

HERMITE ORTHOGONAL RATIONAL FUNCTIONS

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Dedicated to William B. Jones on the occasion of his 70th birthday

ABSTRACT. We recount previous development of d -fold doubling of orthogonal polynomial sequences and give new results on rational function coefficients, recurrence formulas, continued fractions, Rodrigues' type formulas, and differential equations, for the general case and, in particular, for the d -fold Hermite orthogonal rational functions.

1. Introduction. Orthogonal rational functions and related subjects today are active areas of investigation for researchers around the world, yielding theoretical and applicable results spanning a gamut of interests. Included among these are various techniques and methods of numerical integration. [1, 2, 3, 6, 9, 10, 11, 15] are examples. It was shown in [10] that the transformation given in [12, 13] taking systems of orthogonal polynomials to systems of orthogonal Laurent polynomials applies in a more general context of certain function spaces, leading to a recursive construction of hierarchies of rational function spaces. Consequently, Gaussian quadrature rules of a new kind were obtained, ones in which the number of nodes are doubled and redoubled and the abscissas and weights themselves are given by simple recursive formulas, extending the results in [9, 11].

It is our goal here to explicate the family of orthogonal rational function sequences that is constructed from the classical Hermite polynomials, although new and recounted theorems are given for the general case of d -fold doubling of orthogonal polynomial sequences that was introduced in [10].

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2. Recursive transformation. The *doubling transformation* of parameters $\gamma, \lambda > 0$ we denote here by

$$(2.1) \quad v^{(\gamma, \lambda)}(x) := \frac{1}{\lambda} \left(x - \frac{\gamma}{x} \right).$$

Its inverses are

$$(2.2) \quad v_j^{(\gamma, \lambda)}(y) := \frac{\lambda}{2} \left(y + j \sqrt{y^2 + \frac{4\gamma}{\lambda^2}} \right), \quad j = \pm 1.$$

The work in [10] begins with the following very general theorem concerning recursive building of function spaces. The spaces need not be comprised of rational functions and may not be ordered by inclusion.

Theorem 2.1 (Recursive transformation, proof in [10]). *Let m be a non-negative integer and N_m be a positive integer. Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^{\infty}$ be a sequence of pairs of positive real numbers, and let $v^{(\gamma_{m+1}, \lambda_{m+1})}(x)$ and $v_{\pm 1}^{(\gamma_{m+1}, \lambda_{m+1})}(y)$ be given by (2.1) and (2.2), respectively. Suppose*

(i) $I_m^{(\gamma, \lambda)} := \cup_{i=1}^{N_m} (a_{m,i}^{(\gamma, \lambda)}, b_{m,i}^{(\gamma, \lambda)})$ is the union of disjoint intervals $(a_{m,i}^{(\gamma, \lambda)}, b_{m,i}^{(\gamma, \lambda)})$, $i = 1, 2, 3, \dots, N_m$,

(ii) $\mathbf{F}_m^{(\gamma, \lambda)}$ is a real vector space of real, continuous functions on $I_m^{(\gamma, \lambda)}$,

(iii) $w_m^{(\gamma, \lambda)}(x)$ is a non-negative real, continuous function on $I_m^{(\gamma, \lambda)}$ such that $(f, g)_m^{(\gamma, \lambda)} := \int_{I_m^{(\gamma, \lambda)}} f(x)g(x)w_m^{(\gamma, \lambda)}(x) dx$ is an inner product on $\mathbf{F}_m^{(\gamma, \lambda)}$,

(iv) $F_m^{(\gamma, \lambda)} := \{f_{m,n}^{(\gamma, \lambda)}(x)\}_{n=0}^{\infty}$ is an ordered orthogonal basis for $\mathbf{F}_m^{(\gamma, \lambda)}$ under $(\cdot, \cdot)_m^{(\gamma, \lambda)}$.

Set

$$(2.3a) \quad N_{m+1} := 2N_m,$$

$$(2.3b) \quad a_{m+1,i}^{(\gamma, \lambda)} := v_{-1}^{(\gamma_{m+1}, \lambda_{m+1})}(a_{m,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, \dots, N_m,$$

$$(2.3c) \quad a_{m+1, N_m+i}^{(\gamma, \lambda)} := v_1^{(\gamma_{m+1}, \lambda_{m+1})}(a_{m,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, \dots, N_m,$$

$$(2.3d) \quad b_{m+1,i}^{(\gamma, \lambda)} := v_{-1}^{(\gamma_{m+1}, \lambda_{m+1})}(b_{m,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, \dots, N_m,$$

$$(2.3e) \quad b_{m+1, N_m+i}^{(\gamma, \lambda)} := v_1^{(\gamma_{m+1}, \lambda_{m+1})}(b_{m,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, \dots, N_m,$$

$$(2.3f) \quad f_{m+1, 2k}^{(\gamma, \lambda)}(x) := \lambda_{m+1}^k f_{m,k}^{(\gamma, \lambda)}(v^{(\gamma_{m+1}, \lambda_{m+1})}(x)), \quad k = 0, 1, 2, \dots,$$

$$(2.3g) \quad f_{m+1, 2k+1}^{(\gamma, \lambda)}(x) := (-\lambda_{m+1}/\gamma_{m+1})^k (1/x) f_{m,k}^{(\gamma, \lambda)}(v^{(\gamma_{m+1}, \lambda_{m+1})}(x)), \\ k = 0, 1, 2, \dots,$$

$$(2.3h) \quad F_{m+1}^{(\gamma, \lambda)} := \{f_{m+1, n}^{(\gamma, \lambda)}(x)\}_{n=0}^\infty$$

$$(2.3i) \quad \mathbf{F}_{m+1}^{(\gamma, \lambda)} := \text{the real linear span of } F_{m+1}^{(\gamma, \lambda)}$$

$$(2.3j) \quad w_{m+1}^{(\gamma, \lambda)}(x) := w_m^{(\gamma, \lambda)}(v^{(\gamma_{m+1}, \lambda_{m+1})}(x)).$$

Then, with reference to the equations (2.3),

(I) $I_{m+1}^{(\gamma, \lambda)} := \cup_{i=1}^{N_m+1} (a_{m+1, i}^{(\gamma, \lambda)}, b_{m+1, i}^{(\gamma, \lambda)})$ is the union of disjoint intervals $(a_{m+1, i}^{(\gamma, \lambda)}, b_{m+1, i}^{(\gamma, \lambda)})$, $i = 1, 2, 3, \dots, N_m+1$,

(II) $\mathbf{F}_{m+1}^{(\gamma, \lambda)}$ is a real vector space of real, continuous functions on $I_{m+1}^{(\gamma, \lambda)}$,

(III) $w_{m+1}^{(\gamma, \lambda)}(x)$ is a non-negative real, continuous function on $I_{m+1}^{(\gamma, \lambda)}$ such that $(f, g)_{m+1}^{(\gamma, \lambda)} := \int_{I_{m+1}^{(\gamma, \lambda)}} f(x)g(x)w_{m+1}^{(\gamma, \lambda)}(x) dx$ is an inner product on $\mathbf{F}_{m+1}^{(\gamma, \lambda)}$,

(IV) $F_{m+1}^{(\gamma, \lambda)} := \{f_{m+1, n}^{(\gamma, \lambda)}(x)\}_{n=0}^\infty$ is an ordered orthogonal basis for $\mathbf{F}_{m+1}^{(\gamma, \lambda)}$ under $(\cdot, \cdot)_{m+1}^{(\gamma, \lambda)}$.

