ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 35, Number 3, 2005

THE REPRESENTATIONS OF D^1

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ABSTRACT. In this paper we construct explicitly all irreducible representations of the norm one elements group in the quaternion division algebra over a local *p-field* where p is an *odd prime number*.

1. Introduction and notation. In this paper we will construct explicitly all irreducible representations of D^1, *the norm one elements group of* D, where D is the quaternion division algebra over a local p-field for an odd prime number p. Our motivation for finding representations of D^1, in addition to its own interest, is that they are needed to construct the representations of $U(2)$, the nonsplit unitary group in two variables, in relation to the reductive dual pair $(U(1), U(2))$ in the symplectic group $Sp(4)$. Some authors have studied the representations of division algebras in general [**1**]. Here we will be using the method used by Manderscheid [**10**] to construct the representations of $SL(2)$, to parametrize explicitly the representations of D^1. This method was briefly outlined, without details or proofs in [**11**]. We provide here the details and the proofs, getting the explicit inducing data in [**11**]. Although influenced by [**1**], this data does not follow from [**1**].

This paper consists of three sections. The first section is devoted to the basic results about the structure of D^1, its normal subgroups and their characters. In the second section we find all representations of D^1 whose dimensions are bigger than one. Finally in the last section after constructing all one-dimensional representations of D^1 we state and prove Theorem 3.5 which formalizes all the results obtained in Sections 2 and 3.

Let F be a non-Archimedean local p-field where p is an odd prime. Let $O = O_F$ be the ring of integers of F, and let ϖ be a generator of the maximal ideal $P = P_F$ in $O = O_F$. Let $k = k_F$ denote the residual class field O/P, and let q be the cardinality of k.

Received by the editors on December 13, 2000, and in revised form on May 1, 2002.

Let D be the quaternion division algebra over F with the involution $x \to \bar{x}$, $x \in D$. Let $Tr = Tr_{D/F}$ denote the reduced trace map from D to F, and let $\nu = \nu_{D/F}$ denote the reduced norm map from D to F defined by $\nu(x) = x\bar{x}$ and $Tr(x) = x + \bar{x}$, $x \in D$. Also let O_D denote the ring of integers in D, P_D the maximal ideal in O_D, and let $\mathbf{k} = k_D = O_D \diagup P_D$ denote the residual class field of D. We will denote by $v_D(x)$ the order of x in D, and we will normalize the absolute value $|\ |_D$ on D so that $|\, x\, |_D = q^{-2v_D(x)}$. Let π be the prime element in O_D generating P_D and $\pi^2 = \varpi$. For any integer r, P_D^r is defined as $P_D^r = \{x \in D \mid x = a\pi^r, \text{ for some } a \in O_D\}$. P^r in F is defined in the same manner. Let D° denote trace zero elements in D, and let O_{D° denote trace zero elements in O_D. Let χ be a nontrivial character of F^+ of conductor O. The conductor of a character of F^+ is the smallest integer n for which the character is trivial on P^n. Let $D^1 = \{x \in D \mid \nu(x) = 1\}$. Then D^1 is a multiplicative group and we will call it the *norm one elements group* of D. For any positive integer r, set

$$D_r^1 = \left\{x \in D^1 \mid x = 1 + a\pi^r, \text{ for some } a \in O_D\right\}.$$

Then one can check that, for any positive integer r, D_r^1 is a normal subgroup of D^1.

Lemma 1.1. *Let $P_{D^\circ} = O_{D^\circ} \cap P_D$. Then we have $|O_{D^\circ} \diagup P_{D^\circ}| = q$.*

Proof. Define $f : \mathbf{k} \to O_{D^\circ} \diagup P_{D^\circ}$ by $f(a + P_D) = a - \bar{a} + P_{D^\circ}$. As one can check, f is well defined, f is onto by Hilbert's 90, and its kernel is k, so

$$\mathbf{k} \diagup k \cong O_{D^\circ} \diagup P_{D^\circ},$$

which implies that

$$\begin{aligned} |O_{D^\circ} \diagup P_{D^\circ}| &= |\mathbf{k} \diagup k| \\ &= \frac{q^2}{q} \\ &= q. \qquad \square \end{aligned}$$

Lemma 1.2. *Let a be a unit in O_D and r a positive integer. Then there exists a unit in O_D, b say, such that $\nu(b) = 1$ and $b \equiv a \pmod{P_D^r}$ if and only if $\nu(a) \equiv 1 \pmod{P_F^{[(r+1)/2]}}$, where $[\quad]$ is the greatest integer part.*

Proof. Let $\nu(a) \equiv 1 \pmod{P_F^{[(r+1)/2]}}$. Then since $\nu(1+P_D^r) = 1 + P_F^{[(r+1)/2]}$, there exists $g \in 1+P_D^r$ such that $\nu(g) = \nu(a)$. Now set $b = ag^{-1}$. Then one can show that b is what we are looking for. Conversely, let there be an element b with the above mentioned properties. Thus $a^{-1}b \equiv 1 \pmod{P_D^r}$, and $\nu\left(a^{-1}b\right) \equiv 1 \pmod{P_F^{[(r+1)/2]}}$. From

$$\begin{aligned}\nu\left(a^{-1}\right) &= \nu\left(a^{-1}\right)\nu(b)\\ &= \nu\left(a^{-1}b\right)\\ &\equiv 1 \pmod{P_F^{[(r+1)/2]}}\end{aligned}$$

we get the result $\nu(a) \equiv 1 \pmod{P_F^{[(r+1)/2]}}$. □

Lemma 1.3. *Let all notation be as before. Then we have*:

1. *If r is even, then $\left|D_r^1 / D_{r+1}^1\right| = q$.*
2. *If r is odd, then $\left|D_r^1 / D_{r+1}^1\right| = q^2$.*

Proof. 1. Define $f : D_r^1 \to \mathbf{k} = O_D / P_D$ by:

$$f(1+a\pi^r) = a + P_D.$$

Then one can check that f is a homomorphism. Obviously $\ker f = D_{r+1}^1$. Now let $a \in O_D$, then $1+a\pi^r$ is a unit, so by Lemma 1.2 there exists $b \in O_D$ such that $\nu(b) = 1$ and $b \equiv (1+a\pi^r) \pmod{P_D^{r+1}}$ if and only if $\nu(1+a\pi^r) \equiv 1 \pmod{P_F^{[(r+1)/2]}}$. But this condition is the same as:

$$\begin{aligned}\nu(1+a\pi^r) &= (1+a\pi^r)(1+\bar{a}\pi^r)\\ &= 1 + Tr(a)\varpi^{r/2} + \nu(a)\varpi^r\\ &= 1 + \lambda\varpi^{[(r+1)/2]}, \quad \text{for some } \lambda \in O_D.\end{aligned}$$

This equality implies that ϖ must divide $Tr\,(a)$, i.e., $\operatorname{Im} f = O_{D^\circ} / P_D$ which is isomorphic to $O_{D^\circ} / P_{D^\circ}$. Thus

$$D_r^1 / D_{r+1}^1 \cong O_{D^\circ} / P_{D^\circ}.$$

Now apply Lemma 1.1.

