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QUASICONFORMAL MAPS AND POINCARÉ DOMAINS

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ABSTRACT. In this paper we examine the invariance of Poincaré domains under quasi-conformal maps which satisfy a global integrability condition on the Jacobian. We show directly that, under such a map, the image of a John domain or a domain satisfying a quasi-hyperbolic boundary condition is p-Poincaré for $p \geq p_0$, here p_0 depends on the quasi-conformal map. Corresponding results are provided by (q,p)-Poincaré domains. We also provide sufficient conditions for which "rooms and corridors" domains are mapped to Poincaré domains.

1. Introduction. In this paper we examine the invariance of Poincaré domains under quasi-conformal maps which satisfy a global integrability condition on the Jacobian. Sufficient geometric conditions for Poincaré domains abound. Here we consider certain classes of Poincaré domains which include John domains, domains satisfying a quasi-hyperbolic boundary condition, and "rooms and corridors domains."

Hurri [**13**] has shown that Poincaré domains are invariant under locally bi-Lipschitz maps. However, one can see from the Riemann mapping theorem alone that quasi-conformal maps need not preserve Poincaré domains. In fact, a simply connected domain formed by adjoining a sequence of "rooms and corridors" domains [**13**, Example 5.2] to the unit ball can be constructed so that it is not Poincaré for any exponent p. To ensure that the image domain is Poincaré, we utilize an integrability condition introduced by Astala and Koskela [**1**]. Namely, letting D and D^* be domains in $\mathbf{R}^n$ and $f : D \to D^*$ a quasi-conformal

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map, we consider the condition

$$\int_D J_f(x)^{1+\varepsilon}\,dx < \infty \tag{1.1}$$

for some $\varepsilon > 0$, where $J_f(x)$ is the Jacobian of f.

Condition (1.1) can be used in many cases to conclude that a given domain D^* is a Poincaré domain. For example, if D^* is a simply connected John domain in the plane, then the Riemann mapping function $f : B \to D^*$ from the unit disk B satisfies (1.1) with $D = B$, see [**17**]. Our results (see Section 3) now imply that D^* is a Poincaré domain in a certain exponent range with efficient bounds for the Poincaré constant. The necessary definitions and requisite background are stated in Section 2. We commence Section 3 with the result that John domains are mapped to Poincaré domains under quasi-conformal maps satisfying (1.1) and follow this with a corresponding result for domains satisfying a quasi-hyperbolic boundary condition. While Astala and Koskela's result guarantees that the image domain in each of these cases would at least satisfy a quasi-hyperbolic boundary condition, their method of proof is indirect and does not lead to geometric estimates. The methods employed in this paper generate estimates for the Poincaré constants of the image domain in terms of the geometry of the original domain, the quasi-conformality of the map, and the constant ε in (1.1). The invariance of a general class of Poincaré domains under global quasi-conformal maps is examined next. "Rooms and corridors domains," in particular, are contained in this class.

In Section 4 we treat the case of (q,p)-Poincaré domains. We show that if condition (1.1) is satisfied for $\varepsilon \geq \varepsilon_0 > 0$, then the image of a (p_2,p_1)-Poincaré domain is p-Poincaré. For this range of ε this result generalizes the theorems concerning John domains and domains satisfying a quasi-hyperbolic boundary condition. Finally we study a general class of maps related to local Lipschitz maps and prove corresponding results for the image of a (p_2,p_1)-Poincaré domain.

2. Definitions and preliminary results.

Notation. Throughout this paper, we let D, D^*, G and G^* be domains of Euclidean n-space $\mathbf{R}^n$, $n \geq 2$, with finite n-Lebesgue measure. We suppose that $p \in [1, \infty)$ and $q \in [1, \infty)$ unless otherwise stated. We use B^n for the unit ball, $\{x \in \mathbf{R}^n : |x| < 1\}$.

The space $L^p(D)$ is the set of Lebesgue measurable functions u on D for which $||u||^p_{L^p(D)} = \int_D |u(x)|^p\,dx < \infty$. Let $L^p_{\text{loc}}(D)$ denote the space of functions which are locally integrable of order p on D. The space of Lebesgue measurable functions on D with first distributional partial derivatives in $L^p(D)$ is denoted by $L^1_p(D)$. In the Sobolev space, $W^1_p(D) = L^p(D) \cap L^1_p(D)$ we use the norm $||u||_{W^1_p(D)} = ||u||_{L^p(D)} + ||\nabla u||_{L^p(D)}$. Here $\nabla u = (\partial_1 u, \dots, \partial_n u)$ is the distributional gradient of u. We let $W^1_{p,\text{loc}}(D)$ denote the space of functions that lie in $W^1_p(A)$ for every compact subset A of D.

The average of a function u over a domain D with finite Lebesgue measure $|D|$ is $u_D = (1/|D|)\int_D u(x)\,dx$. Let A be a set. The Euclidean distance from $x \in A$ to the boundary of A is written as $d(x, \partial A)$. We let $\text{dia}\,(A)$ denote the diameter of A. We write τQ for the cube with the same center as Q and dilated by a factor $\tau \geq 1$.

The abbreviation $a \lesssim b$ will be used whenever there is a positive constant c_1 which does not depend on a or b such that $a \leq c_1 b$. Similarly, $a \cong b$ means that there are positive constants c_1 and c_2 which do not depend on a or b such that $c_1 a \leq b \leq c_2 a$. We let $c(*, \dots, *)$ denote a constant which depends only on the quantities appearing in the parentheses.

We recall some definitions.

Denote the Jacobian matrix of f at x by $F(x)$ and its determinant by $J_f(x)$. Define

$$|f'(x)| = \sup_{\substack{h \in \mathbf{R}^n \\ |h|=1}} |F(x)h|.$$

A homeomorphism $f : D \to D^*$ is *K-quasi-conformal* if $f \in W^1_{n,\text{loc}}(D)$ and $|f'(x)|^n \leq K J_f(x)$ almost everywhere in D. From this definition of quasi-conformal, it is apparent that condition (1.1) is equivalent to condition (2.1),

$$\int_D |f'(x)|^{n+\delta}\,dx < \infty \tag{2.1}$$

for some $\delta > 0$. In the statements and proofs of theorems, we choose to use whichever condition simplifies the computations.

We begin with the definition of Gehring domains found in [**1**].

Gehring domains. A domain $D^* \subset \mathbf{R}^n$ is a *Gehring domain*, if for all $K \geq 1$ there is a number $\varepsilon = \varepsilon(K) > 0$ such that

$$\int_D |f'(x)|^{n+\varepsilon}\, dx < \infty$$

for each domain D and each K-quasi-conformal map $f : D \to D^*$.

We immediately follow with an equivalent definition, found in [**23**], which we use in this paper.

The domain $D \subset \mathbf{R}^n$ is a *Gehring domain* if and only if, for all $K \geq 1$, there exists a number $\eta = \eta(K) > 0$ such that

$$\int_D (J_g(x))^{-\eta}\, dx < \infty$$

for all K-quasi-conformal maps g defined on D.

Poincaré domains. Let $D \subset \mathbf{R}^n$ be a domain, and let $1 \leq p \leq q < \infty$. If there is a constant $\kappa = \kappa(p,q,D) < \infty$ such that

$$\inf_{a\in\mathbf{R}} ||u-a||_{L^q(D)} \leq \kappa ||\nabla u||_{L^p(D)} \tag{2.2}$$

whenever $u \in L^1_p(D)$, then D is a (q,p)-*Poincaré domain* and we write $D \in \mathcal{P}(q,p)$. If $q = p$, then we say D is a p-*Poincaré domain* and write $D \in \mathcal{P}(p)$. The (q,p)-Poincaré constant, $\kappa_{p,q}(D)$, is the smallest constant κ for which (2.2) holds. If $p = q$, then we write $\kappa_{p,q}(D) = \kappa_p(D)$. Here $1 \leq p \leq q \leq np/(n-p)$ whenever $1 \leq p < n$.

We mention the following monotonicity result for Poincaré domains.

Theorem 2.3 [**10**, Theorem 1.8]. *If D is a p-Poincaré domain with $|D| < \infty$, then D is q-Poincaré whenever $q \geq p$. Explicitly,*

$$\kappa_q(D) \leq 2\frac{q}{p}\kappa_p(D),$$

for $1 \leq p \leq q < \infty$.

