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GENERATION OF ANALYTIC SEMIGROUPS BY DIFFERENTIAL OPERATORS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. We study the generation of analytic semigroups in the space $L^1(\alpha,\beta)$ by a second order operator $Lu = u'' + q_1(x)u' + q_0(x)u$ with mixed non-separated and integral boundary conditions of the form

$$\begin{aligned} B_i(u) \equiv{}& a_i u(\alpha) + b_i u'(\alpha) + c_i u(\beta) + d_i u'(\beta) \\ &+ \int_\alpha^\beta R_i(t)u(t)\,dt + \int_\alpha^\beta S_i(t)u'(t)\,dt \\ ={}& 0, \quad i = 1, 2. \end{aligned}$$

We obtain quite general results that extend previous works by the author (see [**3**]–[**4**]).

The key for showing the generation of analytic semigroups will be an estimate of the form

$$\|R(\lambda : L)\| \leq M|\lambda|^{-1}$$

for the resolvent operator in a suitable sector of the complex plane.

1. Introduction and preliminaries. This work has been inspired by a model arising in optical physics, specifically that in [**8**]–[**9**], where BOITAL (Thermally Induced Optical Bistability with Localized Absorption) multilayer devices are considered and the proposed models numerically analyzed. The authors in [**8**]–[**9**] give a PDE model that, after physical considerations, can be reduced to a finite dimensional ODE model exhibiting similar dynamics. Both in the PDE and ODE models, the boundary conditions that appear are mixed non-separated and integral ones, although a nonlinear function on the boundary is also involved.

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In this paper we consider a linearized version of the boundary conditions mentioned above, and we study the generation of semigroups by the differential operator associated to the system in the space $L^1(\alpha,\beta)$ of integrable functions in the finite interval (α,β). This is the first step in the analytical study of the dynamics of such systems, and it connects directly with previous works by the author [**3**]–[**4**], where the generation of analytic semigroups by differential operators with non-separated or integral boundary conditions were investigated.

In this way the present work should also be viewed as a continuation of our previous papers [**3**]–[**4**]. The results in [**3**], where integral boundary conditions are considered, can be seen as particular cases of the main theorem 7.1 simply by taking $a_i = b_i = c_i = d_i = 0$. The paper [**4**] deals with the case of non-separated boundary conditions (i.e., $R_i \equiv S_i \equiv 0$) and we proved there the generation of analytic semigroups for regular boundary conditions in every space $L^p(\alpha,\beta)$, $1 \le p \le \infty$. For $p = 1$ the results in [**4**] are simply special cases of Theorem 7.1; however, as we will see in Section 8, it is not possible to generalize this theorem to the L^p setting, so [**4**] provides more precise results than Theorem 7.1 in the case of non-separated boundary conditions.

Consider a formal second order operator in the finite interval (α,β), that is,

$$l(u) = u'' + q_1(x)u' + q_0(x)u, \quad x \in (\alpha,\beta)$$

where q_1 and q_0 are regular complex-valued functions. We associate to l two mixed boundary conditions of the form

$$\begin{aligned} B_i(u) \equiv a_i u(\alpha) + b_i u'(\alpha) + c_i u(\beta) + d_i u'(\beta) \\ + \int_\alpha^\beta R_i(t)u(t)\,dt + \int_\alpha^\beta S_i(t)u'(t)\,dt = 0 \end{aligned}$$

for $i = 1, 2$; here, the coefficients a_i, b_i, c_i, d_i are complex and the continuous functions R_i and S_i are complex-valued. Of course, some kind of independence of the boundary conditions should be imposed for avoiding such cases as, for example,

$$\begin{cases} B_1(u) \equiv u(\alpha) - u(\beta) = 0 \\ B_2(u) \equiv \int_\alpha^\beta u'(t)\,dt = 0 \end{cases}$$

where the boundary conditions are formally the same. The way for avoiding such cases will be through the *characteristic determinant*, to be defined in Section 2.

The formal operator l together with the boundary conditions $\{B_1, B_2\}$ define an unbounded linear operator L in $L^1(\alpha,\beta)$ known as the L^1-*realization* of the system:

$$Lu = l(u), \quad D(L) = \{u \in W^{2,1}(\alpha,\beta) : B_1(u) = B_2(u) = 0\}$$

where $W^{2,1}(\alpha,\beta)$ is the Sobolev space of order $(2,1)$. Our objective is to determine the cases, depending on the boundary conditions, for which L is the generator of an analytic semigroup of operators in $L^1(\alpha,\beta)$.

As is well known [**6**], sufficient conditions for assuring that L generates an analytic semigroup of bounded linear operators are:

1. The resolvent set $\rho(L)$ contains a sector of the form

$$\Sigma_{\delta,r} = \{\lambda \in \mathbf{C} : |\arg(\lambda - r)| < \delta, \lambda \neq r\}$$

for some $\delta \in (\pi/2, \pi)$ and $r \in \mathbf{R}$.

2. There exists a constant M such that, for each $\lambda \in \Sigma_{\delta,r}$, the following bound holds:

$$\|R(\lambda : L)\| \leq \frac{M}{|\lambda - r|}$$

where $R(\lambda : L) = (\lambda I - L)^{-1}$ and the norm is the usual one for bounded linear operators in $L^1(\alpha,\beta)$.

It is important to note that the semigroup generated by L is a C_0-semigroup if and only if the domain $D(L)$ is dense in $L^1(\alpha,\beta)$.

For inverting the operator $\lambda I - L$ we prove the existence of an associate Green's function $G(x,s;\lambda)$, so we can express each resolvent operator in the form

$$R(\lambda : L)f = -\int_\alpha^\beta G(\cdot, s; \lambda) f(s)\, ds, \quad f \in L^1(\alpha,\beta).$$

As the Green's function can be given explicitly, this provides us with suitable formulae for bounding $R(\lambda : L)$. The bounds on the resolvent needed for assuring the generation of analytic semigroups will be obtained for a certain class of boundary conditions that we will call *regular*. The main result of the paper is Theorem 7.1, where we state the generation of analytic semigroups of analytic semigroups for regular boundary conditions.

Similar constructions could be made in every space $L^p(\alpha,\beta)$, $1 \le p \le \infty$, but we do not arrive to such precise results as in the L^1 case. We will consider the L^p case in Section 8.

We give a brief outline of the paper. In Section 2 we introduce the characteristic determinant $\Delta(\lambda)$, an entire function that characterizes the spectrum of L, and the associated Green's function $G(x,s;\lambda)$; this allows us to express each resolvent operator $R(\lambda : L)$ in integral form. Section 3 is devoted to give suitable formulae for $\Delta(\lambda)$ and $G(x,s;\lambda)$ that will be used in Section 4 for bounding $R(\lambda : L)$. Here several cases are considered which are analyzed in Section 5; this analysis leads to the definition of regular boundary conditions. In the analysis of cases it is necessary to impose additional regularity conditions on the coefficients R_i and S_i; in Section 6 we will see that this regularity can be relaxed. In Section 7 we state the main result of the paper: for regular boundary conditions the operator L generates an analytic semigroup in $L^1(\alpha,\beta)$; we also consider in this section some interesting examples. Finally, in Section 8 we comment on some results in $L^p(\alpha,\beta)$.

2. Characteristic determinant and spectrum. Consider the operator T given by $Tu = u''$, with domain $D(T) = \{u \in W^{2,1}(0,1) : B_1(u) = B_2(u) = 0\}$. As is well known (see [**4**], [**7**]), we can restrict ourselves to the study of the resolvent of T without loss of generality.

Consider the problem

$$\begin{cases} u'' - \lambda u = f & \text{in } (0,1) \\ B_1(u) = B_2(u) = 0 \end{cases} \tag{2.1}$$

with $f \in L^1(0,1)$ and $\lambda \in \mathbf{C}$. Let $\{u_1, u_2\}$ be a fundamental system of solutions of the equation $u'' - \lambda u = 0$. We define the *characteristic determinant* $\Delta(\lambda)$ to be

$$\Delta(\lambda) = \begin{vmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{vmatrix}. \tag{2.2}$$

It is straightforward to prove that the spectrum of T is given by

$$\sigma(T) = \{\lambda \in \mathbf{C} : \Delta(\lambda) = 0\}$$

and this does not depend on the fundamental system chosen for constructing $\Delta(\lambda)$.

As $\Delta(\lambda)$ is an entire function, the spectrum of T will be at much a denumerable set without finite accumulation points. As we are interested in the case for which the resolvent is not void, we will consider only the cases for which $\Delta(\lambda)$ is not identically zero. This can be interpreted as a kind of independence of the boundary conditions $B_1(u) = 0$ and $B_2(u) = 0$.

Let $\lambda \in \mathbf{C}$ be such that $\Delta(\lambda) \neq 0$ and consider the function $N : [0,1] \times [0,1] \to \mathbf{C}$ defined as

$$N(x,s;\lambda) = \begin{vmatrix} u_1(x) & u_2(x) & g(x,s;\lambda) \\ B_1(u_1) & B_1(u_2) & B_1(g)_x \\ B_2(u_1) & B_2(u_2) & B_2(g)_x \end{vmatrix} \tag{2.3}$$

(the notation $B_i(g)_x$ means that the boundary form B_i is applied to $g(x,s;\lambda)$ on the x variable). The function $g(x,s;\lambda)$ is defined as

$$g(x,s;\lambda) = \pm \frac{1}{2} \frac{u_1(x)u_2(s) - u_1(s)u_2(x)}{u_1'(s)u_2(s) - u_1(s)u_2'(s)} \tag{2.4}$$

where it takes the plus sign for $x > s$ and the minus sign for $x < s$.

The above formulae are based on those of [**1**] for the case of non-separated boundary conditions. It is not difficult to prove that

$$G(x,s;\lambda) = \frac{N(x,s;\lambda)}{\Delta(\lambda)} \tag{2.5}$$

is the Green's function for problem (2.1). Thus, for $\lambda \in \rho(T)$ we can express the resolvent operator $R(\lambda : T)$ as a Hilbert-Schmidt one, as follows:

$$R(\lambda : T)f = -\int_0^1 G(\cdot, s;\lambda) f(s)\, ds, \quad f \in L^1(0,1). \tag{2.6}$$

3. Analysis of $\Delta(\lambda)$ and $N(x,s;\lambda)$. Given an arbitrary $\delta \in (\pi/2, \pi)$, define the sector $\Sigma_\delta = \{\lambda \in \mathbf{C} : |\arg(\lambda)| < \delta, \lambda \neq 0\}$. For $\lambda \in \Sigma_\delta$, let $\rho \in \Sigma_{\delta/2}$ be the square root of λ with positive real part. A fundamental system of solutions of $u''-\lambda u=0$ is given by the functions $u_1(x)=\exp(-\rho x)$ and $u_2(x)=\exp(\rho x)$.

