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LOCALIZED SOLUTIONS OF SUBLINEAR ELLIPTIC EQUATIONS: LOITERING AT THE HILLTOP

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ABSTRACT. We establish existence of infinitely many localized twice-differentiable radial solutions to the equation $\Delta v + f(v) = 0$ in $\mathbf{R}^N$, where f is linearly bounded above. Such equations govern the spatial profiles of solitary-wave solutions to nonlinear wave equations with global regularity of solutions. We use constructive methods to show that there are localized solutions with any prescribed number of nodes.

1. Introduction. We consider the semilinear wave equation

$$u_{tt} - \Delta u = g(u), \tag{1.1}$$

where solutions u are complex-valued functions on spacetime $\mathbf{R}^{N+1}$, with spatial dimension $N \geq 2$, and where the nonlinearity $g : \mathbf{C} \to \mathbf{C}$ has the property that $g(se^{i\psi}) = g(s)e^{i\psi}$ for all real s and ψ. Such a function g is determined by its restriction to the real axis, which is necessarily odd, and which we assume to be real. Let $G(s) \equiv \int_0^s g(s')\,ds'$ be the primitive of g. If $G(s) \leq 0$ for all real s, then conservation of the energy $\mathcal{E}[u, u_t] \equiv \int_{\mathbf{R}^N} \{(1/2)|u_t|^2 + (1/2)|\nabla u|^2 - G(|u|)\}\, d^N x$ implies, under growth conditions on g, that solutions to (1.1) with finite-energy initial data are bounded in bounded regions of spacetime [**8, 10**]. If, on the other hand, the primitive is positive at some amplitudes, then it is possible for singularities to develop. Here we will consider the well-behaved case in which $G(s) \leq 0$ for all s, consistent with global regularity of solutions.

We are interested in standing-wave solutions of the nonlinear wave equation, of the form $u(x, t) = e^{i\omega t} v(x)$, where ω is a real constant. For such a solution, the standing-wave profile $v : \mathbf{R}^N \to \mathbf{C}$ satisfies the associated nonlinear elliptic equation

$$\Delta v + f_\omega(v) = 0 \tag{1.2}$$

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where the nonlinearity $f_\omega : \mathbf{C} \to \mathbf{C}$ is related to g by $f_\omega(z) \equiv g(z)+\omega^2 z$. Under appropriate conditions on f_ω (roughly, that $f'_\omega(0) < 0$ and $F_\omega(s) \equiv \int_0^s f_\omega(s')\,ds' > 0$ for some value of s), there are twice-differentiable solutions to (1.2) that are localized in the sense that $v(x) \to 0$ as $|x| \to \infty$. Such localized classical solutions are of interest in various contexts, and the set of spherically-symmetric (radial) solutions has been extensively studied [**1**–**7**, **9**].

We note that if f_ω results from a nonlinearity g having $G(s) \leq 0$ for all s, then $F_\omega(s) = G(s) + (1/2)\omega^2 s^2$ must be quadratically bounded above. To date, work that establishes the existence of localized spherical solutions either has been variational in character, or has been constructive but applicable only to nonlinearities f_ω with superlinear growth, for which F_ω is not quadratically bounded. In this paper, we use constructive methods to prove the existence of localized solutions to (1.2), for functions f_ω that engender global regularity of solutions to the associated nonlinear wave equation.

We assume that the nonlinearity $f \equiv f_\omega$ in (1.2) is an odd locally Lipschitz-continuous function with $-\infty < -\sigma^2 \equiv \lim_{s\to 0} f(s)/s \leq 0$, and in case $\sigma = 0$ we require that $f(s) < 0$ for small positive s. We furthermore assume that the primitive $F(s) \equiv \int_0^s f(s')\,ds'$ has a positive zero γ, with $f(\gamma) > 0$, and $F(s) < 0$ for all s in the interval $(0, \gamma)$. Under mild growth restrictions on f, these basic hypotheses are sufficient to guarantee the existence of localized radial solutions to (1.2). (See [**1**, **2**, **4**, **9**].)

Constructive methods, which furnish additional information about nodal structure, require additional hypotheses. In this paper, we treat two distinct types of behavior of f for large amplitudes. The first type is linear or sublinear growth, in which $f(s) = \kappa s^p + h(s)$, where κ is a positive constant, $0 < p \leq 1$, and $h(s)/s^p \to 0$ as $s \to \infty$. This discussion, in Appendix A, forms a footnote to [**6**].

Our main results concern a second type of behavior, in which $F(s)$ has a positive local (one-sided) maximum at a value of s larger than γ. Section 2 presents general hypotheses on the shape of F under which the conclusions of the main theorem (below) hold. In Appendix B we show that those hypotheses follow from various more natural assumptions. For example, the conditions of Section 2 are true if f satisfies the following two hypotheses:

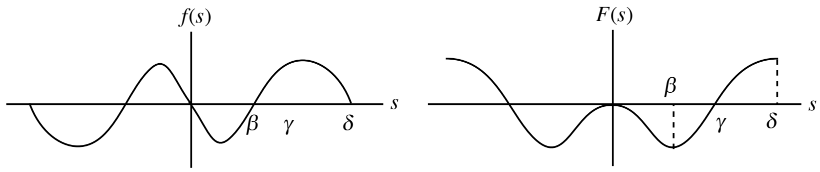

FIGURE 1.

I. There exist β and δ with $0 < \beta < \gamma < \delta$ such that $f(\beta) = f(\delta) = 0$ and $f(s) > 0$ for all s in the interval (β, δ). (Thus F is positive and increasing on (γ, δ), with zero derivative at δ.)

II. F is concave downward near the hilltop at δ, that is, there exists $\varepsilon > 0$ such that $F(t) \leq F(s) + f(s)(t - s)$ for all s and t in the interval $[\delta - \varepsilon, \delta]$.

See Figure 1. We remark that in this case we make no hypotheses on $f(s)$ for arguments with $|s| > \delta$. The behavior of f outside $[-\delta, \delta]$ is irrelevant to our discussion and does not affect the solutions we construct.

The hypotheses of Section 2 are also satisfied in cases where F has a "hilltop at infinity," that is, where F is positive, strictly increasing, and bounded on (γ, ∞), so that $F(\infty) \equiv \lim_{s\to\infty} F(s)$ is finite. If, for example, F additionally satisfies $0 < M_1(f(s))^2 \leq F(\infty) - F(s) \leq M_2 f(s)$ for all s in (γ, ∞), where M_1 and M_2 are constants, then the hypotheses of Section 2 hold.

Our main result is the following:

Main theorem. *Let f satisfy the hypotheses stated in Section* 2. *Then, for each nonnegative integer n, there is a C^2 real-valued solution $v(x)$ to* (1.2), *spherically symmetric with respect to the origin, such that $v(x) \to 0$ as $|x| \to \infty$, and such that $v(x)$ vanishes on exactly n spherical hypersurfaces.*

Because we consider only spherically symmetric solutions $v(x) = w(|x|)$, the main theorem is equivalent to the existence of solutions of the ODE boundary-value problem

$$w'' + \frac{N-1}{r} w' + f(w) = 0 \quad \text{for } r > 0 \tag{1.3}$$

$$w'(0) = 0, \tag{1.4}$$

$$\lim_{r\to\infty} w(r) = 0, \tag{1.5}$$

where $r = |x|$. We use a shooting argument to establish existence. We consider the initial-value problem consisting of (1.3) with initial conditions

$$w(0) = d, \qquad w'(0) = 0, \tag{1.6}$$

and we vary the parameter d to achieve (1.5).