2. Define $f : D_r^1 \to \mathbf{k} = O_D / P_D$ by:

$$f\,(1 + a\pi^r) = a + P_D.$$

By part 1, f is a homomorphism with $\ker f = D_{r+1}^1$. Now we will show that f is onto. Let $a \in O_D$. Then $1 + a\pi^r$ is a unit and because r is odd we have

$$\begin{aligned} \nu\,(1 + a\pi^r) &\equiv 1 \pmod{P_F^{[(r+1)/2]}} \\ &\equiv 1 \pmod{P_F^{[(r+2)/2]}}. \end{aligned}$$

Thus by Lemma 1.2 there exists $b \in O_D$ such that $\nu\,(b) = 1$ and $b \equiv (1 + a\pi^r) \pmod{P_D^{r+1}}$. From here we get $f\,(b) = a + P_D$, i.e., f is onto, and

$$D_r^1 / D_{r+1}^1 \cong \mathbf{k} = O_D / P_D.$$

Thus $\left| D_r^1 / D_{r+1}^1 \right| = |\mathbf{k}| = q^2$. □

Lemma 1.4. *Let h and $h' \in D^1$, and let n be any positive integer. Then $h \equiv h' \pmod{D_n^1}$ if and only if $h - h' \in P_D^n$.*

Proof. Let $h \equiv h' \pmod{D_n^1}$, so $h = h'\,(1 + \delta\pi^n)$ for some $\delta \in O_D$. From here we get $h - h' = \delta\pi^n \in P_D^n$. Conversely let $h - h' \in P_D^n$, so $h - h' = \delta\pi^n$, for some $\delta \in O_D$. From here we get

$$h = h' + \delta\pi^n = h'\left(1 + (h')^{-1}\,\delta\pi^n\right).$$

Since h and h' have norm one so does $1 + (h')^{-1}\,\delta\pi^n$ i.e., $\left(1 + (h')^{-1}\,\delta\pi^n\right) \in D_n^1$. □

Lemma 1.5. *Let n and r be two positive integers with $n/2 \le r < n$, and set $P_{D^\circ}^r = O_{D^\circ} \cap P_D^r$. Then we have:*

$$P_{D^\circ}^r / P_{D^\circ}^n \cong D_r^1 / D_n^1.$$

Proof. Let $a\pi^r \in P^r_{D^\circ}$. Define Cayley transformation $C : P^r_{D^\circ} \to D^1_r/D^1_n$ as follows:

$$C\,(a\pi^r) = \frac{1-a\pi^r}{1+a\pi^r}\, D^1_n.$$

Then C is a homomorphism because by expanding $(1-a\pi^r)/(1+a\pi^r)D^1_n$ and using Lemma 1.4 we get

$$C\,(a\pi^r) = 1 - 2a\pi^r \pmod{P^n_D}.$$

From here we have

$$\begin{aligned} C\,(a\pi^r + b\pi^r) &= C\,((a+b)\,\pi^r) \\ &= 1 - 2\,(a+b)\,\pi^r \pmod{P^n_D} \\ &= \frac{1-(a+b)\,\pi^r}{1+(a+b)\,\pi^r}\, D^1_n \end{aligned}$$

and

$$\begin{aligned} C\,(a\pi^r)\,C\,(b\pi^r) &= (1-a\pi^r)\,(1-b\pi^r) \pmod{P^n_D} \\ &= 1-(a+b)\,\pi^r \pmod{P^n_D} \\ &= C\,((a+b)\,\pi^r) \\ &= \frac{1-(a+b)\,\pi^r}{1+(a+b)\,\pi^r}\, D^1_n, \end{aligned}$$

i.e., $C\,(a\pi^r + b\pi^r) = C\,(a\pi^r)\,C\,(b\pi^r)$. To show that C is onto, let $y = 1 + b\pi^r \in D^1_r$ and take $x = -(b/2)\pi^r \pmod{P^n_D}$. Then one can check that $C\,(x) = y$ and, further,

$$\begin{aligned} Tr\,(x) &= -\frac{b}{2}\,\pi^r + \overline{-\frac{b}{2}\,\pi^r} \\ &= 0 \pmod{P^n_D} \end{aligned}$$

because, since $\nu\,(y) = \nu\,(1+b\pi^r) = 1 + Tr\,(b\pi^r) + \nu\,(b\pi^r) = 1$, we deduce that $(Tr\,(b\pi^r) + \nu\,(b\pi^r))\,/2 = 0$ and $\nu\,(b\pi^r) \in P^n_D$. Therefore the result is obtained. $\square$

For any positive integer, r say, set $P^{-r}_D = \{a\pi^{-r} \mid a \in O_D\}$ and $P^{-r}_{D^\circ} = P^{-r}_D \cap O_{D^\circ}$.

Lemma 1.6. *For any positive integer, r say, we have*:

1. $P_{D^\circ}^{-2r} \diagup P_{D^\circ}^{-2r+1} \cong O_{D^\circ} \diagup P_{D^\circ}$.
2. $P_{D^\circ}^{-(2r+1)} \diagup P_{D^\circ}^{-2r} \cong O_D \diagup P_D$.

Proof. 1. Define $f : P_{D^\circ}^{-2r} \to O_{D^\circ} \diagup P_{D^\circ}$ as follows:

$$f\left(a\pi^{-2r}\right) = a + P_{D^\circ}$$

f is well-defined because $a\pi^{-2r}$ is traceless so a must be traceless, too. And one can check that:

$$\ker f = \left\{a\pi^{-2r} \mid a \in P_{D^\circ}\right\} = P_{D^\circ}^{-2r+1}.$$

f is onto because for any $a \in O_{D^\circ}$, $a\pi^{-2r}$ is also traceless and is in $P_{D^\circ}^{-2r}$ with $f\left(a\pi^{-2r}\right) = a + P_{D^\circ}$. □

2. Define $f : P_{D^\circ}^{-(2r+1)} \to O_D \diagup P_D$ as follows:

$$f\left(a\pi^{-(2r+1)}\right) = a + P_D$$

one can show $\ker f = P_{D^\circ}^{-2r}$. f is onto because for $a \in O_D$, we can write $a = a_\circ + a_1\pi$, for some $a_\circ$ and a_1 in the maximal unramified quadratic extension of F contained in D. Then one can check that $Tr\left(a_\circ\pi\right) = 0$ and $f\left(a_\circ\pi^{-(2r+1)}\right) = a_\circ + P_D = a + P_D$. □

Definition 1.1. Let r be a positive integer, and let φ be a character of D_r^1. The conductor of φ is the smallest integer, l say, for which φ is trivial on D_l^1.

Lemma 1.7. *Let $\alpha \in D^\circ$, with $\upsilon(\alpha) = -(n+1)$ where n is a positive integer. Let r be an integer with $(n/2) \leq r < n$. Define $\chi_\alpha : D_r^1 \to \mathbf{C}^\times$ by*

$$\chi_\alpha(h) = \chi\left(Tr\left(\alpha\left(h-1\right)\right)\right), \quad h \in D_r^1.$$

Then χ_α is a character of D_r^1, with conductor equal to n.