Evaluating the boundary forms B_i in the functions u_j, we obtain, for $i,j=1,2$, the following expression:

$$\begin{aligned}B_i(u_j) &= a_i+(-1)^j b_i\rho+c_i\exp[(-1)^j\rho]+(-1)^j d_i\rho\exp[(-1)^j\rho]\\ &\quad+\int_0^1 R_i(t)\exp[(-1)^j\rho t]\,dt+(-1)^j\rho\int_0^1 S_i(t)\exp[(-1)^j\rho t]\,dt.\end{aligned}$$

Next we substitute the above formula in (2.2). For avoiding complicated formulae we introduce the numbers

$$\Gamma_{xy}=x_1y_2-x_2y_1$$

and the functions

$$\Gamma_{xF}(t)=x_1F_2(t)-x_2F_1(t),$$

where $x,y\in\{a,b,c,d\}$ and $F,G\in\{R,S\}$. We also define

$$\Gamma_R(t,\xi)=R_1(t)R_2(\xi)-R_1(\xi)R_2(t),\quad \Gamma_S(t,\xi)=S_1(t)S_2(\xi)-S_1(\xi)S_2(t)$$

and

$$\Gamma_{RS}(t,\xi)=R_1(t)S_2(\xi)-R_2(t)S_1(\xi).$$

Thus, after a straightforward but long calculation we obtain from (2.2) the following formula:

$$\begin{aligned}\Delta(\lambda) &= 2(\Gamma_{ab}+\Gamma_{cd})\rho\\ &\quad+(-\Gamma_{bd}\rho^2+(\Gamma_{ad}-\Gamma_{bc})\rho+\Gamma_{ac})e^{\rho}\\ &\quad+(\Gamma_{bd}\rho^2+(\Gamma_{ad}-\Gamma_{bc})\rho-\Gamma_{ac})e^{-\rho}\\ &\quad+\rho^2\int_0^1(-\Gamma_{bS}(t)+\Gamma_{dS}(1-t))(e^{\rho t}-e^{-\rho t})\,dt\\ &\quad+\rho\int_0^1(\Gamma_{aS}(t)+\Gamma_{cS}(1-t)-\Gamma)_{bR}(t)\\ &\quad-\Gamma_{dR}(1-t))(e^{\rho t}+e^{-\rho t})\,dt\\ &\quad+\int_0^1(\Gamma_{aR}(t)-\Gamma_{cR}(1-t))(e^{\rho t}-e^{-\rho t})\,dt\\ &\quad+\int_0^1\int_0^1(\Gamma_S(t,\xi)\rho^2+(\Gamma_{RS}(t,\xi)\\ &\quad+\Gamma_{RS}(\xi,t))\rho+\Gamma_R(t,\xi))e^{\rho(\xi-t)}\,d\xi\,dt.\end{aligned}\tag{3.1}$$

Formula (2.4) can be written as

$$g(x,s;\lambda)=\begin{cases}\dfrac{e^{\rho(x-s)}-e^{\rho(s-x)}}{4\rho} & \text{if } x>s\\ \dfrac{e^{\rho(s-x)}-e^{\rho(x-s)}}{4\rho} & \text{if } x<s.\end{cases}$$

Thus, for $i=1,2$, we have

$$\begin{aligned}&B_i(g)_x\\&=(a_i-b_i\rho-c_ie^{-\rho}+d_i\rho e^{-\rho})\frac{e^{\rho s}}{4\rho}+(-a_i-b_i\rho+c_ie^{\rho}+d_i\rho e^{\rho})\frac{e^{-\rho s}}{4\rho}\\&\quad+\left(\int_0^s(R_i(t)-\rho S_i(t)\right)e^{-\rho t}\,dt+\int_s^1(-R_i(t)+\rho S_i(t))e^{-\rho t}\,dt)\frac{e^{\rho s}}{4\rho}\\&\quad+\left(-\int_0^s(R_i(t)+\rho S_i(t)\right)e^{\rho t}\,dt+\int_s^1(R_i(t)+\rho S_i(t))e^{\rho t}\,dt)\frac{e^{-\rho s}}{4\rho}.\end{aligned}$$

Substituting in (2.3) the expressions obtained for $B_i(u_j)$ and $B_i(g)_x$, we have

(3.2)

$$\begin{aligned}&N(x,s;\lambda)\\&=\varphi(x,s;\lambda)+\frac{e^{\rho(x+s)}}{2\rho}\Big[(\Gamma_{bd}\rho^2-(\Gamma_{ad}+\Gamma_{bc})\rho+\Gamma_{ac})e^{-\rho}\\&\quad+\int_s^1(\Gamma_{bS}(t)\rho^2-(\Gamma_{aS}(t)+\Gamma_{bR}(t))\rho+\Gamma_{aR}(t))e^{-\rho t}\,dt\\&\quad+\int_0^s(-\Gamma_{dS}(t)\rho^2+(\Gamma_{cS}(t)+\Gamma_{dR}(t))\rho-\Gamma_{cR}(t))e^{-\rho(t+1)}\,dt\\&\quad+\int_0^s\int_s^1(\Gamma_S(t,\xi)\rho^2+(\Gamma_{RS}(\xi,t)-\Gamma_{RS}(t,\xi))\rho+\Gamma_R(t,\xi))e^{-\rho(t+\xi)}\,d\xi\,dt\Big]\\&\quad+\frac{e^{-\rho(x+s)}}{2\rho}\Big[(\Gamma_{bd}\rho^2+(\Gamma_{ad}+\Gamma_{bc})\rho+\Gamma_{ac})e^{\rho}\\&\quad+\int_s^1(\Gamma_{bS}(t)\rho^2+(\Gamma_{aS}(t)+\Gamma_{bR}(t))\rho+\Gamma_{aR}(t))e^{\rho t}\,dt\end{aligned}$$

$$+ \int_0^s -(\Gamma_{dS}(t)\rho^2 + (\Gamma_{cS}(t) + \Gamma_{dR}(t))\rho + \Gamma_{cR}(t))e^{\rho(t+1)}\,dt$$
$$+ \int_0^s\int_s^1 (\Gamma_S(t,\xi)\rho^2 + (\Gamma_{RS}(t,\xi) - \Gamma_{RS}(\xi,t))\rho + \Gamma_R(t,\xi))e^{\rho(t+\xi)}\,d\xi\,dt\Big]$$

where

$$\varphi(x,s;\lambda) = \begin{cases} \varphi_1(x,s;\lambda) & \text{if } x > s \\ \varphi_2(x,s;\lambda) & \text{if } x < s. \end{cases} \tag{3.3}$$

The functions $\varphi_1(x,s;\lambda)$ and $\varphi_2(x,s;\lambda)$ are defined as

$$\begin{aligned}
\varphi_1(x,s;\lambda) = \frac{e^{\rho(x-s)}}{2\rho}\Big[&(\Gamma_{bd}\rho^2 + (\Gamma_{ad} - \Gamma_{bc})\rho - \Gamma_{ac})e^{-\rho} + 2\Gamma_{ab}\rho \\
&+ \int_0^1 (\Gamma_{bS}(t)\rho^2 + (\Gamma_{aS}(t) - \Gamma_{bR}(t))\rho - \Gamma_{aR}(t))e^{-\rho t}\,dt \\
&+ \int_0^s (-\Gamma_{bS}(t)\rho^2 + (\Gamma_{aS}(t) - \Gamma_{bR}(t))\rho + \Gamma_{aR}(t))e^{\rho t}\,dt \\
&+ \int_0^s (-\Gamma_{dS}(t)\rho^2 + (\Gamma_{cS}(t) - \Gamma_{dR}(t))\rho + \Gamma_{cR}(t))e^{\rho(t-1)}\,dt \\
&+ \int_0^1\int_0^s (-\Gamma_S(t,\xi)\rho^2 + (\Gamma_{RS}(t,\xi) \\
&+ \Gamma_{RS}(\xi,t))\rho + \Gamma_R(t,\xi))e^{\rho(\xi-t)}\,d\xi\,dt\Big] \\
&+ \frac{e^{\rho(s-x)}}{2\rho}\Big[(\Gamma_{bd}\rho^2 + (-\Gamma_{ad} + \Gamma_{bc})\rho - \Gamma_{ac})e^{\rho} + 2\Gamma_{ab}\rho \\
&+ \int_0^1 (\Gamma_{bS}(t)\rho^2 + (-\Gamma_{aS}(t) + \Gamma_{bR}(t))\rho - \Gamma_{aR}(t))e^{\rho t}\,dt \\
&+ \int_0^s (-\Gamma_{bS}(t)\rho^2 + (-\Gamma_{aS}(t) + \Gamma_{bR}(t))\rho + \Gamma_{aR}(t))e^{-\rho t}\,dt \\
&+ \int_0^s (-\Gamma_{dS}(t)\rho^2 + (-\Gamma_{cS}(t) + \Gamma_{dR}(t))\rho + \Gamma_{cR}(t))e^{\rho(1-t)}\,dt \\
&+ \int_0^1\int_0^s (-\Gamma_S(t,\xi)\rho^2 - (\Gamma_{RS}(t,\xi) + \Gamma_{RS}(\xi,t))\rho \\
&+ \Gamma_R(t,\xi))e^{\rho(t-\xi)}\,d\xi\,dt\Big]
\end{aligned}$$

and

$$
\begin{aligned}
&\varphi_2(x,s;\lambda)\\
&= \frac{e^{\rho(x-s)}}{2\rho}\bigg[(\Gamma_{bd}\rho^2 + (-\Gamma_{ad}+\Gamma_{bc})\rho - \Gamma_{ac})e^{\rho} - 2\Gamma_{cd}\rho\\
&\quad + \int_0^1 (-\Gamma_{dS}(t)\rho^2 + (-\Gamma_{cS}(t)+\Gamma_{dR}(t))\rho + \Gamma_{cR}(t))e^{\rho(1-t)}\,dt\\
&\quad + \int_s^1 (\Gamma_{bS}(t)\rho^2 + (-\Gamma_{aS}(t)+\Gamma_{bR}(t))\rho - \Gamma_{aR}(t))e^{\rho t}\,dt\\
&\quad + \int_s^1 (\Gamma_{dS}(t)\rho^2 + (-\Gamma_{cS}(t)+\Gamma_{dR}(t))\rho - \Gamma_{cR}(t))e^{\rho(t-1)}\,dt\\
&\quad + \int_0^1\int_s^1 (\Gamma_S(t,\xi)\rho^2 - (\Gamma_{RS}(t,\xi)+\Gamma_{RS}(\xi,t))\rho - \Gamma_R(t,\xi))e^{\rho(\xi-t)}\,d\xi\,dt\bigg]\\
&\quad + \frac{e^{\rho(s-x)}}{2\rho}\bigg[(\Gamma_{bd}\rho^2 + (\Gamma_{ad}-\Gamma_{bc})\rho - \Gamma_{ac})e^{-\rho} + 2\Gamma_{cd}\rho\\
&\quad + \int_0^1 (-\Gamma_{dS}(t)\rho^2 + (\Gamma_{cS}(t)-\Gamma_{dR}(t))\rho + \Gamma_{cR}(t))e^{\rho(t-1)}\,dt\\
&\quad + \int_s^1 (\Gamma_{bS}(t)\rho^2 + (\Gamma_{aS}(t)-\Gamma_{bR}(t))\rho - \Gamma_{aR}(t))e^{-\rho t}\,dt\\
&\quad + \int_s^1 (\Gamma_{dS}(t)\rho^2 + (\Gamma_{cS}(t)-\Gamma_{dR}(t))\rho - \Gamma_{cR}(t))e^{\rho(1-t)}\,dt\\
&\quad + \int_0^1\int_s^1 (\Gamma_S(t,\xi)\rho^2 + (\Gamma_{RS}(t,\xi)+\Gamma_{RS}(\xi,t))\rho - \Gamma_R(t,\xi))e^{\rho(t-\xi)}\,d\xi\,dt\bigg].
\end{aligned}
$$