John domains. Let $0 < \alpha \leq \beta < \infty$. A domain D in $\mathbf{R}^n$ is called an (α, β)-*John domain*, if there is an $x_0 \in D$ such that every $x \in D$ can be joined to x_0 by a rectifiable path $\gamma : [0, d] \to D$ with arclength as parameter such that $\gamma(0) = x$, $\gamma(d) = x_0$, $d \leq \beta$ and

$$d(\gamma(t), \partial D) \geq \frac{\alpha}{d} t$$

for all $t \in [0, d]$. We write $D \in J(\alpha, \beta)$.

We shall later use the fact that John domains are Gehring domains, [**1**].

Domains satisfying a quasi-hyperbolic boundary condition. The quasi-hyperbolic distance between points x_1 and x_2 in D is given by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d(x, \partial D)},$$

where the infimum is taken over all rectifiable curves γ joining x_1 and x_2 in D, [**6**].

A domain D satisfies a *quasi-hyperbolic boundary condition*, abbreviated $D \in \mathrm{QHBC}\,(a)$, if there exists a point $x_0 \in D$ and a constant $a > 1$ such that

$$k_D(x_0, x) \leq a \log \left(1 + \frac{|x_0 - x|}{\min\{d(x_0, \partial d), d(x, \partial D)\}}\right)$$

for all $x \in D$.

John domains form a proper subclass of domains satisfying a quasi-hyperbolic boundary condition. As forementioned, we always assume that D has finite n-Lebesgue measure. Theorem 3.3 of [**14**] guarantees that if $D \in QHBC(a)$ and $|D| < \infty$, then D is bounded. The constant a in this definition will play a role in estimating the Poincaré constant in Theorem 3.15. We remark here that if D is bounded, then this definition of $\mathrm{QHBC}\,(a)$ gives exactly the same class of domains as those satisfying a quasi-hyperbolic boundary condition as originally defined in [**7**], see [**11**, Theorem 2.6] and [**13**, p. 25].

Gehring has established the following key result on the distortion of Lebesgue measure under quasi-conformal maps; the formulation is due to Reimann.

Theorem 2.4 [**19**]. *Suppose that $f : D \to D^*$ is a K-quasi-conformal map. Then there exists a $\tau = \tau(K, n)$ such that for every cube $Q \subset D$ with $\tau Q \subset D$ and every measurable set $A \subset Q$, we have*

$$\frac{|A^*|}{|Q^*|} \leq c\left(\frac{|A|}{|Q|}\right)^{\delta}.$$

Here $Q^* = f(Q)$, $A^* = f(A)$, $c = c(K, n)$ and $\delta = \delta(K, n)$.

The integrability condition (1.1) of Astala and Koskela [**1**] was first employed to exhibit the precise class of quasi-conformal maps under which quasi-hyperbolic boundary condition domains are invariant.

Theorem 2.5 [**1**]. *Let $D \subset \mathbf{R}^n$ with $|D| < \infty$ satisfy a quasi-hyperbolic boundary condition, and let $f : D \to D^*$ be a K-quasi-conformal mapping. Then $\int_D J_f(x)^{1+\varepsilon}\, dx < \infty$ for some $\varepsilon = \varepsilon(K, n) > 0$ if and only if D^* satisfies a quasi-hyperbolic boundary condition.*

3. p-Poincaré domains. It is well known that John domains are p-Poincaré domains for $p \geq 1$ [**15**], and domains satisfying a quasi-hyperbolic boundary condition are p-Poincaré for $p > p' = n - c_0$, where $c_0 = c_0(a, n) > 0$ [**13**, Theorem 7.12] and [**20**, Corollary 1]. By Astala and Koskela's result, Theorem 2.5, the image of a John domain under a quasi-conformal map with condition (2.1) must at lest satisfy a quasi-hyperbolic boundary condition. From this alone, one can deduce that the image would be p-Poincaré for $p \geq n$. The proof in [**1**] involves establishing the equivalence of the global integrability condition with that of the local Lipschitz function problem studied by Gehring and Martio [**7**]. The techniques they employ do not generate estimates for a quantitative geometric description of the image domain.

In Theorem 3.1 we show directly that the image of a John domain under a quasi-conformal map which satisfies (2.1) is p-Poincaré for $p > p_0$, where the lower bound for the constant p_0 satisfies $p_0 < n$. In an effort to produce efficient bounds for Poincaré constants, we prove each of the cases $p > n$, $p = n$ and $p_0 < p < n$ separately. The first can be proved by relying mainly on a (q, p)-Poincaré estimate for John domains. In the latter two cases we utilize the integral representation theorem for John domains [**15**]. In the final case we use the additional fact that John domains are Gehring. Note that, to prove the image domain is p-Poincaré for $p > p_0$, without regard to Poincaré constants, the final case would be sufficient. We also remark that the image domain here need not itself be John, see Example 3.12.

Theorem 3.1. *Suppose that D and D^* are domains in $\mathbf{R}^n$ such that $D \in J(\alpha, \beta)$ for some α and β, $0 < \alpha \le \beta < \infty$, $D^* = f(D)$ where $f : D \to D^*$ is a K-quasi-conformal map and $\int_D |f'(x)|^{n+\varepsilon}\, dx \le M < \infty$ for some $\varepsilon > 0$. Then $D^* \in \mathcal{P}(p)$ for $p > n$ with Poincaré constant*

$$\kappa_p(D^*) \le c(\varepsilon, n, p) K^{2/p} M^{1/(n+\varepsilon)} \left(\frac{\beta}{\alpha}\right)^{1+n\varepsilon/(p(n+\varepsilon))} |D|^{\varepsilon/(n(n+\varepsilon))}.$$

Further, D^* is n-Poincaré with Poincaré constant

$$\kappa_n(D^*) \le c(n) K^{2/n} M^{1/(n+\varepsilon)} \left(\frac{\beta}{\alpha}\right)^{16n} |D|^{\varepsilon/(n(n+\varepsilon))}.$$

Finally, for the case $p_0 < p < n$, $D^* \in \mathcal{P}(p)$ with Poincaré constant

$$\kappa_p(D^*) \le c(\varepsilon, \eta, n, p)(\beta/\alpha)^{16n} K^{2/p} M^{n/(p(n+\varepsilon))} N^{(n-p)/(p\eta)} \cdot |D|^{\varepsilon/((n+\varepsilon)p)+1/n-(n-p+\eta)/(p\eta)}.$$

Here the constants $\eta > 0$ and N are the Gehring domain constants for D in $\int_D |f'(x)|^{-\eta}\, dx \le N < \infty$, and

$$p_0 = \max\left(n\left(1 - \frac{\varepsilon\eta}{(n+\varepsilon)(n+\eta)}\right), \frac{n+\eta}{1+\eta}\right).$$

Proof of Theorem 3.1. Let $u \in C^1(D^*)$. We will estimate the constant c in

$$\inf_{a\in\mathbf{R}} \left(\int_{D^*} |u(x) - a|^p\, dx\right)^{1/p} \le c\left(\int_{D^*} |\nabla u(x)|^p\, dx\right)^{1/p}.$$

Define $v : D \to \mathbf{R}^n$ as $v(x) = (u \circ f)(x)$, $x \in D$. Since $|(f^{-1})'(y)|^n \le KJ(y, f^{-1})$ almost everywhere for $y \in D^*$, we have

$$\text{(3.2)}\quad \int_{D^*} |u(x) - v_D|^p\, dx \frac{|J(y, f^{-1})|}{|(f^{-1})'(y)|^n}\, dy = K \int_D |v(x) - v_D|^p |f'(x)|^n\, dx.$$

To estimate the righthand side of (3.2), let $f'(x)^t$ denote the adjoint of $f'(x)$; here $f'(x)$ is the derivative of f at x. Then $\nabla v(x) = f'(x)^t \nabla u(f(x))$ and $|f'(x)^t| = |f'(x)|$. Thus,

$$
\begin{aligned}
\int_D |\nabla v(x)|^p |f'(x)|^{n-p}\,dx &= \int_D |\nabla u(f(x))|^p |f'(x)|^n\,dx \\
&\le K \int_D |\nabla u(f(x))|^p J(x,f)\,dx \qquad (3.3)\\
&= K \int_{D^*} |\nabla u(x)|^p\,dx.
\end{aligned}
$$

