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ATOMIC DECOMPOSITION VIA PROJECTIVE GROUP REPRESENTATIONS

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ABSTRACT. In the last few years, representations of functions (or distributions) as sums of "building blocks" have attracted much attention, e.g., time-frequency analysis and wavelet analysis. From a more abstract point of view, the Feichtinger/Gröchenig theory discuss the same problem, with an integrable group representation as the starting point. Here we present a survey of the FG-theory combined with a generalization to projective representations; this makes it directly applicable to Gabor analysis. Furthermore we point out the connections to the existing theory for frame decomposition.

1. Introduction. Consider an integrable representation $(\pi, \mathcal{H})$ of a locally compact group $\mathcal{G}$. It is natural to settle the question of whether there exists a discrete family $\{x_i\}_{i\in I} \subseteq \mathcal{G}$ and $g \in \mathcal{G}$ such that any $f \in \mathcal{H}$ can be written as a superposition of the "building blocks" $\{\pi(x_i)g\}_{i\in I}$.

The Feichtinger-Gröchenig theorem [**6, 7, 11**] gives an answer to this question, not only for elements f in $\mathcal{H}$, but also for elements in the so-called *coorbit spaces*. Thus, FG-theory can be considered as generalized wavelet analysis, and as such it deserves to be known among the wavelet experts. Here we present a survey of the theory, combined with an extension to projective representations. This generalization shows how the theory works and should give the reader more feeling with it. And the generalization is not only of theoretical interest but also important in applications. For example, in $L^2(\mathbf{R})$ building blocks arising by translating and modulating a fixed function are of great importance, leading to the important time-frequency decomposition (discussed, e.g., in [**2, 5, 12**]). Unfortunately, the two operations do not compose to a representation (only to a projective representation). Earlier this problem was solved by extending with a torus component to get the Schrödinger representation, but with the present generalization the theory is directly applicable.

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In Section 2 we give an introduction to the main ingredients in the theory. Section 3 is devoted to an explanation of the usual FG-theory.

In Section 4 the basic facts about projective representations are stated. In particular, we shall see that for each group $\mathcal{G}$ which has a projective representation $(\rho, \mathcal{H})$ it is possible to construct a new group $\tilde{\mathcal{G}}$ which has a representation in the usual sense; this is the starting point for our generalization. In Section 5 we study the connections between the needed Banach spaces related to $\mathcal{G}$ and the corresponding spaces on $\tilde{\mathcal{G}}$.

Finally, in Section 6 we state our version of the atomic decomposition theorem and show how to construct coherent Hilbert frames using the abstract theory. Furthermore, we extend a result of Daubechies about Weyl-Heisenberg coherent states to projective representations.

2. Basic facts. Throughout the paper $\mathcal{G}$ denotes a locally compact σ-compact Hausdorff group. All index sets I will be countable.

2.1. Let π be a unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$; we shall take the inner product $\langle \cdot, \cdot \rangle$ linear in the second entry. Fixing $g \in \mathcal{H}$, we define the corresponding wavelet transform

$$V_g : \mathcal{H} \longrightarrow C(\mathcal{G}), \qquad V_g(f)(x) := \langle \pi(x)g, f \rangle, \quad f \in \mathcal{H},\ x \in \mathcal{G}.$$

Clearly V_g is linear and bounded.

An irreducible unitary continuous representation π is called (*square*) *integrable* if there exists a $g \in \mathcal{H} \backslash \{0\}$ such that $V_g(g)$ is (square) integrable with respect to the left Haar measure dx on $\mathcal{G}$. Clearly, an integrable representation is also square integrable.

2.2. On functions $f : \mathcal{G} \to \mathbf{C}$ the left translation L_x, $x \in \mathcal{G}$, respectively right translation R_x, acts by

$$L_x f(y) = f(x^{-1}y), \qquad y \in \mathcal{G}, \quad (\text{respectively } R_x f(y) = f(yx)).$$

The involution "$\vee$" is defined by

$$f^{\vee}(y) = f(y^{-1}), \qquad y \in \mathcal{G}.$$

If a Banach space Y of functions on $\mathcal{G}$ is left (right) invariant and continuously embedded in L^1_{loc}, then each L_x, respectively R_x, is automatically bounded; the norm will be denoted by $|||L_x|||_Y$ or $|||L_x \mid Y|||$. Sometimes we omit Y in the notation.

By a BF-space we mean a Banach space $(Y, ||\cdot||)$ of (equivalence classes of) measurable functions on $\mathcal{G}$ such that

(i) Y is continuously embedded into $L^1_{\text{loc}}(\mathcal{G})$.

(ii) If $f \in Y$, $g \in L^1_{\text{loc}}$ and $|g(x)| \leq |f(x)|$ for almost every x, then $g \in Y$ and $||g|| \leq ||f||$.

(iii) Y is left and right invariant, and the canonical weight function $w(x) := \max\{|||L_x|||, |||L_{x^{-1}}|||, |||R_x|||, |||R_{x^{-1}}|||\Delta(x^{-1})\}$ is measurable. Here $\Delta(x)$ denotes the modular function

(iv) $Y * L^{1\vee}_w \subseteq Y$ and $||f * g \mid Y|| \leq ||f \mid Y|| \cdot ||g^\vee \mid L^1_w||$ for $f \in Y$, $g \in L^{1\vee}_w$.

(v) There exists a nonempty open set U such that $1_U \in Y$.

(v) excludes some pathological examples. It can be shown that (v) implies the measurability of w if the translation operators are strongly continuous on Y. Furthermore, (ii) and (v) imply that $C_c(\mathcal{G}) \subseteq Y$.

2.3. Let V be a relatively compact neighborhood of the neutral element e in $\mathcal{G}$. A countable family $\{x_i\}_{i\in I}$ is said to be V-separated if $x_iV \cap x_jV = \varnothing$ for $i \neq j$. $\{x_i\}_{i\in I}$ is relatively V-separated if it is a finite union of V-separated sets. Very often the concrete neighborhood V is not relevant and we shall just talk about a relatively separated set.

Given a compact set, U, $\{x_i\}_{i\in I}$ is said to be U-dense if $\cup_i x_iU = \mathcal{G}$.

2.4. To a BF-space Y we can assign a sequence space in the following way: Let $X := \{x_i\}_{i\in I}$ be relatively separated, $W \subseteq \mathcal{G}$ a relatively compact Borel set with nonvoid interior and define

$$Y_d(X) := \left\{ \{\lambda_i\}_{i\in I} \,\middle|\, \sum_{i\in I} |\lambda_i| 1_{x_iW} \in Y \right\}.$$

$Y_d(X)$ is a Banach space with respect to the natural norm. For a fixed set X the definition is independent of W: different choices give the same space and equivalent norms.