A crucial step is the demonstration that the solution to the initial-value problem can be given an arbitrarily large number of zeros by choosing d appropriately. Previous work [**6**] relies on comparison with a scaled version of the differential equation in which the (superlinear) large-$|w|$ behavior of $f(w)$ predominates when d is extremely large. In case f has the behavior illustrated in Figure 1, a quite different mechanism is responsible for the existence of solutions to the initial-value problem with large numbers of zeros.

To discuss this distinction, we interpret (1.3) as an equation of motion for a point with position $w(r)$ at time r, moving in the potential well $F(w)$, subject to the influence of the time-dependent damping force $-((N-1)/r)w'$. According to (1.6) the system is released from rest with initial displacement d, and a solution to the boundary-value problem (1.3)–(1.5) is one that comes to rest at the local maximum of F at the origin, after an infinite time.

Let

$$E(r) \equiv E[w, w'](r) \equiv (1/2)(w'(r))^2 + F(w(r))$$

be the usual energy of the system. If $w(r)$ is a solution to (1.3) then

$$E'(r) = -\frac{N-1}{r}(w'(r))^2,$$

so $E(r)$ is nonincreasing. It follows that if $E[w, w'](r)$ becomes negative at some time r_0, then the solution $w(r)$ has no zeros larger than r_0.

Thus, in order to have many zeros, a solution to the initial-value problem must maintain positive energy a sufficiently long time.

In the situation addressed by previous work, where $\lim_{s\to\infty} f(s)/s^p$ is a positive constant and $1 < p < (N+2)/(N-2)$, solutions with large numbers of zeros are obtained as $d \to \infty$. For such solutions, the initial energy $E(0) = F(d)$ becomes large enough to offset the energy decrease due to damping.

For nonlinearities of the type shown in Figure 1, solutions with large numbers of zeros are obtained as $d \to \delta^-$. The initial energy is no larger than $F(\delta)$, and solutions employ a different mechanism to offset the initial energy decrease due to damping: they are initially essentially constant, and begin to develop non-negligible values of $|w'(r)|$ only after a relatively long time interval. The effect of the damping force $-((N-1)/r)w'$ is thereby mitigated, since $|w'|$ becomes non-negligible only when r is large. This phenomenon of "waiting near the hilltop" allows solutions to retain sufficient energy for large numbers of zero-crossing excursions.

The difference between the two types of behavior is illustrated in Figure 2, which shows numerical solutions of the initial-value problem (1.3) with (1.6) in dimension $N = 3$.

The lower lefthand plot (Figure 2a) shows three solutions in a case illustrative of earlier work, the function $f(w) = -w(1-w^2)$, which is superlinear at large amplitudes. The solutions have, respectively, no zeros ($d = 3$), one zero ($d = 6$), and two zeros ($d = 15$). Note that, as d increases: the location of the first zero moves inward; solutions initially decrease more rapidly; energy loss is compensated by greatly increased initial energy.

The lower righthand plot (Figure 2b) shows three solutions for a nonlinearity illustrative of the hilltop case of our analysis here, the function $f(w) = -w(1-w^2)(4-w^2)$. This function has $\beta = 1$, $\gamma \approx 1.52$, and $\delta = 2$. The solutions have, respectively, no zeros ($d = \delta - 10^{-4}$), one zero ($d = \delta - 10^{-9}$), and two zeros ($d = \delta - 10^{-18}$). Note that, as d increases: the location of the first zero moves outward; solutions initially decrease more slowly; energy loss is minimized by waiting near the hilltop of $F(w)$ at amplitude $w = 2$ until the coefficient $(N-1)/r$ of the damping term in (1.3) is fairly small.

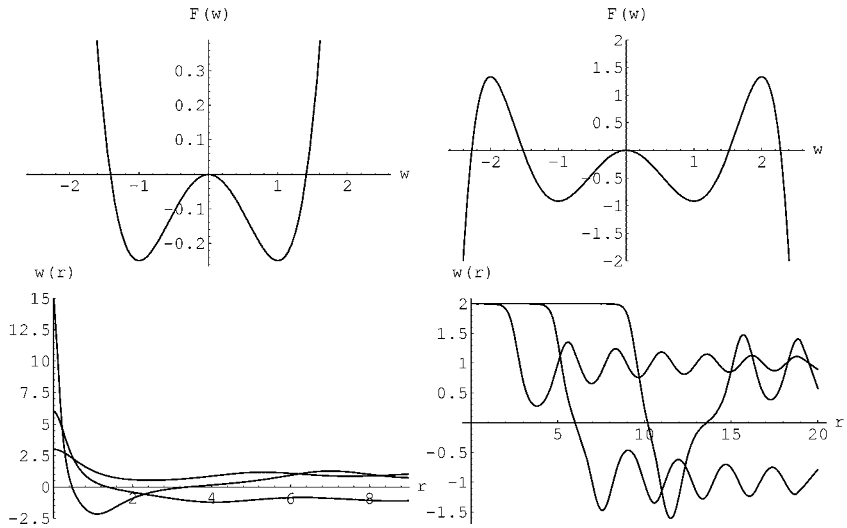

Figure 2a.

$f(w) \equiv -w(1-w^2)$

Figure 2b.

$f(w) \equiv -w(1-w^2)(4-w^2)$

FIGURE 2.

The distinction between these two behaviors is also suggested, albeit less dramatically, by the fact that solutions of (1.3) with (1.6) satisfy

$$w(0) = d, \qquad w'(0) = 0, \qquad w''(0) = (-1/N)f(d).$$

Thus, in the superlinear case, as $d \to \infty$, $w''(0) \to -\infty$, and the function $w(r)$ initially curves downward very sharply. In contrast, in the hilltop case analyzed here, as $d \to \delta^-$, $w''(0) \to 0^-$, and the function $w(r)$ is initially quite flat.

In Section 2, we spell out the (hilltop) hypotheses on the nonlinearity f under which we prove the main theorem. Section 3 establishes some elementary properties of solutions to the initial value problem. Section 4 contains the proof of Theorem 1, which establishes that there are solutions of the initial value problem with arbitrarily many zeros. In Section 5 we sketch the proof of the main theorem, which proceeds, given Theorem 1, by the method employed in [**6**]. In Appendix B, we show that the conditions on f indicated in this section imply the hypotheses of Section 2. As mentioned, Appendix A contains the addendum to [**6**] which establishes there are solutions of the initial value problem with arbitrarily many zeros, in the (non-hilltop) case of linear or sublinear growth of f.

2. Hypotheses for the hilltop. The assertions of the main theorem are a consequence of the following hypotheses.

Let f be an odd locally Lipschitz-continuous function with $-\infty < -\sigma^2 \equiv \lim_{s\to 0} f(s)/s \le 0$. If $\sigma = 0$, we require that $f(s) < 0$ for small positive s. Let $F(s) \equiv \int_0^s f(s')ds'$ be the primitive of f.

We assume that there exist numbers β, γ, and δ with $0 < \beta < \gamma < \delta$ (possibly $\delta = \infty$) such that:

$$\begin{aligned} f(\beta) &= 0, \qquad f(s) > 0 \quad \text{for all } s \in (\beta, \delta), \\ F(\gamma) &= 0, \qquad F(s) < 0 \quad \text{for all } s \in (0, \gamma). \end{aligned}$$

If $\delta = \infty$, we furthermore assume that $F(\infty) \equiv \lim_{s\to\infty} F(s)$ is finite.