Now we need to calculate a bound for the constant c_1 in

$$
\int_D |v(x)-v_D|^p |f'(x)|^n\,dx \le c_1 \int_D |\nabla v(x)|^p |f'(x)|^{n-p}\,dx. \tag{3.4}
$$

First consider the case $p > n$. We prove (3.4) using Hölder's inequality in conjunction with a (q,p)-Poincaré estimate on John domains. First,

$$
\begin{aligned}
&\left(\int_D |v(x)-v_D|^p |f'(x)|^n\,dx\right)^{1/p} \qquad (3.5)\\
&\quad\le \left(\int_D |f'(x)|^{n+\varepsilon}\,dx\right)^{n/(p(n+\varepsilon))} \cdot \left(\int_D |v(x)-v_D|^{p(n+\varepsilon)/\varepsilon}\,dx\right)^{\varepsilon/(p(n+\varepsilon))}\\
&\quad\le M^{n/(p(n+\varepsilon))}\left(\int_D |v(x)-v_D|^{p(n+\varepsilon)/\varepsilon}\,dx\right)^{\varepsilon/(p(n+\varepsilon))}.
\end{aligned}
$$

From [**3**] we obtain

$$
\begin{aligned}
&\left(\int_D |v(x)-v_D|^{p(n+\varepsilon)/\varepsilon}\,dx\right)^{\varepsilon/(p(n+\varepsilon))} \qquad (3.6)\\
&\quad\le c(\varepsilon,n,p)\left(\frac{\beta}{\alpha}\right)^{1+n\varepsilon/(p(n+\varepsilon))} |D|^{1/n+\varepsilon/(p(n+\varepsilon))-\delta/p} \cdot \left(\int_D |\nabla v(x)|^{p/\delta}\,dx\right)^{\delta/p},
\end{aligned}
$$

since $D \in \mathcal{P}(p(n+\varepsilon)/\varepsilon, p/\delta)$, where $p(n+\varepsilon)/\varepsilon \leq (np/\delta)/(n-p/\delta)$ and $p(n+\varepsilon)/\varepsilon \geq p/\delta$, and $1 \leq p/\delta < n$ and $\delta \in (1,\infty)$. We specify how to choose δ later.

Again, by Hölder's inequality,

$$
\begin{aligned}
(3.7)\quad & \left(\int_D |\nabla v(x)|^{p/\delta}\,dx\right)^{\delta/p} \\
&= \left(\int_D |\nabla v(x)|^{p/\delta} |f'(x)|^{(n-p)/\delta} |f'(x)|^{(p-n)/\delta}\,dx\right)^{\delta/p} \\
&\leq \left(\int_D |\nabla v(x)|^{p} |f'(x)|^{n-p}\,dx\right)^{1/p} \\
&\quad \cdot \left(\int_D |f'(x)|^{(p-n)/(\delta-1)}\,dx\right)^{(\delta-1)/p} \\
&\leq |D|^{((n+\varepsilon)(\delta-1)-p+n)/(p(n+\varepsilon))} \left(\int_D |f'(x)|^{n+\varepsilon}\,dx\right)^{(p-n)/(p(n+\varepsilon))} \\
&\quad \cdot \left(\int_D |\nabla v(x)|^{p} |f'(x)|^{n-p}\,dx\right)^{1/p} \\
&\leq M^{(p-n)/(p(n+\varepsilon))} |D|^{(\delta-1)/p-1/(n+\varepsilon)+n/(p(n+\varepsilon))} \\
&\quad \cdot \left(\int_D |\nabla v(x)|^{p} |f'(x)|^{n-p}\,dx\right)^{1/p}.
\end{aligned}
$$

Combining (3.5)–(3.7) yields the constant in (3.4),

$$
c_1^{1/p} = c(\varepsilon,n,p) M^{1/(n+\varepsilon)} \left(\frac{\beta}{\alpha}\right)^{1+n\varepsilon/(p(n+\varepsilon))} |D|^{1/n-1/(n+\varepsilon)}.
$$

These estimates will hold when we choose δ such that $1 < p/n < \delta \leq \varepsilon/(n+\varepsilon) + p/n$. To complete the proof when $p > n$, we need only account for the occurrences of the constant K in (3.2) and (3.3).

Let $p = n$. To calculate the constant $\kappa_n(D^*)$ we apply the integral representation for v in a John domain D,

$$
(3.8)\qquad |v(x) - v_B| \leq c(n)(\beta/\alpha)^{16n} \int_D |x-y|^{1-n} |\nabla v(y)|\,dy;
$$

here $B = B(x_0, c(n)\alpha^4/\beta^5)$, x_0 is a John center of D, [**15**, Theorem 2.2 and Lemma 3.3]. We recall the potential theory estimate

$$\int_D (|x-y|^{1-n})^r \, dy \le c(n,r)(|D|^{1/n})^{n+(1-n)r} \tag{3.9}$$

for all $x \in D$ whenever $r < n/(n-1)$, [**8**, Lemma 7.12].

Using Hölder's inequality with exponents $(n, n/(n-1))$ and (3.8), we obtain

$$\begin{aligned}\int_D |v(x)-v_B|^n |f'(x)|^n \, dx \le c(n)&\bigg(\Big(\frac{\beta}{\alpha}\Big)^{16n}\bigg)^n \\ &\cdot \int_D \bigg(\int_D \Big(|x-y|^{1-n}\Big)^{sn/(n-1)} dy\bigg)^{n-1} \\ &\cdot \int_D (|x-y|^{1-n})^{(1-s)n} |\nabla v(y)|^n \, dy |f'(x)|^n \, dx,\end{aligned}$$

where $s < 1$ will be fixed later. Fubini's theorem, (3.9), and Hölder's inequality with $((n+\varepsilon)/\varepsilon, (n+\varepsilon)/n)$ yield

$$\begin{aligned}\int_D &|v(x) - v_B|^n |f'(x)|^n \, dx \\ &\le c(n,s)\bigg(\Big(\frac{\beta}{\alpha}\Big)^{16n}\bigg)^n |D|^{(1-s)(n-1)} \\ &\quad\cdot \int_D \bigg(\int_D (|x-y|^{1-n})^{(1-s)n(n+\varepsilon)/\varepsilon} \, dx\bigg)^{\varepsilon/(n+\varepsilon)} \\ &\quad\cdot \bigg(\int_D |f'(x)|^{n+\varepsilon} \, dx\bigg)^{n/(n+\varepsilon)} |\nabla v(y)|^n \, dy.\end{aligned}$$

We set $s = 1 - 1/(k(n-1))$, $k > 1$. Then

$$\begin{aligned}\int_D &|v(x) - v_B|^n |f'(x)|^n \, dx \\ &\le c(k,n)\bigg(\Big(\frac{\beta}{\alpha}\Big)^{16n}\bigg)^n M^{n/(n+\varepsilon)} |D|^{\varepsilon/(n+\varepsilon)} \int_D |\nabla v(y)|^n \, dy,\end{aligned}$$

whenever $\varepsilon > n/(k-1)$. Now choose $k > 1$ sufficiently large so that the preceding inequality is true and we have established a modified form of (3.4) for the case $p = n$. We need only invoke the standard estimate,

$$||v - v_D||_{L^p(D)} \le 2||v - c||_{L^p(D)}$$

for any $c \in \mathbf{R}$ to see that

$$\kappa_n(D^*) \leq c(k,n)(\beta/\alpha)^{16n} K^{2/n} |D|^{\varepsilon/(n(n+\varepsilon))} M^{1/(n+\varepsilon)}.$$

The case $p < n$. We use the fact that there exist $\eta > 0$ and N such that

$$\int_D |f'(x)|^{-\eta}\, dx \leq N < \infty. \tag{3.10}$$

We will show that there is a constant c such that

$$\int_D |v(x) - v_B|^p |f'(x)|^n\, dx \leq c \int_D |\nabla v(x)|^p |f'(x)|^{n-p}\, dx. \tag{3.11}$$

We set $\delta = (p(n-p))/(n-p+\eta)$ and choose an s such that

$$1 - \frac{n\varepsilon}{(n-1)(n+\varepsilon)p} < s < \frac{n(p-\delta-1)}{(n-1)(p-\delta)}.$$