Some of the most important properties of $Y_d(X)$ have been stated in [**6**, Lemma 3.5]. We shall use the following two facts:

(i) If the bounded functions with compact support are dense in Y, then the finite sequence is dense in $Y_d(X)$.

(ii) The sequence space corresponding to $L^p(\mathcal{G})$ is $l^p(I)$.

3. Standard FG-theory. In this section we give a survey of the usual FG-theory. For the technical details, we refer to [**6, 7, 8**].

3.1. Let $(\pi, \mathcal{H})$ be an integrable representation of $\mathcal{G}$, and let Y be a BF-space such that

$$\mathcal{A}_w^\pi := \{g \in \mathcal{H} \mid V_g(g) \in L_w^1\} \neq \{0\}.$$

Fixing $g \in \mathcal{A}_w^\pi \backslash \{0\}$, we define

$$\mathcal{H}_w^{1,\pi} := \{f \in \mathcal{H} \mid V_g(f) \in L_w^1\}$$

equipped with the norm $||f \mid \mathcal{H}_w^{1,\pi}|| = ||V_g(f) \mid L_w^1||$.

It is easy to see that $\mathcal{H}_w^{1,\pi}$ is π-invariant, so each f in the antidual $\mathcal{H}_w^{1,\pi\neg}$ of $\mathcal{H}_w^{1,\pi}$ acts on $\pi(x)g$; we shall denote this action by the usual bracket from the inner product and define the extended wavelet transform by

$$V_g(f)(x) := \langle \pi(x)g, f\rangle, \qquad f \in \mathcal{H}_w^{1,\pi\neg},\ x \in \mathcal{G}.$$

The *coorbit space* corresponding to Y is defined as

$$\mathrm{Co}\, Y := \{f \in \mathcal{H}_w^{1,\pi\neg} \mid V_g(f) \in L_w^1\}.$$

$\mathrm{Co}\, Y$ is a Banach space with respect to the norm

$$||f \mid \mathrm{Co}\, Y|| = ||V_g(f) \mid Y||.$$

Furthermore, the definition is independent of g.

$Y = L^2$ is of special interest for us; in [**7**, Corollary 4.4] it is shown that

$$\mathrm{Co}\, L^2 = \mathcal{H}$$

with equivalence of the norms.

3.2. Let $k \in C_c(\mathcal{G})$ be a window function, i.e., a function with $\operatorname{Range} k \subseteq [0,1]$ which is 1 on a compact neighborhood of e. Define

$$\mathcal{B}_w := \left\{ g \in \mathcal{H} \mid \int_{\mathcal{G}} ||(R_x k) V_g(g)||_\infty w(x)\, dx < \infty \right\}.$$

Then $\mathcal{B}_w \subseteq \mathcal{A}_w$ and, in fact, $\mathcal{B}_w$ is a dense subspace of $\mathcal{H}$. If $\mathcal{G}$ is unimodular (or more general, an IN-group), then $\mathcal{B}_w = \mathcal{A}_w$ [**8**, Lemma 7.2].

Remark. In fact, membership of g in $\mathcal{B}_w$ means that the wavelet transform $V_g(g)$ is in the Wiener amalgam space $W^R(C_0, L^1_w)$. For information on this and similar spaces introduced by Feichtinger, we refer to [**7**].

In what follows, we write $||R_x k \cdot V_g(g)||_\infty$ instead of $||(R_x k)V_g(g)||_\infty$.

3.3. The main result of [**7, 11**] is

Theorem 3.1. *Let $g \in \mathcal{B}_w \backslash \{0\}$. There exists a neighborhood $U \in \mathcal{O}(e)$ and two constants $c_1, c_2 > 0$ such that the following is true*:

(i) *For every U-dense and relatively separated set $X = \{x_i\}_{i \in I}$ in $\mathcal{G}$ there is a bounded linear operator*

$$A : \operatorname{Co} Y \longrightarrow Y_d(X)$$

such that

$$||Af \mid Y_d(X)|| \le c_1 ||f \mid \operatorname{Co} Y||, \qquad \forall\, f \in \operatorname{Co} Y$$

and

$$f = \sum_{i \in I} \lambda_i(f) \pi(x_i) g, \qquad \forall\, f \in \operatorname{Co} Y$$

with $\lambda_i(f) := (Af)_i$. The convergence is unconditional and in the norm of $\operatorname{Co} Y$ if the finite sequences are dense in $Y_d(X)$; otherwise, it is in w^*-sense of $\mathcal{H}^{1,\pi\neg}_w$.

(ii) *If* $X = \{x_i\}_{i\in I}$ *is relatively separated, then*

$$\{\lambda_i\}_{i\in I} \longmapsto \sum_{i\in I} \lambda_i \pi(x_i) g$$

defines a bounded linear mapping of $Y_d(X)$ *into* $\mathrm{Co}\, Y$*, with norm less than or equal to* c_2*; the convergence is as above.*

Remarks 3.4. (i) By 2.4(i) the convergence in the theorem is in norm if the set of bounded functions with compact support is dense in Y.

(ii) Weighted L^p-spaces with $1 \le p < \infty$ are BF-spaces in which $C_c(\mathcal{G})$ is dense. Thus, we just have to check that $\mathcal{A}_w \neq \{0\}$ in order to apply the FG-theory.

(iii) In [**11**, Theorem T], Gröchenig has given an explicit condition on U implying that the atomic decomposition works. With

$$G_U^\sharp(x) := \sup_{u\in U} |G(ux) - G(x)|,$$

it says that

$$\int |V_g(g)_U^\sharp(x)| w(x)\, dx < 1.$$

4. Projective representations. Here we collect the needed basic facts about projective representations. As before, let $\mathcal{G}$ be an l.c. group and $\mathcal{H}$ a Hilbert space; the set of unitary operators on $\mathcal{H}$ will be denoted by $\mathcal{U}(\mathcal{H})$.

Definition 4.1. A *projective representation* of $\mathcal{G}$ on $\mathcal{H}$ is a mapping $x \mapsto \rho(x)$ of $\mathcal{G}$ into $\mathcal{U}(\mathcal{H})$ such that

(i) $\rho(e) = I$.

(ii) There exists a continuous function $c : \mathcal{G} \times \mathcal{G} \to \mathbf{C}$ such that

$$\rho(x, y) = c(x, y)\rho(x)\rho(y), \qquad \forall\, x, y \in \mathcal{G}.$$

(iii) The mapping $x \mapsto \langle \rho(x) f_1, f_2\rangle$ is a Borel function on $\mathcal{G}$ for all $f_1, f_2 \in \mathcal{H}$.