We define the following two quantities associated with f:

$$T(y; d) \equiv \int_y^d \frac{dz}{\sqrt{2}\sqrt{F(d) - F(z)}} \quad \text{for } \gamma \le y < d < \delta; \tag{2.1}$$

$$\Delta(\gamma; d) \equiv \sqrt{2}(N-1) \int_\gamma^d \frac{1}{T(z; d)} \sqrt{F(d) - F(z)}\, dz \quad \text{for} \quad \gamma \le d < \delta. \tag{2.2}$$

Our central hypotheses, used in Sections 3, 4 and 5, are that these quantities are well-defined and satisfy:

$$\lim_{d\to\delta^-} T(\gamma; d) = \infty \tag{2.3}$$

and

$$\lim_{d\to\delta^-} \Delta(\gamma; d) = 0. \tag{2.4}$$

We show in Appendix B that these hypotheses hold under a variety of more natural conditions on f that correspond to (one side of) a local maximum of $F(s)$ at $s = \delta$. We will see in the following sections that T is a lower bound on a time delay, and that Δ is an upper bound on initial energy loss.

We will make use of an immediate consequence of these hypotheses. Define

$$\Gamma(d) \equiv \sqrt{2}(N-1) \int_{-d}^{d} \sqrt{F(\delta) - F(y)}\, dy \qquad \text{for } \gamma \le d < \delta. \tag{2.5}$$

Lemma 2.1.

$$\lim_{d \to \delta^-} \frac{\Gamma(d)}{T(\gamma; d)} = 0. \tag{2.6}$$

Proof of Lemma 2.1. In the case when δ is finite, the assertion follows immediately, since then $\Gamma(d) \le \sqrt{2}(N-1) \int_{-\delta}^{\delta} \sqrt{F(\delta) - F(y)}\, dy$, independent of d, and by hypothesis $T(\gamma; d) \to \infty$ as $d \to \delta^-$.

In the case $\delta = \infty$, we have, since F is even, $\Gamma(d) = 2\sqrt{2}(N-1) \int_0^d \sqrt{F(\infty) - F(y)}\, dy$. Thus, for $d \ge \gamma$,

$$\begin{aligned} 0 < \frac{\Gamma(d)}{T(\gamma; d)} = & \frac{2\sqrt{2}(N-1)}{T(\gamma; d)} \int_0^{\gamma} \sqrt{F(\infty) - F(y)}\, dy \\ & + 4(N-1) \frac{\int_\gamma^d \sqrt{F(\infty) - F(y)}\, dy}{\int_\gamma^d [F(d) - F(z)]^{-1/2}\, dz}. \end{aligned} \tag{2.7}$$

The first term goes to zero as $d \to \infty$, by hypothesis (2.3). Because F is monotonically increasing on (γ, ∞), the second term in (2.7) is bounded by

$$4(N-1) \frac{\int_\gamma^d \sqrt{F(\infty) - F(y)}\, dy}{\int_\gamma^d [F(\infty) - F(z)]^{-1/2}\, dz} \le \frac{4(N-1)\sqrt{F(\infty)}\,(d-\gamma)}{\int_{(\gamma+d)/2}^{d} [F(\infty) - F(z)]^{-1/2}\, dz}.$$

Applying the mean value theorem to the integral in the denominator, we see that the second term in (2.7) is bounded by $8(N-1)\sqrt{F(\infty)}\sqrt{F(\infty)-F(z_d)}$, where z_d is some number in the interval $((\gamma+d)/2, d)$. This upper bound tends to zero as $d \to \infty$. Thus $\lim_{d\to\infty} \Gamma(d)/T(\gamma; d) = 0$, as claimed. This completes the proof of Lemma 2.1. $\square$

3. Properties of solutions to the initial value problem.

Lemma 3.1. *If w is a solution of the initial value problem* (1.3) *with* (1.6) *on some interval* $[0, R)$ *with $R \le \infty$ and $\beta < d < \delta$, then $|w| < d$ on* $(0, R)$.

Proof of Lemma 3.1. Suppose by way of contradiction that there is some $r_0 > 0$ such that $|w(r_0)| = d$. Multiplying (1.3) by $w'(r)$, integrating on $(0, r_0)$, and using (1.6) gives

$$\frac{1}{2}w'^2(r_0) + \int_0^{r_0} \frac{N-1}{r} w'^2(r)\, dr + F(w(r_0)) = F(d). \tag{3.1}$$

Since $F(w(r_0)) = F(d)$, it follows from the nonnegativity of the other terms that

$$\int_0^{r_0} \frac{N-1}{r} w'^2(r)\, dr = 0.$$

This implies $w'(r) \equiv 0$ on $(0, r_0)$ and thus $w(r) = d$ for all $r \ge 0$. On the other hand, taking limits in (1.3) gives

$$w''(0) = -\frac{f(d)}{N} < 0 \quad \text{since } \beta < d < \delta, \tag{3.2}$$

showing that $w(r)$ is not constant, a contradiction. Hence, no such r_0 can exist. Thus, we must have $|w| < d$ on $(0, R)$, as claimed. $\square$

The existence for small $r > 0$ of unique solutions to (1.3) with (1.6) can be established by an application of the contraction mapping principle to the map

$$G(w) = d - \int_0^r \frac{1}{t^{N-1}} \int_0^t s^{N-1} f(w)\, ds\, dt,$$

whose fixed points are solutions. To establish existence for all $r > 0$, we recall that on any interval of existence of w, $|w|$ is a priori bounded by $d < \delta$. Thus $|F(w(r))|$ is bounded, since F is continuous. Because the energy $(1/2)(w'(r))^2 + F(w(r))$ is bounded above by its initial value $F(d)$, it follows that $|w'(r)|$ is also bounded. Thus the small-r solution may be continued to all $r > 0$.

Lemma 3.2. *Let w be a solution of* (1.3) *with* (1.6) *where $\gamma < d < \delta$. Then w is decreasing on a nonempty open interval $(0, R_d)$, where either*:

(a) $R_d = \infty$, $\lim_{r\to\infty} w'(r) = 0$, $\lim_{r\to\infty} w(r) = L$ *where* $|L| < d$ *and* $f(L) = 0$, *and* $E(R_d) \equiv \lim_{r\to\infty} E(r) = F(L)$,

or

(b) R_d *is finite*, $w'(R_d) = 0$, $f(w(R_d)) \leq 0$, *and* $E(R_d) = F(w(R_d))$.

In either case, there exists a unique (*finite*) *number $\tau_d \in (0, R_d)$ such that $w(\tau_d) = \gamma$ and such that w is decreasing on $(0, \tau_d]$.*

Proof of Lemma 3.2. Since $w'(0) = 0$ and $w''(0) = -f(d)/N < 0$, we have that w is decreasing for small r.

If w is not everywhere decreasing, then w has a first local minimum at $r = R_d$, with $w'(R_d) = 0$ and $w''(R_d) \geq 0$. It follows from (1.3) that $f(w(R_d)) \leq 0$, and therefore $w(R_d) \leq \beta < \gamma$. Thus, there exists $\tau_d \in (0, R_d)$ with the stated properties.

On the other hand, suppose that $w(r)$ is decreasing for all $r > 0$. We showed in Lemma 3.1 that $|w(r)| < d$ for $r > 0$. Thus $\lim_{r\to\infty} w(r) = L$ with $|L| < d$. We also know that w' is bounded. Therefore, $\lim_{r\to\infty} w'/r = 0$, so from (1.3) we have $\lim_{r\to\infty} w''(r) = -f(L)$.

