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HYPERSURFACES SINGULAR ALONG SMOOTH NONLINEARLY NORMAL CURVES

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ABSTRACT. Let $X \subset \mathbf{P}^n$ be a smooth curve and $X^{(1)}$ the first infinitesimal neighborhood of X in $\mathbf{P}^n$. Here we prove that $X^{(1)}$ has maximal rank for several nonlinearly normal embeddings $X \subset \mathbf{P}^n$.

1. Introduction. Let $X \subset \mathbf{P}^n$ be a smooth curve and $X^{(1)}$ the first infinitesimal neighborhood of X in $\mathbf{P}^n$, i.e., the closed subscheme of $\mathbf{P}^n$ with $(\mathbf{I}_X)^2$ as the ideal sheaf. Thus $X^{(1)}_{\mathrm{red}} = X$. A hypersurface Z of $\mathbf{P}^n$ is singular along X if and only if it contains $X^{(1)}$. Thus the Hilbert function of $X^{(1)}$, i.e., the string of integers $h^0(\mathbf{P}^n, \mathbf{I}_{X^{(1)}}(t))$, $t \geq 0$, is a natural numerical invariant of X. A few papers were devoted to the computation of the Hilbert function of $X^{(1)}$ when X is either a canonically embedded curve or a linearly normal curve of genus g and large degree, say degree $d \geq 2g+3$, [**5–8**]. Here we will consider the case in which C is not linearly normal. Here are our results.

Theorem 1.1. *Fix integers n, d and g, and set $x := d+1-g-n$. Assume $x \geq 2$, $n \geq x+5$, $d-x-1 \geq 2g+3$, $g \leq n-x-2$ and $(n-x)(n-x-1)/2 \geq 2(d-x-2)+1-g$. Let X be a smooth connected projective curve of genus g and $L \in \mathrm{Pic}^d(X)$. Then there is an embedding $j : X \to \mathbf{P}^n$ such that $j^*(\mathbf{O}_{j(X)}(1)) \cong L$ and $h^1(\mathbf{P}^n, \mathbf{I}_{j(X)^{(1)}}(k)) = 0$ for every $k \geq 3$. Furthermore, $h^0(\mathbf{P}^n, \mathbf{I}_{j(X)^{(1)}}(2)) = 0$ and $j(X)^{(1)}$ has maximal rank.*

For instance, if $X \subset \mathbf{P}^n$ is a genus two smooth curve of degree 25, then Theorem 1.1 covers the cases $17 \leq n \leq 22$.

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Theorem 1.2. *Fix integers n, d and g, and set $x := d+1-g-n$. Assume $x \ge 2$, $n \ge x+5$, $d-x-1 \ge 2g+3$, $g \le n-x-2$ and $(n-x)(n-x-1)/2 \ge 2(d-x-2)+1-g$. Then, for the general smooth curve $Y \subset \mathbf{P}^n$ with $\deg(Y) = d$ and $p_a(Y) = g$, we have $h^1(\mathbf{P}^n, \mathbf{I}_{Y^{(1)}}(k)) = 0$ for every $k \ge 3$, $h^0(\mathbf{P}^n, \mathbf{I}_{Y^{(1)}}(2)) = 0$ and $Y^{(1)}$ has maximal rank.*

For several proofs in the quoted references ([**5, 6, 8**], second part of [**7**]) the smoothness of X is essential. Our proofs of Theorems 1.1 and 1.2 will use a degeneration of X to a reducible nodal curve, union of a linearly normal smooth curve C of degree $d-x$ and a smooth rational curve D such that $\deg(D) = x$ and D intersects quasi-transversally C at exactly one point. However, we will apply [**7**, Corollary 3.10] to C and hence, up to now, our method does not give independent proofs or refinements of [**7**].

For every smooth curve $X \subset \mathbf{P}^n$ and every integer $b \ge 0$, let $X^{(b)}$ be the infinitesimal neighborhood of order b of X in $\mathbf{P}^n$, i.e., the closed subscheme of $\mathbf{P}^n$ with $(\mathbf{I}_X)^{b+1}$ as ideal sheaf.