Note that we have taken unitarity of each $\rho(x)$ as part of the definition. The reason for assuming the cocycle c (clearly uniquely determined by (ii)) to be continuous will become clear shortly. It is well known (and easy to check) that c has the following properties:

(i) $|c(x,y)| = 1$ for all x, y.

(ii) $c(x,y)c(xy,z) = c(x,yz)c(y,z)$ for all x, y, z.

(iii) $c(x,e) = c(e,x) = 1$ for all x.

Irreducibility and cyclicity can be defined as usual: ρ is irreducible if the only closed ρ-invariant subspaces are $\mathcal{H}$ and $\{0\}$; ρ is cyclic if there exists a cyclic vector, i.e., a $g \in \mathcal{H}$ such that

$$\overline{\mathrm{Span}}\,\{\rho(x)g \mid x \in \mathcal{G}\} = \mathcal{H}.$$

Lemma 4.2. (i) *If $x \mapsto \rho(x)f$ is continuous at e, then it is continuous at an arbitrary group element y.*

(ii) *Suppose that the mapping $x \mapsto \langle \rho(x)f, f\rangle$ is continuous for all $f \in \mathcal{H}$. Then $x \mapsto \rho(x)f$ is continuous for all $f \in \mathcal{H}$, i.e., ρ is strongly continuous.*

(iii) *ρ is irreducible if and only if each $g \in \mathcal{H}\backslash\{0\}$ is cyclic.*

Proof. (i) For $x \in \mathcal{G}$, we have

$$\begin{aligned} ||\rho(x)f - \rho(y)f|| &= ||\rho(yy^{-1}x)f - \rho(y)f|| \\ &= ||\rho(y)[c(y,y^{-1}x)\rho(y^{-1}x)f - f]|| \\ &= ||c(y,y^{-1}x)\rho(y^{-1}x)f - f|| \\ &\le ||c(y,y^{-1}x)\rho(y^{-1}x)f - c(y,y^{-1}x)f|| \\ &\quad + ||c(y,y^{-1}x)f - f|| \\ &= ||\rho(y^{-1}x)f - f|| + |c(y,y^{-1}x) - 1| \cdot ||f|| \longrightarrow 0 \end{aligned}$$

for $x \to y$.

(ii)

$$\begin{aligned} ||\rho(x)f - \rho(e)f||^2 &= ||\rho(x)f - f||^2 \\ &= ||\rho(x)f||^2 + ||f||^2 \\ &\quad - \langle f, \rho(x)f\rangle - \langle \rho(x)f, f\rangle \longrightarrow 0 \end{aligned}$$

for $x \to e$.

(iii) can be proved exactly in the same way as for the unitary representations. $\square$

Note that the continuity of c has been used in the proof of (i).

It is well known (and the starting point for all the following considerations) that, for each projective representation $(\rho, \mathcal{H})$ of $\mathcal{G}$, it is possible to construct a new group $\tilde{\mathcal{G}}$ which has a usual unitary representation $(\pi, \mathcal{H})$; we just have to define

$$\tilde{\mathcal{G}} := \mathcal{G} \times \Pi$$

(here Π is the torus) with the composition

$$(x, \xi)(y, \eta) = (xy, \xi\eta\overline{c(x, y)})$$

and

$$\pi(x, \xi) := \xi\rho(x).$$

$\tilde{\mathcal{G}}$ is called the *Mackey obstruction group*. It is a locally compact group with respect to the product topology (a fact which depends on the continuity of c). Also the Haar measure is the product measure from $\mathcal{G}$ and Π.

In general $\tilde{\mathcal{G}}$ is not abelian even if $\mathcal{G}$ is. But, as a consequence of the following lemma, $\tilde{\mathcal{G}}$ is unimodular if and only if $\mathcal{G}$ is:

Lemma 4.3. *Denote the modular function on $\tilde{\mathcal{G}}$ by $\tilde{\Delta}$. Then*

$$\tilde{\Delta}(y, \eta) = \Delta(y), \qquad \forall\, y \in \mathcal{G},\ \eta \in \Pi.$$

Proof. Let $f \in C_c(\mathcal{G} \times \Pi)$, $(y, \eta) \in \mathcal{G} \times \Pi$. Then

$$\begin{aligned}
\int_{\mathcal{G}} \int_{\Pi} f[(x, \xi)(y, \eta)]\, d\xi\, dx &= \int_{\mathcal{G}} \int_{\Pi} f(xy, \xi\eta\overline{c(x, y)})\, d\xi\, dx \\
&= \int_{\mathcal{G}} \int_{\Pi} f(xy, \xi)\, d\xi\, dx \\
&= \int_{\Pi} \int_{\mathcal{G}} f(xy, \xi) d\xi\, dx \\
&= \Delta(y)^{-1} \int_{\mathcal{G}} \int_{\Pi} f(x, \xi)\, d\xi\, dx. \qquad \square
\end{aligned}$$

Lemma 4.4. (i) *ρ is irreducible if and only if π is irreducible.*

(ii) *ρ is strongly continuous if and only if π is strongly continuous.*

Proof. (i) Since

$$\begin{aligned}\operatorname{span}\{\rho(x)g \mid x \in \mathcal{G}\} &= \operatorname{span}\{\xi\rho(x)g \mid \xi \in \Pi, x \in \mathcal{G}\}\\ &= \operatorname{span}\{\pi(x,\xi) \mid (x,\xi) \in \tilde{\mathcal{G}}\},\end{aligned}$$

(i) follows from Lemma 4.2.

(ii) Since $\tilde{\mathcal{G}}$ is equipped with the product topology, the continuity of ρ follows directly from that of π. Now suppose ρ to be continuous. Then

$$\begin{aligned}||\pi(x,\xi)f - f|| &= ||\xi\rho(x)f - f||\\ &\le ||\xi\rho(x)f - \rho(x)f|| + ||\rho(x)f - f||\\ &= |1-\xi| \cdot ||f|| + ||\rho(x)f - f||,\end{aligned}$$

from which the result follows. □

Given a projective representation ρ, we denote the wavelet transform corresponding to $g \in \mathcal{H}$ by

$$U_g : \mathcal{H} \longmapsto C^b(\mathcal{G}), \qquad U_g(f)(x) := \langle \rho(x)g, f\rangle, \quad f \in \mathcal{H},\ x \in \mathcal{G}$$

An irreducible continuous projective representation is said to be (*square*) *integrable* if there exists $g \in \mathcal{H}\backslash\{0\}$ such that $U_g(g) \in L^1$ (L^2).

Corollary 4.5. *ρ is* (*square*) *integrable if and only if π is* (*square*) *integrable.*

For the proof, just observe that $|V_g(g)(x,\xi)| = |U_g(g)(x)|$.