3. ORFS constructions and d -fold Gauss quadrature. The theory of Orthogonal Polynomial Sequences (OPS's) is a relatively rich one, and the theories of Orthogonal Laurent Polynomial Sequences (OLPS's) and, still more generally, Orthogonal Rational Function Sequences (ORFS's) are currently experiencing a period of rapid growth. For introductions to these subjects, we suggest [5, 21] for OPS's, [7, 17] for OLPS's and [4] for ORFS's. Recent studies in addition to [12, 13] have examined OLPS's generated from OPS's via transformations. For example, in [8], the doubling transformation (2.1) is used to develop the strong Chebyshev OLPS, and in [18, 19], a closely related

transformation of the form $u(x) = 1/\lambda(\sqrt{x} - \gamma/\sqrt{x})$ is used to generate strong moment distribution functions from symmetric moment distribution functions. Considering the recursive transformation theorem and the work done in [9, 11, 12, 13], we now recount those results given in [10] that show how OPS's can be used to construct not only OLPS's, but ORFS's and associated Gaussian quadrature rules.

We restrict our attention here to continuous, non-negative weight functions $w(x)$ on unions of disjoint intervals $I := \cup_{i=1}^N (a_i, b_i) \subseteq \mathbf{R}$ which give inner products,

$$(f, g) := \int_I f(x)g(x)w(x) dx,$$

on infinite dimensional subspaces \mathbf{P} of the space of real rational functions, where \mathbf{P} has an ordered orthogonal basis $P := \{P_n(x)\}_{n=0}^\infty$, orthogonal with respect to (\cdot, \cdot) . P is called an *Orthogonal Rational Function Sequence* (ORFS) with respect to (\cdot, \cdot) on \mathbf{P} . If \mathbf{P} is the space of real polynomials and the ORFS P is ordered by polynomial degree, P is called an *Orthogonal Polynomial Sequence* (OPS). If \mathbf{P} is the space of real Laurent polynomials with ordered basis $\{1, x^{-1}, x, x^{-2}, x^2, x^{-3}, x^3, \dots\}$, implying the term *L-degree*, and the ORFS P is ordered by L-degree, P is called an *Orthogonal Laurent Polynomial Sequence* (OLPS).

Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$ be a sequence of pairs of positive real numbers, and assume $w_0^{(\gamma, \lambda)}(x)$ is a continuous, non-negative weight function on an interval $I_0^{(\gamma, \lambda)} := (a_{0,1}^{(\gamma, \lambda)}, b_{0,1}^{(\gamma, \lambda)})$, $-\infty \leq a_{0,1}^{(\gamma, \lambda)} < b_{0,1}^{(\gamma, \lambda)} \leq \infty$, giving an inner product,

$$(3.1) \quad (f, g)_0^{(\gamma, \lambda)} := \int_{a_{0,1}^{(\gamma, \lambda)}}^{b_{0,1}^{(\gamma, \lambda)}} f(x)g(x)w_0^{(\gamma, \lambda)}(x) dx,$$

on the space of real polynomials, which we designate by $\mathbf{P}_0^{(\gamma, \lambda)}$. We let $P_0^{(\gamma, \lambda)} := \{P_{0,n}^{(\gamma, \lambda)}(x)\}_{n=0}^\infty$ denote the OPS of monic polynomials. For fixed positive integer n ,

$$(3.2) \quad P_{0,n}^{(\gamma, \lambda)}(x) = \prod_{k=1}^n (x - x_{0,n,k}^{(\gamma, \lambda)}),$$

for distinct real zeros $x_{0,n,1}^{(\gamma,\lambda)}, x_{0,n,2}^{(\gamma,\lambda)}, x_{0,n,3}^{(\gamma,\lambda)}, \dots, x_{0,n,n}^{(\gamma,\lambda)}$. For functions $f(x)$ where the integral exists, Gaussian quadrature, as in [14] for example, then gives unique positive real numbers $w_{0,n,1}^{(\gamma,\lambda)}, w_{0,n,2}^{(\gamma,\lambda)}, w_{0,n,3}^{(\gamma,\lambda)}, \dots, w_{0,n,n}^{(\gamma,\lambda)}$ such that

$$(3.3a) \quad \int_{a_{0,1}^{(\gamma,\lambda)}}^{b_{0,1}^{(\gamma,\lambda)}} f(x)w_0^{(\gamma,\lambda)}(x) dx = \sum_{k=1}^n f(x_{0,n,k}^{(\gamma,\lambda)})w_{0,n,k}^{(\gamma,\lambda)} + E_{0,n}^{(\gamma,\lambda)}[f(x)],$$

where $E_{0,n}^{(\gamma,\lambda)}(x^m) = 0$, for $m = 0, 1, \dots, 2n - 1$, or, more generally,

$$(3.3b) \quad E_{0,n}^{(\gamma,\lambda)}[f(x)] = \frac{f^{(2n)}(\nu)}{(2n)!} (P_{0,n}^{(\gamma,\lambda)}, P_{0,n}^{(\gamma,\lambda)})_0^{(\gamma,\lambda)},$$

for $f(x)$ having a continuous $(2n)$ -th derivative and some $\nu \in (a_{0,1}^{(\gamma,\lambda)}, b_{0,1}^{(\gamma,\lambda)})$.

Applying the recursive transformation theorem, we obtain a sequence $\{\mathbf{P}_d^{(\gamma,\lambda)}\}_{d=0}^\infty$ of function spaces. To begin to describe the spaces $\mathbf{P}_d^{(\gamma,\lambda)}$, let

$$(3.4a) \quad s_{d,0}^{(\gamma,\lambda)} := -\infty, \quad d = 0, 1, 2, \dots,$$

$$(3.4b) \quad s_{0,1}^{(\gamma,\lambda)} := \infty,$$

and define recursively, for $d = 1, 2, 3, \dots$,

$$(3.4c) \quad s_{d,j}^{(\gamma,\lambda)} := v_{-1}^{(\gamma_d, \lambda_d)}(s_{d-1,j}^{(\gamma,\lambda)}), \quad j = 1, 2, 3, \dots, 2^{d-1},$$

$$(3.4d) \quad s_{d,j}^{(\gamma,\lambda)} := v_1^{(\gamma_d, \lambda_d)}(s_{d-1,j-2^{d-1}}^{(\gamma,\lambda)}), \quad j = 2^{d-1} + 1, 2^{d-1} + 2, \dots, 2^d.$$

Form the finite sequences, for $d = 0, 1, 2, \dots$,

$$(3.4e) \quad S_d^{(\gamma,\lambda)} := \{s_{d,j}^{(\gamma,\lambda)}\}_{j=1}^{2^d}.$$

It follows by induction, using monotonicity of $v_{\pm 1}^{(\gamma,\lambda)}(y)$ and the fact that $v_{-1}^{(\gamma,\lambda)}(y) = v_1^{(\gamma,\lambda)}(-y)$, that $S_d^{(\gamma,\lambda)} \setminus \{\infty\}$ is monotone increasing and symmetric about the origin.

Next, form the sets, for $k = 1, 2, 3, \dots$,

$$(3.5a) \quad B_{0,k}^{(\gamma,\lambda)} := \{x^k\},$$

and, for $d = 1, 2, 3, \dots$,

$$(3.5b) \quad B_{d,k}^{(\gamma,\lambda)} := \left\{ \frac{1}{(x - s_{d,1}^{(\gamma,\lambda)})^k}, \frac{1}{(x - s_{d,2}^{(\gamma,\lambda)})^k}, \frac{1}{(x - s_{d,3}^{(\gamma,\lambda)})^k}, \dots, \right. \\ \left. \frac{1}{(x - s_{d,2^d-1}^{(\gamma,\lambda)})^k}, x^k \right\},$$

and finally, for $d = 0, 1, 2, \dots$, set

$$(3.5c) \quad B_d^{N(\gamma,\lambda)} := \{1\} \cup \bigcup_{k=1}^N B_{d,k}^{(\gamma,\lambda)}$$

and

$$(3.5d) \quad B_d^{(\gamma,\lambda)} := \{1\} \cup \bigcup_{k=1}^{\infty} B_{d,k}^{(\gamma,\lambda)}.$$

$B_d^{(\gamma,\lambda)}$ is a linearly independent set of rational functions with the set of poles $S_d^{(\gamma,\lambda)}$. $B_d^{(\gamma,\lambda)}$ forms a basis for a real vector space of rational functions. For example, $B_0^{(\gamma,\lambda)} = \{1, x, x^2, x^3, \dots\}$ is a basis for $\mathbf{P}_0^{(\gamma,\lambda)}$, the space of real polynomials. By the work in [12, 13], $\mathbf{P}_1^{(\gamma,\lambda)}$ is the space of real Laurent polynomials, and hence has basis $B_1^{(\gamma,\lambda)} = \{1, 1/x, x, 1/x^2, x^2, 1/x^3, x^3, \dots\}$. In general, we have the following theorem.