Proof. Let $h_1 = 1 + a_1\pi^r$ and $h_2 = 1 + a_2\pi^r$, then

$$h_1h_2 = 1 + \left(a_1 + a_2\right)\pi^r + a_1\pi^r a_2\pi^r.$$

Now since $r \geq (n/2)$ $a_1\pi^r a_2\pi^r$ is in P_D^n. Thus $Tr\,(a_1\pi^r a_2\pi^r) \in O_D$ and

$$\begin{aligned}\chi_\alpha\,(h_1h_2) &= \chi\,(Tr\,(\alpha\,(a_1+a_2)\,\pi^r)) \\ &= \chi\,(Tr\,(\alpha a_1\pi^r))\,\chi\,(Tr\,(\alpha a_2\pi^r)) \\ &= \chi_\alpha\,(h_1)\,\chi_\alpha\,(h_2)\,.\end{aligned}$$

To show that the conductor is n, note that one can show that h is in the conductor if and only if $Tr\,(\alpha\,(h-1)) \in O$. Since ramification of D is 2, this condition is the same as $\alpha\,(h-1) \in P_D^{-1}$ [**16**]. From here we get $h-1 \in P_D^n$. Thus $h \in (1+P_D^n) \cap D_r^1 = D_n^1$. □

Lemma 1.8. *Notation is as in Lemma* 1.7. *The character* χ_α *is trivial on* D_r^1 *if and only if* $\chi\,(Tr\,(\alpha y)) = 1$, *for any* $y \in P_D^r$.

Proof. If $\chi\,(Tr\,(\alpha y)) = 1$, for any $y \in P_D^r$, then it is clear that χ_α is trivial on D_r^1. Now suppose conversely that χ_α is trivial on D_r^1, and let $y \in P_D^r$. Then $(1+y)/(1+\bar{y}) \in D_r^1$, and one can show that there exists $z \in P_D^{2r}$ such that

$$\frac{1+y}{1+\bar{y}} = 1 + y - \bar{y} + z.$$

From here we get

$$\begin{aligned}1 &= \chi_\alpha\left(\frac{1+y}{1+\bar{y}}\right) \\ &= \chi\,(Tr\,(\alpha\,(y-\bar{y})))\,.\end{aligned} \tag{1}$$

On the other hand, since $y + \bar{y} \in F$ and $Tr\,(\alpha) = 0$, we have

$$\chi\,(Tr\,(\alpha\,(y+\bar{y}))) = 1. \tag{2}$$

From (1) and (2) we will get $\chi\,(Tr\,(2\alpha y)) = 1$. Now since 2 is a unit, we have the result. □

Proposition 1.9. *Let* n *be a given positive integer and let* $r = [(n+1)/2]$, *where* [ ] *denote the greatest integer part function. Any character of* D_r^1 *is in the form* χ_α *for some* $\alpha \in D^\circ$.

Proof. Define $\Lambda : P_{D^\circ}^{-(n+1)} \to \left(D_r^1 \diagup D_n^1\right)^\wedge$ by $\Lambda(\alpha) = \chi_\alpha$ where $(\ )^\wedge$ denote the Pontryagin dual. One can show that Λ is a homomorphism. Using Lemma 1.8 we get:

$$\begin{aligned}\ker \Lambda &= \left\{\alpha \in D^\circ \mid \chi_\alpha(h) = 1,\ \forall\, h \in D_r^1\right\} \\ &= \{\alpha \in D^\circ \mid \chi(Tr(\alpha y)) = 1,\ \forall\, y \in P_D^r\} \\ &= P_D^{-1-r}.\end{aligned}$$

Now since $D_r^1 \diagup D_n^1$ is finite abelian; thus, the cardinality of $\left(D_r^1 \diagup D_n^1\right)^\wedge$, $\left|\left(D_r^1 \diagup D_n^1\right)^\wedge\right|$, is equal to $\left|D_r^1 \diagup D_n^1\right|$. Now Lemmas 1.3, 1.5 and 1.6 complete the proof. □

Lemma 1.10. *Let $\alpha \in D^\circ$, and let $E = F(\alpha)$. Set*

$$E' = \{x \in D \mid Tr(xy) = 0,\ \forall\, y \in E\}.$$

Then $O_D = O_E \oplus O'_E$ where $O'_E = O_D \cap E'$.

Proof. Let $x \in O_E \cap O'_E$. Then $Tr(x) = Tr\left(x^2\right) = 0$. From here we deduce that $x = 0$. Now let $x \in O_D$, and set: $x_1 = Tr(x)/2 + (Tr(x\alpha)/2)\alpha^{-1}$, and $x_2 = x - x_1$. Then one can check that $x_1 \in O_E$, $x_2 \in O'_E$, and $x = x_1 + x_2$. □

Remark 1.1. The following result for $GL(n)$ can be found in [**5**]. We state and prove it here in our notation and our case (division algebra).

Lemma 1.11. *Let $\beta \in D^\circ$, $\beta \neq 0$, with $\beta = \varepsilon\pi^m$, where ε is a unit and m is an integer. Let $E = F(\beta)$. Set*

$$O'_E\pi^m = \{x\pi^m \mid x \in O'_E\}.$$

Define $ad_\beta : O'_E \to O'_E\pi^m$ as follows:

$$ad_\beta(x) = \beta x - x\beta, \quad x \in O'_E.$$

Then ad_β is onto.

Proof. Since $\beta x - x\beta = \left(\beta x\beta^{-1} - x\right)\beta \in O'_E\pi^m$ if and only if $\left(\beta x\beta^{-1} - x\right) \in O'_E$, it is enough to show that $\gamma : O'_E \to O'_E$, defined by $\gamma(x) = \beta x\beta^{-1} - x$ is onto. Let $\Gamma : E' \to E'$ be defined by $\Gamma(x) = \beta x\beta^{-1} - x$. It is easy to show that Γ is an E-linear map. Since $E = F(\beta)$ is a quadratic extension, we may realize D as the cyclic algebra (E, σ, α) where α is an element in $F^\times$ which is not in the image of the norm map $\nu_{E/F}$ from E to F and σ is the nontrivial element of the Galois group $\mathcal{G}(E/F)$, see, e.g., [**16**]. In particular, there exist $\delta \in D^\times$ such that

$$\delta\beta\delta^{-1} = \sigma(\beta) = \bar{\beta} = -\beta$$

and $\delta^2 = \alpha$, and $\{1, \delta\}$ is a basis for D over E. From here, we have

$$\gamma(\delta) = -2\delta.$$

So the eigenvalues of Γ and its determinant are units. Thus Γ and γ are onto as desired. □

Proposition 1.12. *Let $\alpha \in D^\circ$ with $\upsilon(\alpha) = -(n+1)$, where n is a positive integer, and let r be a positive integer with $n/2 \le r < n$. Let χ_α be a character of D^1_r defined as in Lemma* 1.7. *Let D^1 act on $\left(D^1_r\right)^\wedge$ by conjugation. Then the stabilizer of χ_α in D^1 is $E^1 D^1_{n-r}$ where E^1 is the norm one elements group of $E = F(\alpha)$.*