Such an s exists provided that $p > n(1 - \varepsilon\eta/((n+\varepsilon)(n+\eta)))$. The integral representation and Hölder's inequality with $(p-\delta, (p-\delta)/(p-\delta-1))$ yield

$$\begin{aligned}
&\int_D |v(x)| - v_B|^p |f'(x)|^n\, dx \\
&\quad \leq c(n,p)\left(\frac{\beta}{\alpha}\right)^{16np} \\
&\qquad \cdot \int_D \left(\int_D |x-y|^{1-n} |\nabla v(y)|\, dy\right)^p |f'(x)|^n\, dx \\
&\quad \leq c(n,p)\left(\frac{\beta}{\alpha}\right)^{16np} \\
&\qquad \cdot \int_D \left(\int_D (|x-y|^{1-n})^{s(p-\delta)/(p-\delta-1)}\, dy\right)^{p(p-\delta-1)/(p-\delta)} \\
&\qquad \cdot \left(\int_D (|x-y|^{1-n})^{(1-s)(p-\delta)} |\nabla v(y)|^{p-\delta}\, dy\right)^{p/(p-\delta)} |f'(x)|^n\, dx \\
&\quad \leq c(\delta,n,p,s)\left(\frac{\beta}{\alpha}\right)^{16np} (|D|^{1/n})^{np(p-\delta-1)/(p-\delta)+(1-n)sp} \\
&\qquad \cdot \int_D \left(\int_D (|x-y|^{1-n})^{(1-s)(p-\delta)}\right. \\
&\qquad\qquad \left. \cdot |\nabla v(y)|^{p-\delta} |f'(x)|^{n(p-\delta)/p}\, dy\right)^{p/(p-\delta)} dx.
\end{aligned}$$

Here the constraint $p-\delta>1$ from Hölder's inequality is equivalent to $p>(n+\eta)/(1+\eta)$. We apply Minkowski's inequality, [**24**, p. 271], to obtain

$$\begin{aligned}\int_D |v(x)-v_B|^p|f'(x)|^n\,dx \\ &\le c(\delta,n,p,s)\left(\frac{\beta}{\alpha}\right)^{16np}(|D|^{1/n})^{np(p-\delta-1)/(p-\delta)+(1-n)sp} \\ &\quad\cdot\left(\int_D|\nabla v(y)|^{p-\delta}\left(\int_D(|x-y|^{1-n})^{(1-s)p}\right.\right. \\ &\qquad\qquad\left.\left.\cdot|f'(x)|^n\,dx\right)^{(p-\delta)/p}dy\right)^{p/(p-\delta)}.\end{aligned}$$

Here, by Hölder's inequality with $((n+\varepsilon)/\varepsilon,(n+\varepsilon)/n)$

$$\begin{aligned}\int_D(|x-y|^{1-n})^{(1-s)p}|f'(x)|^n\,dx \\ &\le c(\varepsilon,n,p)M^{n/(n+\varepsilon)}(|D|^{1/n})^{n\varepsilon/(n+\varepsilon)+(1-n)(1-s)p}.\end{aligned}$$

Thus,

$$\begin{aligned}\int_D|v(x)-v_B|^p|f'(x)|^n\,dx \\ &\le c(\delta,\varepsilon,n,p,s)\left(\frac{\beta}{\alpha}\right)^{16np}M^{n/(n+\varepsilon)}|D|^{\varepsilon/(n+\varepsilon)+p/n-p/(p-\delta)} \\ &\quad\cdot\left(\int_D|\nabla v(y)|^{p-\delta}\,dy\right)^{p/(p-\delta)}.\end{aligned}$$

Hölder's inequality with exponents $(p/(p-\delta),p/\delta)$ implies

$$\begin{aligned}\int_D|\nabla v(y)|^{p-\delta}\,dy \\ &=\int_D|\nabla v(y)|^{p-\delta}|f'(y)|^{(n-p)(p-\delta)/p}|f'(y)|^{(p-n)(p-\delta)/p}\,dx \\ &\le\left(\int_D|\nabla v(y)|^p|f'(y)|^{n-p}\,dy\right)^{(p-\delta)/p} \\ &\quad\cdot\left(\int_D|f'(y)|^{(p-n)(p-\delta)/\delta}\,dy\right)^{\delta/p}.\end{aligned}$$

Note that our choice of δ gives $(p-n)(p-\delta)/\delta = -\eta$. By combining the above inequalities, we obtain (3.11) with

$$c = c(\delta,\varepsilon,n,p,s)\left(\frac{\beta}{\alpha}\right)^{16np} M^{n/(n+\varepsilon)} N^{\delta/(p-\delta)} |D|^{\varepsilon/(n+\varepsilon)+p/n+p/(p-\delta)}$$

and hence

$$\kappa_p(D^*) \leq 2c(\varepsilon,\eta,n,p)\left(\frac{\beta}{\alpha}\right)^{16n} K^{2/p} M^{n/(p(n+\varepsilon))} N^{(n-p)/p\eta} \cdot |D|^{\varepsilon/((n+\varepsilon)p)+1/n-(n-p+\eta)/(p\eta)},$$

whenever

$$p > p_0 = \max\left(n\left(1-\frac{\varepsilon\eta}{(n+\varepsilon)(n+\eta)}\right), \frac{n+\eta}{1+\eta}\right). \qquad \square$$

Remark. We note that the case $p > n$ can be proved using the integral representation (3.8). The Poincaré constant estimate would then be

$$\kappa_p(D) \leq c(n)\left(\frac{\beta}{\alpha}\right)^{16n} K^{2/p} M^{1/(n+\varepsilon)} |D|^{\varepsilon(n+p)/(np(n+\varepsilon))}$$

which is not as sharp as the estimate given in Theorem 3.1 in the exponent of β/α.

For the calculation we fix $0 < s < n\varepsilon/((n-1)(n+\varepsilon)p) < 1$. The integral representation and Hölder's inequality with $(p, p/(p-1))$ yield

$$\begin{aligned}\int_D |v(x)-v_B|^p |f'(x)|^n\,dx \\ &\leq c(n,p)\left(\frac{\beta}{\alpha}\right)^{16np} \int_D |\nabla u(y)|^p |f'(y)|^{n-p} \\ &\quad \cdot \int_D (|x-y|^{1-n})^{ps} |f'(x)|^n\,dx\,dy \\ &\quad \cdot \left(\int_D (|x-y|^{1-n})^{p(1-s)/(p-1)} |f'(y)|^{(p-n)/(n-1)}\,dy\right)^{p-1}.\end{aligned}$$

Using Hölder's inequality with $((n+\varepsilon)/n, (n+\varepsilon)/\varepsilon)$ and with

$$\left(\frac{(n+\varepsilon)(p-1)}{p-n}, \frac{(n+\varepsilon)(p-1)}{(n+\varepsilon)(p-1)-(p-n)}\right)$$

we obtain

$$\int_D |v(x)| - v_B|^p |f'(x)|^n \, dx \\ \leq c(n,p)\left(\frac{\beta}{\alpha}\right)^{16np} M^{p/(n+\varepsilon)} |D|^{\varepsilon(n+p)/(n(n+\varepsilon))}.$$

The method which we used in the proof for Theorem 3.1 yields that $\kappa_p(D)$ will be as given above.

Martio and Sarvas [**16**] have shown that (α,β)-John domains are preserved under global quasi-conformal maps. However, as the next example demonstrates, the class of John domains need not be invariant under the quasi-conformal maps of Theorem 3.1.

Example 3.12. If $D \in J(\alpha,\beta)$ and $f : D \to D^*$ is a K-quasi-conformal mapping satisfying (1.1), then D^* need not be an (α',β')-John domain for any α' and β'. For example, let D^* be a domain as in [**13**, Example 7.15] and consider a Riemann map $f : B^2(0,1) \to D^*$. From [**13**] we know $D^* \in \mathcal{P}(p)$ for $p > 2 - c_1$, and D^* is not in $\mathcal{P}(p)$ whenever $p \in [1, 2-c_2)$, here $c_2 > c_1$. Thus, D^* is not an (α',β')-John domain for any α' and β', whereas $B^2(0,1)$ is clearly a John domain.

Hurri [**13**] has given estimates for the p-Poincaré constants of (α,β)-John domains. We consider the following John domain and its image and compute the p-Poincaré constants generated by Theorem 3.1.