Theorem 3.1 (Proof in [10]). *Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^{\infty}$ be a sequence of pairs of positive real numbers, and let d be any non-negative integer. $B_d^{(\gamma,\lambda)}$ given in (3.5) is a basis for $\mathbf{P}_d^{(\gamma,\lambda)}$, the d -fold doubling of the space $\mathbf{P}_0^{(\gamma,\lambda)}$ of real polynomials, given by Theorem 2.1. Hence, $S_d^{(\gamma,\lambda)}$ given in (3.4) is the set of poles of the rational functions of $\mathbf{P}_d^{(\gamma,\lambda)}$.*

Remark 3.2. We suppress the explicit notational reference to the dependence on the sequence of parameters (γ, λ) at times, hereafter, when the relief from notation is deemed desirable for expositional purposes and there is little loss of clarity.

In the cases $d = 0, 1$, if we consider B_d as an ordered set, with ordering as implied in the definition (3.5), then P_d is obtained by applying the Gram-Schmidt process to B_d , using $(\cdot, \cdot)_d$ provided by the recursive transformation theorem. However, a close reading of the argument in [10] that $\mathbf{B}_d \subset \mathbf{P}_d$ reveals that this is not true for $d = 2, 3, 4, \dots$. Next, we collect some additional facts pertaining to the structure of the ordered orthogonal basis P_d of \mathbf{P}_d before turning our attention to related quadrature rules. The next theorem is an easy consequence of the previous two.

Theorem 3.3 (*d-Fold doubling of Orthogonal Polynomial Sequences*, proof in [10]). *Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$ be a sequence of pairs of positive real numbers, and let d be any positive integer. Suppose $P_0^{(\gamma, \lambda)} := \{P_{0,n}^{(\gamma, \lambda)}(x)\}_{n=0}^\infty$ is an OPS with respect to $(f, g)_0^{(\gamma, \lambda)} := \int_{a_{0,1}^{(\gamma, \lambda)}}^{b_{0,1}^{(\gamma, \lambda)}} f(x)g(x)w_0^{(\gamma, \lambda)}(x) dx$ on the space $\mathbf{P}_0^{(\gamma, \lambda)}$ of real polynomials, where $w_0^{(\gamma, \lambda)}(x)$ is a continuous, non-negative weight function defined on an interval $I_0^{(\gamma, \lambda)} := (a_{0,1}^{(\gamma, \lambda)}, b_{0,1}^{(\gamma, \lambda)})$, $-\infty \leq a_{0,1}^{(\gamma, \lambda)} < b_{0,1}^{(\gamma, \lambda)} \leq \infty$.*

Then $P_d^{(\gamma, \lambda)}$ is an ORFS with respect to $(f, g)_d^{(\gamma, \lambda)}$, given by the continuous, non-negative weight function $w_d^{(\gamma, \lambda)}(x)$ defined on the union $I_d^{(\gamma, \lambda)}$ of disjoint intervals, on the space $\mathbf{P}_d^{(\gamma, \lambda)}$ of real rational functions with basis $B_d^{(\gamma, \lambda)}$ given in (3.5), where

$$(3.6a) \quad P_{d,2k}^{(\gamma, \lambda)}(x) := \lambda_d^k P_{d-1,k}^{(\gamma, \lambda)}(v^{(\gamma_d, \lambda_d)}(x)), \quad k = 0, 1, 2, \dots,$$

$$(3.6b) \quad P_{d,2k+1}^{(\gamma, \lambda)}(x) := (-\lambda_d/\gamma_d)^k (1/x) P_{d-1,k}^{(\gamma, \lambda)}(v^{(\gamma_d, \lambda_d)}(x)), \\ k = 0, 1, 2, \dots,$$

$$(3.6c) \quad P_d^{(\gamma, \lambda)} := \{P_{d,n}^{(\gamma, \lambda)}(x)\}_{n=0}^\infty,$$

$$(3.7a) \quad a_{d,i}^{(\gamma, \lambda)} := v_{-1}^{(\gamma_d, \lambda_d)}(a_{d-1,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, \dots, 2^{d-1},$$

$$(3.7b) \quad a_{d,2^{d-1}+i}^{(\gamma, \lambda)} := v_1^{(\gamma_d, \lambda_d)}(a_{d-1,i}^{(\gamma, \lambda)}), \quad i = 1, 2, 3, 2^{d-1},$$

$$\begin{aligned}
 (3.7c) \quad & b_{d,i}^{(\gamma,\lambda)} := v_{-1}^{(\gamma_d,\lambda_d)}(b_{d-1,i}^{(\gamma,\lambda)}), \quad i = 1, 2, 3, \dots, 2^{d-1}, \\
 (3.7d) \quad & b_{d,2^{d-1}+i}^{(\gamma,\lambda)} := v_1^{(\gamma_d,\lambda_d)}(b_{d-1,i}^{(\gamma,\lambda)}), \quad i = 1, 2, 3, \dots, 2^{d-1}, \\
 (3.7e) \quad & I_d^{(\gamma,\lambda)} := \bigcup_{j=1}^{2^d} (a_{d,j}^{(\gamma,\lambda)}, b_{d,j}^{(\gamma,\lambda)}), \\
 (3.7f) \quad & w_d^{(\gamma,\lambda)}(x) := w_{d-1}^{(\gamma,\lambda)}(v^{(\gamma_d,\lambda_d)}(x)), \\
 (3.7g) \quad & (f, g)_d^{(\gamma,\lambda)} := \int_{I_d^{(\gamma,\lambda)}} f(x)g(x)w_d^{(\gamma,\lambda)}(x) dx,
 \end{aligned}$$

for $v^{(\gamma_d,\lambda_d)}(x)$ and $v_{\pm 1}^{(\gamma_d,\lambda_d)}(y)$ given by (2.1) and (2.2), respectively.

The next two results follow by induction arguments on d .