Example 4.6. Let $\mathcal{H} = L^2(R)$. For $x, y \in R$ we define the translation and modulation operators on $\mathcal{H}$ by

$$(L_x f)(z) := f(z-x), \qquad f \in \mathcal{H},\ x \in R$$

and

$$(M_y f)(z) = e^{iyz} f(z), \qquad f \in \mathcal{H},\ x \in R.$$

Then $\rho(x, y) := L_x M_y$ defines a projective representation of R^2 since

$$\rho(x_1 + x_2, y_1 + y_2) = e^{-iy_1 x_2} \rho(x_1, y_1)\rho(x_2, y_2) \\ \forall\, (x_1, y_1), (x_2, y_2) \in R^2.$$

The corresponding Mackey obstruction group is $R \times R \times \Pi$ with the composition

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 t_2 e^{iy_1 x_2}),$$

i.e., the Heisenberg group; our construction gives us the representation

$$\begin{aligned}[\pi(x, y, t) f](z) &= [t\rho(x, y) f](z) \\ &= t e^{iy(z-x)} f(z - x), \qquad f \in \mathcal{H},\ z \in R\end{aligned}$$

i.e., the Schrödinger representation.

More generally, let $\mathcal{G}$ be a locally compact Abelian group with dual group $\mathcal{G}$. Then

$$\begin{aligned}[\rho(x, \gamma) f](z) &:= [L_2(\gamma f)](z) \\ &= \gamma(z - x) f(z - x), \qquad x, z \in \mathcal{G},\ \gamma \in \hat{\mathcal{G}}\end{aligned}$$

defines a projective representation of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$. The Mackey group is $\mathcal{G} \times \hat{\mathcal{G}} \times \Pi$ with the composition

$$(x_1, \gamma_1, t_1)(x_2, \gamma_2, t_2) = (x_1 + x_2, \gamma_1 + \gamma_2, t_1 t_2 \gamma_1(x_2)),$$

and the corresponding representation is

$$[\pi(x, \gamma, t) f](z) = t\gamma(z - x) f(z - x), \qquad z \in \mathcal{G},\ f \in L^2(\mathcal{G}).$$

5. Banach spaces corresponding to $\tilde{\mathcal{G}}$. In light of Section 4, a natural way to obtain a theory for atomic decomposition using projective representations is the following: take the corresponding representation of the Mackey group and transfer all relevant conditions and results back to ρ. As we shall see in the next two section, this principle works quite well and gives the desired results.

5.1. Let Y be a BF-space on a group $\mathcal{G}$ which has a projective representation ρ. Define a space of functions on $\tilde{\mathcal{G}}$ by

$$\tilde{Y} := \left\{ f : \mathcal{G} \times \Pi \longmapsto \mathbf{C} \mid f \text{ measurable}, \int_{\Pi} |f(\cdot, \eta)| \, d\eta \in Y \right\}.$$

This construction already appeared in [**9**].

$\tilde{Y}$ is a Banach space with respect to the norm

$$||f \mid \tilde{Y}|| = || \int |f(\cdot, \eta)| \, d\eta \mid Y||.$$

Lemma 5.1. *$\tilde{Y}$ is left invariant and $|||L_{x,\xi} \mid \tilde{Y}||| = |||L_x \mid Y|||$, for all $(x, \xi) \in \mathcal{G} \times \Pi$.*

Proof. Let $(x, \xi) \in \mathcal{G} \times \Pi$. If $f \in \tilde{Y}$, then $g := \int |f(\cdot, \eta)| \, d\eta \in Y$ and

$$\begin{aligned} \int |L_{x,\xi} f(y, \eta)| \, d\eta &= \int |f[(x, \xi)^{-1}(y, \eta)]| \, d\eta \\ &= \int |f(x^{-1}y, \eta c(x, x^{-1}) \overline{\xi c(x^{-1}, y)})| \, d\eta \\ &= \int |f(x^{-1}y, \eta)| \, d\eta = L_x g(y). \end{aligned}$$

Therefore, $L_{x,\xi} f \in \tilde{Y}$ and

$$\begin{aligned} ||L_{x,\xi} f \mid \tilde{Y}|| &= ||L_x g \mid Y|| \\ &\le |||L_x \mid Y||| \cdot ||g \mid Y|| \\ &= |||L_x \mid Y||| \cdot ||f \mid \tilde{Y}||. \end{aligned}$$

Thus $|||L_{x,\xi}||| \le |||L_x|||$. Let us now prove the opposite inequality. First, observe that for each $g \in Y$ we can define a function $f \in \tilde{Y}$ by

$$f(y, \eta) := g(y), \qquad (y, \eta) \in \tilde{\mathcal{G}}.$$

Then $||f \mid \tilde{Y}|| = ||g \mid Y||$. Furthermore, a calculation similar to the one given at the beginning of the proof applies to f, g and gives that

$$||L_{x,\xi} f \mid \tilde{Y}|| = ||L_x g \mid Y||.$$

Therefore,

$$\begin{aligned} |||L_x \mid Y||| &= \sup\{||L_x g \mid Y|| \mid ||g \mid Y|| = 1\} \\ &\leq \sup\{||L_{x,\xi} f \mid \tilde{Y}|| \mid ||f \mid \tilde{Y}|| = 1\} \\ &= |||L_{x,\xi} \mid \tilde{Y}||| \end{aligned}$$

as desired. $\square$

A similar calculation shows that $\tilde{Y}$ is right invariant and that

$$|||R_{x,\xi}||| = |||R_x|||.$$

In particular, our calculations (together with Lemma 4.3) show that the canonical weight function $\tilde{w}$ corresponding to $\tilde{Y}$ is

$$\tilde{w}(x,\xi) = w(x).$$

We have now checked that $\tilde{Y}$ satisfies the third condition in the definition of a BF-space (see Section 2.2); in fact, the same type of argument enables us to prove that $\tilde{Y}$ satisfies all of them. We shall not go into the details here.