Example 3.13. Let $\delta \in (0,1)$. Set $D = B^2 \backslash \{B(0,1/3) \cup [2/3,1]\}$ and define the conformal map $f : B^2 \to B^2$ such that

$$f(z) = \frac{z-1+\delta}{1+(-1+\delta)z}.$$

Both D and $f(D) = D^*$ are annuli with a slit removed. Since $D \subset B^2$, we have the estimate

$$\int_D |f'(z)|^{2+\varepsilon} \, dz \leq \frac{\pi(2-\delta)^{2+\varepsilon}}{\delta^{2+\varepsilon}} = M.$$

Hence by Theorem 3.1, $\kappa_p(D^*) \lesssim 1/\delta$ for $p \geq n = 2$.

For the case $p < 2$, we also employ the estimate

$$\int_D |f'(z)|^{-\eta}\, dz \leq \pi(2-\delta)^{-\eta}\delta^{-\eta}2^{2\eta} = N,$$

for any $\eta > 0$. From Theorem 3.1, we then have $\kappa_p(D^*) \lesssim (1/\delta)^{(4-p)/p}$ for $1 < p < 2$.

On the other hand, D^* is an (α^*, β^*)-John domain where $\alpha^* \approx \delta$ and $\beta^* \approx 1$. Theorem 8.5 of [**13**] provides the estimate

$$\kappa_p(D^*) \leq c(n,p)(\beta^*/\alpha^*)^{1+n/p}|D^*|^{1/n}$$

for $p \geq 1$. For the domain D^*, here, we would then obtain $\kappa_p(D^*) \lesssim (1/\delta)^{1+2/p}$. Thus the constant provided by Theorem 3.1 gives a sharper estimate than that given by Theorem 8.5 of [**13**], for all $p > 1$.

We conjecture that, in general, Theorem 8.5 of [**13**] could be improved upon to give the estimate $\kappa_p(D) \leq c(n,p)(\beta/\alpha)|D|^{1/n}$ for an arbitrary (α, β)-John domain D.

Smith and Stegenga proved that all domains satisfying a quasi-hyperbolic boundary condition are n-Poincaré, but this fails if n is replaced by anything smaller [**21**]. In particular, the class of QHBC$\,(a)$ domains is p-Poincaré for $p > p'$, where p' satisfies $p' \to n$ as $a \to \infty$. In the next example, we show that their result implies the constant p_0 of Theorem 3.1 must satisfy $p_0 \to n$ as $\varepsilon \to 0$, thus making our estimate for p_0 asymptotically sharp. The authors would like to thank the referee for suggesting this example.

Example 3.14. Let D be the unit disk, and let f be a univalent analytic function on D which extends to be Hölder continuous with exponent α on the closed disk. Then $D^* = f(D) \in$ QHBC$\,(a)$, where $a = a(\alpha)$ satisfies $a \to \infty$ as $\alpha \to 0$, [**2, 21**]. By the result of Smith and Stegenga, $D^* \in \mathcal{P}(p)$ for $p > p'$, where $p' \to n$ as $\alpha \to 0$.

However, Theorem 3.1 can also be applied. Theorem 4.4 of [**1**] shows that $\int_D |f'(x)|^{2+\varepsilon}\, dx < \infty$ for some $\varepsilon > 0$, where $\varepsilon \to 0$ as $\alpha \to 0$. Also, the Koebe distortion theorem [**18**] guarantees that

$\int_D |f'(x)|^{-\eta}\, dx < \infty$, for any $\eta < 1$. Thus, Theorem 3.1 asserts that $D^* \in \mathcal{P}(p)$ for $p > p_0 = p_0(n, \varepsilon, \eta)$. By the preceding discussions, p_0 must satisfy $p_0 \to n$ as $\varepsilon \to 0$.

We present a theorem for domains satisfying a quasi-hyperbolic boundary condition, Theorem 3.15. The proof is similar to the proof for Theorem 3.1 and thus we will omit it.

Theorem 3.15. *Suppose that D and D^* are domains in $\mathbf{R}^n$ such that $D \in \mathrm{QHBC}\,(a)$, $D^* = f(D)$ where $f : D \to D^*$ is a K-quasi-conformal map and $\int_D |f'(x)|^{n+\varepsilon}\, dx \le M < \infty$ for some ε. Then $D^* \in \mathcal{P}(p)$ for $p > n$ with Poincaré constant*

$$\kappa_p(D^*) = c(\varepsilon, a, n, p) K^{2/p} M^{1/(n+\varepsilon)} |D|^{\varepsilon/(n(n+\varepsilon)) + \varepsilon(n-\lambda)(1-a)/(2ap(n+\varepsilon))}.$$

Here λ is the constant associated with the Whitney cube #-number condition for D, $\#\{Q \,|\, |Q|^{1/n} \approx 2^{-i}|D|^{1/n}\} \le c(D) 2^{\lambda i}$.

We recall the following decomposition theorem on Poincaré domains from [**13**, Theorem 4.11].

Theorem 3.16 [**13**]. *Let a domain G be the union of domains $D_i \in \mathcal{P}_p$ with $\kappa_p(D_i) \le c_1 < \infty$, $i = 1, 2, \dots$, such that each domain D_i lies in a cube Q_i with the following three properties.*

There are constants N and s such that

$$\sum_{j=1}^{\infty} \chi_{Q_j}(x) \le N_{\chi \cup_{j=1}^{\infty} Q_j}(x) \tag{3.17}$$

for all $x \in \mathbf{R}^n$,

$$Q_i \subset sQ_j \tag{3.18}$$

where $j = 1, 2, \dots, i$, and

$$\kappa_p(D_i)^p |Q_i| \le c_1 \min\{|D_i \cap D_{i-1}|, |D_i \cap D_{i+1}|\}. \tag{3.19}$$

Then G is a p-Poincaré domain.

This result led to a precise characterization of "rooms and corridors" domains, [**13**, 5.9]; thus, we are especially interested to see how the domains of this theorem would be affected by a quasi-conformal map. Note that since the standard techniques involved in such decomposition theorems involve a covering argument, we must consider global quasi-conformal maps, $f : \mathbf{R}^n \to \mathbf{R}^n$, which would then be defined on all of the cubes Q_i in the cover as well.

Theorem 3.20. *Suppose that $G = \cup_{i=1}^{\infty} D_i$ is a p-Poincaré domain by means of Theorem* 3.16 *above. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a K-quasi-conformal map, and denote $f(D_j) = D_j^*$. Suppose that, for some q and for a fixed $\delta = \delta(K,n)$ and for all $D_j^* = f(D_j)$,*

$$\kappa_p(D_j)^{-p/\delta} \kappa_q(D_j^*)^q \le c_2. \tag{3.21}$$

Then $f(G) = G^*$ is a q-Poincaré domain.

Proof of Theorem 3.20. For our domain G^* we use the decomposition $G^* = \cup_{i=1}^{\infty} D_i^*$, here each $D_i^* \subset Q_i^* = f(Q_i)$. Note that now whereas Q_i^* need not be a cube, it still possesses key geometric properties as the quasi-conformal image of a cube. We follow the method of [**13**, Theorem 4.11] and establish the corresponding conditions of (3.17)–(3.19) for G^*. Since f is a homeomorphism,

$$\sum \chi_{f(Q_j)}(x) \le N \chi_{\cup_{j=1}^{\infty} f(Q_j)}(x) \tag{3.22}$$

follows directly. We prove that the image domains Q_j^* have the engulfing property:

$$Q_i^* \subset s' Q_j^*, \qquad s' = s'(K,n,s), \quad j = 1, 2, \dots, i. \tag{3.23}$$

We first fix i and then fix $j \le i$.

Condition (3.18) states $Q_i \subset sQ_j$ for $j = 1, 2, \dots, i$. Inner and outer cubes comparable in size to Q_j^*, Q_i^* and $(sQ_j)^*$ are defined using ideas from [**5, 19**] and [**22**].