Conjecture 1.3. *For all integers n, b and g such that $n \ge 3$, $b \ge 0$ and $g \ge 0$, there is an integer $d(n,g,b) \ge 2g+n+2b$ such that for all integers $d \ge d(n,g,b)$ the curve $X^{(b)}$ has maximal rank, where $X \subset \mathbf{P}^n$ is a general degree d embedding in $\mathbf{P}^n$ of a general smooth curve of genus g.*

2. The proofs. Let $X \subset \mathbf{P}^n$ be a smooth curve and $\mathbf{I}_X$ its ideal sheaf. Set $d := \deg(X)$ and $g := 1 - \chi(\mathbf{O}_X)$. For all integers t we have the exact sequences

$$0 \longrightarrow (\mathbf{I}_X)^2(t) \longrightarrow \mathbf{I}_X(t) \longrightarrow \mathbf{I}_X/(\mathbf{I}_X)^2(t) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow \mathbf{I}_X/(\mathbf{I}_X)^2(t) \longrightarrow \mathbf{O}_{X^{(1)}}(t) \longrightarrow \mathbf{O}_X(t) \longrightarrow 0. \tag{2}$$

The sheaf $\mathbf{I}_X/(\mathbf{I}_X)^2$ is a rank $n-1$ vector bundle on X isomorphic to the conormal bundle N_X^* of X in $\mathbf{P}^n$. From the exact sequence

$$0 \longrightarrow N_X^* \longrightarrow \Omega_{\mathbf{P}^n}|X \longrightarrow \Omega_X \longrightarrow 0 \tag{3}$$

we obtain $\operatorname{rank}(N_X^*) = n-1$ and $\deg(N_X^*) = -d(n+1)-2g+2$. Thus, $\chi(\mathbf{O}_{X^{(1)}}(t)) = \chi(N_{X^*}(t))+\chi(\mathbf{O}_X(t)) = -d(n+1)-2g+2+(n-1)\,dt+(n-1)(1-g)+td+1-g = n\,dt-dn-d+(n+2)(1-g)$.

Proof of Theorem 1.1. Fix $P \in X$, and take a hyperplane H of $\mathbf{P}^n$. Since $d-x \geq 2g+1$ and $n = d-x+1-g$, there is a linearly normal embedding $i : X \to H$ such that $i^*(\mathbf{O}_H(1)) \cong L(-xP)$. Set $C := i(X)$. Since $d-x \geq 2g+2$, C is projectively normal in H. Since $d-x \geq 2g+3$ we may apply [**7**, Corollary 3.10], to the curve C and obtain that the first infinitesimal neighborhood $C_H^{(1)}$ of $i(X)$ in H satisfies $H^1(H, \mathbf{I}_{C_H^{(1)}}(3)) = 0$. Set $Q := i(P) \in C$. Let $D \subset \mathbf{P}^n$ be a general smooth rational curve of degree x passing through Q. Hence D spans a linear space M of dimension x such that $M \cap C = \{Q\}$ scheme-theoretically and D is a rational normal curve of M. By [**3**] and the assumption $n \geq 4$, there is a flat family of smooth projective curves $\{Z_t \subset \mathbf{P}^n\}_{t\in U}$, U smooth and connected affine curve, $o \in U$, such $Z_o = C \cup D$ (scheme-theoretically), Z_t embedded in $\mathbf{P}^n$ by a linear subspace $V \subseteq H^0(X, L)$ with $\dim(V) = n+1$. Since Z_o is a locally complete intersection, the family of all conormal bundles $\{N^*Z_t\}_{t\in U}$ is a flat family of vector bundles on the family of curves $\{Z_t \subset \mathbf{P}^n\}_{t\in U}$. By (2) we obtain that $\{Z_t^{(1)} \subset \mathbf{P}^n\}_{t\in U}$ is a flat family of curves. Hence, by semi-continuity, to prove $h^1(\mathbf{P}^n, \mathbf{I}_{Z_t^{(1)}}(3)) = 0$, and hence the case $k = 3$ of Theorem 1.1, it is sufficient to prove that $h^1(\mathbf{P}^n, \mathbf{I}_{Z_0^{(1)}}(3)) = 0$, i.e., that $h^1(\mathbf{P}^n, \mathbf{I}_{(C\cup D)^{(1)}}(3)) = 0$. Since C and D are quasi-transversal at Q and $C \cap D = \{Q\}$, a local calculation shows that $(C\cup D)^{(1)} = C^{(1)} \cup D^{(1)}$. A local calculation shows that the residual scheme $\operatorname{Res} H(C^{(1)} \cup D^{(1)})$ of $C^{(1)} \cup D^{(1)}$ with respect to the Cartier divisor H of $\mathbf{P}^n$ is $C \cup D^{(1)}$. We have $(C^{(1)} \cup D^{(1)}) \cap H = C_H^{(1)} \cup (D^{(1)} \cap H)$. Thus, for every integer t, we have an exact sequence

$$0 \to \mathbf{I}_{\operatorname{Res} H(C^{(1)}\cup D^{(1)})}(t-1) \to \mathbf{I}_{C^{(1)}\cup D^{(1)}}(t) \to \mathbf{I}_{(C^{(1)}\cup D^{(1)})\cap H, H}(t) \to 0 \tag{4}$$