5.2. A BF-space Y is said to have *absolutely continuous norm* [**13**, Chapter 15] if

$$\begin{aligned} &[\{f_n\}_{n=1}^{\infty}, \quad \forall\, x : |f_n(x)| \searrow 0, \quad \text{for } n \to \infty] \\ &\qquad\qquad \Longrightarrow ||f_n \mid Y|| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty. \end{aligned}$$

As we shall see now, our construction assures that $\tilde{Y}$ inherits this property from Y:

Proposition 5.2. *If Y has absolutely continuous norm, then so does $\tilde{Y}$.*

Proof. Let $\{f_n\} \subseteq \tilde{Y}$ be a set such that $|f_n(x,\eta)| \searrow 0$, for all $(x,\eta) \in \tilde{\mathcal{G}}$. By assumption $\int |f_1(\cdot,\eta)|\,d\eta \in Y$, so $\int |f_1(x,\eta)|\,d\eta < \infty$ for almost every $x \in \mathcal{G}$, say for $x \in \mathcal{G} - N$. This means that $[\eta \mapsto f_1(x,\eta)] \in L^1(\Pi)$ for $x \in \mathcal{G} - N$. Since $|f_1| \geq |f_2| \geq \cdots$ and $\lim f_n = 0$, the Lebesgue dominated convergence theorem implies that $\int |f_n(x,\eta)|\,d\eta \searrow 0$ for all $x \in \mathcal{G} - N$. But $\int |f_n(\cdot,\eta)|\,d\eta \in Y$ for all n, so our assumptions on Y imply that $||f_n \mid \tilde{Y}|| = ||\int |f_n(\cdot,\eta)|\,d\eta|| \searrow 0$ for $n \to \infty$. $\square$

Corollary 5.3. *If Y has absolutely continuous norm, then $C_c(\tilde{\mathcal{G}})$ is dense in Y.*

Proof. By [**4**, Proposition 1.4], $C_c(\tilde{\mathcal{G}})$ is dense in every BF-space with absolutely continuous norm. $\square$

Remark. In general $(\tilde{L}^p)(\mathcal{G}) \neq L^p(\tilde{\mathcal{G}})$ so in the case of an L^p-space on $\mathcal{G}$ our construction of a space on $\tilde{\mathcal{G}}$ is not the 'natural.' In fact, all the following also works perfectly for L^p-spaces with the natural definition; nevertheless, our definition has been chosen since it gives us a more general recipe to get a BF-space on $\tilde{\mathcal{G}}$ from an arbitrary BF-space on $\mathcal{G}$.

5.3. Let $X = \{x_i\}_{i\in I} \subseteq \mathcal{G}$ be relatively separated and $W \subseteq \mathcal{G}$ a relatively compact Borel set with nonvoid interior. As we have seen in Section 2, to each BF-space Y on $\mathcal{G}$ we can assign a sequence space

$$Y_d(\{x_i\}) = \left\{ \{\lambda_i\}_i \,\middle|\, \sum_i |\lambda_i| 1_{x_i W} \in Y \right\}.$$

If $\{y_j\}_{j=1}^n \subseteq \Pi$ is finite, then $\{(x_i,y_j)\}_{i,j} \subseteq \tilde{\mathcal{G}}$ is relatively separated; $W \times \Pi \subseteq \tilde{\mathcal{G}}$ is relatively compact, so the sequence space corresponding to $\tilde{Y}$ is

$$\tilde{Y}_d(\{(x_i,y_j)\}) = \left\{ \{\lambda_{i,j}\}_{i,j} \,\middle|\, \sum_{i,j} |\lambda_{i,j}| 1_{(x_i,y_i)W\times\Pi} \in \tilde{Y} \right\}$$

with the natural norm.

Lemma 5.4. *In the situation above,*

$$\{\lambda_{i,j}\}_{i,j} \in \tilde{Y}_d(\{(x_i, y_j)\}) \Longleftrightarrow \left\{ \sum_{j=1}^{n} |\lambda_{i,j}| \right\}_i \in Y_d(\{x_i\}).$$

Furthermore,

$$||\{\lambda_{i,j}\}_{i,j} \mid \tilde{Y}_d(\{(x_i, y_j)\})|| = \left\| \left\{ \sum_{j=1}^{n} |\lambda_{i,j}| \right\}_i \middle| Y_d(\{x_i\}) \right\|.$$

Proof. First, observe that

$$\begin{aligned} 1_{(x_i,y_j)W\times\Pi}(x,\xi) \neq 0 &\Longleftrightarrow (x_i, y_j)^{-1}(x,\xi) \in W \times \Pi \\ &\Longleftrightarrow x_i^{-1}x \in W \\ &\Longleftrightarrow 1_{x_iW}(x) \neq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \tilde{Y}_d(\{(x_i, y_j)\}) &= \left\{ \{\lambda_{i,j}\} \middle| [(x,\xi) \longmapsto \sum_{i,j} |\lambda_{i,j}| 1_{x_iW}(x)] \in \tilde{Y} \right\} \\ &= \left\{ \{\lambda_{i,j}\}_{i,j} \middle| \sum_i \sum_j |\lambda_{i,j}| 1_{x_iW} \in Y \right\} \\ &= \left\{ \{\lambda_{i,j}\}_{i,j} \middle| \left\{ \sum_{j=1}^{n} |\lambda_{i,j}| \right\}_i \in Y_d(\{x_i\}) \right\} \end{aligned}$$

with the corresponding norm estimate. □

Remark. As a special case of the lemma, we obtain that

$$\{\lambda_i\}_i \in \tilde{Y}_d(\{(x_i, 1)\}_i) \Longleftrightarrow \{\lambda_i\}_i \in Y_d(\{x_i\}),$$

i.e., that

$$\tilde{Y}_d(\{(x_i, 1)\}_i) = Y_d(\{x_i\}).$$

Corollary 5.5. *If the finite sequences are dense in $Y_d(\{x_i\})$, then the same holds for $\tilde{Y}_d\{(x_i, y_j)\})$.*

Proof. Let $\{\lambda_{i,j}\} \in \tilde{Y}_d(\{(x_i, y_j)\})$. Then $\{\sum_{j=1}^n |\lambda_{i,j}|\}_i \in Y_d(\{x_i\})$. Given $\varepsilon > 0$, there is a finite set $K \subseteq I$ such that

$$\left\| \left\{ \sum_{j=1}^n |\lambda_{i,j}| \right\}_{i \in I \setminus K} | Y_d(\{x_i\}) \right\| < \varepsilon$$

which means that

$$\|\{\lambda_{i,j}\}_{(i,j)\in I\times\{1,2,.n\}} - \{\lambda_{i,j}\}_{(i,j)\in K\times\{1,2,.n\}} | \tilde{Y}_d(\{(x_i, y_j)\})\| < \varepsilon. \qquad \square$$

5.3. We shall now assume $(\rho, \mathcal{H})$ to be a projective *integrable* representation of $\mathcal{G}$. As before, the corresponding representation of the Mackey group will be denoted by $(\pi, \mathcal{H})$. Given a BF-space Y with canonical weight w, we define

$$\mathcal{A}_w^\rho := \{g \in \mathcal{H} \mid U_g(g) \in L_w^1\}.$$

As we have seen in Section 5.1, the weight function $\tilde{w}$ corresponding to $\tilde{Y}$ is $w(x, \xi) = w(x)$, and $\tilde{Y}$ can be used in the atomic decomposition with respect to π if and only if

$$\mathcal{A}_{\tilde{w}}^\pi \neq \{0\}.$$

Now

$$\begin{aligned}\mathcal{A}_{\tilde{w}}^\pi &= \left\{ g \in \mathcal{H} \,\middle|\, \int_\Pi \int_\mathcal{G} |\langle \pi(x, \xi) g, g\rangle| \tilde{w}(x, \xi)\, dx\, d\xi < \infty \right\} \\ &= \left\{ g \in \mathcal{H} \,\middle|\, \int_\mathcal{G} |\langle \rho(x) g, g\rangle| w(x)\, dx < \infty \right\} = \mathcal{A}_w^\rho.\end{aligned}$$

