \leq n$, then our assumption (3) and Proposition 2.4 imply

$$b_i = 0 = x_i \quad \text{or} \quad b_i < x_i, \qquad \forall\, m < i \leq n.$$

If $1 \leq j \leq m < i \leq n$, then $\widehat{\beta}_{i,j}(x_j) = \widehat{\beta}_{i,j}(b_j) = \widehat{\beta}_{i,j}\widehat{\beta}_{j,1}(x_1) \leq \widehat{\beta}_{i,1}(x_1) = b_i$, and it follows that

$$\widehat{\beta}_{i,j}(x_j) = 0 = x_i \quad \text{or} \quad \widehat{\beta}_{i,j}(x_j) < x_i, \\ \forall\, 1 \leq j \leq m < i \leq n.$$

Further, we have

$$x_j < \breve{\alpha}_{j,i}(x_i), \qquad \forall\, 1 \le j \le m < i \le n.$$

Indeed, if $\widehat{\beta}_{i,j}(x_j) < x_i$, then $x_i > 0$ and $0 < x_j < 1$, hence Proposition 2.4 gives the equivalent inequality $x_j < \breve{\alpha}_{j,i}(x_i)$. On the other hand, assume that $x_i = 0$ for some $m < i$. Then for $2 \le j \le m$ we have $\widehat{\beta}_{i,j}(x_j) = 0 = x_i$, thus also $\beta_{i,j}(x_j) = 0 \le x_i$, and it follows that $x_j \le \alpha_{j,i}(x_i) = \alpha_{j,i}(0) = \breve{\alpha}_{j,i}(x_i)$. If actually $x_j = \breve{\alpha}_{j,i}(x_i)$, we could conclude that $\breve{\alpha}_{1,j}(x_j) = \breve{\alpha}_{1,j}\breve{\alpha}_{j,i}(x_i) \ge \breve{\alpha}_{1,i}(x_i) > x_1$, a contradiction to Proposition 4.8 (2). Hence we have $x_j < \breve{\alpha}_{j,i}(x_i)$ even if $x_i = 0$. This inequality also holds for $j = 1$ by our hypothesis (3).

Moreover, since $0 < x_i < 1$ for $i \le m$, we also obtain from Proposition 2.4 and Proposition 4.8 (1) that

$$x_j < \breve{\alpha}_{j,1}(x_1), \qquad \forall\, 2 \le j \le m.$$

We now start our induction on m. First, assume that $m = 2$. Then for all $i > 2$ we have

$$\widehat{\beta}_{1,i}(x_i) < x_1 < \breve{\alpha}_{1,i}(x_i).$$

And for $i = 2$ we obtain

$$\widehat{\beta}_{1,2}(x_2) < \breve{\alpha}_{1,2}(x_2) < x_1 < \alpha_{1,2}(x_2).$$

Moreover, as discussed before, we have

$$x_2 < \breve{\alpha}_{2,i}(x_i), \qquad i \ne 2.$$

Now x_2 is the infimum of elements r_λ such that $\alpha_{1,2}$ is continuous at each r_λ and such that $r_\lambda < \breve{\alpha}_{2,i}(x_i)$ for $i \ne 2$. For $i = 1$, this last inequality is equivalent to $\widehat{\beta}_{1,2}(r_\lambda) < x_1$. For each λ, the element $(x_1, r_\lambda, x_3, \dots, x_n)$ belongs to L, and

$$\widehat{\beta}_{1,2}(r_\lambda) < x_1 < \alpha_{1,2}(x_2) \le \alpha_{1,2}(r_\lambda) = \breve{\alpha}_{1,2}(r_\lambda).$$

Hence, if those elements eventually belong to ∂L, then it follows from Proposition 4.6 that each of them is in the closure of the $\mathcal{C}_1$-points of L, hence $(x_1, \dots , x_n)$ belongs to the closure of the $\mathcal{C}_1$-points. Thus, we may assume that for each λ,

$$(x_1, r_\lambda, x_3, \dots , x_n) \in L^\circ.$$

Then there is a number $\varepsilon > 0$ so that

$$x_2 < x_2' < x_2 + \varepsilon \Longrightarrow (x_1, x_2', x_3, \dots , x_n) \in L^\circ.$$