In the following we are going to bound $|N(x,s;\lambda)|$ from (3.2)–(3.3). For the sake of simplicity we will denote with the same symbol $\|\cdot\|$ the supremum norm in one and two variables, so

$$
\begin{aligned}
\|F(t)\| &= \sup\{|F(t)| : 0 \le t \le 1\},\\
\|F(t,\xi)\| &= \sup\{|F(t,\xi)| : 0 \le t,\xi \le 1\}.
\end{aligned}
$$

Also, let $\Re(\rho)$ denote the real part of ρ. As $\rho \in \Sigma_{\delta/2}$, then $\Re(\rho) > 0$.

Using the triangle inequality and performing the resulting integrals, we obtain

$$
\begin{aligned}
&|N(x,s;\lambda)| \\
&\le |\varphi(x,s;\lambda)| + \frac{e^{x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2 + |\Gamma_{ad}+\Gamma_{bc}||\rho| + |\Gamma_{ac}|)e^{(s-1)\Re(\rho)} \\
&\quad + (\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS}+\Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|)\,\frac{1-e^{(s-1)\Re(\rho)}}{\Re(\rho)} \\
&\quad + (\|\Gamma_{dS}\||\rho|^2 + \|\Gamma_{cS}+\Gamma_{dR}\||\rho| + \|\Gamma_{cR}\|)\,\frac{e^{(s-1)\Re(\rho)}-e^{-\Re(\rho)}}{\Re(\rho)} \\
&\quad + (\|\Gamma_{S}\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_{R}\|)\,\frac{1-e^{(s-1)\Re(\rho)}-e^{-s\Re(\rho)}+e^{-\Re(\rho)}}{\Re(\rho)^2}\Bigg] \\
&\quad + \frac{e^{-x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2 + |\Gamma_{ad}+\Gamma_{bc}||\rho| + |\Gamma_{ac}|)^{e^{(1-s)}\Re(\rho)} \\
&\quad + (\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS}+\Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|)\,\frac{e^{(1-s)\Re(\rho)}-1}{\Re(\rho)} \\
&\quad + (\|\Gamma_{dS}\||\rho|^2 + \|\Gamma_{cS}+\Gamma_{dR}\||\rho| + \|\Gamma_{cR}\|)\,\frac{e^{\Re(\rho)-e^{(1-s)}\Re(\rho)}}{\Re(\rho)} \\
&+ (\|\Gamma_{S}\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_{R}\|)\,\frac{1-e^{(1-s)\Re(\rho)-e^{s\Re(\rho)}+e^{\Re(\rho)}}}{\Re(\rho)^2}\Bigg].
\end{aligned}
\tag{3.4}
$$

From (3.3) we have

$$
|\varphi(x,s;\lambda)| = \begin{cases} |\varphi_1(x,s;\lambda)| & \text{if } x > s \\ |\varphi_2(x,s;\lambda)| & \text{if } x < s \end{cases}
\tag{3.5}
$$

where

$$
\begin{aligned}
&|\varphi_1(x,s;\lambda)| \\
&\le \frac{e^{x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2+|\Gamma_{ad}-\Gamma_{bc}||\rho|+|\Gamma_{ac}|)e^{-(s+1)\Re(\rho)}+2|\Gamma_{ab}||\rho|e^{-s\Re(\rho)} \\
&\quad + (\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS}-\Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|)\,\frac{1-e^{-(s+1)\Re(\rho)}}{\Re(\rho)}
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&+(\|\Gamma_{dS}\||\rho|^2+\|\Gamma_{cS}-\Gamma_{dR}\||\rho|+\|\Gamma_{cR}\|)\,\frac{e^{-\Re(\rho)}-e^{-(s+1)\Re(\rho)}}{\Re(\rho)}\\
&+(\|\Gamma_S\||\rho|^2+2\|\Gamma_{RS}\||\rho|+\|\Gamma_R\|)\,\frac{1-e^{-\Re(\rho)}+e^{-(s+1)\Re(\rho)}-e^{-s\Re(\rho)}}{\Re(\rho)^2}\Bigg]\\
&+\frac{e^{-x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2+|\Gamma_{ad}+\Gamma_{bc}||\rho|+\Gamma_{ac}|)e^{(s+1)\Re(\rho)}\\
&+2|\Gamma_{ab}||\rho|e^{s\Re(\rho)}+(\|\Gamma_{bS}\||\rho|^2+\|\Gamma_{aS}-\Gamma_{bR}\||\rho|+\|\Gamma_{aR}\|)\,\frac{e^{(s+1)\Re(\rho)}-1}{\Re(\rho)}\\
&+(\|\Gamma_{dS}\||\rho|^2+\|\Gamma_{cS}-\Gamma_{dR}\||\rho|+\|\Gamma_{cR}\|)\,\frac{e^{(s+1)\Re(\rho)}-e^{\Re(\rho)}}{\Re(\rho)}\\
&+(\|\Gamma_S\||\rho|^2+2\|\Gamma_{RS}\||\rho|+\|\Gamma_R\|)\,\frac{1+e^{(s+1)\Re(\rho)}-e^{s\Re(\rho)}-e^{\Re(\rho)}}{\Re(\rho)^2}\Bigg].
\end{aligned}
$$

and

(3.7)

$$
\begin{aligned}
&|\varphi_2(x,s;\lambda)|\\
&\le \frac{e^{x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2+|\Gamma_{ad}-\Gamma_{bc}||\rho|+|\Gamma_{ac}|)e^{(1-s)\Re(\rho)}+2|\Gamma_{cd}||\rho|e^{-s\Re(\rho)}\\
&\quad+(\|\Gamma_{bS}\||\rho|^2+\|\Gamma_{aS}-\Gamma_{bR}\||\rho|+\|\Gamma_{aR}\|)\,\frac{e^{(1-s)\Re(\rho)}-1}{\Re(\rho)}\\
&\quad+(\|\Gamma_{dS}\||\rho|^2+\|\Gamma_{cS}-\Gamma_{dR}\||\rho|+\|\Gamma_{cR}\|)\,\frac{e^{(1-s)\Re(\rho)}-e^{-\Re(\rho)}}{\Re(\rho)}\\
&\quad+(\|\Gamma_S\||\rho|^2+2\|\Gamma_{RS}\||\rho|+\|\Gamma_R\|)\frac{e^{(1-s)\Re(\rho)}-e^{-s\Re(\rho)}+e^{-\Re(\rho)}-1}{\Re(\rho)^2}\Bigg]\\
&\quad+\frac{e^{-x\Re(\rho)}}{2|\rho|}\Bigg[(|\Gamma_{bd}||\rho|^2+|\Gamma_{ad}+\Gamma_{bc}||\rho|+|\Gamma_{ac}|)e^{(s-1)\Re(\rho)}+2|\Gamma_{cd}||\rho|e^{s\Re(\rho)}\\
&\quad+(\|\Gamma_{bS}\||\rho|^2+\|\Gamma_{aS}-\Gamma_{bR}\||\rho|+\|\Gamma_{aR}\|)\,\frac{1-e^{(s-1)\Re(\rho)}}{\Re(\rho)}\\
&\quad+(\|\Gamma_{dS}\||\rho|^2+\|\Gamma_{cS}-\Gamma_{dR}\||\rho|+\|\Gamma_{cR}\|)\,\frac{e^{\Re(\rho)}-e^{(s-1)\Re(\rho)}}{\Re(\rho)}\\
&\quad+(\|\Gamma_S\||\rho|^2+2\|\Gamma_{RS}\||\rho|+\|\Gamma_R\|)\,\frac{e^{\Re(\rho)}-e^{s\Re(\rho)}+e^{(s-1)\Re(\rho)}-1}{\Re(\rho)^2}\Bigg].
\end{aligned}
$$

4. Bounds in $L^1(0,1)$. Take an arbitrary $0 \neq f \in L^1(0,1)$ and suppose that $\Delta(\lambda) \neq 0$. From (2.6) we have

$$\|R(\lambda : T)f\|_{L^1(0,1)} \leq \left(\sup_{0 \leq s \leq 1} \int_0^1 |G(x,s;\lambda)|\, dx \right) \|f\|_{L^1(0,1)}$$

so

(4.1)
$$\|R(\lambda : T)\| \leq \sup_{0 \leq s \leq 1} \int_0^1 |G(x,s;\lambda)|\, dx = \frac{1}{|\Delta(\lambda)|} \sup_{0 \leq s \leq 1} \int_0^1 |N(x,s;\lambda)|\, dx.$$

It will then be necessary to bound $\int_0^1 |N(x,s;\lambda)|\, dx$ appropriately.