Proof. Let $h \in D^1$ be in the stabilizer of χ_α in D^1. Write $h = 1 + y$ and $h^{-1} = 1 + z$, for some y, and $z \in O_D$. Here h^{-1} denote the inverse of h. Then for any $h_r = (1 + x) \in D^1_r$ we must have:

$$\chi_\alpha\left(h^{-1}h_r h\right) = \chi_\alpha(h_r),$$

which is the same as:

$$\chi\left(Tr\left(\alpha\left(h^{-1}h_r h - 1\right)\right)\right) = \chi\left(Tr\left(\alpha\left(h_r - 1\right)\right)\right)$$

or

$$\begin{aligned}\chi\left(Tr\left(\alpha h^{-1}xh\right)\right) &= \chi\left(Tr(\alpha x)\right)\\ &= \chi\left(Tr\left(\alpha x h h^{-1}\right)\right)\\ &= \chi\left(Tr\left(h^{-1}\alpha x h\right)\right)\end{aligned}$$

and this is the same as:

$$\chi\left(Tr\left(\left(\alpha h^{-1}-h^{-1}\alpha\right)xh\right)\right)=1$$

or

$$\chi\left(Tr\left(h\left(\alpha h^{-1}-h^{-1}\alpha\right)x\right)\right)=1.$$

Now since h is a unit and Tr induces a nondegenerate bilinear form we must have:

$$\left(\alpha h^{-1}-h^{-1}\alpha\right)x\equiv 0 \pmod{P_D^{-1}} \quad \forall\, x\in P_D^r,$$

which is the same as:

$$\left(\alpha h^{-1}-h^{-1}\alpha\right)\equiv 0 \pmod{P_D^{-1-r}}.$$

Now note that $\left(\alpha h^{-1}-h^{-1}\alpha\right)=\alpha z-z\alpha=0$ if and only if $z\in E$. Using Lemma 1.10, we can write $z=z_1+z_2$, for some $z_1\in O_E$, and $z_2\in O'_E$. Then we have:

$$\begin{aligned}\left(\alpha h^{-1}-h^{-1}\alpha\right)&=\alpha z-z\alpha\\&=\alpha z_2-z_2\alpha.\end{aligned}$$

Now by Lemma 1.11 there exists $z_3\in O'_E$, such that $\alpha z_2-z_2\alpha=\alpha z_3-z_3\alpha$. On the other hand, since $\alpha z_2-z_2\alpha\in P_D^{-1-r}$ and $\upsilon(\alpha)=-(1+n)$, thus $\upsilon(z_3)=n-r$. Now we have:

$$\alpha z_2-z_2\alpha=\alpha z_3-z_3\alpha,$$

which is the same as:

$$\alpha\left(z_2-z_3\right)=\left(z_2-z_3\right)\alpha.$$

This gives us $(z_2-z_3)\in O_E$. But we know that $(z_2-z_3)\in O'_E$, hence $(z_2-z_3)=0$, i.e. $z_2=z_3\in P_D^{n-r}$. From here we get:

$$\begin{aligned}h^{-1}&=1+z\\&=1+z_1+z_2\\&=(1+z_1)\left(1+(1+z_1)^{-1}z_2\right),\end{aligned}$$

which is an element in $E^1 D^1_{n-r}$. Now since $E^1 D^1_{n-r}$ obviously is contained in the stabilizer, we have the result. □

Lemma 1.13. *Let E/F be an unramified quadratic extension of F. Let n and r be two positive integers with $n/2 \leq r < n$. If r is even, then:*

$$\left(E^1 D^1_r\right) / D^1_n = \left(E^1 D^1_{r+1}\right) / D^1_n.$$

Proof. Let k' denote the residual class field of E. Then, since E/F is unramified, $k' = \mathbf{k}$. Now let $h = (1 - a\pi^r)/(1 + a\pi^r) D^1_n$ be an element of D^1_r / D^1_n. Write $a = a_\circ + a_1\pi$ where $a_\circ, a_1 \in O_E$. Now we have:

$$\begin{aligned} h &= \frac{1 - a\pi^r}{1 + a\pi^r} D^1_n = \frac{1 - (a_\circ + a_1\pi)\,\pi^r}{1 + (a_\circ + a_1\pi)\,\pi^r} D^1_n \\ &= \frac{1 - (a_\circ + a_1\pi)\,\pi^r}{1 + (a_\circ + a_1\pi)\,\pi^r} \cdot \frac{1 + (a_\circ + a_1\pi)\,\pi^r}{1 - (a_\circ + a_1\pi)\,\pi^r} \\ &\quad \times \frac{1 + (a_\circ + a_1\pi)\,\pi^r + a_\circ a_1 \pi^{2r+1}}{1 - (a_\circ + a_1\pi)\,\pi^r + a_\circ a_1 \pi^{2r+1}} D^1_n \\ &= \frac{1 - a_\circ\pi^r}{1 + a_\circ\pi^r} \cdot \frac{1 - a_1\pi^{r+1}}{1 + a_1\pi^{r+1}} D^1_n. \end{aligned}$$

Since $a_\circ\pi^r \in E$ from the last equality we get:

$$\begin{aligned} E^1 h &= E^1 \frac{1 - a_\circ\pi^r}{1 + a_\circ\pi^r} \cdot \frac{1 - a_1\pi^{r+1}}{1 + a_1\pi^{r+1}} D^1_n \\ &= E^1 \frac{1 - a_1\pi^{r+1}}{1 + a_1\pi^{r+1}} D^1_n. \end{aligned}$$

Thus

$$\left(E^1 D^1_r\right) / D^1_n \subset \left(E^1 D^1_{r+1}\right) / D^1_n.$$

Since we always have

$$\left(E^1 D^1_{r+1}\right) / D^1_n \subset \left(E^1 D^1_r\right) / D^1_n;$$

thus, the result. □

Lemma 1.14. *For any character χ_α of D^1_r there is a character φ_α of $E^1 D^1_r$ such that $\varphi_{\alpha|D^1_r} = \chi_\alpha$.*

Proof. $\chi_{\alpha|E^1\cap D_r^1}$ is a character of $E^1 \cap D_r^1$ as a subgroup of E^1. Thus there exists $\varphi \in \left(E^1\right)^\wedge$ such that $\chi_{\alpha|E^1\cap D_r^1} = \varphi_{|E^1\cap D_r^1}$. Now define

$$\varphi_\alpha : E^1 D_r^1 \to \mathbf{C}^\times$$

by

$$\varphi_\alpha(eh) = \varphi(e)\,\chi_\alpha(h), \quad e \in E^1,\ h \in D_r^1.$$

Then one can check that φ_α is a well-defined character and that $\varphi_{\alpha|D_r^1} = \chi_\alpha$. □

Remark 1.2. Since φ in above lemma is not unique, we set:

$$\Phi(\alpha) = \left\{\varphi \in \left(E^1\right)^\wedge \mid \varphi = \chi_\alpha \text{ on } E^1 \cap D_r^1\right\}.$$

Thus for any $\varphi \in \Phi(\alpha)$ we have a character φ_α of $E^1 D_r^1$ such that $\varphi_{\alpha|D_r^1} = \chi_\alpha$.

Lemma 1.15. *Let n be a positive odd integer such that $r = (n+1)/2$ is even. Set*:

$$H_{r-1} = \left\{x \in D_{r-1}^1 \diagup D_n^1 \mid x = \frac{1 - a\pi^{r-1}}{1 + a\pi^{r-1}}\,D_n^1,\ a \in O = O_F\right\}.$$

Then H_{r-1} is a subgroup of $D_{r-1}^1 \diagup D_n^1$.