Theorem 3.4 (*d*-Fold doubling of zeros, see [10]). *Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$ be a sequence of pairs of positive real numbers, and let d be any positive integer. Suppose that $\{P_{0,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ is an OPS, as in Theorem 3.3, and that $P_{0,n}^{(\gamma,\lambda)}(x)$, for $n \geq 1$, has zeros $\{x_{0,n,k}^{(\gamma,\lambda)}\}_{k=1}^n$ satisfying*

$$\begin{aligned}
 -\infty &=: s_{0,0}^{(\gamma,\lambda)} \leq a_{0,1}^{(\gamma,\lambda)} < x_{0,n,1}^{(\gamma,\lambda)} < x_{0,n,2}^{(\gamma,\lambda)} < \dots < x_{0,n,n}^{(\gamma,\lambda)} < b_{0,1}^{(\gamma,\lambda)} \leq s_{0,1}^{(\gamma,\lambda)} \\
 &:= \infty.
 \end{aligned}$$

For $m = 0, 1, 2, \dots, 2^d - 1$, $P_{d,2^d n+m}^{(\gamma,\lambda)}(x)$ given in (3.6) has $2^d n$ real, distinct, simple zeros, denoted by $\{x_{d,n,k}^{(\gamma,\lambda)}\}_{k=1}^{2^d n}$, which satisfy

$$\begin{aligned}
 (3.8a) \quad & x_{d,n,i}^{(\gamma,\lambda)} = v_{-1}^{(\gamma_d,\lambda_d)}(x_{d-1,n,i}^{(\gamma,\lambda)}), \quad i = 1, 2, 3, \dots, 2^{d-1}n, \\
 (3.8b) \quad & x_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)} = v_1^{(\gamma_d,\lambda_d)}(x_{d-1,n,i}^{(\gamma,\lambda)}), \quad i = 1, 2, 3, \dots, 2^{d-1}n,
 \end{aligned}$$

for $v_{\pm 1}^{(\gamma_d,\lambda_d)}(y)$ given by (2.2), and satisfy

$$\begin{aligned}
 (3.9) \quad & s_{d,j-1}^{(\gamma,\lambda)} \leq a_{d,j}^{(\gamma,\lambda)} < x_{d,n,n(j-1)+1}^{(\gamma,\lambda)} < x_{d,n,n(j-1)+2}^{(\gamma,\lambda)} < \dots < x_{d,n,nj}^{(\gamma,\lambda)} < b_{d,j} \\
 & \leq s_{d,j}^{(\gamma,\lambda)}
 \end{aligned}$$

for $j = 1, 2, 3, \dots, 2^d$, where $s_{d,j}^{(\gamma,\lambda)}$ is given by (3.4) and $a_{d,j}^{(\gamma,\lambda)}, b_{d,j}^{(\gamma,\lambda)}$ by (3.7).

Theorem 3.5 (See [10]). *Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$ be a sequence of pairs of positive real numbers, and let d be any positive integer. For any positive integer n , $P_{d,2^d n}^{(\gamma,\lambda)}(x)$, defined in (3.6), is given by*

$$(3.10) \quad P_{d,2^d n}^{(\gamma,\lambda)}(x) = \prod_{k=1}^{2^d n} (x - x_{d,n,k}^{(\gamma,\lambda)}) \prod_{j=1}^{2^d-1} \left(\frac{1}{x - s_{d,j}^{(\gamma,\lambda)}} \right)^n,$$

where $s_{d,j}^{(\gamma,\lambda)}$ is given by (3.4) and $x_{d,n,k}^{(\gamma,\lambda)}$ by (3.8).

We turn our attention now to Gaussian Quadrature.

Theorem 3.6 (*d*-Fold Gauss quadrature, proof in [10]). *Proceed under the assumptions and notation of Theorem 3.3 and Theorem 3.4. For $k = 1, 2, \dots, 2^d n$, set*

$$(3.11) \quad w_{d,n,k}^{(\gamma,\lambda)} := \int_{I_d^{(\gamma,\lambda)}} l_{d,n,k}^{(\gamma,\lambda)}(x) w_d^{(\gamma,\lambda)}(x) dx,$$

where

$$(3.12) \quad l_{d,n,k}^{(\gamma,\lambda)}(x) := \frac{P_{d,2^d n}^{(\gamma,\lambda)}(x)}{(x - x_{d,n,k}^{(\gamma,\lambda)}) P'_{d,2^d n}^{(\gamma,\lambda)}(x_{d,n,k}^{(\gamma,\lambda)})}.$$

Then, for every rational function $q(x)$ in the real linear span of $B_d^{2n(\gamma,\lambda)} \setminus \{x^{2n}\}$,

$$(3.13) \quad \int_{I_d^{(\gamma,\lambda)}} q(x) w_d^{(\gamma,\lambda)}(x) dx = \sum_{k=1}^{2^d n} q(x_{d,n,k}^{(\gamma,\lambda)}) w_{d,n,k}^{(\gamma,\lambda)}.$$

We call (3.13) the $(2^d n)$ -point d -fold Gauss quadrature formula of parameters $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$.

Theorem 3.7 (*d*-Fold Gauss quadrature weights, proof in [10]).
 Let d and n be positive integers, and let $x_{d,n,k}^{(\gamma,\lambda)}$ and $w_{d,n,k}^{(\gamma,\lambda)}$, for $k = 1, 2, 3, \dots, 2^d n$, be the abscissas and weights, respectively, in the $(2^d n)$ -point d -fold Gauss quadrature formula of parameters $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$, formula (3.13). Then

$$(3.14a) \quad w_{d-1,n,i}^{(\gamma,\lambda)} = v^{(\gamma_d, \lambda_d)}(x_{d,n,i}^{(\gamma,\lambda)}) w_{d,n,i}^{(\gamma,\lambda)}, \quad i = 1, 2, 3, \dots, 2^{d-1}n,$$

$$(3.14b) \quad w_{d-1,n,i}^{(\gamma,\lambda)} = v^{(\gamma_d, \lambda_d)}(x_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)}) w_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)}, \\ i = 1, 2, 3, \dots, 2^{d-1}n,$$

where $w_{d-1,n,i}^{(\gamma,\lambda)}$, for $i = 1, 2, 3, \dots, 2^{d-1}n$, are the weights in the $(2^{d-1}n)$ -point $(d-1)$ -fold Gauss quadrature formula of parameters $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$.

Theorem 3.8 (*d*-Fold Gauss quadrature error, proof in [10]). Proceed under the assumptions and notation of Theorems 3.3, 3.4 and 3.6. Suppose

$$(3.15) \quad Q(x) := q(x) \prod_{j=1}^{2^d-1} (x - s_{d,j})^{2^n}$$

has a continuous $(2^{d+1}n)$ -th derivative on $I_d^{(\gamma,\lambda)}$. Then there exists ε_j in $(a_{d,1}^{(\gamma,\lambda)}, b_{d,2^d}^{(\gamma,\lambda)})$, for $j = 1, 2, \dots, 2^d$, such that

$$(3.16) \quad \int_{I_d^{(\gamma,\lambda)}} q(x) w_d^{(\gamma,\lambda)}(x) dx = \sum_{k=1}^{2^d n} q(x_{d,n,k}^{(\gamma,\lambda)}) w_{d,n,k}^{(\gamma,\lambda)} + E_{d,n}^{(\gamma,\lambda)}[q(x)],$$

where

$$(3.17) \quad E_{d,n}^{(\gamma,\lambda)}[q(x)] = \sum_{j=1}^{2^d} \frac{Q^{(2^{d+1}n)}(\varepsilon_j)}{(2^{d+1}n)!} \int_{a_{d,j}^{(\gamma,\lambda)}}^{b_{d,j}^{(\gamma,\lambda)}} [P_{d,2^d n}^{(\gamma,\lambda)}(x)]^2 w_d(x) dx.$$

These results concerning d -fold doubling and Gaussian quadrature constituted the successful completion of the main goal of the work in

[10] and give a template from which special case studies can begin. As we shall see in the next section, an OPS with support $I_0 = (-\infty, \infty)$, like the Hermite polynomials, offers an opportunity for useful refinements of the general d -fold Gauss quadrature presented above. In addition, in any case, taking (γ, λ) to be a constant sequence of parameters leads to simpler descriptions and formulas, as illustrated by the next theorem. Theorem 3.9 also gives some justification in calling \mathbf{P}_d the d -fold doubling of \mathbf{P}_0 and in the use of similar terminology in the more general context of the recursive transformation theorem.