If $f(L)$ is nonzero, then $w''(r)$ is bounded away from zero for r in the interval (r_0, ∞) for some r_0, which implies that $w'(r)$ is unbounded as $r \to \infty$, contradicting the fact that w' is bounded. Thus $f(L) = 0$, and so $L \leq \beta$. Thus there exists a finite τ_d with the stated properties.

The fact that $\lim_{r\to\infty} w'(r) = 0$ can be seen as follows. The energy $E(r) \equiv (1/2)(w'(r))^2 + F(w(r))$ is decreasing and bounded below, and so has a limit as $r \to \infty$. Since $w(r)$ has a limit, the term $F(w(r))$ also has a limit as $r \to \infty$. Thus $\lim_{r\to\infty}(1/2)(w'(r))^2$ exists, so $w'(r)$ has a limit, which must be zero because w is bounded. This completes the proof of Lemma 3.2. ◻

4. Solutions with many zeros. Let w_d be the solution of (1.3) with (1.6) where $\gamma < d < \delta$, and let τ_d be, as in Lemma 3.2, the smallest value of r for which $w_d(r) = \gamma$. Since w_d is decreasing on $(0, \tau_d]$, the inverse of w_d is well-defined on $[\gamma, d)$ and $w_d^{-1} : [\gamma, d) \to (0, \tau_d]$. In addition, we have the following estimate.

Lemma 4.1. *For $y \in [\gamma, d)$,*

$$w_d^{-1}(y) \geq T(y; d), \tag{4.1}$$

where $T(y; d)$ is defined by equation (2.1).

Proof of Lemma 4.1. Since the energy $E(r)$ is nonincreasing, the solution $w = w_d$ satisfies

$$\frac{1}{2} w'^2 + F(w) \leq F(d).$$

Since w is decreasing on $[0, \tau_d]$, we have

$$-w' \leq \sqrt{2}\sqrt{F(d) - F(w)} \quad \text{for } r \in [0, \tau_d]. \tag{4.2}$$

Hence,

$$\int_{w(r)}^{d} \frac{dz}{\sqrt{F(d) - F(z)}} = \int_0^r \frac{-w'(t)\, dt}{\sqrt{F(d) - F(w(t))}} \leq r\sqrt{2}$$
$$\text{for } r \in [0, \tau_d].$$

Letting $r = w^{-1}(y)$ gives

$$w^{-1}(y) \geq \int_y^d \frac{dz}{\sqrt{2}\sqrt{F(d) - F(z)}} = T(y; d),$$

for $y \in [\gamma, d)$, as claimed. ◻

Remark. We note that $T(y; d)$ is the time at which the solution to the (undamped) initial value problem

$$u'' + f(u) = 0, \qquad u(0) = d, \qquad u'(0) = 0 \tag{4.3}$$

first reaches position y, that is, $u(T(y;d)) = y$.

Lemma 4.2.
$$\lim_{d\to\delta^-} \tau_d = \infty.$$

Proof of Lemma 4.2. Since $\tau_d = w^{-1}(\gamma)$, setting $y = \gamma$ in equation (4.1) gives
$$\tau_d \geq T(\gamma; d).$$
By the hypotheses of Section 2, $\lim_{d\to\delta^-} T(\gamma;d) = \infty$. Thus $\lim_{d\to\delta^-} \tau_d = \infty$, as claimed.

Lemma 4.3.
$$\lim_{d\to\delta^-} [E(0) - E(\tau_d)] = 0.$$

Proof of Lemma 4.3. Since $(d/dr)E(r) = (-(N-1)/r)(w'(r))^2$, we have
$$0 \leq E(0) - E(\tau_d) = (N-1)\int_0^{\tau_d} \frac{1}{r}(w'(r))^2\,dr.$$
As in Lemma 4.1, we have
$$0 \leq -w'(r) = |w'(r)| \leq \sqrt{2}\sqrt{F(d) - F(w(r))} \tag{4.4}$$
for $r \in [0, \tau_d]$. Making the change of variables $z = w(r)$, and using Lemma 4.1, we obtain
$$\begin{aligned} 0 \leq E(0) - E(\tau_d) &\leq \sqrt{2}(N-1)\int_\gamma^d \frac{1}{w^{-1}(z)}\sqrt{F(d) - F(z)}\,dz \\ &\leq \sqrt{2}(N-1)\int_\gamma^d \frac{1}{T(z;d)}\sqrt{F(d) - F(z)}\,dz = \Delta(\gamma; d). \end{aligned}$$
By hypothesis, the quantity $\Delta(\gamma;d)$ defined by equation (2.2) has limit zero as $d \to \delta^-$. Thus $\lim_{d\to\delta^-}[E(0) - E(\tau_d)] = 0$, as claimed. □

Remark. We note that $\Delta(y;d)$ is the energy lost during the excursion from $w(0) = d$ to $w(r) = y$, but computed by replacing w with the solution u to the undamped initial value problem (4.3).

Lemma 4.4. *Let w be the solution of the initial value problem* (1.3) *with* (1.6)*, where $|d| < \delta$. Suppose that $w(r)$ is monotonic on the interval (R_1, R_2), where $R_1 > \tau_d$. Then the energy loss $E(R_1) - E(R_2)$ on that interval satisfies the upper bound*

$$0 \leq E(R_1) - E(R_2) \leq \frac{\Gamma(d)}{R_1},$$

where $\Gamma(d)$ is defined by equation (2.5)*.*

Furthermore,

$$\frac{\Gamma(d)}{\tau_d} \longrightarrow 0 \quad \text{as } d \longrightarrow \delta^-,$$

so that

$$[E(R_1) - E(R_2)] \to 0 \quad \text{as } d \to \delta^-.$$

Proof of Lemma 4.4. Since $(d/dr)E(r) = (-(N-1)/r)(w'(r))^2$, we have

$$\begin{aligned} 0 \leq E(R_1) - E(R_2) &= (N-1) \int_{R_1}^{R_2} \frac{1}{r}(w'(r))^2 \, dr \\ &\leq \frac{N-1}{R_1} \int_{R_1}^{R_2} |w'(r)||w'(r)| \, dr. \end{aligned}$$

Using the fact that

$$\begin{aligned} |w'(r)| &= \sqrt{2}\sqrt{E(r) - F(w(r))} \leq \sqrt{2}\sqrt{F(d) - F(w(r))} \\ &\leq \sqrt{2}\sqrt{F(\delta) - F(w(r))}, \end{aligned}$$

and making the change of variable $y = w(r)$, we have

$$E(R_1) - E(R_2) \leq \frac{\sqrt{2}(N-1)}{R_1} \left| \int_{w(R_1)}^{w(R_2)} \sqrt{F(\delta) - F(y)} \, dy \right|.$$

Since $|w(r)| < d$ for all $r > 0$, we have

$$E(R_1) - E(R_2) \leq \frac{\sqrt{2}(N-1)}{R_1} \int_{-d}^{d} \sqrt{F(\delta) - F(y)} \, dy = \frac{\Gamma(d)}{R_1},$$

as claimed.