Proof. Let $h, h' \in H_{r-1}$. By Lemma 1.4 we can write:

$$h = \frac{1 - a\pi^{r-1}}{1 + a\pi^{r-1}}\,D_n^1 \equiv 1 - 2a\pi^{r-1} + 2a^2\pi^{2(r-1)} \pmod{P_D^n} \tag{3}$$

and

$$h' = \frac{1 - a'\pi^{r-1}}{1 + a'\pi^{r-1}}\,D_n^1 \equiv 1 - 2a'\pi^{r-1} + 2a'^2\pi^{2(r-1)} \pmod{P_D^n}$$

for some a and $a' \in O$. Then we have

$$\begin{aligned} hh' &\equiv 1 - 2(a + a')\pi^{r-1} + 2(a + a')^2\pi^{2(r-1)} \pmod{P_D^n} \\ &= \frac{1 - (a + a')\pi^{r-1}}{1 + (a + a')\pi^{r-1}}\,D_n^1. \end{aligned}$$

Thus $hh' \in H_{r-1}$. $\square$

Lemma 1.16. *Let H_{r-1} be as in Lemma* 1.15*. Then, for h and $h' \in H_{r-1}$, we have $h = h'$ if and only if $a - a' \in P_F^{r/2}$.*

Proof. Let:

$$h \equiv 1 - 2a\pi^{r-1} + 2a^2\pi^{2(r-1)} \pmod{P_D^n}$$

and

$$h' \equiv 1 - 2a'\pi^{r-1} + 2a'^2\pi^{2(r-1)} \pmod{P_D^n}.$$

be two elements in H_{r-1}. Then we have

$$h - h' = -2\,(a - a')\,\pi^{r-1} + 2\left(a^2 - a'^2\right)\pi^{2(r-1)} \in P_D^n$$

From here we get

$$-2\,(a - a') + 2\left(a^2 - a'^2\right)\pi^{r-1} \in P_D^r.$$

Thus $\pi^{r-1} \mid (a - a')$. So $(a - a') \in P_D^{r-1} \cap O = P_F^{r/2}$. $\square$

Lemma 1.17. *Let $n, r,$ and H_{r-1} be as in Lemma* 1.15*. Then $|H_{r-1}| = q^{r/2}$.*

Proof. Define $f : H_{r-1} \to O/P_F^{r/2}$ by $f(h) = a + P_F^{r/2}$ for any $h = 1 - a\pi^{r-1}/1 + a\pi^{r-1}D_n^1 \in H_{r-1}$. Then by Lemma 1.16, f is well defined and, by Lemma 1.15, f is a homomorphism. Obviously f is onto with $\ker f = \{1\}$. Thus, $|H_{r-1}| = \left|O/P_F^{r/2}\right| = q^{r/2}$. $\square$

Lemma 1.18. *The notation is as in Lemma* 1.15*. Then we have*

$$\left(D_r^1/D_n^1\right) \cap H_{r-1} = \left\{h = \frac{1 - a\pi^{r-1}}{1 + a\pi^{r-1}}\,D_n^1 \mid a \in P = P_F\right\}.$$

Proof. Let $h \in \left(D_r^1/D_n^1\right) \cap H_{r-1}$. Then for some a and $b \in O = O_F$ we have:

$$h = \frac{1 - a\pi^{r-1}}{1 + a\pi^{r-1}}\,D_n^1 = (1 + b\pi^r)\,D_n^1.$$

Thus

$$\frac{1 - a\pi^{r-1}}{1 + a\pi^{r-1}} \equiv (1 + b\pi^r) \pmod{P_D^n}.$$

From here one can show that $\pi \mid a$, so $a \in P = P_F$. □

Lemma 1.19. *The notation is as in Lemma* 1.15*. Set* $\mathfrak{D}_{r-1} = \left(D_r^1/D_n^1\right) H_{r-1}$*. Then* $\mathfrak{D}_{r-1}$ *is a subgroup of* D_{r-1}^1/D_n^1*.*

Proof. This is true because D_r^1/D_n^1 and H_{r-1} are subgroups of D_{r-1}^1/D_n^1, and D_r^1/D_n^1 is normal in D_{r-1}^1/D_n^1. □

Lemma 1.20. $\left|\left(D_r^1/D_n^1\right) \cap H_{r-1}\right| = q^{(r/2)-1}$.

Proof. The same map and argument as in Lemma 1.17 work. □

If G is a group and G_1 and G_2 are subgroups of G, write $[G : G_1]$ for the number of left G_1-cosets in G and $[G_1 : G : G_2]$ for the number of (G_1, G_2)-double cosets in G.

Lemma 1.21. *Notations are as above. We have*

$$\left[\mathfrak{D}_{r-1} : D_r^1/D_n^1\right] = \left[D_{r-1}^1/D_n^1 : \mathfrak{D}_{r-1}\right] = q.$$

Proof. By definition we have

$$\begin{aligned}\left[\mathfrak{D}_{r-1} : D_r^1/D_n^1\right] &= \frac{|\mathfrak{D}_{r-1}|}{|D_r^1/D_n^1|}\\ &= \frac{\left|D_r^1/D_n^1\right| \cdot |H_{r-1}| \,/\, \left|D_r^1/D_n^1 \cap H_{r-1}\right|}{|D_r^1/D_n^1|}\\ &= q.\end{aligned}$$

Similar computations work for the second part. □

Lemma 1.22. *Let* $E_1^1 = F^\times (1 + P_E) \cap E^1$*. Then, for any* $h \in H_{r-1}$ *and any* $\lambda \in E_1^1$*, we have* $h\,\lambda\,h^{-1} \in \left(E_1^1 D_r^1\right)/D_n^1$*.*

Proof. Let:

$$h \equiv 1 - 2a\pi^{r-1} + 2a^2\pi^{2(r-1)} \pmod{P_D^n} \in H_{r-1}.$$

Since $\nu(h) = 1$, so $h^{-1} = \bar{h}$. Thus we have:

$$\begin{aligned} h^{-1} &= \bar{h} \\ &\equiv 1 + 2a\pi^{r-1} + 2a^2\pi^{2(r-1)} \pmod{P_D^n}. \end{aligned}$$

Now, for $\lambda = f + e\pi^2 \in E_1^1$, $f \in O$, $e \in O_E$, we have:

$$\begin{aligned} h\lambda h^{-1} &= \lambda - 2a\bar{e}\pi^{r+1} \\ &\equiv \lambda\left(1 - 2a\bar{e}\bar{\lambda}\pi^{r+1}\right) \pmod{P_D^n}. \end{aligned}$$

From here we get:

$$h\lambda h^{-1} \in \left(E_1^1 D_{r+1}^1\right)/D_n^1 \subset \left(E_1^1 D_r^1\right)/D_n^1. \qquad \square$$

Corollary 1.23. *$E_1^1\mathfrak{D}_{r-1} = \left(E_1^1 D_r^1\right)/D_n^1$ is a subgroup of $\left(E^1 D_{r-1}^1\right)/D_n^1$.*

Lemma 1.24. *Let α and χ_α be as in Lemma 1.7. Then for any $h \in H_{r-1} \cap \left(D_r^1/D_n^1\right)$ we have $\chi_\alpha(h) = 1$.*

Proof. Let $h \in H_{r-1} \cap \left(D_r^1/D_n^1\right)$. Then, as a result of Lemma 1.18, we can write

$$h \equiv \left(1 - 2a\pi^{r+1} + 2a^2\pi^{2(r+1)}\right) \pmod{P_D^n}, \quad \text{for some } a \in O.$$

From here and by definition of χ_α we have

$$\begin{aligned} \chi_\alpha(h) &= \chi\left(Tr\alpha\left(h-1\right)\right) \\ &= \chi\left(Tr\big(-2\alpha a\pi^{r+1}\big)\right)\chi\left(Tr\big(2\alpha a^2\pi^{2(r+1)}\big)\right) \\ &= \chi(0)\,\chi(0) \\ &= 1. \qquad \square \end{aligned}$$

Lemma 1.25. *Let α and φ_α be as in Lemma* 1.14. *Define*

$$\widetilde{\varphi}_\alpha : \left(E_1^1 D_r^1 / D_n^1\right) H_{r-1} \longrightarrow \mathbf{C}^\times$$

by

$$\widetilde{\varphi}_\alpha(\gamma h) = \varphi_\alpha(\gamma), \quad \forall\, \gamma \in \left(E_1^1 D_r^1\right)/D_n^1,\ \forall\, h \in H_{r-1}.$$

Then $\widetilde{\varphi}_\alpha$ is a character of $\left(E_1^1 D_r^1 / D_n^1\right) H_{r-1}$.