We work with concentric cubes P, P_j, S_j with $S_j \subset Q_j^* \subset P_j$ and $(sQ_j)^* \subset P$ which satisfy

$$|P_j| \le c_3 |Q_j^*| \le c_4 |S_j| \tag{3.24}$$

and

$$|P| \le c_3 |(sQ_j)^*|. \tag{3.25}$$

We also use the concentric cubes P_i and S_i for which $S_i \subset Q_i^* \subset P_i$ and

$$|P_i| \le c_3 |Q_i^*| \le c_4 |S_i|. \tag{3.26}$$

Here c_3 and c_4 depend only on K and n, and we may assume that $P_i, P_j \subset P$. Also note that the A_∞-measure result of Theorem 2.4 applied to f^{-1} gives

$$\frac{|Q_j^*|}{|(sQ_j)^*|} \ge \left(\frac{1}{c}\right)^{1/\delta} \left(\frac{1}{s}\right)^{n/\delta} = c(K, n, s). \tag{3.27}$$

Since P and P_j are concentric, and $|P_j| \le |P|$, and

$$|P_j| \ge |Q_j^*| \ge c(K, n, s)|(sQ_j)^*| \ge c_3 c(K, n, s)|P|$$

by (3.25) and (3.27), we can see that there is $\sigma = \sigma(K, n, s)$ for which $P \subset \sigma P_j$.

Moreover, there is a $\tau = \tau(K, n)$ such that $P_j \subset \tau S_j$. Hence, $\sigma\tau S_j$ satisfies

$$Q_i^* \subset P_i \subset \sigma P_j \subset \sigma\tau S_j \subset \sigma\tau Q_j^*.$$

Thus, $Q_i^* \subset s' Q_j^*$, here $s' = s'(K, n, s)$ and $j = 1, \dots, i$.

Using Reimann's result, Theorem 2.4, for the K-quasi-conformal map f^{-1}, we determine the condition corresponding to (3.19) which G^* satisfies. We rewrite (3.19) as

$$\min\left\{\frac{|D_i \cap D_{i-1}|}{|Q_i|}, \frac{|D_i \cap D_{i+1}|}{|Q_i|}\right\} \ge \frac{(\kappa_p(D_i))^p}{c_1}.$$

Then we obtain

$$\min\left\{\frac{|f(D_i \cap D_{i-1})|}{|f(Q_i)|}, \frac{|f(D_i \cap D_{i+1})|}{|f(Q_i)|}\right\} \ge \left(\frac{1}{cc_1}\right)^{1/\delta} (\kappa_p(D_i))^{p/\delta}, \tag{3.28}$$

where $c = c(K,n)$ and $\delta = \delta(K,n)$ are the constants from Theorem 2.4.

We need next to examine Boman's theorem [**4**] for these domains Q_j^*. Let $\mathcal{F}^* = \{Q_j^*\}$ and $1 < q < \infty$. We prove

$$\left\| \sum_{Q_j^* \in \mathcal{F}^*} a_{Q_j^*} \chi_{s'Q_j^*} \right\|_{L^q(\mathbf{R}^n)} \le c(s')^n \left\| \sum_{Q_j^* \in \mathcal{F}^*} a_{Q_j^*} \chi_{Q_j^*} \right\|_{L^q(\mathbf{R}^n)}, \tag{3.29}$$

for any coefficients $a_{Q_j^*}$ in $\mathbf{R}$. Consider the conjugate exponent q' for q. We use the fact that $||M\varphi||_{q'} \le c_{q'} ||\varphi||_{q'}$, here $M\varphi$ stands for the standard Hardy-Littlewood maximal function.

Consider any $\varphi \in L_{q'}(\mathbf{R}^n)$. With only minor modifications, we can follow Boman's proof

$$\begin{aligned}
\left| \int_{\mathbf{R}^n} \sum_{Q_j^*} a_{Q_j^*} \chi_{s'Q_j^*} \varphi \, dx \right| &= \left| \sum_{Q_j^*} a_{Q_j^*} (s')^n |Q_j^*| \frac{1}{|s'Q_j^*|} \int_{s'Q_j^*} \varphi \, dx \right| \\
&\le (s')^n \left| \sum_{Q_j^*} a_{Q_j^*} |Q_j^*| c_3 \frac{1}{|s'P_j|} \int_{s'P_j} \varphi \, dx \right| \\
&\le (s')^n \left| \sum_{Q_j^*} a_{Q_j^*} c_3 \int_{Q_j^*} M\varphi \, dx \right| \\
&\le (s')^n c_3 \left| \int_{\mathbf{R}^n} \sum_{Q_j^*} a_{Q_j^*} Q_j^* M\varphi \, dx \right| \\
&\le (s')^n c_3 \left\| \sum a_{Q_j^*} \chi_{Q_j^*} \right\|_q ||M\varphi||_{q'} \\
&\le (s')^n c_3 c_{q'} \left\| \sum a_{Q_j^*} \chi_{Q_j^*} \right\|_q ||\varphi||_{q'}
\end{aligned}$$

and (3.29) follows. Now we make use of conditions (3.22), (3.23), (3.28) and (3.29) to show that G^* is q-Poincaré. We have

$$
\begin{aligned}
(3.30)\qquad \int_{G^*} |v(y) - v_{D_1^*}|^q\,dy
&\le \sum_{i=1}^{\infty} \int_{D_i^*} |v(y) - v_{D_1^*}|^q\,dy \\
&\le 2^{q-1} \sum_{i=1}^{\infty} \int_{D_i^*} |v(y) - v_{D_i^*}|^q\,dy \\
&\quad + 2^{q-1} \sum_{i=1}^{\infty} \int_{D_i^*} \bigg(\sum_{j=1}^{i-1} |v_{D_j^*} - v_{D_{j+1}^*}| \bigg)^q\,dy \\
&\le 2^{q-1} \sum_{i=1}^{\infty} (\kappa_q(D_i^*))^q \int_{D_i^*} |\nabla v(y)|^q\,dy \\
&\quad + 2^{q-1} \sum_{i=1}^{\infty} \int_{D_i^*} \bigg(\sum_{j=1}^{i-1} |v_{D_j^*} - v_{D_{j+1}^*}| \bigg)^q\,dy.
\end{aligned}
$$

To estimate the last term,

$$
\begin{aligned}
|v_{D_j^*} - v_{D_{j+1}^*}|^q &= \frac{1}{|D_j^* \cap D_{j+1}^*|} \\
&\quad \cdot \int_{D_j^* \cap D_{j+1}^*} |v_{D_j^*} - v_{D_{j+1}^*}|^q\,dy \\
&\le \frac{2^{q-1}}{|D_j^* \cap D_{j+1}^*|} \sum_{h=j}^{j+1} \int_{D_h^*} |v(y) - v_{D_h^*}|^q\,dy.
\end{aligned}
$$

Now we apply condition (3.28),

$$
\begin{aligned}
|v_{D_j^*} - v_{D_{j+1}^*}|^q &\le 2^{q-1} (cc_1)^{1/\delta} \kappa_p(D_j)^{-p/\delta} \\
&\quad \cdot \sum_{h=j}^{j+1} \frac{1}{|Q_h^*|} \int_{D_h^*} |v(y) - v_{D_h^*}|^q\,dy \\
&\le 2^{q-1} (cc_1)^{1/\delta} \kappa_p(D_j)^{-p/\delta} \kappa_q(D_j^*)^q \\
&\quad \cdot \sum_{h=j}^{j+1} \frac{1}{|Q_h^*|} \int_{D_h^*} |\nabla v(y)|^q\,dy.
\end{aligned}
$$

The engulfing property in condition (3.23) and the following inequality

$$\sum_{j=1}^{i-1}(t_j+t_{j+1})^{1/q} \le 2\sum_{j=1}^{i} t_j^{1/q}, \qquad t_j \ge 0,$$

yield

$$\begin{aligned}\sum_{j=1}^{i-1}|v_{D_j^*}-v_{D_{j+1}^*}|\chi_{D_j^*}(x) &\le 4(cc_1)^{1/q\delta}\kappa_p(D_j)^{-p/q\delta}\kappa_q(D_j^*)\\ &\quad\cdot\sum_{j=1}^{i}\left(\frac{1}{|Q_j^*|}\int_{D_j^*}|\nabla v(y)|^q\,dy\chi_{s'Q_j^*}(x)\right)^{1/q}.\end{aligned}$$

Hence, by (3.29) and (3.22)