Theorem 3.9 (Proof in [10]). *Let $(\gamma, \lambda) := \{(\gamma_d, \lambda_d)\}_{d=1}^\infty$ be a sequence of pairs of positive real numbers, and let $\{\mathbf{P}_d^{(\gamma, \lambda)}\}_{d=0}^\infty$ be the sequence of real inner product spaces of rational functions such that $\mathbf{P}_d^{(\gamma, \lambda)}$ has basis $B_d^{(\gamma, \lambda)}$ given in (3.5). For any $d \geq 2$, $\mathbf{P}_0^{(\gamma, \lambda)} \subset \mathbf{P}_1^{(\gamma, \lambda)} \subset \mathbf{P}_d^{(\gamma, \lambda)}$. However, $\mathbf{P}_0^{(\gamma, \lambda)} \subset \mathbf{P}_1^{(\gamma, \lambda)} \subset \mathbf{P}_2^{(\gamma, \lambda)} \subset \dots$ if and only if (γ, λ) is a constant sequence.*

Remark 3.10. Henceforth, at times we will find it convenient to identify a constant sequence (γ, λ) with the ordered pair of the same name, and vice versa.

Remark 3.11. As mentioned above, technically, we are concerned with \mathbf{P}_d as real-valued functions defined on a union of disjoint intervals I_d . I_d is at most the real numbers without the $2^d - 1$ points of $S_d \setminus \{\infty\}$. However, the proof and the conclusions of the present theorem, in particular, are presented from the point of view of largest domain of definition for the rational functions involved.

4. d -Fold Hermite rationals and Hermite-Gauss quadrature.

Let $\{H_n(x)\}_{n=0}^\infty$ be the sequence of monic Hermite polynomials, orthogonal with respect to

$$(4.1) \quad (f(x), g(x))_H := \int_{-\infty}^\infty f(x)g(x) e^{-x^2} dx.$$

Using the known formula

$$(4.2) \quad (H_n(x), H_n(x))_H = \frac{n!}{2^n} \sqrt{\pi},$$

we write the quadrature (3.3) in the compact form

$$(4.3) \quad \int_{-\infty}^{\infty} f(x)e^{-x^2} dx = \sum_{k=1,2,\dots,n} f(h_{0,n,k}^{(\gamma,\lambda)}) H_{0,n,k}^{(\gamma,\lambda)} + \frac{f^{(2n)}(\nu)}{(2n)!} \frac{n!}{2^n} \sqrt{\pi},$$

for $f(x)$ having a continuous $(2n)$ th derivative, and some $\nu \in (-\infty, \infty)$. We have thus denoted the zeros of $H_n(x)$ by $h_{0,n,k}^{(\gamma,\lambda)}$ and the corresponding quadrature weights by $H_{0,n,k}^{(\gamma,\lambda)}$, for $k = 1, 2, \dots, n$.

Now, let γ and λ be constant positive numbers, and set

$$(4.4a) \quad v^{[0](\gamma,\lambda)}(x) := x$$

and

$$(4.4b) \quad v^{[d](\gamma,\lambda)}(x) := v^{[d-1](\gamma,\lambda)}(v^{(\gamma,\lambda)}(x))$$

for $d = 1, 2, 3, \dots$. $v^{[d](\gamma,\lambda)}(x)$ is the d -fold composition of the doubling transformation $v^{(\gamma,\lambda)}(x)$ given in Definition (2.1). Since

$$(4.5) \quad \lim_{x \rightarrow s_{d,j}^{(\gamma,\lambda)}} (x - s_{d,j}^{(\gamma,\lambda)})^m e^{-[v^{[d](\gamma,\lambda)}(x)]^2} = 0,$$

for $j = 1, 2, \dots, 2^d - 1$, $m = 0, \pm 1, \pm 2, \dots$ and $s_{d,j}^{(\gamma,\lambda)}$ given by (3.4), we can then write (3.7g), in this case, as

$$(4.6) \quad (f(x), g(x))_{H_d^{(\gamma,\lambda)}} := \int_{-\infty}^{\infty} f(x)g(x)e^{-[v^{[d](\gamma,\lambda)}(x)]^2} dx,$$

as an inner product for the space $\mathbf{P}_d^{(\gamma,\lambda)}$ of real rational functions with basis B_d defined in (3.5).

Denote by $\{H_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^{\infty}$ the ORFS for $\mathbf{P}_d^{(\gamma,\lambda)}$ with respect to (4.6), yielded by Theorem 3.3 applied to the Hermite OPS $\{H_n(x)\}_{n=0}^{\infty}$. We call the functions $H_{d,n}^{(\gamma,\lambda)}(x)$ the d -fold Hermite rationals of parameters $\gamma, \lambda > 0$. By (3.8), the zeros, which we denote by $h_{d,n,k}^{(\gamma,\lambda)}$ for $k = 1, 2, \dots, 2^d n$, of $H_{d,2^d n}^{(\gamma,\lambda)}(x)$, $n \geq 1$, satisfy

$$(4.7a) \quad h_{d,n,i}^{(\gamma,\lambda)} = \frac{\lambda}{2} \left(h_{d-1,n,i}^{(\gamma,\lambda)} - \sqrt{(h_{d-1,n,i}^{(\gamma,\lambda)})^2 + \frac{4\gamma}{\lambda^2}} \right), \\ i = 1, 2, 3, \dots, 2^{d-1}n,$$

$$(4.7b) \quad h_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)} = \frac{\lambda}{2} \left(h_{d-1,n,i}^{(\gamma,\lambda)} + \sqrt{(h_{d-1,n,i}^{(\gamma,\lambda)})^2 + \frac{4\gamma}{\lambda^2}} \right),$$

$$i = 1, 2, 3, \dots, 2^{d-1}n.$$

The corresponding weights, which we denote by $H_{d,n,k}^{(\gamma,\lambda)}$ for $k = 1, 2, \dots, 2^d n$, considering Theorem 3.7, we can write as

$$(4.8a) \quad H_{d,n,i}^{(\gamma,\lambda)} = \frac{\lambda(h_{d,n,i}^{(\gamma,\lambda)})^2}{(h_{d,n,i}^{(\gamma,\lambda)})^2 + \gamma} H_{d-1,n,i}^{(\gamma,\lambda)}, \quad i = 1, 2, 3, \dots, 2^{d-1}n,$$

$$(4.8b) \quad H_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)} = \frac{\lambda(h_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)})^2}{(h_{d,n,2^{d-1}n+i}^{(\gamma,\lambda)})^2 + \gamma} H_{d-1,n,i}^{(\gamma,\lambda)},$$

$$i = 1, 2, 3, \dots, 2^{d-1}n.$$

The following theorem gives the $(2^d n)$ -Point d -fold Hermite-Gauss quadrature formula of parameters $\gamma, \lambda > 0$.

Theorem 4.1 (d -Fold Hermite-Gauss quadrature, see [10]). *Let $\gamma, \lambda > 0$, and let d and n be positive integers. With $h_{d,n,k}^{(\gamma,\lambda)}$ and $H_{d,n,k}^{(\gamma,\lambda)}$ given in (4.7) and (4.8), respectively,*

$$(4.9) \quad \int_{-\infty}^{\infty} q(x) e^{-[v^{[d]}(\gamma,\lambda)(x)]^2} dx = \sum_{k=1}^{2^d n} q(h_{d,n,k}^{(\gamma,\lambda)}) H_{d,n,k}^{(\gamma,\lambda)} + E_{d,n}^{(\gamma,\lambda)}[q(x)]$$

with

$$(4.10) \quad E_{d,n}^{(\gamma,\lambda)}[q(x)] := \frac{Q^{(2^{d+1}n)}(\varepsilon)}{(2^{d+1}n)!} \frac{n!}{2^n} \sqrt{\pi} \lambda^{d(2n+1)},$$

for some ε in $(-\infty, \infty)$, provided $Q(x) := q(x) \prod_{j=1}^{2^d-1} (x - s_{d,j}^{(\gamma,\lambda)})^{2n}$ has a continuous $(2^{d+1}n)$ -th derivative, $s_{d,j}^{(\gamma,\lambda)}$ defined in (3.4).