Thus, the two spaces of admissible vectors coincide. Fix now $g \in \mathcal{A}_w^\rho \setminus \{0\}$. Exactly the same argument shows that (with the obvious definitions)

$$\mathcal{H}_w^{1,\rho} = \mathcal{H}_{\tilde{w}}^{1,\pi}.$$

Also the natural definition of a coorbit space with respect to ρ and Y gives the same as the old one with respect to π and $\tilde{Y}$; denoting the wavelet transform corresponding to π by V_g, we have

$$\begin{aligned}\mathrm{Co}_\pi(\tilde{Y}) &= \{f \in \mathcal{H}_{\tilde{w}}^{1,\pi\neg} | V_g(f) \in \tilde{Y}\} \\ &= \left\{ f \in \mathcal{H}_{\tilde{w}}^{1,\pi\neg} \,\middle|\, \left[x \longmapsto \int_\Pi |\langle \xi \rho(x) g, f\rangle|\, d\xi \right] \in Y \right\} \\ &= \{f \in \mathcal{H}_w^{1,\rho\neg} \mid U_g(f) \in Y\} = \mathrm{Co}_\rho Y,\end{aligned}$$

again with the same norm. $\square$

From now on, we just write $\operatorname{Co} Y$ for both of them.

Our definition of $\mathcal{B}_w$ from Section 3.2 also makes sense for a projective representation ρ; in fact, as we shall show in the lemma below, we get the same space as for the corresponding representation π:

Lemma 5.6. $\mathcal{B}_w^\rho = \mathcal{B}_{\tilde{w}}^\pi$.

Proof. Let $k \in C_c(\mathcal{G})$ be a window function; then

$$\mathcal{B}_w^\rho = \left\{ g \in \mathcal{H} \Big| \int_{\mathcal{G}} \|R_x k \cdot U_g(g)\|_\infty w(x)\, dx < \infty \right\}.$$

As the window function on $\tilde{\mathcal{G}}$, needed for the description of $\mathcal{B}_{\tilde{w}}^\pi$, it is appropriate to take $\tilde{k}(x,\xi) := k(x), (x,\xi) \in \tilde{\mathcal{G}}$; with this choice we obtain

$$\begin{aligned}\|R_{x,\xi}\tilde{k} \cdot V_g(g)\|_\infty &= \sup_{(y,\eta)\in\tilde{\mathcal{G}}} |\tilde{k}[(y,\eta)(x,\xi)] V_g(g)(y,\eta)| \\ &= \sup_{(y,\eta)\in\tilde{\mathcal{G}}} |k(yx)\langle \eta\rho(y)g, g\rangle| \\ &= \|R_x k \cdot U_g(g)\|_\infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{B}_{\tilde{w}}^\pi &= \left\{ g \in \mathcal{H} \Big| \int_{\mathcal{G}} \int_{\Pi} \|R_{x,\xi}\tilde{k} \cdot V_g(g)\|_\infty \tilde{w}(x,\xi)\, d\xi\, dx < \infty \right\} \\ &= \left\{ g \in \mathcal{H} \Big| \int_{\mathcal{G}} \|R_x k \cdot U_g(g)\|_\infty w(x)\, dx < \infty \right\} = \mathcal{B}_w^\rho.\end{aligned}$$

From now on, we denote both of them just by $\mathcal{B}_w$.

6. The atomic decomposition theorem. We are now able to state and prove the main theorem. We keep the situation as in Section 5, i.e., $(\rho, \mathcal{H})$ is a projective representation of $\mathcal{G}$ and Y is a BF-space such that $\mathcal{A}_w \neq 0$. As we have seen, in this case $\mathcal{B}_w$ is a dense subspace of $\mathcal{H}$.

Theorem 6.1. *Let $g \in \mathcal{B}_w \backslash \{0\}$. There exists a neighborhood $U \in \mathcal{O}(e)$ and two constants $c_1, c_2 > 0$ such that*

(i) *For every U-dense and relatively separated family $X = \{x_i\}_{i\in I} \subseteq \mathcal{G}$ there is a bounded linear operator*

$$A : \operatorname{Co} Y \longmapsto Y_d(\{x_i\})$$

such that

$$\|Af \mid Y_d(\{x_i\})\| \le c_1 \|f \mid \operatorname{Co} Y\|, \qquad \forall\, f \in \operatorname{Co} Y$$

and

$$f = \sum_{i\in I} \lambda_i(f)\rho(x_i)g, \qquad \forall\, f \in \operatorname{Co} Y$$

with $\lambda_i(f) := (Af)_i$.

The convergence is in the norm of $\operatorname{Co} Y$ if the finite sequences are dense in $Y_d(\{x_i\})$, otherwise it is in w^*-sense; in both cases, the convergence is unconditional.

(ii) *Conversely, if $X = \{x_i\}_{i\in I} \subseteq \mathcal{G}$ is relatively separated, then*

$$\{\lambda_i\}_{i\in I} \longmapsto \sum_{i\in I} \lambda_i \rho(x_i) g$$

defines a bounded linear mapping of $Y_d(\{x_i\})$ into $\operatorname{Co} Y$, with norm less than or equal to c_2; the convergence is as above.

Proof. By Corollary 4.5, the representation $(\pi, \mathcal{H})$ corresponding to $(\rho, \mathcal{H})$ is integrable; furthermore, $\tilde{Y}$ can be used in the atomic decomposition with respect to π since

$$\mathcal{A}_{\tilde{w}}^{\pi} = \mathcal{A}_w^{\rho} \neq \{0\}.$$

Now, let $g \in \mathcal{B}_w \backslash \{0\}$. Take the neighborhood $\tilde{U} \in \mathcal{O}(e, 1)$ in $\tilde{\mathcal{G}}$ and the two constants c_1, c_2 as in Theorem 3.1.

$\tilde{\mathcal{G}}$ is equipped with the product topology, so we can choose neighborhoods $U \in \mathcal{O}(e)$ in $\mathcal{G}$ and $V \in \mathcal{O}(1)$ in Π such that $U \times V \subseteq \tilde{U}$.