From (3.4) we obtain, after performing the integrals, the following inequality:

$$\begin{aligned}
\int_0^1 |N(x,s;\lambda)|\, dx \leq \int_0^1 & |\varphi(x,s;\lambda)|\, dx \\
& + \frac{1}{|\rho|\Re(\rho)} \Bigg[(\Gamma_{bd}||\rho|^2 + |\Gamma_{ad} + \Gamma_{bc}||\rho| + |\Gamma_{ac}|) \\
& \cdot (\sinh[s\Re(\rho)] + \sinh[(1-s)\Re(\rho)]) \\
& + (\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS} + \Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|) \\
& \cdot \frac{\cosh[\Re(\rho)] - \cosh[s\Re(\rho)] + \cosh[(1-s)\Re(\rho)] - 2}{\Re(\rho)} \\
& + (\|\Gamma_{dS}\||\rho|^2 + \|\Gamma_{cS} + \Gamma_{dR}\||\rho| + \|\Gamma_{cR}\|) \\
& \cdot \frac{\cosh[\Re(\rho)] + \cosh[s\Re(\rho)] - \cosh[(1-s)\Re(\rho)] - 2}{\Re(\rho)} \\
& + 2(\|\Gamma_S\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_R\|) \\
& \cdot \frac{\sinh[\Re(\rho)] - \sinh[s\Re(\rho)] - \sinh[(1-s)\Re(\rho)]}{\Re(\rho)^2} \Bigg].
\end{aligned}$$

In order to evaluate $\int_0^1 |\varphi(x,s;\lambda)|\, dx$, from (3.5) we write

$$\int_0^1 |\varphi(x,s;\lambda)|\, dx = \int_0^s |\varphi_2(x,s;\lambda)|\, dx + \int_s^1 |\varphi_1(x,s;\lambda)|\, dx.$$

From (3.7) and (3.6) we have, respectively,

$$\begin{aligned}
&\int_0^s |\varphi_2(x,s;\lambda)|\,dx \\
&\qquad \le \frac{1}{|\rho|\Re(\rho)}\biggl[2|\Gamma_{cd}||\rho|\sinh[s\Re(\rho)] \\
&\qquad\quad + (|\Gamma_{bd}||\rho|^2 + |\Gamma_{ad}-\Gamma_{bc}||\rho| + |\Gamma_{ac}|) \\
&\qquad\quad \cdot (\sinh[\Re(\rho)] - \sinh[(1-s)\Re(\rho)]) \\
&\qquad\quad + (\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS}-\Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|) \\
&\qquad\quad \cdot \frac{\cosh[\Re(\rho)] - \cosh[s\Re(\rho)] - \cosh[(1-s)\Re(\rho)] + 2}{\Re(\rho)} \\
&\qquad\quad + 2(\|\Gamma_{dS}\||\rho|^2 + \|\Gamma_{cS}-\Gamma_{dR}\||\rho| + \|\Gamma_{cR}\|) \\
&\qquad\quad \cdot \frac{\cosh[\Re(\rho)] - \cosh[(1-s)\Re(\rho)]}{\Re(\rho)} \\
&\qquad\quad + 2(\|\Gamma_{S}\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_{R}\|) \\
&\qquad\quad \cdot \frac{\sinh[\Re(\rho)] - \sinh[s\Re(\rho)] - \sinh[(1-s)\Re(\rho)]}{\Re(\rho)^2}\biggr]
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^1 |\varphi_1(x,s;\lambda)|\,dx \\
&\qquad \le \frac{1}{|\rho|\Re(\rho)}\biggl[2|\Gamma_{ab}||\rho|\sinh[(1-s)\Re(\rho)] \\
&\qquad\quad + (|\Gamma_{bd}||\rho|^2 + |\Gamma_{ad}-\Gamma_{bc}||\rho| + |\Gamma_{ac}|) \\
&\qquad\quad \cdot (\sinh[\Re(\rho)] - \sinh[s\Re(\rho)]) \\
&\qquad\quad + 2(\|\Gamma_{bS}\||\rho|^2 + \|\Gamma_{aS}-\Gamma_{bR}\||\rho| + \|\Gamma_{aR}\|) \\
&\qquad\quad \cdot \frac{\cosh[\Re(\rho)] - \cosh[s\Re(\rho)]}{\Re(\rho)} \\
&\qquad\quad + (\|\Gamma_{dS}\||\rho|^2 + \|\Gamma_{cS}-\Gamma_{dR}\||\rho| + \|\Gamma_{cR}\|) \\
&\qquad\quad \cdot \frac{\cosh[\Re(\rho)] - \cosh[s\Re(\rho)] - \cosh[(1-s)\Re(\rho)] + 2}{\Re(\rho)} \\
&\qquad\quad + 2(\|\Gamma_{S}\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_{R}\|) \\
&\qquad\quad \cdot \frac{\sinh[\Re(\rho)] - \sinh[s\Re(\rho)] - \sinh[(1-s)\Re(\rho)]}{\Re(\rho)^2}\biggr].
\end{aligned}$$

Adding up the inequalities above, we get

$$\begin{aligned}
&\int_0^1 |N(x,s;\lambda)|\,dx \\
&\quad \le \frac{2}{|\rho|\Re(\rho)}\Bigg[(|\Gamma_{bd}||\rho|^2+(|\Gamma_{ad}|+|\Gamma_{bc}|)|\rho|+|\Gamma_{ac}|)\sinh[\Re(\rho)] \\
&\qquad + |\Gamma_{ab}||\rho|\sinh[(1-s)\Re(\rho)] + |\Gamma_{cd}||\rho|\sinh[s\Re(\rho)] \\
&\qquad + 2(\|\Gamma_{bS}\||\rho|^2 + (\|\Gamma_{aS}\| + \|\Gamma_{bR}\|)|\rho| + \|\Gamma_{aR}\|) \\
&\qquad \cdot \frac{\cosh[\Re(\rho)] - \cosh[s\Re(\rho)]}{\Re(\rho)} \\
&\qquad + 2(\|\Gamma_{dS}\||\rho|^2 + (\|\Gamma_{cS}\| + \|\Gamma_{dR}\|)|\rho| + \|\Gamma_{cR}\|) \\
&\qquad \cdot \frac{\cosh[\Re(\rho)] - \cosh[(1-s)\Re(\rho)]}{\Re(\rho)} \\
&\qquad + 3(\|\Gamma_S\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_R\|) \\
&\qquad \cdot \frac{\sinh[\Re(\rho)] - \sinh[s\Re(\rho)] - \sinh[(1-s)\Re(\rho)]}{\Re(\rho)^2}\Bigg].
\end{aligned}$$

Note that, as $\rho \in \Sigma_{\delta/2}$, we have $\Re(\rho) \ge \cos(\delta/2)|\rho|$. Then, taking the supremum and eliminating the negative terms, we obtain the following inequality:

$$\begin{aligned}
&\sup_{0\le s\le 1}\int_0^1 |N(x,s;\lambda)|\,dx \\
&\le \frac{e^{\Re(\rho)}}{\cos(\delta/2)|\rho|^2}\,(|\Gamma_{bd}||\rho|^2 + (|\Gamma_{ab}| + |\Gamma_{ad}| + |\Gamma_{bc}| + |\Gamma_{cd}|)|\rho| + |\Gamma_{ac}|) \\
&\quad + \frac{2e^{\Re(\rho)}}{\cos^2(\delta/2)|\rho|^3}\,\big[(\|\Gamma_{bS}\| + \|\Gamma_{dS}\|)|\rho|^2 \\
&\quad + (\|\Gamma_{aS}\|+\|\Gamma_{bR}\| + \|\Gamma_{cS}\| + \|\Gamma_{dR}\|)|\rho| + \|\Gamma_{aR}\| + \|\Gamma_{cR}\|\,\big] \\
&\quad + \frac{3e^{\Re(\rho)}}{\cos^3(\delta/2)|\rho|^4}\,(\|\Gamma_S\||\rho|^2 + 2\|\Gamma_{RS}\||\rho| + \|\Gamma_R\|).
\end{aligned}$$

Taking $m := \max[(1/\cos(\delta/2)), (2/\cos^2(\delta/2)), (3/\cos^3(\delta/2))]$ and, grouping the terms of the same order, we finally obtain

$$\sup_{0\le s\le 1}\int_0^1 |G(x,s;\lambda)|\,dx \le m\,\frac{H(\rho)}{|\rho|^2} = m\,\frac{H(\rho)}{|\lambda|} \tag{4.2}$$

where
(4.3)

$$H(\rho)=\frac{e^{\Re(\rho)}}{|\Delta(\lambda)|}\Big[|\Gamma_{bd}||\rho|^2+(|\Gamma_{ab}|+|\Gamma_{ad}|+|\Gamma_{bc}|+|\Gamma_{cd}|+\|\Gamma_{bS}\|+\|\Gamma_{dS}\|)$$
$$\cdot|\rho|+(|\Gamma_{ac}|+\|\Gamma_{aS}\|+\|\Gamma_{bR}\|+\|\Gamma_{cS}\|+\|\Gamma_{dR}\|+\|\Gamma_S\|)$$
$$+\frac{\|\Gamma_{aR}\|+\|\Gamma_{cR}\|+2\|\Gamma_{RS}\|}{|\rho|}+\frac{\|\Gamma_R\|}{|\rho|^2}\Big].$$

Now we must analyze the function $H(\rho)$ in order to determine the cases for which it is bounded in the sector $\Sigma_{\delta/2}$.

It will be convenient to write the formula (3.1) for the characteristic determinant in a slightly different form, as follows:
(4.4)

$$\begin{aligned}\Delta(\lambda)=e^{\rho}\bigg[&(-\Gamma_{bd}\rho^2+(\Gamma_{ad}-\Gamma_{bc})\rho+\Gamma_{ac})\\&+(\Gamma_{bd}\rho^2+(\Gamma_{ad}-\Gamma_{bc})\rho-\Gamma_{ac})e^{-2\rho}+2(\Gamma_{ab}+\Gamma_{cd})\rho e^{-\rho}\\&+\rho^2\int_0^1(-\Gamma_{bS}(t)+\Gamma_{dS}(1-t))(e^{\rho(t-1)}-e^{-\rho(t+1)})\,dt\\&+\rho\int_0^1(\Gamma_{aS}(t)+\Gamma_{cS}(1-t)-\Gamma_{bR}(t)\\&-\Gamma_{dR}(1-t))(e^{\rho(t-1)}+e^{-\rho(t+1)})\,dt\\&+\int_0^1(\Gamma_{aR}(t)-\Gamma_{cR}(1-t))(e^{\rho(t-1)}-e^{-\rho(t+1)})\,dt\\&+\int_0^1\int_0^1(\Gamma_S(t,\xi)\rho^2+(\Gamma_{RS}(t,\xi)+\Gamma_{RS}(\xi,t))\rho\\&+\Gamma_R(t,\xi))e^{\rho(\xi-t-1)}\,d\xi\,dt\bigg].\end{aligned}$$

Note that the second line in the above formula can be made arbitrarily small for $|\rho|$ sufficiently large. We can then choose r_0 large enough so that for

$$(4.5)\quad |(\Gamma_{bd}\rho^2+(\Gamma_{ad}-\Gamma_{bc})\rho-\Gamma_{ac})e^{-2\rho}+2(\Gamma_{ab}+\Gamma_{cd})\rho e^{-\rho}|<\frac{1}{|\rho|^2}$$

holds for every $\rho\in\Sigma_{\delta/2}$ with $|\rho|>r_0$.