Examples, comparisons and observations concerning d -Fold Hermite-Gauss Quadrature are given in [10], along with the following application to the computation of special functions. Consider the system of

differential equations

$$(4.11) \quad \begin{aligned} x X_m''(x) - (m - 1/2)X_m'(x) - X_m(x) &= 0, \\ m &= 0, \pm 1, \pm 2, \dots, \end{aligned}$$

which might arise in the study by separation and superposition of the wave equation

$$x U_{xx} + U_{xt} = U;$$

for example, $U(x, t) = \sum_{m=1}^{\infty} X_m(x) e^{-(m-1/2)t}$.

A solution to (4.11) is

$$(4.12) \quad X_m(x) := e^{-2\sqrt{x}} \int_{-\infty}^{\infty} \tau^{2m} e^{-[v^{[1]}(\sqrt{x}, 1)(\tau)]^2} d\tau, \quad x > 0,$$

and d -fold Hermite-Gauss quadrature, Theorem 4.1, yields

$$(4.13) \quad X_m(x) = e^{-2\sqrt{x}} \sum_{k=1}^{2n} (h_{1,n,k}^{(\sqrt{x}, 1)})^{2m} H_{1,n,k}^{(\sqrt{x}, 1)},$$

for any positive integer n with $n > m \geq 0$.

In order to see that (4.12) satisfies (4.11), note that

$$X_m(x) = \int_0^{\infty} u^{m-1/2} e^{-u-x/u} du,$$

by the substitution $u = \tau^2$ in (4.12). From this last equation, we obtain

$$(4.14) \quad X_m'(x) = -X_{m-1}(x),$$

and, by the integration by parts formula,

$$(4.15) \quad X_m(x) = (m - 1/2)X_{m-1}(x) + xX_{m-2}(x).$$

The differential equation (4.11) results by combining these last two equations.

From Equation (4.13), we obtain

$$(4.16) \quad X_0(x) = \sqrt{\pi} e^{-2\sqrt{x}}$$

and

$$(4.17) \quad X_1(x) = (1/2 + \sqrt{x})\sqrt{\pi} e^{-2\sqrt{x}}.$$

These expressions for $X_0(x)$ and $X_1(x)$ and the formula (4.15) can then be used recursively to find expressions for $X_m(x)$, for all other integers m .

5. Results and observations. We first focus on $P_d^{(\gamma,\lambda)} := \{P_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$, the general d -fold doubling of an OPS $P_0^{(\gamma,\lambda)}$, before turning to the Hermite special case. A weaker version of the following inner product relation was used to obtain the error formula, equation (4.10). The proof, which we omit here, parallels that of Theorem 2.2.8 in [12].

Theorem 5.1. *Let $d \geq 1$, and let k be a non-negative integer. Then*

$$\begin{aligned} \left(P_{d,2k}^{(\gamma,\lambda)}, P_{d,2k}^{(\gamma,\lambda)}\right)_d^{(\gamma,\lambda)} &= \gamma_d^{2k+1} \left(P_{d,2k+1}^{(\gamma,\lambda)}, P_{d,2k+1}^{(\gamma,\lambda)}\right)_d^{(\gamma,\lambda)} \\ &= \lambda_d^{2k+1} \left(P_{d-1,k}^{(\gamma,\lambda)}, P_{d-1,k}^{(\gamma,\lambda)}\right)_{d-1}^{(\gamma,\lambda)}. \end{aligned}$$

In practice, one may require the partial fraction decomposition

$$(5.1) \quad P_{d,n}^{(\gamma,\lambda)}(x) = \sum_{i=0}^{\lceil \frac{n}{2^d} \rceil} a_{d,n,i}^{(\gamma,\lambda)} x^i + \sum_{i=1}^{\lceil \frac{n}{2^d} \rceil} \sum_{j=1}^{2^{d-1}} b_{d,n,i,j}^{(\gamma,\lambda)} \left(x - s_{d,j}^{(\gamma,\lambda)}\right)^{-i},$$

for real coefficient $a_{d,n,i}^{(\gamma,\lambda)}$ and $b_{d,n,i,j}^{(\gamma,\lambda)}$. Here, $\lceil n/2^d \rceil$ is the least integer greater than $n/2^d$. Having $P_{d,n}^{(\gamma,\lambda)}(x)$ in the form given by the definition, (3.6), an algorithm for finding these coefficients is given in Figure 1.

1. For $i = \lceil \frac{n}{2^d} \rceil, \lceil \frac{n}{2^d} \rceil - 1, \lceil \frac{n}{2^d} \rceil - 2, \dots, 1$, in that order, calculate

$$b_{d,n,i,j}^{(\gamma,\lambda)} = \lim_{x \rightarrow s_{d,j}^{(\gamma,\lambda)}} (x - s_{d,j}^{(\gamma,\lambda)})^i \left(P_{d,n}^{(\gamma,\lambda)}(x) - \sum_{k=0}^{\lceil \frac{n}{2^d} \rceil - i - 1} \frac{b_{d,n,\lceil \frac{n}{2^d} \rceil - k, j}^{(\gamma,\lambda)}}{(x - s_{d,j}^{(\gamma,\lambda)})^{\lceil \frac{n}{2^d} \rceil - k}} \right),$$

for each j from 1 to $2^d - 1$.

2. For each i from 0 to $\lceil \frac{n}{2^d} \rceil$, calculate

$$a_{d,n,i}^{(\gamma,\lambda)} = \lim_{x \rightarrow 0} \frac{d^i}{dx^i} \left(P_{d,n}^{(\gamma,\lambda)}(x) - \sum_{k=1}^{\lceil \frac{n}{2^d} \rceil} \sum_{j=1}^{2^d - 1} \frac{b_{d,n,k,j}^{(\gamma,\lambda)}}{(x - s_{d,j}^{(\gamma,\lambda)})^k} \right).$$

FIGURE 1. A partial fraction decomposition algorithm.

Orthogonal rational function sequences, under certain regularity conditions, satisfy three-term recurrence, see [4]. Consider the possibility that $P_d^{(\gamma,\lambda)}$ satisfies a three-term recurrence of the form $P_{d,n}^{(\gamma,\lambda)}(x) = q_n(x)P_{d,n-1}^{(\gamma,\lambda)}(x) - r_n(x)P_{d,n-2}^{(\gamma,\lambda)}(x)$, for some $q_n(x)$ and $r_n(x)$ in the space $\mathbf{P}_d^{(\gamma,\lambda)}$. Since $P_{d,2k-2}^{(\gamma,\lambda)}(x) = (-\gamma_d)^{k-1}xP_{d,2k-1}^{(\gamma,\lambda)}(x)$, we would then have $P_{d,n}^{(\gamma,\lambda)}(x) = s_n(x)P_{d,n-1}^{(\gamma,\lambda)}(x)$, for $s_n(x)$ in $\mathbf{P}_d^{(\gamma,\lambda)}$; thus, if $P_d^{(\gamma,\lambda)}$ satisfies a three-term formula, then it satisfies a two-term formula. In fact, each odd indexed function is, by Definition (3.6), given in terms of the preceding even indexed one. However, one observes from the d -fold doubling of zeros theorem (3.4) that $P_{d,2^d j}^{(\gamma,\lambda)}(x)$ has 2^d more distinct zeros than does $P_{d,2^d j-1}^{(\gamma,\lambda)}(x)$. Hence, we can see that $s_{2^d j}(x) = P_{d,2^d j}^{(\gamma,\lambda)}(x)/P_{d,2^d j-1}^{(\gamma,\lambda)}(x)$ is not in $\mathbf{P}_d^{(\gamma,\lambda)}$, and we have the following.