To show that $\Gamma(d)/\tau_d \to 0$ as $d \to \delta^-$, we recall that $\tau_d \geq T(\gamma; d)$, see Lemma 4.2. Thus $\Gamma(d)/\tau_d \leq \Gamma(d)/T(\gamma; d)$ and, by Lemma 2.1, $\Gamma(d)/T(\gamma; d) \to 0$ as $d \to \delta^-$. This concludes the proof of Lemma 4.4. $\square$

Lemma 4.5. *Let $w_d(r)$ be the solution of* (1.3) *with* (1.6). *If $d \in (\gamma, \delta)$ is sufficiently close to δ, then w_d has a first turning point at $r = R_1(d) > 0$ with $-d < w_d(R_1(d)) < -\gamma$ (and $w_d'(R_1(d)) = 0$). Furthermore,* $\lim_{d\to\delta^-} R_1(d) = \infty$ *and* $\lim_{d\to\delta^-} E(R_1(d)) = F(\delta)$.

Proof of Lemma 4.5. Let $(0, R_d)$ be the maximal interval of (initial) monotonicity of w_d that is established by Lemma 3.2. (If $R_d = \infty$, then the statements below refer to the appropriate limiting values.) We have

$$\begin{aligned} F(\delta) - E(R_d) &= E(0) - E(R_d) \\ &= (E(0) - E(\tau_d)) + (E(\tau_d) - E(R_d)) \\ &\leq [E(0) - E(\tau_d)] + \frac{\Gamma(d)}{\tau_d}, \end{aligned}$$

by Lemma 4.4. As $d \to \delta^-$, the term in square brackets vanishes by virtue of Lemma 4.3, and the last term vanishes by virtue of Lemma 4.4. Thus $\lim_{d\to\delta^-} E(R_d) = F(\delta)$. Therefore, for d sufficiently close to δ, $E(R_d) > 0$, so that $|w_d(R_d)| > \gamma$. This implies R_d is finite, because, by Lemma 3.2, if $R_d = \infty$ then $w(R_d) = L$ where L is a zero of f with $|L| < \delta$, and the only such zeros have $|L| < \gamma$.

Thus, again by Lemma 3.2, w_d has a local minimum at $r = R_1(d) \equiv R_d > 0$, and w_d is decreasing on $(0, R_1(d))$, and $f(w_d(R_1(d))) \leq 0$. Thus $w_d(R_1(d)) \leq \beta$ and $|w_d(R_1(d))| > \gamma$, which implies $-d < w_d(R_1(d)) < -\gamma$.

Finally, $R_1(d) \equiv R_d > \tau_d$ by Lemma 3.2, so that $\lim_{d\to\delta^-} R_1(d) = \infty$ by virtue of Lemma 4.2. This concludes the proof of Lemma 4.5. $\square$

Theorem 4.1. *Suppose f satisfies the hypotheses given in Section* 2. *Let integer $m \geq 1$ be given. There is a number d_m with $\gamma < d_m < \delta$ such that for all d between d_m and δ, the solution of the initial value problem* (1.3) *with* (1.6) *has at least m positive zeros.*

Proof of Theorem 4.1. From Lemma 4.5 we know that for $d \in (\gamma, \delta)$ sufficiently close to δ, the solution $w_d(r)$ to (1.3) with (1.6) has a first turning point at $r = R_1(d)$, with $-d < w_d(R_1(d)) < -\gamma$. Thus w_d has a zero in $(0, R_1(d))$, and the assertion of the theorem is established in the case when $m = 1$. We henceforth assume that $m \geq 2$.

Lemma 4.5 shows that $\lim_{d\to\delta^-} E(R_1(d)) = F(\delta)$. Let $d_* \in (\gamma, \delta)$ be such that $E(R_1(d)) > (1/2)F(\delta)$ for all $d \in (d_*, \delta)$. Since $R_1(d) > \tau_d$, Lemma 4.4 shows that $\lim_{d\to\delta^-} \Gamma(d)/R_1(d) = 0$. Given $m \geq 2$, choose $d_m \in (d_*, \delta)$ so close to δ that $\Gamma(d)/R_1(d) \leq F(\delta)/(4m)$ for all $d \in (d_m, \delta)$.

Consider an interval $(R_1(d), R_2)$ on which $w_d(r)$ is monotonic. By Lemma 4.4, we have

$$\begin{aligned} E(R_2) &= E(R_1(d)) - [E(R_1(d)) - E(R_2)] \\ &\geq E(R_1(d)) - \frac{\Gamma(d)}{R_1(d)} \\ &> \frac{1}{2}F(\delta) - \frac{1}{4m}F(\delta) \\ &= \frac{1}{2}\left(1 - \frac{1}{2m}\right)F(\delta) > 0. \end{aligned}$$

Since $E(R_2)$ is thus positive and bounded away from zero, it follows that $w_d(r)$ has a turning point at $r = R_2(d) > R_1(d)$ with w_d monotonically increasing on $(R_1(d), R_2(d))$ and $\gamma < |w_d(R_2(d))| < \delta$. (Here w_d cannot be monotonic on $(R_1(d), \infty)$ since, if it were, then (as in the proof of Lemma 3.2) its limit value L would be a zero of f with $|L| < \delta$, and then $E(r) \to F(L) < 0$ as $r \to \infty$.) Since the differential equation (1.3) gives $f(w_d(R_2(d))) = -w_d''(R_2(d)) > 0$ at the local maximum $R_2(d)$, we see that $w_d(R_2(d)) > \gamma$, so w_d has a second zero in $(R_1(d), R_2(d))$.

Next (if $m \geq 3$), consider an interval $(R_2(d), R_3)$ on which w_d is monotonic. We have

$$\begin{aligned} E(R_3) &= E(R_2(d)) - [E(R_2(d)) - E(R_3)] \\ &\geq E(R_2(d)) - \frac{\Gamma(d)}{R_2(d)} \\ &\geq E(R_2(d)) - \frac{\Gamma(d)}{R_1(d)} \end{aligned}$$

$$\begin{aligned}
&> \frac{1}{2}\left(1-\frac{1}{2m}\right)F(\delta)-\frac{1}{4m}F(\delta)\\
&= \frac{1}{2}\left(1-\frac{2}{2m}\right)F(\delta)>0.
\end{aligned}$$

Thus again $w_d(r)$ has a turning point at $r = R_3(d) > R_2(d)$ with w_d monotonically decreasing on $(R_2(d), R_3(d))$ and $w_d(R_3(d)) < -\gamma$. Therefore, w_d has a third zero in $(R_2(d), R_3(d))$.

We may continue in this way to find m successive turning points of $w_d(r)$, at $r = R_j(d)$, $j = 1, 2, \dots, m$, with $E(R_j(d)) \geq (1/2)(1-(j-1)/(2m))F(\delta) \geq (1/4)F(\delta) > 0$. Therefore, for all $d \in (d_m, \delta)$, the solution $w_d(r)$ has at least m zeros.

This concludes the proof of Theorem 1. $\square$

5. Proof of the main theorem. The main theorem can now be proven by the method employed in [**6**], which we sketch here. Exactly as in that paper, we can establish the following:

Lemma 5.1. *Suppose $w_{d_*}(r)$ is a solution to the initial value problem* (1.3) *with* (1.6)*, where $d = d_* \in (\beta, \delta)$, such that w_{d_*} has exactly m zeros, and such that $\lim_{r\to\infty} w_{d_*}(r) = 0$. If d is sufficiently close to d_*, then the solution $w_d(r)$ has at most $(m+1)$ zeros.*

To prove the main theorem, we define

$$A_j \equiv \{d \in (\beta, \delta) \mid w_d(r) \text{has exactly } j \text{ positive zeros}\}$$

and we set $d_j \equiv \sup A_j$ for $j = 0, 1, 2, \dots$.