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{D_i^*}\left(\sum_{j=1}^{i-1}|v_{D_j^*}-v_{D_{j+1}^*}|\chi_{D_j^*}(x)\right)^q dx\\ &\quad\le 4^q(cc_1)^{1/\delta}\kappa_p(D_j)^{-p/\delta}\kappa_q(D_j^*)^q\\ &\qquad\cdot\int_{\mathbf{R}^n}\left(\sum_{j=1}^{\infty}\left(\frac{1}{|Q_j^*|}\int_{D_j^*}|\nabla v(y)|^q\,dy\chi_{s'Q_j^*}(x)\right)^{1/q}\right)^q dx\\ &\quad\le 4^q(cc_1)^{1/\delta}\kappa_p(D_j)^{-p/\delta}\kappa_q(D_j^*)^q(s')^{nq}\\ &\qquad\cdot\int_{\mathbf{R}^n}\left(\sum_{j=1}^{\infty}\left(\frac{1}{|Q_j^*|}\int_{D_j^*}|\nabla v(y)|^q\,dy\chi_{Q_j^*}(x)\right)^{1/q}\right)^q dx\\ &\quad\le 4^q(cc_1)^{1/\delta}\kappa_p(D_j)^{-p/\delta}\kappa_q(D_j^*)^q(s')^{nq}N^q\\ &\qquad\cdot\int_{G^*}|\nabla v(y)|^q\,dy.\end{aligned}$$

In order for G^* to be a q-Poincaré domain we need to require

$$\kappa_p(D_j)^{-p/\delta}\kappa_q(D_j^*)^q \le c_2 \tag{3.21}$$

as well as $\kappa_q(D_j^*) \le c_5$ in (3.30). Note that the latter condition follows immediately from (3.21). □

Herron and Koskela [**12**] employed capacity techniques to study the case when n-Poincaré domains are preserved under a given map f.

They exhibited, [**12**, Example 3.6], a specific n-Poincaré "rooms and corridors" domain, $G \subset \mathbf{R}^n$, for which $f(G)$ was not n-Poincaré. We state here the relevant definition and result.

Sobolev capacity mappings. A homeomorphism $f : D \to D^*$ is a K-Sobolev capacity mapping if

$$\begin{aligned} K^{-1}(s - \operatorname{cap}(E, F; D)) &\leq s - \operatorname{cap}(f(D), f(F); D^*) \\ &\leq K(s - \operatorname{cap}(E, F; D)) \end{aligned}$$

for each pair of disjoint compact sets $E, F \subset D$; here $D, D^* \subset R^n$ and

$$s - \operatorname{cap}(E, F; D) = \inf_{u \in \mathcal{W}} \int_D (|u|^n + |\nabla u|^n)$$

is the Sobolev capacity of E and F relative to D,

$$\begin{aligned} \mathcal{W} = \mathcal{W}(E, F; D) = \{u \in W_n^1(D) \cap C(D \cup E \cup F) \\ : u|_E \leq c, u|_F \geq 1 + c, c \in \mathbf{R}\}. \end{aligned}$$

Theorem 3.31 [**12**]. *If $f : D \to D^*$ is a K-Sobolev capacity mapping and D is an n-Poincaré domain, then D^* is an n-Poincaré domain.*

In the next example, we apply Theorem 3.20 to a class of "rooms and corridors" domains. In particular, one can deduce from this computation that the domain $f(G)$ of [**12**, Example 3.6] is q-Poincaré for some $q > n$.

Example 3.32. We consider a domain G as in [**13**, 5.9] which can be written as a union of sequence of "rooms and corridors" positioned symmetrically about the x_1-axis, namely, $G = \cup_{i=1}^{\infty} D_i$. Here the domains D_{2i-1} are cubes with sidelength $M^{-(2i-1)}$, $M > 1$, and the domains D_{2i} are "corridors" of dimension $M^{-2i} + M^{-2i-2}$ along the x_1-axis and of dimension bM^{-2ai}, $b > 0$, $a > 1$, in the $n-1$ directions orthogonal to the x_1-axis. We position D_{2i} so that the length of the x_1-axis contained in each of $D_{2i} \cap D_{2i-1}$ and $D_{2i} \cap D_{2i+1}$ is $(M^{-2i-2})/2$. In addition, to ease computations, we assume that the limit point on the x_1-axis of the sequence of domains, $\{D_i\}$, is situated at the origin. By [**13**], this domain is p-Poincaré for $p \geq (n-1)(a-1)$, where convex Poincaré domain constants are used to establish (3.19).

We let $f : \mathbf{R}^n \to \mathbf{R}^n$ be the radial stretching map, $f(x) = |x|^{K-1}x$. This map f is K'-quasi-conformal, $K' = K'(K, n)$. In order to apply Theorem 3.20 we need only examine how f distorts each D_i and show that (3.21) is satisfied.

For cubes $D_{2i-1}, f(D_{2i-1}) = D^*_{2i-1}$ is an (α, β)-John domain, with $\alpha = \alpha(K, n)$, $\beta = \beta(K, n)$, [**16**]. Thus (3.21) follows immediately in this case.

For corridors, D_{2i}, first note that $f(D_{2i}) = D^*_{2i}$ is star shaped with respect to $f(x_{2i})$, where x_{2i} is the center of D_{2i}. Using the notation and estimates of [**13**, 3.1], for D^*_{2i}, we have $L_{2i} \approx M^{-2iK}$ and $l_{2i} \approx M^{-2iK-2ai}$. It follows that $\kappa_q(D^*_{2i})^q \lesssim M^{-2iKq+2ai(n-1)}$.

Now we estimate (3.21) using the fact that the p-Poincaré constant $\kappa_p(D_{2i}) \approx M^{-2i}$ for the convex domain D_{2i}.

A simple calculation shows that (3.21) is satisfied when

$$q \geq \frac{p/\delta + a(n-1)}{K}.$$

Thus, $f(G)$ is q-Poincaré for all such q. $\square$

4. (q,p)-Poincaré domains. We begin with a result on (p_2, p_1)-Poincaré domains. For appropriate values of p_1 and p_2, this class of domains includes John domains and domains satisfying a quasi-hyperbolic boundary condition. But there are other domains, too; for example, star shaped domains with respect to a point, see [**14**].

Theorem 4.1. *Suppose that D and D^* are domains in $\mathbf{R}^n$ such that $D \in \mathcal{P}(p_2, p_1)$ for some $p_1 < p_2$. Suppose also that $f : D \to D^*$ is a K-quasi-conformal mapping and $\int_D |f'(x)|^{n+\varepsilon}\, dx \leq M < \infty$ for $\varepsilon > n^2/(p_2 - n) > 0$. Then $D^* \in \mathcal{P}(p)$ whenever $1 \leq p_1 < n \leq p$.*

Proof of Theorem 4.1. Let $u \in C^1(D^*)$. We will calculate the constant

c in

$$\inf_{a\in\mathbf{R}}\bigg(\int_{D^*}|u(x)-a|^p\,dx\bigg)^{1/p}\le c\bigg(\int_{D^*}|\nabla u(x)|^p\,dx\bigg)^{1/p}.$$

The main ideas in the proof are similar to those in Theorem 3.1 and we provide only necessary modifications. Define $v : D \to \mathbf{R}^n$ as $v(x) = (u\circ f)(x)$, $x \in D$. As before, we obtain (3.2) and (3.3) and we seek a bound for c_1 in (3.4). Again, we need to apply Hölder's inequality with $(p_2/p, p_2/(p_2-p))$ and with $(r = (n+\varepsilon)(p_2-p)/(np_2), r/(r-1))$. First

$$\begin{aligned}(4.2)\quad \bigg(\int_D|v(x)-v_D|^p|f'(x)|^n\,dx\bigg)^{1/p}&\\ \le\bigg(\int_D|f'(x)|^{np_2/(p_2-p)}\,dx\bigg)^{(p_2-p)/(pp_2)}&\\ \cdot\bigg(\int_D|v(x)-v_D|^{p_2}\,dx\bigg)^{1/p_2}&\\ \le M^{n/(p(n+\varepsilon))}|D|^{1/p-1/p_2-n/(p(n+\varepsilon))}&\\ \cdot\bigg(\int_D|v(x)-v_D|^{p_2}\,dx\bigg)^{1/p_2};&\end{aligned}$$

here we require $p < \varepsilon p_2/(n+\varepsilon)$ to assure $r > 1$. Since $D\in\mathcal{P}(p_2,p_1)$ with Poincaré constant $\kappa_{p_2,p_1}(D)$, where $p_1 < p_2 \le np_1/(n-p_1)$ and $1\le p_1 < n$, we obtain

$$(4.3)\quad \bigg(\int_D|v(x)-v_D|^{p_2}\,dx\bigg)^{1/p_2}\le\kappa_{p_2,p_1}(D)\bigg(\int_D|\nabla v(x)|^{p_1}\bigg)^{1/p_1}.$$

If $p = n$ we proceed from (4.3) using Hölder's inequality with exponents $(n/p_1, n/(n-p_1))$, then

$$\bigg(\int_D|\nabla v(x)|^{p_1}\,dx\bigg)^{1/p_1}\le|D|^{(n-p_1)/(np_1)}\bigg(\int_D|\nabla v(x)|^n\,dx\bigg)^{1/n}.$$

Thus (3.4) holds for $p = n$ provided that $n = p < \varepsilon p_2/(n+\varepsilon)$. This constraint forces the lower bound for ε, $\varepsilon > n^2/(p_2-n) > 0$.