Proof. From Lemma 1.24 one can check that $\widetilde{\varphi}_\alpha$ is well defined. Moreover $\widetilde{\varphi}_\alpha$ is a homomorphism because for any γh and $\gamma' h' \in \left(E_1^1 D_r^1 / D_n^1\right) H_{r-1}$ by Corollary 1.23 we have

$$\begin{aligned}\widetilde{\varphi}_\alpha(\gamma h \gamma' h') &= \widetilde{\varphi}_\alpha\left(\gamma h \gamma' \bar{h} h h'\right)\\ &= \varphi_\alpha\left(\gamma h \gamma' \bar{h}\right)\\ &= \varphi_\alpha(\gamma)\,\varphi_\alpha\left(h \gamma' \bar{h}\right).\end{aligned}$$

Now, since $\left(E_1^1 D_{r-1}^1\right)/D_n^1$ is in the stabilizer of χ_α, from Lemma 1.24 and Corollary 1.23 we get

$$\varphi_\alpha\left(h\gamma'\bar{h}\right) = \varphi_\alpha(\gamma');$$

so

$$\widetilde{\varphi}_\alpha(\gamma h \gamma' h') = \widetilde{\varphi}_\alpha(\gamma h)\,\widetilde{\varphi}_\alpha(\gamma' h'). \qquad \square$$

2. Representations of D^1. Let $\alpha \in D^\circ$ with $v_D(\alpha) = -n-1$, $n > 0$. Put $r = [(n+1)/2]$, and let $E = F(\alpha)$ be a quadratic extension of F contained in D.

Corollary 2.1. *By Proposition* 1.12, *we have*:

1. *The stabilizer of χ_α in D^1 is $E^1 D_r^1$ when n is even,*
2. *The stabilizer of χ_α in D^1 is $E^1 D_{r-1}^1$ when n is odd.*

Theorem 2.2. *Let $\alpha \in D^\circ$ with $v_D(\alpha) = -n-1$, $n > 0$, and $r = [(n+1)/2]$. All other notations are as before. Then*:

1. *If n is even, let φ_α be a character of $E^1D^1_r$ defined in Lemma* 1.14 *and set*:

$$\rho(\alpha,\varphi) = Ind\left(D^1, E^1D^1_r, \varphi_\alpha\right).$$

Then $\rho(\alpha,\varphi)$ is an irreducible representation of D^1.

2. *If n and $r = [(n+1)/2]$ are odd, then by Lemma* 1.13 *we have*:

$$\left(E^1D^1_{r-1}\right)/D^1_n = \left(E^1D^1_r\right)/D^1_n.$$

Thus any character of $\left(E^1D^1_{r-1}\right)/D^1_n$ is a character of $\left(E^1D^1_r\right)/D^1_n$ and vice versa. In this case again let φ_α be a character of $E^1D^1_r$ determined by Lemma 1.14 *and set*:

$$\rho(\alpha,\varphi) = Ind\left(D^1, E^1D^1_{r-1}, \varphi_\alpha\right).$$

Then $\rho(\alpha,\varphi)$ is an irreducible representation of D^1.

3. *If n is odd and $r = [(n+1)/2]$ is even, then for any $\varphi \in \Phi(\alpha)$ there is a unique q-dimensional irreducible representation, $\tau_2(\alpha,\varphi)$, say, of $E^1D^1_{r-1}$ such that its restriction to $E^1D^1_r$ is a direct sum of φ_α's. Now set*:

$$\rho(\alpha,\varphi) = Ind\left(D^1, E^1D^1_{r-1}, \tau_2(\alpha,\varphi)\right).$$

Then $\rho(\alpha,\varphi)$ is an irreducible representation of D^1.

Proof. 1. By Corollary 2.1 the stabilizer of χ_α in D^1 is $E^1D^1_r$. Now apply Clifford theory and Theorem $(45.2)'$ in [**2**].

2. In this case by Corollary 2.1 the stabilizer of χ_α in D^1 is $E^1D^1_{r-1}$. Again, Clifford theory, Theorem $(45.2)'$ in [**2**] and Lemma 1.13 give the result.

3. To prove this part we need some more results. □

Proposition 2.3. *Let $\tau(\alpha,\varphi) = Ind\left(E^1_1D^1_{r-1}/D^1_n,\ E^1_1\mathfrak{D}_{r-1},\ \widetilde{\varphi}_\alpha\right)$. Then $\tau(\alpha,\varphi)$ is an irreducible representation of dimension q.*

Proof. This result follows from Lemma 1.21 and Theorem $(45.2)'$ in [**2**]. □

Lemma 2.4. *If x is any element of $\left(E^1D^1_{r-1}\right)/D^1_n$ which does not lie in $\left(E^1D^1_r\right)/D^1_n$, then $x^{-1}\left(E^1D^1_r/D^1_n\right)x \cap \left(E^1D^1_r\right)/D^1_n = E^1_1D^1_r/D^1_n$.*

Proof. Since D_r^1 is normal in D^1 it is enough to take $x = \left(1 + a\pi^{r-1}\right) \pmod{D_n^1}$, where a is a unit. Since $\nu(x) = 1 \pmod{D_n^1}$, we have $x^{-1} = \bar{x} = \left(1 - \pi^{r-1}\bar{a}\right) \pmod{D_n^1}$. Now let $h = \lambda(1 + b\pi^r) \pmod{D_n^1}$ be an element in $\left(E^1 D_r^1\right) / D_n^1$, where b is O_D and $\lambda \in E^1$ and also note that r is even. Then we have

$$\begin{aligned}
\bar{x}hx &= \left(1 - \pi^{r-1}\bar{a}\right)\lambda(1 + b\pi^r)\left(1 + a\pi^{r-1}\right) \pmod{D_n^1} \\
&= \lambda\left(1 - \bar{\lambda}\pi^{r-1}\bar{a}\lambda\right)\left(1 + a\pi^{r-1} + b\pi^r\right) \pmod{D_n^1} \\
&= \lambda\left(1 + a\pi^{r-1} + b\pi^r - \bar{\lambda}\pi^{r-1}\bar{a}\lambda - \bar{\lambda}\pi^{r-1}\bar{a}\lambda a\pi^{r-1}\right) \pmod{D_n^1}.
\end{aligned}$$

Now note that $\bar{x}hx \in \left(E^1 D_r^1\right) / D_n^1$ if and only if $\left(a\pi^{r-1} - \bar{\lambda}\pi^{r-1}\bar{a}\lambda\right) \in D_r^1$. We can write a as $\alpha + \beta\pi$ where α, β are in E and α is a unit because a is a unit. From here we get

$$\begin{aligned}
a\pi^{r-1} - \bar{\lambda}\pi^{r-1}\bar{a}\lambda &= \alpha\pi^{r-1} + \beta\pi^r - \bar{\lambda}\pi^{r-1}\bar{\alpha}\lambda + \bar{\lambda}\pi^{r-1}\beta\pi\lambda \\
&= \alpha\pi^{r-1} - \bar{\lambda}^2\alpha\pi^{r-1} + \beta\pi^r + \bar{\beta}\pi^r \\
&= \alpha\left(1 - \bar{\lambda}^2\right)\pi^{r-1} + \left(\beta + \bar{\beta}\right)\pi^r.
\end{aligned}$$