Especially, all coordinates are strictly between 0 and 1. Moreover, we have

$$\widehat{\beta}_{i,j}(x_j) < x_i, \qquad i \neq 2, i \neq j.$$

Indeed, for $j \neq 2$, this inequality follows from $(x_1, x_2+\varepsilon/2, x_3, \dots , x_n) \in L^\circ$, and the same inclusion yields $\widehat{\beta}_{i,2}(x_2) \leq \widehat{\beta}_{i,2}(x_2 + \varepsilon/2) < x_i$. For a similar reason, we have

$$x_i < \breve{\alpha}_{i,j}(x_j), \qquad i \neq 2 \neq j, i \neq j.$$

If we actually had $x_{i_0} \leq \breve{\alpha}_{i_0,2}(x_2)$ for some $i_0 > 2$, then we could find a sequence $r_\lambda < x_{i_0}$ such that $\lim_{\lambda\to\infty} r_\lambda = x_{i_0}$ and such that

$$\widehat{\beta}_{i_0,i}(x_i) < r_\lambda < \breve{\alpha}_{i_0,i}(x_i), \qquad i \neq i_0.$$

Since $x_2 = \beta_{2,1}(x_1)$, we conclude that

$$(x_1, \dots , x_{i_0-1}, r_\lambda, x_{i_0+1}, \dots , x_n) \in \partial L$$

for all λ. It follows from Proposition 4.6 that each of those points belongs to the closure of the $\mathcal{C}_1$-points of ∂L, and hence x would belong to the closure of the $\mathcal{C}_1$-points. Hence, from now on we also may assume that

$$x_i > \breve{\alpha}_{i,2}(x_2), \qquad i \neq 2.$$

For each $i > 2$ we can approximate x_i from below by elements x_i' so that $(x_1, x_3', \dots , x_n') \in (L^2)^\circ$ and so that $x_2 < \breve{\alpha}_{2,i}(x_i')$ and $x_i' \leq \alpha_{i,2}(x_2)$.

The element $(x_1, x_2, x_3', \dots, x_n')$ then belongs to ∂L, and hence we may assume without loss of generality that

$$\breve{\alpha}_{i,2}(x_2) < x_i < \alpha_{i,2}(x_2), \qquad \forall\, i \neq 2.$$

Now pick $\varepsilon > 0$ so small that $|x_i - y_i| < \varepsilon$ and $i \neq 2$ imply that

1. $\breve{\alpha}_{i,2}(x_2) < y_i < \alpha_{i,2}(x_2)$,
2. $(y_1, y_3, \dots, y_n) \in (L^2)^\circ$.

Note that $\beta_{2,i}$ is constant on the open interval $]\breve{\alpha}_{i,2}(x_2), \alpha_{i,2}(x_2)[$. Hence (the dual of) Proposition 4.4 implies that y is a $\mathcal{C}_1$-point of ∂L. This completes the proof for $m = 2$.

We now proceed with the induction step. Let $m > 2$. Use Lemma 4.2 on the coordinates $2, \dots, m$ in order to find a coordinate $j_0 \in \{2, \dots, m\}$ so that for all $i \in \{2, \dots, m\} \backslash \{j_0\}$ we have $x_{j_0} < \breve{\alpha}_{j_0,i}(x_i)$. We renumber our coordinates so that $j_0 = m$. Utilizing one of our previous inequalities (the one saying that $x_j < \breve{\alpha}_{j,i}(x_i)$ for all $1 \leq j \leq m < i \leq n$), we obtain

$$x_m < \breve{\alpha}_{m,i}(x_i), \qquad \forall\, i \neq m.$$

We now can approximate x_m from above by elements y_m so that

1. $\widehat{\beta}_{1,m}(y_m) < x_1$,
2. $x_m < y_m < \breve{\alpha}_{m,i}(x_i)$ for all $i \neq m$,
3. $\alpha_{1,m}$ is continuous at y_m.

Then $x_1 < \alpha_{1,m}(x_m) \leq \alpha_{1,m}(y_m) = \breve{\alpha}_{1,m}(y_m)$. Moreover, $(x_1, \dots, x_{m-1}, y_m, x_{m+1}, \dots, x_n) \in L$, actually, since $x_2 = \beta_{2,1}(x_1)$, this point belongs to the boundary of L, and it satisfies the hypotheses of the proposition with $m-1$ instead of m. By the induction hypothesis, all the points belong to the closure of the $\mathcal{C}_1$-points. We finally conclude that $(x_1, \dots, x_n)$ belongs to the closure of the $\mathcal{C}_1$-points of ∂L. □

We finally have completed the proof by induction of

Theorem 4.9. *Let $L \subseteq \mathbf{R}^n$ be a full sublattice. Then the $\mathcal{C}_1$-points of ∂L are dense in ∂L.*

Proof. By the results of Section 3 the statement holds for $n = 2$. The results of this section, especially the part from Propositions 4.6 through 4.8 establish the induction step from dimension n to dimension $n + 1$. □

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