Theorem 5.2. *If $d \geq 1$, then $P_d^{(\gamma,\lambda)}$ does not satisfy a 3-term recursion of the form $P_{d,n}^{(\gamma,\lambda)}(x) = q_n(x)P_{d,n-1}^{(\gamma,\lambda)}(x) - r_n(x)P_{d,n-2}^{(\gamma,\lambda)}(x)$, where $q_n(x), r_n(x) \in \mathbf{P}_d^{(\gamma,\lambda)}$.*

It is well known that monic orthogonal polynomial sequences, in our notation $P_d^{(\gamma,\lambda)}$ with $d = 0$, always satisfy a 3-term recurrence formula,

$$(5.2) \quad P_{0,n}^{(\gamma,\lambda)}(x) = (x - a_n)P_{0,n-1}^{(\gamma,\lambda)}(x) - b_n P_{0,n-2}^{(\gamma,\lambda)}(x),$$

a_n and b_n constants. It was shown in [12] that $P_d^{(\gamma,\lambda)}$ with $d = 1$, the Laurent polynomial case, satisfies a four-term recurrence relation with Laurent polynomial coefficients. The next theorem extends these results to $P_d^{(\gamma,\lambda)}$ for general d . It says that subsequences of $P_d^{(\gamma,\lambda)}$ of period 2^d are given by three-term recursion. A proof by induction on d is straightforward from the definition of d -fold doubling (3.6) and formula (5.2), and we will omit it.

Theorem 5.3. *Let $d \geq 0$ and $j \in \{0, 1, 2, \dots, 2^d - 1\}$. For $n = 0, 1, 2, \dots$, set*

$$(5.3) \quad R_{d,j,n}^{(\gamma,\lambda)}(x) := P_{d,2^d n+j}^{(\gamma,\lambda)}(x).$$

Then, for $n \geq 2$, there are rational functions $q_{d,j,n}^{(\gamma,\lambda)}(x)$ and $r_{d,j,n}^{(\gamma,\lambda)}(x)$ in $\mathbf{P}_d^{(\gamma,\lambda)}$ such that

$$(5.4) \quad R_{d,j,n}^{(\gamma,\lambda)}(x) = q_{d,j,n}^{(\gamma,\lambda)}(x)R_{d,j,n-1}^{(\gamma,\lambda)}(x) - r_{d,j,n}^{(\gamma,\lambda)}(x)R_{d,j,n-2}^{(\gamma,\lambda)}(x).$$

As a consequence of the theorem, there are 2^d continued fractions associated with $P_d^{(\gamma,\lambda)}$, in that each of the 2^d -periodic subsequences indexed by j is given as n th denominators of the continued fraction

$$\mathbf{K} \left(\frac{-r_{d,j,n}^{(\gamma,\lambda)}(x)}{q_{d,j,n}^{(\gamma,\lambda)}(x)} \right).$$

It's convenient here to renormalize $P_d^{(\gamma,\lambda)}$ by setting

$$(5.5) \quad p_{0,n}^{(\gamma,\lambda)}(x) := P_{0,n}^{(\gamma,\lambda)}(x), \quad n = 0, 1, 2, \dots,$$

and, for $d \geq 1$, defining recursively

$$(5.6a) \quad p_{d,2n}^{(\gamma,\lambda)}(x) := p_{d-1,n}^{(\gamma,\lambda)}(v^{(\gamma_d,\lambda_d)}(x)), \quad n = 0, 1, 2, \dots$$

Then it's clear that $p_{d,n}^{(\gamma,\lambda)}(x)$ and $P_{d,n}^{(\gamma,\lambda)}(x)$ differ by a nonzero multiplicative constant and, hence, $\{p_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ is an ORFS with respect to $(\cdot, \cdot)_d^{(\gamma,\lambda)}$ on $\mathbf{P}_d^{(\gamma,\lambda)}$. Also, note that

$$(5.7) \quad p_{d,2^d n}^{(\gamma,\lambda)}(x) = \left(\prod_{k=1}^d \lambda_k^n \right) P_{d,2^d n}^{(\gamma,\lambda)}(x).$$

In terms of this new ORFS, Theorem 5.3 can readily be reformulated as follows, recalling $v^{[d](\gamma,\lambda)}(x)$ given in Equations (4.4).

Theorem 5.4. *Suppose $P_0^{(\gamma,\lambda)} := \{P_{0,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ is a monic OPS such that*

$$P_{0,n}^{(\gamma,\lambda)}(x) = (x - a_n)P_{0,n-1}^{(\gamma,\lambda)}(x) - b_n P_{0,n-2}^{(\gamma,\lambda)}(x), \quad n = 1, 2, 3, \dots,$$

for constants a_n and b_n , and where we set $P_{0,-1}^{(\gamma,\lambda)}(x) := 0$.

Then, for $d \geq 0$ and $j \in \{0, 1, 2, \dots, 2^d - 1\}$,

$$(5.8) \quad p_{d,2^d n+j}^{(\gamma,\lambda)}(x) = (v^{[d](\gamma,\lambda)}(x) - a_n)p_{d,2^d(n-1)+j}^{(\gamma,\lambda)}(x) - b_n p_{d,2^d(n-2)+j}^{(\gamma,\lambda)}(x),$$

for $n = 1, 2, 3, \dots$, where we define $p_{d,2^d(-1)+j}^{(\gamma,\lambda)}(x) := 0$.

We can now see that $\{p_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ and, thus, $P_d^{(\gamma,\lambda)} := \{P_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ can be associated with the n th denominators of

$$\mathbf{K} \left(\frac{-b_n}{v^{[d](\gamma,\lambda)}(x) - a_n} \right).$$

In several special cases, the monic OPS $P_0^{(\gamma,\lambda)}$ is given by a formula of the type

$$(5.9) \quad P_{0,n}^{(\gamma,\lambda)}(x) = \frac{1}{K_n w_0^{(\gamma,\lambda)}(x)} \left(\frac{d}{dx} \right)^n \left(\rho^n(x) w_0^{(\gamma,\lambda)}(x) \right), \quad n = 0, 1, 2, \dots,$$

where K_n is independent of x and $\rho(x)$ is a polynomial independent of n . See [5], page 146. Formulas of this kind are called *Rodrigues' type formulas*. The next theorem extends the result for the $d = 1$, Laurent polynomial case in [12] to d -fold doubling. The proof again is direct from the definitions and is omitted.

Theorem 5.5. *Suppose $P_0^{(\gamma,\lambda)} := \{P_{0,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ is given by a Rodrigues' type formula (5.9). Then, for $d \geq 0$ and $n = 0, 1, 2, \dots$,*

$$(5.10) \quad p_{d,n}^{(\gamma,\lambda)}(x) = \frac{1}{K_n w_{d,n}^{(\gamma,\lambda)}(x)} \left(\frac{d}{dv^{[d](\gamma,\lambda)}(x)} \right)^{\lfloor n/2^d \rfloor} \cdot \left(\rho^{\lfloor n/2^d \rfloor} (v^{[d](\gamma,\lambda)}(x)) w_d^{(\gamma,\lambda)}(x) \right),$$

where we define

$$(5.11a) \quad w_{0,n}^{(\gamma,\lambda)}(x) := w_0^{(\gamma,\lambda)}(x), \quad n = 0, 1, 2, \dots,$$

and, recursively for $d \geq 1$, we set

$$(5.11b) \quad w_{d,2m}^{(\gamma,\lambda)}(x) := w_{d-1,m}^{(\gamma,\lambda)}(v^{(\gamma_d,\lambda_d)}(x))$$

and

$$(5.11c) \quad w_{d,2m+1}^{(\gamma,\lambda)}(x) := x w_{d-1,m}^{(\gamma,\lambda)}(v^{(\gamma_d,\lambda_d)}(x)),$$

and where $\lfloor n/2^d \rfloor$ denotes the greatest integer less than $n/2^d$.