It will also be necessary to know how the integral terms depend on ρ. Using that $\Re(\rho) \geq \cos(\delta/2)|\rho|$, it is easy to see that

$$\begin{aligned}(4.6\text{a})\quad &\left|\rho^2 \int_0^1 (-\Gamma_{bS}(t) + \Gamma_{dS}(1-t))(e^{\rho(t-1)} - e^{-\rho(t+1)})\,dt\right| \\ &\qquad\leq \frac{\|\Gamma_{bS}\| + \|\Gamma_{dS}\|}{\cos(\delta/2)}|\rho|,\end{aligned}$$

(4.6b)

$$\begin{aligned}&\left|\rho \int_0^1 (\Gamma_{aS}(t) + \Gamma_{cS}(1-t) - \Gamma_{bR}(t) - \Gamma_{dR}(1-t))(e^{\rho(t-1)} + e^{-\rho(t+1)})\,dt\right| \\ &\qquad\leq \frac{\|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\|}{\cos(\delta/2)},\end{aligned}$$

$$\begin{aligned}(4.6\text{c})\quad &\left|\int_0^1 (\Gamma_{aR}(t) - \Gamma_{cR}(1-t))(e^{\rho(t-1)} - e^{-\rho(t+1)})\,dt\right| \\ &\qquad\leq \frac{\|\Gamma_{aR}\| + \|\Gamma_{cR}\|}{\cos(\delta/2)|\rho|},\end{aligned}$$

$$\begin{aligned}(4.6\text{d})\quad &\Bigg|\int_0^1\int_0^1 (\Gamma_S(t,\xi)\rho^2 + (\Gamma_{RS}(t,\xi) + \Gamma_{RS}(\xi,t))\rho + \Gamma_R(t,\xi)) \\ &\qquad e^{\rho(\xi-t-1)}\,d\xi\,dt\Bigg| \\ &\qquad\leq \frac{1}{\cos^2(\delta/2)}\left(\|\Gamma_S\| + 2\frac{\|\Gamma_{RS}\|}{|\rho|} + \frac{\|\Gamma_R\|}{|\rho|^2}\right).\end{aligned}$$

5. Analysis of cases. We are now ready to analyze the function $H(\rho)$. In the following $E(\rho)$ will denote a bounded function of ρ that could differ from one case to another.

Case 1. Suppose that $\Gamma_{bd} \neq 0$. From (4.4) we can write the characteristic determinant as

$$\Delta(\lambda) = \Gamma_{bd}\rho^2 e^{\rho}(-1 + E(\rho))$$

for a certain function $E(\rho)$. Using (4.5)–(4.6), we see that $E(\rho)$ can be bounded as

$$\begin{aligned}|E(\rho)| \le \frac{|\Gamma_{ad}-\Gamma_{bc}|}{|\Gamma_{bd}|\rho|} + \frac{|\Gamma_{ac}|}{|\Gamma_{bd}||\rho|^2} + \frac{1}{|\Gamma_{bd}||\rho|^4} + \frac{\|\Gamma_{bS}\|+\|\Gamma_{dS}\|}{\cos(\delta/2)|\Gamma_{bd}||\rho|} \\ + (\|\Gamma_{bR}\|+\|\Gamma_{dR}\|+\|\Gamma_{aS}\|+\|\Gamma_{cS}\|)\,\frac{1}{\cos(\delta/2)|\Gamma_{bd}||\rho|^2} \\ + \frac{\|\Gamma_{aR}\|+\|\Gamma_{cR}\|}{\cos(\delta/2)|\Gamma_{bd}||\rho|^3} \\ + \left(\frac{\|\Gamma_S\|}{|\rho|^2} + 2\frac{\|\Gamma_{RS}\|}{|\rho|^3} + \frac{\|\Gamma_R\|}{|\rho|^4}\right)\frac{1}{\cos^2(\delta/2)|\Gamma_{bd}|}\end{aligned}$$

so we can choose r_0 sufficiently large for $|E(\rho)| \le 1/2$ if $|\rho| > r_0$. Thus, for $\rho \in r_0 + \Sigma_{\delta/2}$, we have

$$|\Delta(\lambda)| \ge |\Gamma_{bd}||\rho|^2 e^{\Re(\rho)}(1-|E(\rho)|) \ge \frac{|\Gamma_{bd}|}{2}|\rho|^2 e^{\Re(\rho)}.$$

Finally, from (4.3) we obtain

$$\begin{aligned}H(\rho) \le \frac{2}{|\Gamma_{bd}|}\big[1+r_0^{-1}(|\Gamma_{ab}|+|\Gamma_{ad}|+|\Gamma_{bc}|+|\Gamma_{cd}|+\|\Gamma_{bS}\|+\|\Gamma_{dS}\|) \\ + r_0^{-2}(|\Gamma_{ac}| + \|\Gamma_{aS}\| + \|\Gamma_{bR}\| + \|\Gamma_{cS}\| + \|\Gamma_{dR}\| + \|\Gamma_S\|) \\ + r_0^{-3}(\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\|) + r_0^{-4}\|\Gamma_R\|\big] =: H_0\end{aligned}$$

which proves that $H(\rho)$ is bounded by a constant H_0 in the sector $r_0 + \Sigma_{\delta/2}$. □

From now on, we will suppose that $\Gamma_{bd} = 0$; then, the dominant term in $H(\rho)$ will be $|\Gamma_{ab}| + |\Gamma_{ad}| + |\Gamma_{bc}| + |\Gamma_{cd}| + \|\Gamma_{bS}\| + \|\Gamma_{dS}\|$.

Case 2. Suppose that $|\Gamma_{ab}|+|\Gamma_{ad}|+|\Gamma_{bc}|+|\Gamma_{cd}|+\|\Gamma_{bS}\|+\|\Gamma_{dS}\| \ne 0$. Taking into account (4.4) and (4.6a), we see that the dominant term in $\Delta(\lambda)$ is

$$(\Gamma_{ad}-\Gamma_{bc})\rho + \rho^2 \int_0^1\int_0^1 (-\Gamma_{bS}(t) + \Gamma_{dS}(1-t))(e^{\rho(t-1)} - e^{-\rho(t+1)})\,dt.$$

In order to get appropriate bounds, we should give the above term a more suitable form. For doing that we suppose that $S_1, S_2 \in C^1([0,1];\mathbf{C})$ (eventually, we will see that this will not rest generality to our results) and perform an integration by parts. Then the above term can be written as

$$\rho\bigg[(\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0))+(\Gamma_{dS}(0)-\Gamma_{bS}(1))e^{-2\rho} + 2(\Gamma_{bS}(0)-\Gamma_{dS}(1))e^{-\rho}+\int_0^1(\Gamma_{bS'}(t)(t)+\Gamma_{dS'}(1-t)) \cdot (e^{\rho(t-1)}+e^{-\rho(t+1)})\,dt\bigg].$$

If $m_0 := \max(\|S_1\|_\infty, \|S_2\|_\infty)$ and r_0 is large enough, we have for $|\rho|>r_0$ that

$$\left|(\Gamma_{ad}-\Gamma_{bc})\rho+\rho^2\int_0^1\!\!\int_0^1(-\Gamma_{bS}(t)+\Gamma_{dS}(1-t))(e^{\rho(t-1)}-e^{-\rho(t+1)})\,dt\right| \leq |\rho|\bigg(|\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0)|+\frac{1}{|\rho|^2}+\frac{m_0}{\cos(\delta/2)|\rho|}\bigg).$$

We have two subcases to consider.

Case 2.1. $\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0)\neq 0$. For $|\rho|>r_0$ we have

$$|\Delta(\lambda)|\geq|\rho|e^{\Re(\rho)}(|\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0)|-|E(\rho)|)$$

where

$$|E(\rho)|\leq\frac{2}{|\rho|^2}+\frac{|\Gamma_{ac}|}{|\rho|}+\frac{m_0+\|\Gamma_{bR}\|+\|\Gamma_{dR}\|+\|\Gamma_{aS}\|+\|\Gamma_{cS}\|}{\cos(\delta/2)|\rho|} + \frac{\|\Gamma_{aR}\|+\|\Gamma_{cR}\|}{\cos(\delta/2)|\rho|^2}+\frac{1}{\cos^2(\delta/2)}\bigg(\frac{\|\Gamma_S\|}{|\rho|}+2\,\frac{\|\Gamma_{RS}\|}{|\rho|^2}+\frac{\|\Gamma_R\|}{|\rho|^3}\bigg).$$

This last term can be done less than $\frac{1}{2}|\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0)|$ by choosing r_0 large enough. Thus, we have for $|\rho|>r_0$ that

$$|\Delta(\lambda)|\geq\frac{|\rho|}{2}\,e^{\Re(\rho)}|\Gamma_{ad}-\Gamma_{bc}-\Gamma_{bS}(1)+\Gamma_{dS}(0)|$$

so

$$\begin{aligned}H(\rho) \leq{}& \frac{2}{|\Gamma_{ad} - \Gamma_{bc} - \Gamma_{bS}(1) + \Gamma_{dS}(0)|} \\ &\cdot \big[|\Gamma_{ab}| + |\Gamma_{ad}| + |\Gamma_{bc}| + |\Gamma_{cd}| + \|\Gamma_{bS}\| + \|\Gamma_{dS}\| \\ &+ r_0^{-1}(|\Gamma_{ac}| + \|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\| + \|\Gamma_S\|) \\ &+ r_0^{-2}(\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\|) + r_0^{-3}\|\Gamma_R\| r_0^3\big] =: H_0.\end{aligned}$$

This shows that $H(\rho)$ is bounded in $r_0 + \Sigma_{\delta/2}$. □

Case 2.2. $\Gamma_{ad} - \Gamma_{bc} - \Gamma_{bS}(1) + \Gamma_{dS}(0) = 0$. Note that

$$|\Delta(\lambda)| \leq e^{\Re(\rho)}|\rho E(\rho)|$$

where $E(\rho)$ is the same function as in Case 2.1, and $|\rho E(\rho)|$ can be bounded by a certain constant $c_0 > 0$. Thus, we have

$$\begin{aligned}H(\rho) \geq c_0^{-1}\bigg[&(|\Gamma_{ab}| + |\Gamma_{ad}| + |\Gamma_{bc}| + |\Gamma_{cd}| + \|\Gamma_{bS}\| + \|\Gamma_{dS}\|)|\rho| \\ &+ (|\Gamma_{ac}| + \|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\| + \|\Gamma_S\|) \\ &+ \frac{\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\|}{|\rho|} + \frac{\|\Gamma_R\|}{|\rho|^2}\bigg]\end{aligned}$$

which shows that $H(\rho)$ is not bounded. □

In the following we will suppose that $\Gamma_{ab} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bS} \equiv 0$ and $\Gamma_{dS} \equiv 0$. Now the behavior of $H(\rho)$ depends on the coefficient $|\Gamma_{ac}| + \|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\| + \|\Gamma_S\|$.