Since α is a unit we deduce that $\left(1 - \bar{\lambda}^2\right) \in P_D \cap E$, and this forces that $\lambda \in D_1^1 \cap E^1 = E_1^1$. □

Lemma 2.5. *Let H and K be two finite subgroups of a group G. Then, for any $g \in G$, the order of a double coset HgK is $|H|\left[K : g^{-1}Hg \cap K\right]$.*

Proof. This is easily verified if it is not well known. □

Lemma 2.6. *All notations are as before.*

1. $\left[E^1 : E_1^1\right] = (q+1)/2$.
2. $\left[E^1 D_r^1 / D_n^1 : E^1 D_{r-1}^1 / D_n^1 : E^1 D_r^1 / D_n^1\right] = 2q - 1$.
3. $\left[E^1 D_r^1 / D_n^1 : E^1 D_{r-1}^1 / D_n^1 : E_1^1 D_r^1 / D_n^1\right] = q^2$.
4. $\left[E_1^1 D_r^1 / D_n^1 : E^1 D_{r-1}^1 / D_n^1 : E_1^1 D_r^1 / D_n^1\right] = q^2[(q+1)/2]$.

Proof. 1. Let $g = a + b\varepsilon \in E^1$such that $a^2 - b^2\varepsilon^2 = 1$. Now let $b = b_\circ + b_1\varpi$, where $b_\circ \in \Re$ and $\Re$ is the set of representative elements of k in O. Then since $1+b^2\epsilon^2 \left(= a^2\right)$ is a square, $1+b_\circ^2\epsilon^2$ is a square too (Hensel's lemma). Thus there exists $a_\circ \in \Re$ such that $a_\circ^2 = 1 + b_\circ^2\epsilon^2$. One can show that $a = a_\circ + a_1\varpi$ for some $a_1 \in O$. Now let $g_1 = a_\circ + b_\circ\epsilon$. Then $g_1 \in E^1$ and

$$\begin{aligned} g_1^{-1}g &= (a_\circ - b_\circ\epsilon)\,(a + b\epsilon) \\ &= a_\circ a + a_\circ b\epsilon - ab_\circ\epsilon - b_\circ b\epsilon^2 \\ &= \left(a_\circ a - b_\circ b\epsilon^2\right) + (a_\circ b - ab_\circ)\,\epsilon. \end{aligned}$$

From $a^2 - b^2\epsilon^2 = 1 = a_\circ^2 - b_\circ^2\epsilon^2$, one can check that $\varpi \mid (a_\circ b - ab_\circ)$, i.e., $g_1^{-1}g \in E_1^1$ and this implies $g \in g_1E_1^1$. It is easy to show that $a_\circ + b_\circ\epsilon \in E_1^1$ if and only if $b_\circ = 0$. Thus

$$\left\{a_\circ + b_\circ\epsilon \mid b_\circ \in \Re,\ \text{and } a_\circ^2 = 1 + b_\circ^2\epsilon^2\right\}$$

is a set of representatives of cosets of E_1^1 in E^1. Since $b_\circ^2 = (-b_\circ)^2$, so there are only $(q-1)/2 + 1 = (q+1)/2$ distinct cosets.

2. Let m be the number of double cosets, let x_i, $1 \le i \le m$ be the double cosets representatives, and let $x_m = 1$. Then we can write:

$$\begin{aligned} \left(E^1D_{r-1}^1\right)\!/D_n^1 &= \bigcup_{i=1}^{m} \left(\left(\left(E^1D_r^1\right)\!/D_n^1\right)x_i\left(\left(E^1D_r^1\right)\!/D_n^1\right)\right) \\ &= \left[\bigcup_{i=1}^{m-1} \left(\left(\left(E^1D_r^1\right)\!/D_n^1\right)x_i\left(\left(E^1D_r^1\right)\!/D_n^1\right)\right)\right] \\ &\quad \cup \left(E^1D_r^1\right)\!/D_n^1, \end{aligned}$$

where $x_i \notin \left(E^1D_r^1\right)\!/D_n^1$ for $1 \le i \le m-1$. Now by Lemmas 2.4 and 2.5 we get:

$$\begin{aligned} \left|\left(E^1D_{r-1}^1\right)\!/D_n^1\right| = (m-1) \cdot \left|\left(E^1D_r^1\right)\!/D_n^1\right| \frac{q+1}{2} \\ + \left|\left(E^1D_r^1\right)\!/D_n^1\right|. \end{aligned}$$

Dividing both sides by $\left|\left(E^1D_r^1\right)\!/D_n^1\right|$, we get

$$q^2 = (m-1) \cdot \frac{q+1}{2} + 1.$$

Thus

$$m = 2q - 1.$$

3. The same argument as in part 2 and the fact that:

$$\left[\left(E_1^1 D_r^1\right)\diagup D_n^1 : x^{-1}\left(\left(E^1 D_r^1\right)\diagup D_n^1\right) x\right] = 1,$$
$$\text{for any } x \in \left(E^1 D_r^1\right)\diagup D_n^1$$

yield the result.

4. The same argument as in parts 2 and 3 gives us

$$\left(E^1 D_{r-1}^1\right)\diagup D_n^1 = \bigcup_{i=1}^{m}\left(\left(\left(E^1 D_r^1\right)\diagup D_n^1\right) x_i \left(\left(E^1 D_r^1\right)\diagup D_n^1\right)\right).$$

From here we get

$$\left|\left(E^1 D_{r-1}^1\right)\diagup D_n^1\right| = m \cdot \left|\left(E^1 D_r^1\right)\diagup D_n^1\right|.$$

Thus

$$m = \frac{\left|\left(E^1 D_{r-1}^1\right)\diagup D_n^1\right|}{\left|\left(E^1 D_r^1\right)\diagup D_n^1\right|} = q^2 \cdot \left(\frac{q+1}{2}\right). \qquad \square$$

Proposition 2.7. *For $\varphi \in \Phi(\alpha)$, let φ'_α be the restriction of φ_α to $E_1^1 D_r^1$, and let*

$$\tau_1(\alpha, \varphi) = Ind\left(E_1^1 D_{r-1}^1, E_1^1 D_r^1, \varphi'_\alpha\right).$$

Then $\tau_1(\alpha, \varphi)$ is a direct sum of q copies of $\tau(\alpha, \varphi)$.

Proof. Since for $\widetilde{\varphi}_\alpha$, defined in Lemma 1.25 we have $\varphi'_\alpha = \widetilde{\varphi}_\alpha$ on $E_1^1 D_r^1$, so $\tau_1(\alpha, \varphi)$ will be equivalent to $\left[E_1^1 \mathfrak{D}_{r-1} : \left(E_1^1 D_r^1\right)\diagup D_n^1\right]$ copies of $\tau(\alpha, \varphi)$. Now apply Lemma 2.6. $\square$

The following lemma is the key to the construction and motivated by Lemma 2.7 in [**10**].