Now let $\{x_i\} \subseteq \mathcal{G}$ be U-dense and relatively separated. Choose a finite V-dense set $\{y_j\}_{j=1}^n \subseteq \Pi$.

It is easy to show that $\{(x_i, y_i)\} \subseteq \tilde{\mathcal{G}}$ is $\tilde{U}$-dense and relatively separated. Thus, by Theorem 3.1 there is a bounded linear operator

$$\tilde{A} : \mathrm{Co}\, Y \longmapsto \tilde{Y}_d(\{(x_i, y_j)\})$$

with norm less than c_1 and with the property that each $f \in \mathrm{Co}\, Y$ can be written as

$$f = \sum_{i,j} \lambda_{i,j}(f) \pi(x_i, y_j) g$$

where $\lambda_{i,j}(f) := (\tilde{A} f)_{i,j}$.

The series converges unconditionally, so

$$\begin{aligned} f &= \sum_{i,j} \lambda_{i,j}(f) y_j \rho(x_i) g \\ &= \sum_i \left[\sum_{j=1}^n \lambda_{i,j}(f) y_j \right] \rho(x_i) g, \qquad \forall\, f \in \mathrm{Co}\, Y. \end{aligned}$$

The coefficient sequence $\{\sum_{j=1}^n \lambda_{i,j}(f) y_j\}_i$ is in $Y_d(\{x_i\})$; the reason is that $|\sum_{j=1}^n \lambda_{i,j}(f) y_j| \leq \sum_{j=1}^n |\lambda_{i,j}(f)| =: s_i$, where $\{s_i\} \in Y_d(\{x_i\})$ by Lemma 5.4.

Therefore, we can define a linear mapping

$$\begin{aligned} \phi : \tilde{Y}_d(\{(x_i, y_j)\}) &\longmapsto Y_d(\{x_i\}), \\ \phi(\{\lambda_{i,j}\}_{i,j}) &:= \left\{ \sum_{j=1}^n \lambda_{i,j} y_j \right\}_i. \end{aligned}$$

ϕ is bounded with norm ≤ 1:

$$\begin{aligned} \|\phi(\{\lambda_{i,j}\}_{i,j})\| &= \left\| \left\{ \sum_{j=1}^n \lambda_{i,j} y_j \right\}_i \middle| Y_d(\{x_i\}) \right\| \\ &= \left\| \sum_i \left| \sum_{j=1}^n \lambda_{i,j} y_j \right| 1_{x_i W} \middle| Y \right\| \\ &\leq \left\| \sum_i \sum_j |\lambda_{i,j}| 1_{x_i W} \middle| Y \right\| \\ &= \left\| \left\{ \sum_j |\lambda_{i,j}| \right\}_i \middle| Y_d(\{x_i\}) \right\| \\ &= \|\{\lambda_{i,j}\}_{i,j} \mid \tilde{Y}_d(\{(x_i, y_j)\})\| \end{aligned}$$

for all $\{\lambda_{i,j}\} \in \tilde{Y}_d(\{(x_i, y_j)\})$; we have again used Lemma 5.4. Thus, (i) is satisfied with

$$A := \phi \circ \tilde{A}.$$

The type of convergence follows from Theorem 3.1 and Corollary 5.5.

(ii) Let $\{x_i\} \subseteq \mathcal{G}$. By Lemma 5.4, $Y_d(\{x_i\}) = \tilde{Y}_d(\{(x_i, 1)\})$, so by Theorem 3.1 each $\{\lambda_i\}_i \in Y_d(\{x_i\})$ defines an element

$$\sum_i \lambda_i \pi(x_i, 1) g = \sum_i \lambda_i \rho(x_i) g$$

in $\operatorname{Co} Y$. Furthermore, the mapping

$$\{\lambda_i\}_i \longmapsto \sum_i \lambda_i \rho(x_i) g$$

is bounded with norm less than or equal to c_2. $\square$

Corollary 6.2. *Take $U \in \mathcal{O}(e)$ as in Theorem 6.1. Then, given a U-dense and relatively separated set $\{x_i\}_i \in \mathcal{G}$, we have*

$$\operatorname{Co} Y = \left\{ \sum_i \lambda_i \rho(x_i) g \,\middle|\, \{\lambda_i\}_i \in Y_d(\{x_i\}) \right\}.$$

Remarks. (i) As we have seen, the theorem can be formulated exactly as the original, and the constants c_1, c_2 can be taken from the decomposition theorem applied to $\pi, \tilde{Y}$.

(ii) Part (ii) generalizes Proposition 2 in [**10**]. Furthermore, our proof is different since we are working with another definition of $\tilde{Y}$.

(iii) Our generalization of the FG-theory does not augment the class of spaces which can be decomposed; the point is that it is now possible to work more straightforwardly than before.

(iv) Using Remark (iii) after Theorem 3.1, we can state a sufficient condition on $\tilde{U}$ from the proof, namely that

$$\|V_g(g)^\sharp_{\tilde{U}} \mid L^1_{\tilde{w}}\| = \int_{\tilde{\mathcal{G}}} |V_g(g)^\sharp_{\tilde{U}}(x, \xi) \mid w(x)\, dx\, d\xi < 1$$

where
$$G^{\sharp}_{\tilde{U}}(x,\xi) = \sup_{(y,\eta)\in\tilde{U}} |\, G[(y,\eta)(x,\xi)] - G(x,\xi)|.$$

6.2. A countable set $\{f_i\}_{i\in I}$ of elements in a Hilbert space $\mathcal{H}$ is called a *frame* (see [**3**] or the survey paper [**12**]) if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad \forall\, f \in \mathcal{H}.$$

It can be shown that if $\{f_i\}_{i\in I}$ is a frame then

$$S : \mathcal{H} \longmapsto \mathcal{H}, \qquad Sf := \sum_{i\in I} \langle f_i, f\rangle f_i$$

defines a bounded invertible operator, the series being unconditionally convergent; therefore, any frame gives us an atomic decomposition of $\mathcal{H}$:
$$f = SS^{-1}f = \sum_{i\in I} \langle f_i, S^{-1}f\rangle f_i, \qquad f \in \mathcal{H}.$$
In [**11**], Gröchenig has introduced a generalization of the frame concept to the setting of Banach spaces. Also, he has shown that the assumptions in Theorem 3.1 imply that $\{\pi(x_i)g\}_{i\in I}$ is a Banach frame for $\mathrm{Co}\, Y$.

It is not difficult to show that the same is true for $\{\rho(x_i)g\}_{i\in I}$ under the assumptions in Theorem 6.1. Nevertheless we shall restrict our attention to the Hilbert space $\mathcal{H}$ corresponding to ρ and give a direct proof.

The next result will be stated without proof; it is a part of Theorem 8.1 in [**8**].