Case 3. Suppose that $|\Gamma_{ac}| + \|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\| + \|\Gamma_S\| \neq 0$. Now the dominant term in $\Delta(\lambda)$ is

$$\begin{aligned}\Gamma_{ac} + \rho\int_0^1 &(\Gamma_{aS}(t) + \Gamma_{cS}(1-t) - \Gamma_{bR}(t) - \Gamma_{dR}(1-t)) \\ &\cdot (e^{\rho(t-1)} + e^{-\rho(t+1)})\,dt + \rho^2\int_0^1\int_0^1 \Gamma_S(t,\xi)e^{\rho(\xi-t-1)}\,d\xi\,dt.\end{aligned}$$

As in Case 2, we must give to this term a more suitable form. Supposing that also R_1 and R_2 are in $C^1([0,1];\mathbf{C})$ and performing an integration by parts, we can write the expression above in the following form:

$$\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1) + \frac{\tilde{E}(\rho)}{\rho}$$

where $\tilde{E}(\rho)$ is a bounded function. We can rewrite the characteristic determinant as

$$\Delta(\lambda) = e^{\rho}\left(\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1) + \frac{E(\rho)}{\rho}\right)$$

with $E(\rho)$ another bounded function. We must distinguish two subcases.

Case 3.1. $\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1) \neq 0$. Choosing $|\rho| > r_0$ large enough so that

$$\left|\frac{E(\rho)}{\rho}\right| \leq \frac{1}{2}\left|\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1)\right|$$

we have

$$|\Delta(\lambda)| \geq \frac{1}{2}|\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1)|e^{\Re(\rho)}.$$

This shows that $H(\rho)$ is bounded. □

Case 3.2. $\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1) = 0$. We have that

$$|\Delta(\lambda)| = \frac{E(\rho)}{|\rho|}\, e^{\Re(\rho)} \leq \frac{c_0}{|\rho|}\, e^{\Re(\rho)}$$

for a certain constant $c_0 > 0$. Thus,

$$\begin{aligned} H(\rho) \leq c_0^{-1}\bigg[&(|\Gamma_{ac}| + \|\Gamma_{bR}\| + \|\Gamma_{dR}\| + \|\Gamma_{aS}\| + \|\Gamma_{cS}\| + \|\Gamma_S\|)|\rho| \\ &+ \|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\| + \frac{\|\Gamma_R\|}{|\rho|}\bigg] \end{aligned}$$

so $H(\rho)$ cannot be bounded. $\square$

From now on, we suppose that $\Gamma_{ab} = \Gamma_{ac} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bR} \equiv 0$, $\Gamma_{dR} \equiv 0$, $\Gamma_{aS} \equiv 0$, $\Gamma_{cS} \equiv 0$, $\Gamma_{bS} \equiv 0$, $\Gamma_{dS} \equiv 0$ and $\Gamma_S \equiv 0$. Then $H(\rho)$ and $\Delta(\lambda)$ can be written in simplified form as

$$H(\rho) = \frac{e^{\Re(\rho)}}{|\Delta(\lambda)|}\left(\frac{\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\|}{|\rho|} + \frac{\|\Gamma_R\|}{|\rho|^2}\right)$$

and

$$\begin{aligned}\Delta(\lambda) = e^{\rho}\bigg[\int_0^1 (\Gamma_{aR}(t) - \Gamma_{cR}(1-t))(e^{\rho(t-1)} - e^{-\rho(t+1)})\,dt \\ + \int_0^1\int_0^1 ((\Gamma_{RS}(t,\xi) + \Gamma_{RS}(\xi,t))\rho + \Gamma_R(t,\xi))e^{\rho(\xi-t-1)}\,d\xi\,dt\bigg].\end{aligned}$$

Case 4. Suppose that $\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\| \neq 0$. The dominant term in $\Delta(\lambda)$ is, in this case,

$$\begin{aligned}\int_0^1 (\Gamma_{aR}(t) - \Gamma_{cR}(1-t))(e^{\rho(t-1)} - e^{-\rho(t+1)})\,dt \\ + \rho\int_0^1\int_0^1 (\Gamma_{RS}(t,\xi) + \Gamma_{RS}(\xi,t))e^{\rho(\xi-t-1)}\,d\xi\,dt\end{aligned}$$

that, after an integration by parts, can be written as

$$\frac{1}{\rho}\left(\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0)\right) + \frac{\tilde{E}(\rho)}{\rho^2}$$

where $\tilde{E}(\rho)$ is bounded. Thus we have

$$\Delta(\lambda) = \frac{e^{\rho}}{\rho}\left(\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0) + \frac{E(\rho)}{\rho}\right).$$

We consider two subcases.

Case 4.1. $\Gamma_{aR}(1)-\Gamma_{cR}(0)+\Gamma_{RS}(0,1)+\Gamma_{RS}(1,0) \neq 0$. We can choose r_0 sufficiently large so that

$$|\Delta(\lambda)| \geq \frac{e^{\Re(\rho)}}{2|\rho|}\,|\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0)|$$

holds for $|\rho| > r_0$, so

$$H(\rho) \leq 2\,\frac{\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\| + \|\Gamma_R\| r_0^{-1}}{|\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0)|} =: H_0$$

which shows that $H(\rho)$ is bounded in the sector $r_0 + \Sigma_{\delta/2}$. $\square$

Case 4.2. $\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0) = 0$. We have that

$$|\Delta(\lambda)| = \left|\frac{e^\rho}{\rho^2}\, E(\rho)\right| \leq c_0\, \frac{e^{\Re(\rho)}}{|\rho|^2}$$

for a certain constant $c_0 > 0$. Thus,

$$H(\rho) \geq \frac{|\rho|}{r_0}\left(\|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\| + \frac{\|\Gamma_R\|}{|\rho|}\right).$$

This shows that $H(\rho)$ is not bounded. $\square$

Finally, we also suppose that $\Gamma_{aR} \equiv 0$, $\Gamma_{cR} \equiv 0$ and $\Gamma_{RS} \equiv 0$. Thus,

$$H(\rho) = \|\Gamma_R\|\, \frac{e^{\Re(\rho)}}{|\Delta(\lambda)||\rho|^2}, \quad \Delta(\lambda) = \int_0^1\!\int_0^1 \Gamma_R(t,\xi) e^{\rho(\xi - t)}\, d\xi\, dt.$$

Integrating by parts, the characteristic determinant can be written as

$$\Delta(\lambda) = \frac{e^\rho}{\rho^2}\left(\Gamma_R(0,1) + \frac{E(\rho)}{\rho}\right)$$

for a certain bounded function $E(\rho)$.

Case 5. Suppose that $\Gamma_R(0,1) \neq 0$. Then it is possible to take r_0 large enough so that

$$|\Delta(\lambda)| \geq |\Gamma_R(0,1)|\, \frac{e^{\Re(\rho)}}{2|\rho|^2}$$

holds for $|\rho| > r_0$. Thus, $H(\rho)$ can be bounded as

$$H(\rho) \leq 2\,\frac{\|\Gamma_R\|}{|\Gamma_R(0,1)|} =: H_0$$

in the sector $r_0 + \Sigma_{\delta/2}$. $\square$

Case 6. Suppose that $\Gamma_R(0,1) = 0$. Then for some constant $c_0 > 0$, we have

$$|\Delta(\lambda)| = \left| e^\rho \, \frac{E(\rho)}{\rho^3} \right| \le c_0 \, \frac{e^{\Re(\rho)}}{|\rho|^3}$$

so

$$H(\rho) \ge c_0^{-1} \|\Gamma_R\| |\rho|.$$

This shows that $H(\rho)$ is not bounded. $\square$

We have see that $H(\rho)$ is bounded by a constant $H_0 > 0$ in a sector of the form $r_0 + \Sigma_{\delta/2}$, only in the following five cases: 1, 2.1, 3.1, 4.1 and 5. This leads to the following definition (note that some redundancies have been avoided):

Definition 5.1. Suppose that $R_i, S_i \in C([0,1]; \mathbf{C})$ for $i = 1, 2$. The boundary conditions $\{B_1, B_2\}$ are *regular* if they verify one of the following conditions:

1. $\Gamma_{bd} \neq 0$.
2. $\Gamma_{bd} = 0$ and $\Gamma_{ad} - \Gamma_{bc} - \Gamma_{bS}(1) + \Gamma_{dS}(0) \neq 0$.
3. $\Gamma_{ab} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bS} \equiv 0$, $\Gamma_{dS} \equiv 0$ and

$$\Gamma_{ac} + \Gamma_{aS}(1) + \Gamma_{cS}(0) - \Gamma_{bR}(1) - \Gamma_{dR}(0) + \Gamma_S(0,1) \neq 0.$$

4. $\Gamma_{ab} = \Gamma_{ac} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bR} \equiv 0$, $\Gamma_{dR} \equiv 0$, $\Gamma_{aS} \equiv 0$, $\Gamma_{bS} \equiv 0$, $\Gamma_{cS} \equiv 0$, $\Gamma_{dS} \equiv 0$, $\Gamma_S \equiv 0$ and

$$\Gamma_{aR}(1) - \Gamma_{cR}(0) + \Gamma_{RS}(0,1) + \Gamma_{RS}(1,0) \neq 0.$$

5. $a_i = b_i = c_i = d_i = 0$, $S_i \equiv 0$ for $i = 1, 2$ and $\Gamma_R(0,1) \neq 0$.

It is not difficult to see that Definition 5.1 does not depend on possible elementary simplifications on the boundary conditions or possible integrations by parts.

From (4.1) and (4.2), we deduce that, in the case of regular boundary conditions with C^1 coefficients, the sector $r_0^2 + \Sigma_\delta$ is contained in

$\rho(T)$ and there is a constant $M_0 := mH_0$ such that $\|R(\lambda : T)\| \leq (M_0/|\lambda|)$ for every $\lambda \in r_0^2 + \Sigma_\delta$. Defining $r := (r_0^2/\sin(\delta))$ and $M := M_0[1 + (1/\sin(\delta))]$, we have that $\Sigma_{\delta,r} \equiv r + \Sigma_\delta \subset \rho(T)$ and

$$\|R(\lambda : T)\| \leq \frac{M}{|\lambda - r|}, \quad \lambda \in \Sigma_{\delta,r}. \tag{5.1}$$

As a consequence, in this case T is the generator of an analytic semigroup of bounded linear operators in $L^1(0,1)$; in general, this semigroup will not be a C_0-semigroup.

6. Approximation. At some point in the analysis of cases made in the previous section, we needed to impose some regularity conditions of the functions R_i and S_i, specifically, that they were of class C^1. In this section we will show that it is sufficient with supposing continuity. The idea is to use the well-known approximation results of Kato ([**5**], Chapter 9).

Suppose that boundary conditions $\{B_1, B_2\}$ are regular (note that we only suppose $R_i, S_i \in C([0,1]; \mathbf{C})$ for $i = 1, 2$). We can build two sequences $\{R_i^n\}$ and $\{S_i^n\}$ in $C^1([0,1]; \mathbf{C})$ such that

1. The sequences $\{R_i^n\}, \{S_i^n\}, \{(R_i^n)'\}$ and $\{(S_i^n)'\}$ are uniformly bounded.

2. $\{R_i^n\}$ and $\{S_i^n\}$ converge uniformly to R and S, respectively.