Concerning A_0, we note that A_0 is nonempty because $(\beta, \gamma) \subset A_0$ (since negative initial energy precludes zeros). Also, $d_0 < \delta$, since for d sufficiently near δ, w_d has arbitrarily many zeros, as established by Theorem 1. As in [**6**], we can use the continuous dependence of solutions on initial conditions to establish that $w_{d_0}(r) > 0$ and $w'_{d_0}(r) \leq 0$ for all $r \geq 0$, and $w_{d_0}(r) \to 0$ as $r \to \infty$. Thus $v(x) = w_{d_0}(|x|)$ is the nodeless localized solution whose existence is asserted in the main theorem.

Concerning A_1, Lemma 5.1 establishes that A_1 is nonempty and, by Theorem 1, $d_1 < \delta$. We can again use the continuous dependence of solutions on initial conditions to establish that w_{d_1} has exactly one zero and $w_{d_1}(r) \to 0$ as $r \to \infty$, so that $v(x) = w_{d_1}(|x|)$ is the single-node localized solution whose existence is asserted in the main theorem.

The proof for larger numbers of nodes proceeds in the same way.

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Appendix

A. Linear and sublinear growth. This is an addendum to the work of McLeod, Troy, and Weissler [**6**] on localized radial solutions in $\mathbf{R}^N$ of (1.2). That paper assumed superlinear growth of the nonlinearity: $f(w) = \kappa^2|w|^{p-1}w + g(w)$ where κ^2 is a positive constant, $\lim_{w\to\infty} |g(w)|/|w|^p = 0$, and $1 < p < (N+2)/(N-2)$. Under this assumption the authors proved the existence of an infinite number of localized radial solutions of (1.2). An important lemma in their paper established that the solution of (1.3) with (1.6) has arbitrarily many zeros for sufficiently large values of d. Their method, a rescaling argument in which the large-$|w|$ behavior of $f(w)$ predominates, does not apply if f grows linearly or sublinearly.

In this appendix, we extend their work to the case where the nonlinearity either grows linearly or sublinearly. That is, we assume the hypotheses of [**6**], with the exception of their hypothesis (f4), which we replace with the assumption that

$$\begin{aligned} & f(w) = \kappa^2|w|^{p-1}w + g(w), \\ & \text{where} \quad 0 < p \le 1 \text{ and } \lim_{w\to\infty} \frac{|g(w)|}{|w|^p} = 0 \end{aligned} \tag{A.1}$$

for some positive constant κ^2. Here we establish that also in this case, the initial value problem (1.3) with (1.6) has arbitrarily large numbers of zeros for sufficiently large values of d. The rest of the theorems in their paper hold without modification.

Case I. *Linear growth.* Assume (A.1) with $p = 1$. Suppose $w_d(r)$ is a solution to (1.3) with (1.6). Let $z_d(r) \equiv (1/d)w_d(r)$. Then z_d satisfies

$$z_d'' + \frac{N-1}{r} z_d' + \frac{f(dz_d)}{d} = 0 \tag{A.2}$$

and

$$z_d(0) = 1, \qquad z_d'(0) = 0. \tag{A.3}$$

It then follows, as in [**6**], that $|z_d|$ and $|z_d'|$ are bounded independently of d, and hence that (a subsequence of) z_d converges uniformly on compact sets to the function z that solves

$$z'' + \frac{N-1}{r} z' + \kappa^2 z = 0 \tag{A.4}$$

subject to (A.3). Setting $y(r) \equiv r^{(1/2)(N-2)} z(r)$ yields Bessel's equation

$$y'' + \frac{1}{r} y' + \left(\kappa^2 - \frac{((1/2)(N-2))^2}{r^2} \right) y = 0,$$

whose solutions have infinitely many positive zeros. Thus the solution z to (A.4) with (A.3) has an infinite number of positive zeros. Since $z_d \to z$ uniformly on compact sets, we see that z_d will have as many zeros as desired for large enough values of d.

Case II. *Sublinear growth.*

Assume (A.1) with $0 < p < 1$. Now suppose w_λ is a solution to (1.3) with $w_\lambda(0) = \lambda^{2/(1-p)}$ and $w_\lambda'(0) = 0$. Let $z_\lambda(r) = \lambda^{-2/(1-p)} w(\lambda r)$. (Note that the scaling of the argument is inverse to that in [**6**].) Then z_λ satisfies:

$$z_\lambda'' + \frac{N-1}{r} z_\lambda' + \lambda^{-2p/(1-p)} f(\lambda^{2/(1-p)} z_\lambda) = 0 \tag{A.5}$$

$$z_\lambda(0) = 1, \qquad z_\lambda'(0) = 0. \tag{A.6}$$

It then follows, as in [**6**], that $|z_\lambda|$ and $|z_\lambda'|$ are uniformly bounded, and hence that (a subsequence of) z_λ converges uniformly on compact sets to a function z that satisfies

$$z'' + \frac{N-1}{r} z' + \kappa^2 |z|^{p-1} z = 0 \tag{A.7}$$

$$z(0) = 1, \qquad z'(0) = 0. \tag{A.8}$$

Remark. It is known that solutions of this initial value problem exist for all $r > 0$. Existence for small r follows by the contraction mapping principle. Existence for all $r > 0$ follows from the fact that the decreasing energy bounds z and z'.

Lemma A.1. *Suppose z satisfies* (*A*.7) *with* (*A*.8), *where* $0 < p < 1$. *Then z has an infinite number of isolated zeros.*

Proof of Lemma A.1. We assume for purposes of contradiction that $z > 0$ for all $r > 0$. Multiplying equation (A.7) by r^{N-1} and integrating on $(0, r)$ gives

$$-r^{N-1} z' = \kappa^2 \int_0^r s^{N-1} |z|^{p-1} z \, ds. \tag{A.9}$$

Since $z > 0$ by assumption, the righthand side of (A.9) is positive. Hence $z' < 0$ and so z is a decreasing function. Thus we can estimate the righthand side of (A.9) as follows:

$$-r^{N-1} z' = \kappa^2 \int_0^r s^{N-1} z^p \, ds \geq \kappa^2 z^p \int_0^r s^{N-1} \, ds = \frac{\kappa^2 r^N}{N} z^p.$$

Thus,

$$-z^{-p} z' \geq \kappa^2 r N. \tag{A.10}$$

Integrating (A.10) on $(0, r)$ gives

$$z^{1-p} \leq 1 - \frac{(1-p)\kappa^2}{2N} r^2. \tag{A.11}$$

By assumption $z > 0$ and so the righthand side of (A.11) is positive. On the other hand, $0 < p < 1$ and the righthand side goes to $-\infty$ as $r \to \infty$ and so we obtain a contradiction. Thus z must have a first zero. $\square$

Denote by q_1 the first zero of z. We have $z > 0$ on $[0, q_1)$. Hence, the righthand side of (A.9) with $r = q_1$ is positive, and therefore $z'(q_1) < 0$.

We next show that z has a negative minimum. Again we prove this by contradiction. Suppose that z is decreasing for all $r > q_1$. From above we know that z is bounded for all $r > 0$, hence we must have that

$$\lim_{r\to\infty} z(r) = L.$$

Further, $L < 0$ since $z(q_1) = 0$ and $z'(q_1) < 0$. We also know that z' is bounded, so taking limits in the differential equation gives

$$\lim_{r\to\infty} z''(r) = -\kappa^2 |L|^{p-1} L.$$

Since z' is bounded, the only possible limit is $L = 0$. But we have $L < 0$ and so we have a contradiction. Thus, z must have a first minimum which we denote m_1.