Lemma 6.3. *Let $g \in \mathcal{B}_w$. If $\{(x_i, y_j)\}_{(i,j)\in I\times\{1,\dots,n\}} \subseteq \tilde{\mathcal{G}}$ is relatively separated, then*

$$\{V_g(g)(x_i, y_j)\}_{i,j} \in \tilde{Y}_d(\{(x_i, y_j)\}), \qquad \forall\, f \in \mathrm{Co}\, Y$$

and

$$\exists c > 0 : ||\{V_g(g)(x_i, y_j)\}_{i,j} \mid \tilde{Y}_d(\{(x_i, y_j)\})|| \leq c||f \mid \operatorname{Co} Y||,$$
$$\forall\, f \in \operatorname{Co} Y.$$

Let us look at the special case $Y = L^2(\mathcal{G})$; as we have seen in Sections 2.4 and 3.1, $\operatorname{Co} L^2 = \mathcal{H}$ and $L^2_d(\{x_i\}_{i\in I}) = l^2(I)$. Now, if $\{x_i\}_{i\in I}$ is relatively separated in $\mathcal{G}$, then $\{(x_i, 1)\}_{i\in I}$ is relatively separated in $\tilde{\mathcal{G}}$, and from the remark after Lemmas 5.4 and 6.3, we get that

$$\{V_g(f)(x_i, 1)\}_{i\in I} \in Y_d(\{x_i\})$$

and

$$\|\{V_g(f)(x_i, 1)\}_{i\in I} \mid Y_d(\{x_i\})\| \leq c\|f \mid \operatorname{Co} Y\|;$$

thus, there exists a constant $B > 0$ with

$$\sum_{i\in I} |\langle \rho(x_i)g, f\rangle|^2 \leq B\|f\|^2, \qquad \forall\, f \in \mathcal{H}.$$

i.e., the upper frame condition is satisfied.

Corollary 6.4. *Let $g \in \mathcal{B}_w \backslash \{0\}$. With $U \in \mathcal{O}(e)$ as in Theorem* 6.1*, $\{\rho(x_i)g\}_{i\in I}$ is a frame for $\mathcal{H}$ for all relatively separated and U-dense sets $\{x_i\}_{i\in I}$ in $\mathcal{G}$.*

Proof. By Theorem 6.1 any $f \in \mathcal{H}$ can be written as $f = \sum_{i\in I} \lambda_i(f)\rho(x_i)g$, with coefficient functionals satisfying

$$\|\{\lambda_i(f)\} \mid l^2(I)\| \leq k\|f\|$$

for some constant k independent of f. Therefore,

$$\begin{aligned}
\|f\|^4 &= \left[\sum_{i\in I} \lambda_i(f)\langle \rho(x_i)g, f\rangle\right]^2 \\
&\leq \sum_{i\in I} |\lambda_i(f)|^2 \cdot \sum_{i\in I} |\langle \rho(x_i)g, f\rangle|^2
\end{aligned}$$

and

$$k^{-2}\|f\|^2 \leq \sum_{i\in I} |\langle \rho(x_i)g, f\rangle|^2, \qquad \forall\, f \in \mathcal{H}.$$

The upper estimate was established after Lemma 6.3. □

Let us end this section by a remark about the connections between frames and projective representations. Let $H = \{h_i\}_i$ be a subgroup of $\tilde{\mathcal{G}}$ and $\{e_j\}_{j=1}^{l}$ a finite set of elements in the Hilbert space $\mathcal{H}$ corresponding to the projective representation ρ. Consider the set

$$E = \{\rho(h)e_j \mid h \in H,\ 1 \leq j \leq l\}.$$

If

$$\sum_{1\leq j\leq l} \sum_{h\in H} |\langle f, \rho(h)e_j\rangle|^2 < \infty, \qquad \forall\, f \in \mathcal{H},$$

then the corresponding frame operator

$$S : \mathcal{H} \longrightarrow \mathcal{H}, \quad Sf = \sum_{i,j} \langle \rho(h_i)e_j, f\rangle \rho(h_i)e_j$$

converges unconditionally for all $f \in \mathcal{H}$.

Lemma 6.5. *$S\rho(h) = \rho(h)S$ for all $h \in H$.*

Proof. Let $h \in H$. Then

$$\begin{aligned}
\rho(h)Sf &= \sum_{i,j} \langle \rho(h_i)e_j, f\rangle \rho(h)\rho(h_i)e_j \\
&= \sum_{i,j} \langle \rho(h)\rho(h_i)e_j, \rho(h)f\rangle \rho(h)\rho(h_i)e_j \\
&= \sum_{i,j} \langle \rho(hh_i)e_j, \rho(h)f\rangle \rho(hh_i)e_j \\
&= \sum_{i,j} \langle \rho(h_i)e_j, \rho(h)f\rangle \rho(h_i)e_j \\
&= S\rho(h)f. \qquad \square
\end{aligned}$$

Theorem 6.6. *If E is a frame, then*

(i) $f = \sum_{i,j} \langle \rho(h_i) S^{-1} e_j, f \rangle \rho(h_i) e_{j,}$.

(ii) $f = \sum_{i,j} \langle \rho(h_i) e_j, f \rangle \rho(h_i) S^{-1} e_{j,}$.

Proof. If E is a frame, then S has a bounded inverse defined on all of $\mathcal{H}$. As a consequence of Lemma 6.5,

$$S^{-1}\rho(h) = \rho(h)S^{-1}, \qquad \forall\, h \in H.$$

(i) now follows from

$$\begin{aligned} f = SS^{-1}f &= \sum_{i,j} \langle \rho(h_i)e_j, S^{-1}f\rangle \rho(h_i)e_j \\ &= \sum_{i,j} \langle S^{-1}\rho(h_i)e_j, f\rangle \rho(h_i)e_j \\ &= \sum_{i,j} \langle \rho(h_i)S^{-1}e_j, f\rangle \rho(h_i)e_j. \end{aligned}$$

Similarly, (ii) is a consequence of $f = S^{-1}Sf$. □

Theorem 6.6 generalizes a result of I. Daubechies about Weil-Heisenberg coherent states [**2**]. Daubechies also explains why the result is important for applications. The main point is that $S^{-1}\rho(h_i)e_j = \rho(h_i)S^{-1}e_j$. Thus, in order to make frame decompositions using $\{\rho(h_i)e_j\}$ it is not necessary to compute the whole family

$$\{S^{-1}\rho(h_i)e_j\}_{i,j}\ ;$$

we only need to calculate the finite family $\{S^{-1}e_j\}_{j=1}^{l}$ and then apply the projective representation. For a further discussion of the result (in the context of Gabor expansions), we refer to [**1**].

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