3. $R_i^n(0) = R(0)$, $R_i^n(1) = R(1)$, $S_i^n(0) = S(0)$ and $S_i^n(1) = S(1)$ for each $n \in \mathbf{N}$. If $S_i \equiv 0$ we take $S_i^n \equiv 0$, and the same for R_i.

For $i = 1, 2$, consider the boundary conditions

$$\begin{aligned} B_i^n(u) \equiv a_i u(0) + b_i u'(0) + c_i u(1) + d_i u'(1) \\ + \int_0^1 R_i^n(t) u(t)\, dt + \int_0^1 S_i^n(t) u'(t)\, dt = 0 \end{aligned}$$

and let T_n be the associated operator in $L^1(0,1)$, i.e.,

$$T_n u = u'', \quad D(T_n) = \{u \in W^{2,1}(0,1) : B_1^n(u) = B_2^n(u) = 0\}.$$

It is clear for construction that $\{B_1^n, B_2^n\}$ verify the same regularity condition as $\{B_1, B_2\}$. Thus, there exist constants r_n and M_n such that the sector Σ_{δ,r_n} is contained in $\rho(T_n)$ and $\|(R(\lambda:T_n)\| \leq M_n/(|\lambda - r_n|)$ holds for every $\lambda \in \Sigma_{\delta,r_n}$. From 1–3 and the analysis of cases made in Section 5, it is not difficult to see that the constants r_n and M_n can be chosen in a uniform way. Thus we have constants r and M such that

$$(6.1) \qquad \|R(\lambda:T_n)\| \leq \frac{M}{|\lambda - r|}, \quad \lambda \in \rho(T_n) \subset \Sigma_{\delta,r}, \quad n \in \mathbf{N}.$$

Let $\Delta_n(\lambda)$ and $\Delta(\lambda)$ be the characteristic determinants associated to T_n and T, respectively. As $\Delta(\lambda)$ has at most a denumerable number of zeros, we can choose $\lambda_0 \in \Sigma_{\delta,r} \subset \rho(T_n)$ such that $\Delta(\lambda_0) \neq 0$ so $\lambda_0 \in \rho(T)$.

Take an arbitrary $f \in L^1(0,1)$. Then we have

$$\begin{aligned} &\|R(\lambda_0:T_n)f - R(\lambda_0:T)f\|_{L^1(0,1)} \\ &\qquad \leq \left(\sup_{0\leq s\leq 1} \int_0^1 |G(x,s;\lambda_0) - G_n(x,s;\lambda_0)|\,dx\right)\|f\|_{L^1(0,1)} \end{aligned}$$

where $G(x,s;\lambda)$ and $G_n(x,s;\lambda)$ are the Green's functions associated to T and T_n, respectively. Using formulae (2.2)–(2.5), it is easy to see that the right member in the above inequality goes to zero as $n \to \infty$. This proves that $R(\lambda_0:T_n)f$ converges to $R(\lambda:T)f$ in $L^1(0,1)$ as $n \to \infty$, for every $f \in L^1(0,1)$.

From [**5**], Chapter 9, we deduce that $\Sigma_{\delta,r} \subset \rho(T)$ and $R(\lambda:T_n)$ converges to $R(\lambda:T)$ strongly in $L^1(0,1)$ as $n \to \infty$, for every $\lambda \in \Sigma_{\delta,r}$. This implies that (6.1) holds for the operator T.

We can resume this section with the following result:

Proposition 6.1. *Suppose that the boundary conditions* $\{B_1, B_2\}$ *are regular. Fix an arbitrary* $\delta \in (\pi/2, \pi)$. *Then there exist constants* $r \in \mathbf{R}$ *and* $M \geq 0$ *such that the sector* $\Sigma_{\delta,r}$ *is contained in* $\rho(T)$ *and the following bound holds*:

$$\|R(\lambda:T)\| \leq \frac{M}{|\lambda - r|}$$

for every $\lambda \in \Sigma_{\delta,r}$. *As a consequence, the operator* T *generates an analytic semigroup of bounded linear operators in* $L^1(0,1)$.

7. Generation of analytic semigroups. As we commented in Section 2, it is possible to extend Proposition 6.1 to the more general operator L by means of some standard transformations, (see [**4**] or [**7**] for the details). It is a simple exercise to show that the regularity of the boundary conditions is not affected by such transformations.

We can now state the main result of this paper:

Theorem 7.1. *Consider the second-order differential system*

$$\begin{cases} l(u) = u'' + q_1(x)u' + q_0(x)u & \textit{in } (\alpha,\beta) \\ B_1(u) = B_2(u) = 0 \end{cases}$$

where $q_1 \in C^1([\alpha,\beta];\mathbf{C})$ *and* $q_0 \in C([\alpha,\beta];\mathbf{C})$. *For* $i = 1,2$, *the boundary conditions are mixed non-separated and integral ones*:

$$B_i(u) \equiv a_i u(\alpha) + b_i u'(\alpha) + c_i u(\beta) + d_i u'(\beta) + \int_\alpha^\beta R_i(t)u(t)\,dt + \int_\alpha^\beta S_i(t)u'(t)\,dt = 0,$$

where $a_i, b_i, c_i, d_i \in \mathbf{C}$ *and* $R_i, S_i \in C([\alpha,\beta];\mathbf{C})$. *Suppose that the boundary conditions are regular, i.e., their coefficients verify one of the following conditions*:

1. $\Gamma_{bd} \neq 0$.
2. $\Gamma_{bd} = 0$ *and* $\Gamma_{ad} - \Gamma_{bc} - \Gamma_{bS}(\beta) + \Gamma_{dS}(\alpha) \neq 0$.
3. $\Gamma_{ab} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bS} \equiv 0$, $\Gamma_{dS} \equiv 0$ *and*

$$\Gamma_{ac} + \Gamma_{aS}(\beta) + \Gamma_{cS}(\alpha) - \Gamma_{bR}(\beta) - \Gamma_{dR}(\alpha) + \Gamma_S(\alpha,\beta) \neq 0,$$

4. $\Gamma_{ab} = \Gamma_{ac} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bR} \equiv 0$, $\Gamma_{dR} \equiv 0$, $\Gamma_{aS} \equiv 0$, $\Gamma_{bS} \equiv 0$, $\Gamma_{cS} \equiv 0$, $\Gamma_{dS} \equiv 0$, $\Gamma_S \equiv 0$ *and*

$$\Gamma_{aR}(\beta) - \Gamma_{cR}(\alpha) + \Gamma_{RS}(\alpha,\beta) + \Gamma_{RS}(\beta,\alpha) \neq 0.$$

5. $a_i = b_i = c_i = d_i = 0$, $S_i \equiv 0$ *for* $i = 1,2$, *and* $\Gamma_R(\alpha,\beta) \neq 0$.

Consider the L^1-realization of the differential system, that is, the unbounded linear operator $L : D(L) \subset L^1(\alpha, \beta) \to L^1(\alpha, \beta)$ defined as

$$Lu = u'' + q_1(x)u' + q_0(x)u$$

with domain $D(L) = \{u \in W^{2,1}(\alpha, \beta) : B_1(u) = B_2(u) = 0\}$. Then L is the generator of an analytic semigroup $\{e^{tL}\}_{t\geq 0}$ of bounded linear operators in $L^1(\alpha, \beta)$. When the domain $D(L)$ is dense in $L^1(\alpha, \beta)$, the analytic semigroup is a C_0-semigroup.

If the domain $D(L)$ is not dense in $L^1(\alpha, \beta)$, it is possible to obtain a C_0-semigroup on a subspace of $L^1(\alpha, \beta)$. Define X_0 as the closure of $D(L)$ in $L^1(\alpha, \beta)$. Let L_0 be the part of L in X_0, that is, $D(L_0) = \{u \in D(L) : Lu \in X_0\}$ and $L_0 u = Lu$ for $u \in D(L_0)$. Then the operator L_0 verifies the hypotheses of Theorem 7.1 and its domain $D(L_0)$ is dense in X_0. Thus L_0 generates an analytic C_0-semigroup $\{e^{tL_0}\}_{t\geq 0}$ on X_0 and the following relation holds:

$$e^{tL_0}u = e^{tL}u, \quad u \in X_0, \quad t \geq 0.$$

We conclude this section with some interesting cases of regular mixed boundary conditions.

Example 7.1 (Non-separated boundary conditions). Consider the conditions

$$\begin{cases} B_1(u) \equiv a_1 u(\alpha) + b_1 u'(\alpha) + c_1 u(\beta) + d_1 u'(\beta) = 0 \\ B_2(u) \equiv a_2 u(\alpha) + b_2 u'(\alpha) + c_2 u(\beta) + d_2 u'(\beta) = 0, \end{cases}$$

which are supposed to be linearly independent. The regularity conditions are, in this case,

1. $\Gamma_{bd} \neq 0$.
2. $\Gamma_{bd} = 0$ and $\Gamma_{ad} - \Gamma_{bc} \neq 0$.
3. $\Gamma_{ab} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$ and $\Gamma_{ac} \neq 0$.

In all cases the domain $D(L)$ is dense in $L^1(\alpha, \beta)$, so the analytic semigroup generated by L is a C_0-semigroup.

Regular conditions $\{B_1, B_2\}$ are known as *Birkhoff-regular* boundary conditions, and they were introduced by Birkhoff in his early paper

[1] for obtaining asymptotic expansions for the eigenvalues of the associated operator. The spectral theory of non-separated boundary conditions has been widely investigated: see [7] and the references therein. In our paper [4] we stated that it is also possible to obtain generation of analytic semigroups in every space $L^p(\alpha,\beta)$, $1 \le p \le \infty$ (see Section 8).

Classical examples of Birkhoff-regular boundary conditions are the separated and periodic ones:

$$\begin{cases} a_1 u(\alpha) + b_1 u'(\alpha) = 0 \\ c_2 u(\beta) + d_2 u'(\beta) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} u(\alpha) = r u'(\alpha) \\ u(\beta) = r u'(\beta) \end{cases} \qquad r \neq 0.$$

As an example of Birkhoff-irregular boundary conditions, we can consider the initial value conditions: $u(\alpha) = u'(\alpha) = 0$; it is not difficult to prove that in this case we do not obtain generation of analytic semigroups. $\square$

Example 7.2 (Integral boundary conditions). Consider the conditions

$$\begin{cases} B_1(u) \equiv \displaystyle\int_\alpha^\beta R_1(t) u(t)\, dt + \int_\alpha^\beta S_1(t) u'(t)\, dr = 0 \\ B_2(u) \equiv \displaystyle\int_\alpha^\beta R_2(t) u(t)\, dt + \int_\alpha^\beta S_2(t) u'(t)\, dr = 0, \end{cases}$$

with $R_i, S_i \in C([\alpha,\beta]; \mathbf{C})$, $i = 1, 2$. These kinds of integral boundary conditions have been widely studied in our paper [3], the main results of which can be considered as special cases of Theorem 7.1.