Next, we will show that z has a second zero q_2 with $q_2 > m_1$. So we assume for purposes of contradiction that $z < 0$ for $r > m_1$. Letting $w = -z$, we have that $w > 0$ for $r > m_1$ and that

$$w'' + \frac{N-1}{r} w' + \kappa^2 w^p = 0 \tag{A.12}$$

$$w(m_1) > 0, \qquad w'(m_1) = 0. \tag{A.13}$$

Arguing as above gives

$$-r^{N-1} w' = \int_{m_1}^{r} \kappa^2 s^{N-1} w^p \, ds. \tag{A.14}$$

The righthand side of (A.14) is positive. Therefore, w is decreasing. Thus, for all $r > m_1$,

$$-r^{N-1} w' \geq \kappa^2 w^p \frac{(r^N - m_1^N)}{N}.$$

Choose r large enough so that $r^N - m_1^N \geq (1/2) r^N$, that is, $r \geq r_0 \equiv 2^{1/N} m_1$. Then for $r > r_0$ we have

$$-w^{-p} w' \geq \frac{\kappa^2 r}{2N}. \tag{A.15}$$

Therefore,

$$w^{-p+1} \leq w^{-p+1}(r_0) - \frac{(1-p)\kappa^2}{4N}(r^2 - r_0^2). \tag{A.16}$$

The lefthand side of (A.16) is positive and the righthand side approaches $-\infty$ as $r \to \infty$, and so we obtain a contradiction. Thus, w and hence z must have a zero q_2 with $q_2 > m_1 > q_1$. Further,

$$-q_2^{N-1} z'(q_2) = \int_{m_1}^{q_2} \kappa^2 s^{N-1} |z|^{p-1} z\, ds. \tag{A.17}$$

As above, $z < 0$ on $[m_1, q_2)$ hence the righthand side of (A.17) is negative. Hence, $z'(q_2) > 0$.

As before, it can be shown that z has a local maximum $m_2 > q_2$ and a subsequent zero $q_3 > m_2$. Similarly, it can be shown that z has an infinite number of isolated zeros on $(0, \infty)$. This completes the proof of the lemma. □

B. Verification of Hilltop time-delay and energy-loss limits. In this appendix, we show that the general (hilltop) hypotheses on f given in Section 2 are consequences of some more special assumptions that are easily verified.

We always assume that f is an odd locally Lipschitz function, with primitive F. We assume that there are numbers β, γ, and δ with $0 < \beta < \gamma < \delta \leq \infty$ such that:

$$\begin{aligned} f(\beta) &= 0, \qquad f(s) > 0 \quad \text{for all } s \in (\beta, \delta), \\ F(\gamma) &= 0, \qquad F(s) < 0 \quad \text{for all } s \in (0, \gamma). \end{aligned}$$

If $\delta = \infty$, we furthermore assume that $F(\infty) \equiv \lim_{s\to\infty} F(s)$ is finite.

We begin by showing that the bounds

$$0 < \frac{F(\delta) - F(s)}{f(s)} \leq M_2 < \infty \tag{B.1}$$

and

$$0 < M_1 \leq \frac{\sqrt{F(\delta) - F(s)}}{f(s)} \tag{B.2}$$

for all $s \in [\gamma, \delta)$, where M_1 and M_2 are (finite, positive) constants, are sufficient to guarantee that the conditions

$$\lim_{d \to \delta^-} T(\gamma; d) = \infty \tag{2.3}$$

and

$$\lim_{d \to \delta^-} \Delta(\gamma; d) = 0 \tag{2.4}$$

hold.

For brevity in the following, we define the auxiliary quantities

$$\rho(y; d) \equiv \left[\frac{F(\delta) - F(y)}{F(\delta) - F(d)}\right]^{1/2}, \tag{B.3}$$

and

$$\Phi(r) \equiv \int_1^r \frac{d\phi}{\sqrt{\phi^2 - 1}} = \log(r + \sqrt{r^2 - 1}). \tag{B.4}$$

Recall that $T(y; d)$ is defined for $\gamma \le y < d < \delta$ by

$$T(y; d) \equiv \int_y^d \frac{dz}{\sqrt{2}\sqrt{F(d) - F(z)}}. \tag{2.1}$$

Lemma B.1. *Suppose f satisfies the bound (B.2). Then for $\gamma \le y < d < \delta$,*

$$T(y; d) \ge \sqrt{2} M_1 \Phi(\rho(y; d)). \tag{B.5}$$

Proof of Lemma B.1. We may rewrite (2.1) to obtain

$$T(y; d) = \frac{1}{\sqrt{2}\sqrt{F(\delta) - F(d)}} \int_y^d \frac{dz}{\sqrt{(F(\delta) - F(z))/(F(\delta) - F(d)) - 1}}.$$

Changing from integration variable z to variable $\phi = \rho(z; d)$, we compute

$$dz = -2[F(\delta) - F(d)]^{1/2} \frac{[F(\delta) - F(z)]^{1/2}}{f(z)} \, d\phi,$$

and we have

$$\begin{aligned} T(y; d) &= \sqrt{2} \int_1^{\rho(y;d)} \left\{ [F(\delta) - F(z(\phi))]^{1/2} / f(z(\phi)) \right\} \frac{d\phi}{\sqrt{\phi^2 - 1}} \\ &\geq \sqrt{2} M_1 \Phi(\rho(y; d)), \end{aligned}$$

upon using (B.2), as was to be shown. $\square$

Corollary B.2. *If f satisfies* (B.2)*, then* $\lim_{d \to \delta^-} T(\gamma; d) = \infty$.

Proof of Corollary B.2. Note that $\rho(\gamma; d) \to \infty$ as $d \to \delta^-$. Since $\Phi(r)$ diverges as $r \to \infty$, the assertion follows from Lemma (B.1). $\square$

This shows that hypothesis (2.3) about $T(y; d)$ follows from condition (B.2).

We now investigate hypothesis (2.4) about the quantity $\Delta(\gamma; d)$, defined for $\gamma \leq d < \delta$ by

$$\Delta(\gamma; d) \equiv \sqrt{2}(N - 1) \int_\gamma^d \frac{1}{T(z; d)} \sqrt{F(d) - F(z)} \, dz. \tag{2.2}$$

Lemma B.3. *Suppose f satisfies the bounds* (B.1) *and* (B.2)*. Then for $\gamma \leq d < \delta$,*

$$\Delta(\gamma; d) \leq 2(N - 1) \frac{M_2}{M_1} \sqrt{F(\delta) - F(d)} \int_1^{\rho(\gamma;d)} \frac{\sqrt{\phi^2 - 1}}{\phi \Phi(\phi)} \, d\phi. \tag{B.6}$$

Proof of Lemma B.3. We may rewrite (2.2) as

$$\begin{aligned} \Delta(\gamma; d) = \sqrt{2}(N - 1) &\sqrt{F(\delta) - F(d)} \\ &\cdot \int_\gamma^d \frac{1}{T(z; d)} \sqrt{\frac{F(\delta) - F(z)}{F(\delta) - F(d)} - 1} \, dz. \end{aligned}$$

Using the lower bound (B.5) for $T(y;d)$, we obtain

$$\Delta(\gamma;d) \le \frac{(N-1)}{M_1}\sqrt{F(\delta)-F(d)}\int_\gamma^d \frac{\sqrt{\rho^2(z;d)-1}}{\Phi(\rho(z;d))}\,dz.$$