We conclude that $D^* \in \mathcal{P}(n)$; hence, $D^* \in \mathcal{P}(p)$ for all $p \geq n$ by Theorem 2.3. $\square$

Corollary 4.4. *Suppose that D and D^* are domains in $\mathbf{R}^n$ such that D is a Gehring domain and (p_2, p_1)-Poincaré for some $p_1 < p_2$. Suppose also that $f : D \to D^*$ is a K-quasi-conformal mapping and $\int_D |f'(x)|^{n+\varepsilon}\, dx \leq M < \infty$ for $\varepsilon > n^2/(p_2 - n) > 0$. Then $D^* \in \mathcal{P}(p)$ whenever $p > p_1(n+\eta)/(p_1+\eta)$; here the constant $\eta > 0$ is the Gehring constant for D in $\int_D |f'(x)|^{-\eta}\, dx < \infty$.*

Proof of Corollary 4.4. As a result of Theorem 4.1, it suffices to prove the case $p < n$. Starting from estimates 4.2 and 4.3, we seek a suitable upper bound for $\int_D |\nabla v(x)|^{p_1}\, dx$. Hölder's inequality with $(p/p_1, p/(p-p_1))$ and with

$$\left(\frac{\eta(p-p_1)}{p_1(n-p)}, \frac{\eta(p-p_1)}{\eta(p-p_1) - p_1(n-p)}\right)$$

yield

$$\begin{aligned}
&\left(\int_D |\nabla v(x)|^{p_1}\, dx\right)^{1/p_1} \\
&\quad = \left(\int_D |\nabla v(x)|^{p_1} |f'(x)|^{(n-p)p_1/p} |f'(x)|^{(p-n)p_1/p}\, dx\right)^{1/p_1} \\
&\quad \leq \left(\int_D |\nabla v(x)|^{p} |f'(x)|^{n-p}\, dx\right)^{1/p} \\
&\qquad \cdot \left(\int_D |f'(x)|^{p_1(p-n)/(p-p_1)}\, dx\right)^{(p-p_1)/(p_1 p)} \\
&\quad \leq |D|^{1/p_1 - 1/p - n/(\eta p) + 1/\eta} \left(\int_D |f'(x)|^{-\eta}\, dx\right)^{(n-p)/(\eta p)} \\
&\qquad \cdot \left(\int_D |\nabla v(x)|^{p} |f'(x)|^{n-p}\, dx\right)^{1/p}.
\end{aligned}$$

The claim follows whenever $\eta(p-p_1)/(p_1(n-p)) > 1$. This holds true for $p > p_1(\eta + n)/(\eta + p_1)$. $\square$

It is known that Poincaré domains are preserved under a locally bi-Lipschitz homeomorphism [**13**]. We recall this definition and result.

Let D and D^* be domains in $\mathbf{R}^n$. A homeomorphism $f : D \to D^*$ is locally bi-Lipschitz or an $L - BLD$ homeomorphism, if for all $x \in D$,

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \leq L$$
$$\liminf_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \geq \frac{1}{L}.$$

Theorem 4.5 [**13**, Theorem 2.1]. *Suppose that $D \in \mathcal{P}(p)$ with Poincaré constant $\kappa_p(D)$. Suppose that f is an L-BLD homeomorphism of D onto a domain D^*. Then $D^* \in \mathcal{P}(p)$ with Poincaré constant $2\kappa_p(D)L^{2(n/p)+1}$.*

We now define a class of maps which is related to locally Lipschitz homeomorphisms.

Let $f : D \to D^*$ be a homeomorphism. Then define

$$K_p(x, f) = \limsup_{r \to 0} \left(\sup_{y \in S(x,r)} \frac{|f(x) - f(y)|}{|x - y|} \right)^p \cdot \left(\frac{|f(B(x,r))|}{|B(x,r)|} \right)^{-1}$$

and set

$$K_p(f) = \sup_{x \in D} K_p(x, f).$$

We consider maps f for which $K_p(f) < \infty$. These maps were examined by Gol'dshtein, Gurov and Romanov in [**9**], and we refer the reader there for a detailed study of their properties. For $p = n$ this class coincides with the class of quasi-conformal homeomorphisms. In addition, whenever $K_p(f) < \infty$, f is differentiable almost everywhere and, for $p > 1$, such a homeomorphism f is ACL. If $p > n$, then f is locally Lipschitz. If $p < n$, then for every compact set B,

$$|f(B)|/|B| \geq (K_p(f))^{n/(p-n)}.$$

If $K_p(f) < \infty$, then $K_{p_1}(f) < \infty$ whenever $p_1 > p > n$ or $p_1 < p < n$.

Theorem 4.6. *Suppose that $f : D \to D^*$ is a homeomorphism with $K_p(f) < \infty$, $D \in \mathcal{P}(kp, p)$, and $J_f^{1+\varepsilon} \in L^1(D)$ with $\varepsilon = 1/(k-1) > 0$ and $k > 1$. Then $D^* \in \mathcal{P}(p)$.*

Proof of Theorem 4.6. For any $u \in W_p^1(D^*)$ we first establish

$$\left(\int_D |v(x) - v_D|^{kp}\,dx\right)^{1/kp} \le c(k,n,p,|D|)\left(\int_{D^*} |\nabla u(x)|^p\,dx\right)^{1/p} \tag{4.7}$$

where $v(x) = u \circ f(x)$. Since $D \in \mathcal{P}(kp, p)$,

$$\left(\int_D |v(x) - v_D|^{kp}\,dx\right)^{1/kp}\left(\int_D |\nabla u(f(x))|^p|f'(x)|^p\,dx\right)^{1/p}.$$

By hypothesis, $K_p(f) < \infty$, thus

$$\begin{aligned}\left(\int_D |v(x) - v_D|^{kp}\,dx\right)^{1/kp}&\\ &\le c(k,n,p,|D|)K_p(f)\left(\int_D |\nabla u(f(x))|^p J_f(x)\,dx\right)^{1/p},\\ &= c(k,n,p,|D|)K_p(f)\left(\int_{D^*} |\nabla u(x)|^p\,dx\right)^{1/p}.\end{aligned}$$

Now we need only show

$$\left(\int_D |v(x) - v_D|^{kp}\,dx\right)^{1/kp} \ge \left(\int_{D^*} |u(x) - b|^p\,dx\right)^{1/p} \tag{4.8}$$

for some $b \in \mathbf{R}$, to prove that D^* is p-Poincaré. Using Hölder's inequality with exponents $(k, k/(k-1))$,

$$\begin{aligned}\int_{D^*}|u(y)-b|^p\,dy &= \int_D |v(x)-b|^p J_f(x)\,dx\\ &\le \left(\int_D |v(x)-b|^{kp}\,dx\right)^{1/k}\left(\int_D J_f(x)^{k/(k-1)}\,dx\right)^{(k-1)/k}\end{aligned}$$

and (4.8) follows whenever $b = v_D$.

Theorem 4.9. *Suppose that $f : D \to D^*$ is a homeomorphism with $K_p(f) < \infty$. Suppose that $D \in \mathcal{P}(p)$ with $|J_f| \leq M$. Then $D^* \in \mathcal{P}(p)$.*

The proof for Theorem 4.9 is similar to the proof of Theorem 4.6.

Remark. If we assume above in Theorem 4.6 that $k = n/(n-p)$, then Theorem 4.6 reads as follows: *Suppose that $K_p(f) < \infty$, $D \in \mathcal{P}(np/(n-p), p)$ and $J_f^{1+1/p} \in L^1(D)$. Then D^* is a p-Poincaré domain.*

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