The conditions for regularity are

1. $\Gamma_S(\alpha,\beta) \neq 0$.

2. $\Gamma_S \equiv 0$ and $\Gamma_{RS}(\alpha,\beta) + \Gamma_{RS}(\beta,\alpha) \neq 0$. This condition can be separated into two subcases:

(a) $S_1 \equiv 0$ and $R_1(\alpha)S_2(\beta) + R_1(\beta)S_2(\alpha) \neq 0$.

(b) $S_2 \equiv 0$ and $R_2(\alpha)S_1(\beta) + R_2(\beta)S_1(\alpha) \neq 0$.

3. $S_1 \equiv 0$, $S_2 \equiv 0$ and $\Gamma_R(\alpha,\beta) \neq 0$.

The domain $D(L)$ could not be dense in $L^1(\alpha,\beta)$. To see that, consider the following examples:

$$\begin{cases} \displaystyle\int_\alpha^\beta e^t u(t)\,dt = 0 \\ \displaystyle\int_\alpha^\beta t u'(t)\,dt = 0 \end{cases} \quad \text{and} \quad \begin{cases} \displaystyle\int_\alpha^\beta u'(t)\,dt = 0 \\ \displaystyle\int_\alpha^\beta u(t)\,dt + \int_\alpha^\beta t u'(t)\,dt = 0. \end{cases}$$

In both cases the conditions are regular. In the first one the domain $D(L)$ is not dense in $L^1(\alpha,\beta)$, so the semigroup generated by L is not a C_0-semigroup. However, the second example can be written as

$$\begin{cases} u(\alpha) - u(\beta) = 0 \\ \alpha u(\alpha) - \beta u(\beta) = 0 \end{cases}$$

so $D(L)$ is dense in $L^1(\alpha,\beta)$ and the analytic semigroup is a C_0-semigroup. $\square$

8. The L^p case. Consider the case of non-separated boundary conditions

$$\begin{cases} B_1(u) \equiv a_1 u(\alpha) + b_1 u'(\alpha) + c_1 u(\beta) + d_1 u'(\beta) = 0 \\ B_2(u) \equiv a_2 u(\alpha) + b_2 u'(\alpha) + c_2 u(\beta) + d_2 u'(\beta) = 0. \end{cases}$$

For $1 \le p \le \infty$, consider the linear operator $L_p : L^p(\alpha,\beta) \to L^p(\alpha,\beta)$ defined as $L_p u = l(u)$, $D(L_p) = \{u \in W^{2,p}(\alpha,\beta) : B_1(u) = B_2(u) = 0\}$. In our paper [**4**] we proved that, for Birkhoff-regular boundary conditions (see Example 7.1), the operator L_p is the generator of an analytic semigroup in $L^p(\alpha,\beta)$ (if $p \neq \infty$ the semigroup is also a C_0-semigroup). For proving this result we obtained bounds of the form $M/|\lambda|$ for the resolvent operators $R(\lambda : L_p)$, both in the spaces $L^1(\alpha,\beta)$ and $L^\infty(\alpha,\beta)$; then, by interpolation, we deduced the same kind of bounds in all the scale of spaces $L^p(\alpha,\beta)$, $1 \le p \le \infty$. The case $p = 1$ can be viewed as a particular case of Theorem 7.1.

A natural question arises: is it possible to generalize Theorem 7.1 to the L^p setting, as in the case of non-separated boundary conditions? We will give a partial answer.

Consider the L^p-realization L_p with mixed boundary conditions $\{B_1, B_2\}$ as in Theorem 8.1. Of course, a direct approach for bounding $R(\lambda : L_p)$ from formulae (2.2)–(2.6) is not possible if $1 < p < \infty$.

Instead, we should try to bound the resolvent in $L^\infty(\alpha,\beta)$ in order to interpolate. However, even for regular boundary conditions we do not arrive to bounds of the form $M/|\lambda|$ for $R(\lambda : L_\infty)$, as the following example shows:

Example 8.1. Consider the boundary conditions

$$\begin{cases} B_1(u) \equiv \displaystyle\int_0^1 u'(t)\,dt = 0 \\ B_2(u) \equiv \displaystyle\int_0^1 e^t u(t)\,dt = 0 \end{cases}$$

that verify condition 4 of regularity. Note also that the coefficients are of class C^1, so we have stronger conditions than mere regularity.

Fix $M > 0$ and take $f_0 \equiv 1$. If $\lambda = \rho^2 \in \Sigma_{\delta,r}$ with r sufficiently large, we have

$$\begin{aligned} \|R(\lambda : T_\infty)\| &\geq \|R(\lambda : T_\infty) f_0\|_{L^\infty(0,1)} = \sup_{0\leq x\leq 1} |R(\lambda : T_\infty) f_0(x)| \\ &= \sup_{0\leq x\leq 1} \left| \int_0^1 G(x,s;\lambda)\,ds \right| = \sup_{0\leq x\leq 1} \left| \int_0^1 \frac{N(x,s;\lambda)}{\Delta(\lambda)}\,ds \right|. \end{aligned}$$

The characteristic determinant is, in this case,

$$\Delta(\lambda) = \frac{e^{-\rho}-1}{\rho^2-1}\,[\rho(e+1)(e^\rho-1)-(e-1)(e^\rho+1)].$$

After some calculations, we obtain

$$\int_0^1 N(x,s;\lambda)\,ds = (e-1)\,\frac{e^\rho-1}{\rho(\rho^2-1)}\,[-\rho(1+e^{-\rho})+(1-e^{-\rho})]$$

so

$$\begin{aligned} &\sup_{0\leq x\leq 1} \left| \int_0^1 \frac{N(x,s;\lambda)}{\Delta(\lambda)}\,ds \right| \\ &\qquad\geq \left| \frac{(e-1)(e^\rho-1)[(1-e^{-\rho})-\rho(1+e^{-\rho})]}{\rho(e^{-\rho}-1)[\rho(1+e)(e^\rho-1)-(e-1)(e^\rho+1)]} \right|. \end{aligned}$$

The second member can be made greater than $M/|\rho|^2$, taking $|\rho| > r$ large enough.

We have seen that, for every $M > 0$, we can take $r > 0$ such that

$$\|R(\lambda : T_\infty)\| > \frac{M}{|\lambda|} > \frac{M}{|\lambda - r|}, \quad \lambda \in \Sigma_{\delta,r}$$

so T_∞ cannot be the generator of an analytic semigroup. □

What can be then said in the L^∞ case? First of all, note that sections 2–4 are valid in every space $L^p(\alpha, \beta)$, with the obvious modifications. Thus, for the operator T_∞, we have that

$$\begin{aligned} \|R(\lambda : T_\infty)\| &\le \sup_{0\le x\le 1} \int_0^1 |G(x,s;\lambda)|\, ds \\ &= \frac{1}{|\Delta(\lambda)|} \sup_{0\le x\le 1} \int_0^1 |N(x,s;\lambda)|\, ds \\ &\le \frac{m}{|\lambda|} H_\infty(\rho). \end{aligned}$$

The function $H_\infty(\rho)$ is obtained from (3.4)–(3.7) after a long calculation:

$$\begin{aligned} H_\infty(\rho) = \frac{e^{\Re(\rho)}}{|\Delta(\lambda)|} \Big[&(|\Gamma_{bd}| + \|\Gamma_{bS}\| + \|\Gamma_{dS}\|)|\rho|^2 \\ &+ (|\Gamma_{ab}| + |\Gamma_{ad}| + |\Gamma_{bc}| + |\Gamma_{cd}| + \|\Gamma_{aS}\| \\ &+ \|\Gamma_{bR}\| + \|\Gamma_{cS}\| + \|\Gamma_{dR}\| + \|\Gamma_S\|)|\rho| \\ &+ |\Gamma_{ac}| + \|\Gamma_{aR}\| + \|\Gamma_{cR}\| + 2\|\Gamma_{RS}\| + \frac{\|\Gamma_R\|}{|\rho|} \Big]. \end{aligned}$$

(Observe that $H_\infty(\rho)$ is not the same function $H(\rho)$ obtained in the L^1 case.) Making an analysis of cases similar to that in Section 5, we obtain that $H_\infty(\rho)$ is bounded in $r_0 + \Sigma_{\delta/2}$ only in the following three cases:

1. $\Gamma_{bd} \neq 0$.
2. $\Gamma_{bd} = 0$, $\Gamma_{ad} - \Gamma_{bc} \neq 0$ and $\Gamma_{bS} \equiv \Gamma_{dS} \equiv 0$.
3. $\Gamma_{ab} = \Gamma_{ad} = \Gamma_{bc} = \Gamma_{bd} = \Gamma_{cd} = 0$, $\Gamma_{bR} \equiv \Gamma_{dR} \equiv \Gamma_{aS} \equiv \Gamma_{bS} \equiv \Gamma_{cS} \equiv \Gamma_{dS} \equiv \Gamma_S \equiv 0$ and $\Gamma_{ac} \neq 0$.

We say that the boundary conditions $\{B_1, B_2\}$ are L^∞-*regular* if they verify one of the conditions above.

The first case $\Gamma_{bd} \neq 0$ corresponds to both regular and L^∞-regular boundary conditions. Then, by means of the Riesz-Thorin interpolation theorem [**2**], we obtain the following result:

Theorem 8.1. *If the mixed boundary conditions verify $\Gamma_{bd} \neq 0$, then the operator L_p generates an analytic semigroup of bounded linear operators in $L^p(\alpha, \beta)$ for every $1 \leq p \leq \infty$.*

What happens with cases 2 and 3 of L^∞-regular boundary conditions? It is clear that they are particular cases of regular boundary conditions; however, they are not well-defined, as the following example shows:

Example 8.2. Suppose $(\alpha, \beta) = (0, 1)$ and consider the boundary conditions

$$\begin{cases} B_1(u) \equiv u(0) - u(1) = 0 \\ B_2(u) \equiv u'(1) + \int_0^1 e^t u(t)\, dt = 0, \end{cases}$$

that verify condition 2 of L^∞-regularity. But condition B_1 could be written as:

$$B_1(u) \equiv \int_0^1 u'(t)\, dt = 0,$$

which leads to L^∞-irregular boundary conditions. This shows that the definition of L^∞-regularity is not consistent. $\square$

Example 8.3. In the case of non-separated boundary conditions, the function $H_\infty(\rho)$ is exactly the same as $H(\rho)$, so the analysis of cases made in Section 5 is valid also in $L^\infty(\alpha, \beta)$. We have the following result:

Theorem 8.2. *Let $\{B_1, B_2\}$ be Birkhoff-regular non-separated boundary conditions. Then, for $1 \leq p \leq \infty$, the operator L_p generates an analytic semigroup in $L^p(\alpha, \beta)$. If $p \neq \infty$, the semigroup is also a C_0-semigroup.*

The above theorem was previously proved in our paper [**4**]. $\square$

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