We now introduce integration variable $\phi = \rho(z;d)$ as before, to get

$$\begin{aligned}\Delta(\gamma;d) \le {}& \frac{2(N-1)}{M_1}\sqrt{F(\delta)-F(d)} \\ & \cdot \int_1^{\rho(\gamma;d)} \left\{\frac{F(\delta)-F(z(\phi))}{f(z(\phi))}\right\} \frac{\sqrt{\phi^2-1}}{\phi\Phi(\phi)}\,d\phi.\end{aligned}$$

Finally, using the inequality (B.1), we obtain

$$\Delta(\gamma;d) \le 2(N-1)\frac{M_2}{M_1}\sqrt{F(\delta)-F(d)}\int_1^{\rho(\gamma;d)} \frac{\sqrt{\phi^2-1}}{\phi\Phi(\phi)}\,d\phi,$$

as claimed. $\square$

Lemma B.4. *If f satisfies* (B.1) *and* (B.2) *then*

$$\lim_{d\to\delta^-}\Delta(\gamma;d) = 0.$$

Proof of Lemma B.4. We estimate the expressions in the upper bound (B.6) for $\Delta(\gamma;d)$:

Using (B.3) and (B.4) in the right side of (B.6), we obtain

$$\Delta(\gamma;d) \le 2(N-1)\frac{M_2}{M_1}\sqrt{F(\delta)}\frac{1}{\rho(\gamma;d)}\int_1^{\rho(\gamma;d)} \frac{\sqrt{\phi^2-1}}{\phi\log(\phi+\sqrt{\phi^2-1})}\,d\phi.$$

Since $\rho(\gamma;d)\to\infty$ as $d\to\delta^-$, it suffices to compute

$$\lim_{r\to\infty}\frac{\int_1^r(\sqrt{\phi^2-1}/(\phi\log(\phi+\sqrt{\phi^2-1})))\,d\phi}{r}.$$

Using L'Hopital's Rule we see that this last limit is zero. Thus,

$$\lim_{d\to\delta^-}\Delta(\gamma;d) = 0,$$

as claimed.

This shows that hypothesis (2.4) about $\Delta(\gamma; d)$ follows from conditions (B.1) and (B.2). $\square$

We now present specific conditions on f that guarantee that the conditions (2.3) and (2.4) hold. We discuss separately the cases of hilltops at finite and infinite values of δ.

I. Hilltop at finite δ. Suppose that δ is finite, and $f(\delta) = 0$. Assume also that F is concave downward near the hilltop, that is, there is $\varepsilon > 0$ such that $F(t) \leq F(s) + f(s)(t-s)$ for all s and t in the interval $[\delta - \varepsilon, \delta]$.

Under these assumptions it is straightforward to show that the bound (B.1) holds. To see this, note first that, because $f(s) > 0$ for all $s \in [\gamma, \delta)$, the function f is bounded away from 0 on the interval $[\gamma, \delta - \varepsilon]$, so the inequality $F(\delta) - F(s) \leq M_3 f(s)$ clearly holds for $s \in [\gamma, \delta - \varepsilon]$ with some finite constant M_3. On the other hand, for s in the interval $[\delta - \varepsilon, \delta]$, the concavity of F implies

$$F(\delta) - F(s) \leq (\delta - s) f(s) \leq \varepsilon f(s).$$

Thus $F(\delta) - F(s) \leq M_2 f(s)$ for all $s \in [\gamma, \delta]$, where $M_2 \equiv \max\{M_3, \varepsilon\}$, and this establishes (B.1).

The Lipschitz continuity of f yields the other bound (B.2). This fact is a consequence of the following lemma:

Lemma B.5. *Suppose f is Lipschitz on $[\gamma, \delta]$ and such that $f(s) > 0$ for all $s \in (\gamma, \delta)$, and $f(\delta) = 0$. Then there is a positive constant C such that*

$$F(\delta) - F(s) \geq C(f(s))^2$$

for all $s \in [\gamma, \delta]$, where F is any antiderivative of f.

Proof of Lemma B.5. Let $K \in (0, \infty)$ be the Lipschitz constant for f: $|f(s) - f(t)| \leq K|s - t|$ for all s and t in $[\gamma, \delta]$. Extend f to all of $\mathbf{R}$

by defining

$$\tilde{f}(s) \equiv \begin{cases} f(\gamma) & \text{if } s < \gamma \\ f(s) & \text{if } s \in [\gamma, \delta] \\ f(\delta) & \text{if } s > \delta \end{cases}$$

We note that $\tilde{f}$ is also Lipschitz with constant K, on all of $\mathbf{R}$. Let $\phi \in C_0^\infty(\mathbf{R})$ be nonnegative and such that $\int_{-\infty}^{\infty} \phi(s)\,ds = 1$, and define $\phi_\varepsilon(t) \equiv (1/\varepsilon)\phi(t/\varepsilon)$ for $\varepsilon \neq 0$. Define the C^∞ function f_ε by $f_\varepsilon \equiv \phi_\varepsilon * \tilde{f}$. Then, as $\varepsilon \to 0$, f_ε converges uniformly on $[\gamma, \delta]$ to f.

Let F_ε be any antiderivative of f_ε. It follows that $F_\varepsilon(\delta) - F_\varepsilon(s)$ converges uniformly to $F(\delta) - F(s)$ for $s \in [\gamma, \delta]$, as $\varepsilon \to 0$.

Now,

$$\begin{aligned} |f_\varepsilon(s) - f_\varepsilon(t)| &\leq \int_{-\infty}^{\infty} |\tilde{f}(s-r) - \tilde{f}(t-r)| \phi_\varepsilon(r)\,dr \\ &\leq \int_{-\infty}^{\infty} K|s-t|\phi_\varepsilon(r)\,dr = K|s-t|, \end{aligned}$$

so we see that $|f_\varepsilon'(t)| \leq K$, independent of ε and of t. In particular, $-(1/K)f_\varepsilon'(t) \leq 1$, so that (since f_ε is nonnegative)

$$\begin{aligned} & -\frac{2}{2K} f_\varepsilon(t) f_\varepsilon'(t) \leq f_\varepsilon(t) \\ \Longrightarrow \quad & -\frac{1}{2K} \int_s^\delta \frac{d}{dt}(f_\varepsilon(t))^2\,dt \leq \int_s^\delta f_\varepsilon(t)\,dt \\ \Longrightarrow \quad & \frac{1}{2K}[(f_\varepsilon(s))^2 - (f_\varepsilon(\delta))^2] \leq F_\varepsilon(\delta) - F_\varepsilon(s), \end{aligned}$$

for all $s \in [\gamma, \delta]$, and all $\varepsilon \neq 0$. Taking the limit as $\varepsilon \to 0$, we conclude that

$$F(\delta) - F(s) \geq \frac{1}{2K}(f(s))^2$$

for all $s \in [\gamma, \delta]$, as claimed. □

We have thus established that bounds (B.1) and (B.2) hold for the case of a hilltop at finite δ. Lemma B.2 and Lemma B.4 hence show that, for this case, the hypotheses of Section 2 hold.

II. Hilltop at infinity. Suppose $\delta = \infty$. If $f(s)$ has exponential decay for large s, then the bounds (B.1) and (B.2) hold, and Lemmas B.2 and B.4 establish the hypotheses of Section 2.

If, on the other hand, $f(s)$ is asymptotic to a multiple of s^{-p} for some constant $p > 1$, the bounds (B.1) and (B.2) do not hold. Nevertheless, straightforward computation of $T(\gamma; d)$ and $\Delta(\gamma; d)$ shows that the hypotheses of Section 2 hold in this case as well.

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