Lemma 2.8. *Let ξ be the character of $Ind\left(E^1 D_{r-1}^1, E^1 D_r^1, \varphi_\alpha\right)$, and let ξ_1 be the character of $Ind\left(E^1 D_{r-1}^1, E_1^1 D_r^1, \varphi'_\alpha\right)$. Then $\eta =$*

$2q^{-1}\xi_1 - \xi$ *is the character of an irreducible representation,* $\tau_2(\alpha, \varphi)$, *say, of* $E^1 D^1_{r-1}$ *whose restriction to* $E^1_1 D^1_{r-1}$ *is* $\tau(\alpha, \varphi)$.

Proof. Let $\langle , \rangle$ denote the usual scaler product on $L^2\left(\left(E^1 D^1_{r-1}\right)\big/ D^1_r\right)$. By Lemma 2.6 and Mackey's theorem, we get:

$$\begin{aligned}\langle \eta, \eta \rangle &= 4q^{-2}\langle \xi_1, \xi_1\rangle - 4q^{-1}\langle \xi, \xi_1\rangle + \langle \xi, \xi\rangle \\ &= 4q^{-2}\left(q^2 \cdot \left(\frac{q+1}{2}\right)\right) - 4q^{-1}\left(q^2\right) 2q - 1 \\ &= 2(q+1) - 4q + 2q - 1 \\ &= 1.\end{aligned}$$

Thus η is a character of an irreducible representation of $E^1 D^1_{r-1}$. Now since $\xi_1(1) = q^2 \cdot [(q+1)/2]$ and $\xi(1) = q^2$, we get $\eta(1) = q$. Thus η is the character of an irreducible representation of $E^1_1 D^1_{r-1}$ having dimension q, call it $\tau_2(\alpha, \varphi)$. The multiplicity of $\tau_2(\alpha, \varphi)$ in $\tau(\alpha, \varphi)$ induced to $E^1 D^1_{r-1}$ is $\left\langle \eta, q^{-1}\xi_1\right\rangle = 1$. So, by Frobenius reciprocity, the restriction of $\tau_2(\alpha, \varphi)$ to $E^1_1 D^1_{r-1}$ is equivalent to $\tau(\alpha, \varphi)$. □

Proof of part 3 *of Theorem* 2.2. Since $\tau_2(\alpha, \varphi)$ is an extension of $\tau(\alpha, \varphi)$ by Theorem 51.7 in [**2**], every irreducible summand of $Ind\left(E^1 D^1_{r-1}, E^1_1 D^1_r, \tau(\alpha, \varphi)\right)$ is equivalent to some $\tau_2(\alpha, \varphi) \otimes \psi$ where ψ is a representation of E^1, which is trivial on E^1_1. Thus by Theorem 38.5 in [**2**] and Lemma 2.8 in this paper, it follows that:

$$\tau_2(\alpha, \varphi) \otimes \psi \cong \tau_2(\alpha, \varphi\psi).$$

Now apply Clifford's theorem [**2**].

2. Characters (one-dimensional representations) of D^1. We can obtain almost all representations of D^1 from Theorem 2.2; however, we cannot deduce one-dimensional representations of D^1 from this theorem. We will determine these as follows.

Lemma 3.1. *The commutator group of* D^1 *is equal to* D^1_1 *where* D^1_1 *is*

$$D^1_1 = \left\{x \in D^1 \mid x - 1 \in P_D\right\}.$$

Proof. See [**14**]. □

Lemma 3.2. *$D^1 \diagup D_1^1$ is a cyclic group of order $q+1$.*

Proof. Define $f : D^1 \diagup D_1^1 \to \mathbf{k}^\times$ by $f\left(\delta D_1^1\right) = \delta + P_D$, $\delta \in D^1$. Then one can check that f is a well-defined homomorphism. f is one-to-one because if $\delta + P_D = 1$ then $\delta - 1 \in P_D$, thus $\delta \in D_1^1$. It is easy to see the image of f is equal to:

$$\mu_{q+1} = \left\{a \in \mathbf{k}^\times \mid \bar{\nu}\,(a) = 1\right\},$$

where $\bar{\nu}$ is the map induced by norm map on residual field $\mathbf{k}$ defined as $\bar{\nu}\,(a + P_D) = \nu\,(a) + P$. So $D^1 \diagup D_1^1 \cong \mu_{q+1}$. This group is cyclic because $\mathbf{k}^\times$ is a multiplicative subgroup of a finite field. The next lemma shows that μ_{q+1} has $q+1$ elements. □

Lemma 3.3. *The group μ_{q+1} in Lemma 3.2 has $q+1$ elements.*

Proof. Define $f : \mathbf{k}^\times \to \mu_{q+1}$ by $f\,(a) = a/\bar{a}$. Hilbert's 90 shows that f is onto, and one can show $\ker f = k^\times$. Hence $\mathbf{k}^\times \diagup k^\times \cong \mu_{q+1}$, and from here we get $|\mu_{q+1}| = |\mathbf{k}^\times \diagup k^\times| = q^2 - 1/q - 1 = q + 1$. □

Theorem 3.4. *Any character of D^1 is a character of $D^1 \diagup D_1^1$ and vice versa.*

Proof. Let ψ be a character of D^1. Then since D_1^1 is the commutator group of D^1, ψ will be trivial on D_1^1. Conversely let $\bar{\psi}$ be a character of $D^1 \diagup D_1^1$, then $\psi\,(\delta) = \bar{\psi}\left(\delta D_1^1\right)$ is a character of D^1. □

Convention. From now on an irreducible representation of D^1 determined by part i, $1 \leq i \leq 3$, in Theorem 2.2 will be called of type i, and any one-dimensional representation of D^1 will be called a character.

Theorem 3.5. *Any irreducible representation of D^1 is either one of those determined in Theorem 2.2 or is a character. Further, they enjoy the following equivalencies.*

1. *A representation of the type i never is equivalent to a representation of type j, $i \neq j$, $1 \leq i, j \leq 3$.*

2. *A representation of the type i, $1 \leq i \leq 3$, never is equivalent to a character.*

3. *Two representations $\rho(\alpha, \varphi), \rho(\alpha', \varphi')$ of type i, $1 \leq i \leq 3$, are equivalent if and only if*

- *they have same conductor, n say,*
- *there exists $g \in D^1$ such that $\alpha' - g\alpha g^{-1} \in P_D^{n-r}$ where $r = [(n+1)/2]$,*
- $\varphi'(e') = \varphi\left(geg^{-1}\right)$, *$e' \in E' = F(\alpha')$, $e \in E = F(\alpha)$,*
- *and $E' = F(\alpha') = gEg^{-1}$.*

Proof. Let ρ be a nontrivial irreducible representation of D^1. Since D^1 is compact there exists an integer $n \geq 1$ such that the restriction of ρ to D_n^1, $\rho_{|D_n^1}$, is trivial. Let n be the least integer with this property. Then, if $n = 1$, by Theorem 3.4, ρ is a character. If $n > 1$, then the restriction of ρ to D_r^1 where $r = [(n+1)/2]$ can be considered as a representation χ_α on D_r^1 / D_n^1 so it is the direct sum of χ_α for some α, because D_r^1 / D_n^1 is abelian. Thus ρ is one of those determined by Theorem 2.2. Statements 1 and 2 are obvious. For 3, consider the restriction of $\rho(\alpha, \varphi)$ and $\rho(\alpha', \varphi')$ to D_r^1 where $r = [(n+1)/2]$ and then apply Clifford's theorem [**2**]. □

Acknowledgments. This paper is a revision of part of our doctoral thesis at the University of Iowa [**13**]. I would like to thank David Manderscheid for introducing me to this work. I would like also to thank the referee for valuable comments and information.

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