Hence, if $P_0^{(\gamma,\lambda)}$ is given by a Rodrigues' type formula (5.9), then $\{p_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ and $P_d^{(\gamma,\lambda)} := \{P_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ are given by generalized Rodrigues' type formulas. We also have, as a corollary, using (5.7),

$$(5.12) \quad P_{d,2^d n}^{(\gamma,\lambda)}(x) = \frac{\prod_{k=1}^d \lambda_k^n}{K^n w_d^{(\gamma,\lambda)}(x)} \left(\frac{d}{dv^{[d](\gamma,\lambda)}(x)} \right)^n \left(\rho^n (v^{[d](\gamma,\lambda)}(x)) w_d^{(\gamma,\lambda)}(x) \right),$$

for $d \geq 1$ and $n = 0, 1, 2, \dots$, if $P_0^{(\gamma,\lambda)}$ is given by formula (5.9).

Closely related to Rodrigues' formulas are second order differential equations. For example, there are monic OPS's which are the eigenfunctions for certain self-adjoint differential operators. See [5], page 148.

Theorem 5.6. *Let $d \geq 1$, and let n be a non-negative integer. Suppose $y = P_{d-1,n}^{(\gamma,\lambda)}(t)$ satisfies*

$$(5.13) \quad \frac{d}{dt} \left(k(t) \frac{dy}{dt} \right) - b_n w(t) y = 0,$$

for all t in $I_{d-1}^{(\gamma,\lambda)}$. Then $Y = P_{d,2n}^{(\gamma,\lambda)}(x)$ satisfies

$$(5.14) \quad \frac{d}{dx} \left(K(x) \frac{dY}{dx} \right) - b_n W(x) Y = 0,$$

for all x in $I_d^{(\gamma,\lambda)}$, where $K(x) := k(v^{(\gamma_a, \lambda_a)}(x)) / [dv^{(\gamma_a, \lambda_a)}(x)/dx]$ and $W(x) := w(v^{(\gamma_a, \lambda_a)}(x)) [dv^{(\gamma_a, \lambda_a)}(x)/dx]$.

Proof. For $x \in I_d^{(\gamma,\lambda)}$ and $t = v^{(\gamma_a, \lambda_a)}(x)$, we have $t \in I_{d-1}^{(\gamma,\lambda)}$, $Y(x) = \lambda_d^n y(t)$, and

$$\begin{aligned} & \frac{d}{dx} \left(K(x) \frac{dY}{dx} \right) - b_n W(x) Y \\ &= \lambda_d^n (dv^{(\gamma_a, \lambda_a)}(x)/dx) \left(\frac{d}{dt} \left(k(t) \frac{dy}{dt} \right) - b_n w(t) y \right) \\ &= \lambda_d^n (dv^{(\gamma_a, \lambda_a)}(x)/dx) (0) \\ &= 0. \quad \square \end{aligned}$$

Hence, if $P_0^{(\gamma,\lambda)} := \{P_{0,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ satisfies a system of differential equations of the form (5.13), then so do $\{p_{d,2^n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ and $\{P_{d,2^n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$.

Returning now to the d -fold Hermite Rationals $H_{d,n}^{(\gamma,\lambda)}(x)$, the monic Hermite polynomials $H_{0,n}^{(\gamma,\lambda)}(x)$ are given by a Rodrigues' type formula

(see [5, p. 145])

$$(5.15) \quad H_{0,n}^{(\gamma,\lambda)}(x) = \frac{1}{(-2)^n e^{-x^2}} \left(\frac{d}{dx}\right)^n e^{-x^2}, \quad n = 0, 1, 2, \dots,$$

and are eigenfunctions for a self-adjoint differential operator, in that, for all x ,

$$(5.16) \quad \frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} H_{0,n}^{(\gamma,\lambda)}(x) \right) + 2n e^{-x^2} H_{0,n}^{(\gamma,\lambda)}(x) = 0, \quad n = 0, 1, 2, \dots$$

Hence, by Theorem 5.5, $\{H_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ is given by a generalized Rodrigues' type formula, and, by Theorem 5.6, $\{H_{d,2^d n}^{(\gamma,\lambda)}(x)\}_{n=0}^\infty$ satisfies a system of differential equations of the form (5.14). In particular, for $d \geq 1$ and $n = 0, 1, 2, \dots$,

$$(5.17) \quad H_{d,2^d n}^{(\gamma,\lambda)}(x) = \frac{\prod_{k=1}^d \lambda_k^n}{(-2)^n e^{-[v^{[d]}(\gamma,\lambda)(x)]^2}} \left(\frac{d}{dv^{[d]}(\gamma,\lambda)(x)}\right)^n \left(e^{-[v^{[d]}(\gamma,\lambda)(x)]^2}\right),$$

and, for all x not in the set of singularities $S_d^{(\gamma,\lambda)}$,

$$\begin{aligned} \frac{d}{dx} \left(\frac{e^{-[v^{[d]}(\gamma,\lambda)(x)]^2}}{[v^{[d]}(\gamma,\lambda)(x)]'} \frac{d}{dx} H_{d,2^d n}^{(\gamma,\lambda)}(x) \right) \\ + 2n e^{-[v^{[d]}(\gamma,\lambda)(x)]^2} [v^{[d]}(\gamma,\lambda)(x)]' H_{d,2^d n}^{(\gamma,\lambda)}(x) = 0. \end{aligned}$$

Finally, the fundamental recurrence formula for the monic Hermite polynomials is

$$(5.19) \quad H_{0,n}^{(\gamma,\lambda)}(x) = x H_{0,n-1}^{(\gamma,\lambda)}(x) - \frac{n-1}{2} H_{0,n-2}^{(\gamma,\lambda)}(x), \quad n = 1, 2, 3, \dots,$$

where $H_{0,-1}^{(\gamma,\lambda)}(x) := 0$ [5, p. 158]. If we define

$$(5.20) \quad h_{0,n}^{(\gamma,\lambda)}(x) := H_{0,n}^{(\gamma,\lambda)}(x), \quad n = 0, 1, 2, \dots,$$

and, for $d \geq 1$, set

$$(5.21a) \quad h_{d,2n}^{(\gamma,\lambda)}(x) := h_{d-1,n}^{(\gamma,\lambda)}(v^{(\gamma,\lambda_d)}(x)), \quad n = 0, 1, 2, \dots$$

and

$$(5.21b) \quad h_{d,2n+1}^{(\gamma,\lambda)}(x) := \frac{1}{x} h_{d-1,n}^{(\gamma,\lambda)}(v^{(\gamma,\lambda_d)}(x)), \quad n = 0, 1, 2, \dots,$$

then the recurrence formula (5.8) translates as

(5.22)

$$h_{d,2^d n+j}^{(\gamma,\lambda)}(x) = v^{[d](\gamma,\lambda)}(x) h_{d,2^d(n-1)+j}^{(\gamma,\lambda)}(x) - \frac{n-1}{2} h_{d,2^d(n-2)+j}^{(\gamma,\lambda)}(x),$$

for $n = 1, 2, 3, \dots$, $j = 0, 1, 2, \dots, 2^d - 1$, and all $d \geq 0$, and where we define $h_{d,2^d(-1)+j}^{(\gamma,\lambda)}(x) := 0$. In this sense, we see that the d -fold Hermite ORFS $\{H_{d,n}^{(\gamma,\lambda)}(x)\}_{n=0}^{\infty}$ is associated with the n th denominators of the continued fraction

$$\mathbf{K} \left(\frac{-(n-1)/2}{v^{[d](\gamma,\lambda)}(x)} \right).$$

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