(Horace lemma). Hence, $h^1(\mathbf{P}^n, \mathbf{I}_{Z_0^{(1)}}(3)) \leq h^1(H,_{(C^{(1)}\cup D^{(1)})\cap H,H}(3)) + h^1(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(2))$. Call $T \subset \mathbf{P}^{n-x-1}$ the image of C by the linear projection from M. By the generality of M with the only restriction that $Q \in M$, the very ampleness of the line bundle $L(-(x+1)P)$ and the assumption $n-x-1 \geq 3$, T is a smooth nondegenerate curve of degree $d-x-1$ corresponding to an embedding of X by a general linear subspace W of $H^0(X, L(-(x+1)P))$ with $\dim(W) = n-x$. A quadric hypersurface is singular along D if and only if M is contained in its vertex. Hence $h^0(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(2)) = h^0(\mathbf{P}^{n-x-1}, \mathbf{I}_T(2))$. Since $2(d-x-1)+1-g \leq (n-x+1)(n-x)/2 = h^0(\mathbf{P}^{n-x-1}, \mathbf{O}_{\mathbf{P}^{n-x-1}}(2))$ and $g \leq n-x-1$, we have $h^1(\mathbf{P}^{n-x-1}, \mathbf{I}_T(2)) = 0$ ([**3**] for $n-x-1 \geq 4$, [**2**] for $n-x-1 = 3$). Hence, we obtain $h^0(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(2)) = (n-x+1)(n-x)/2+g-1-2(d-x-1)$, i.e., $h^1(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(2)) = 0$. Now we will check the vanishing of $h^1(H, \mathbf{I}_{(C^{(1)}\cup D^{(1)})\cap H, H}(3))$. Since $CH^{(1)}$ contains the first infinitesimal neighborhood of Q in H, $(C^{(1)}\cup D^{(1)})\cap H$ is the union of $C_H^{(1)}$ and the union, A, of $x-1$ general double points of H. By [**7**, Corollary 3.10], and the assumption $d-x \geq 2g+3$, we have $h^1(H, \mathbf{I}_{C_H^{(1)}}(3)) = 0$. Let E be a hyperplane of H. As in the first part we degenerate C to the union T of a linearly normal curve $F \subset E$, $E \cong X$, with $\deg(E) = d-x-1$ and a line R meeting F at one point and general with this property. We apply the first part of the proof to $T^{(1)} \cup A$. The residual scheme of $T^{(1)} \cup A$ with respect to the Cartier divisor E of H is just $T \cup R^{(1)} \cup A$. Since $\dim(\langle R \cup A_{\text{red}}\rangle) = x+1$, we conclude as in the first part. Now we check that $h^0(\mathbf{P}^n, \mathbf{I}_{j(X)^{(1)}}(2)) = 0$. Since $j(X)$ is nondegenerate and the singular locus of a quadric hypersurface is a linear space, we have $h^0(\mathbf{P}^n, \mathbf{I}_{j(X)^{(1)}}(2)) = 0$. Fix an integer $k \geq 4$. By [**7**, Corollary 3.10], we have $h^1(\mathbf{P}^n, \mathbf{I}_{C^{(1)}}(k)) = 0$ for every $k \geq 4$. Hence, using again the Horace lemma and (if $x \geq 2$) a further degeneration, to prove the vanishing of $h^1(\mathbf{P}^n, \mathbf{I}_{j(X)^{(1)}}(k))$, it is sufficient to prove that $h^1(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(k)) = 0$. This is true by Castelnuovo-Mumford's lemma because $h^2(\mathbf{P}^n, \mathbf{I}_{C\cup D^{(1)}}(k-1)) = 0$, but it may also be proved degenerating D to a union of lines and then applying the Horace method. Hence $j(X)^{(1)}$ has maximal rank.

Proof of Theorem 1.2. Take the curve $C \cup D$ as in the proof of Theorem 1.2 but with C of general degree $d-x$ embedding a general smooth curve of genus g. Notice that T is a general smooth curve of degree $d-x-1$ and genus g in $\mathbf{P}^{n-x-1}$. Instead of applying [**3**] or [**2**], apply respectively [**4**] (case $n-x-1 \geq 4$) or [**1**] (case $nx-1 = 3$).

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REFERENCES

1. E. Ballico and Ph. Ellia, *The maximal rank conjecture for nonspecial curves in* $\mathbf{P}^3$, Invent. Math. **79** (1985), 541–555.

2. ———, *On the projection of a general curve in* $\mathbf{P}^3$, Ann. Mat. Pura Appl. **142** (1985), 15–48.

3. ———, *On the postulation of a general projection of a curve in* $\mathbf{P}^N$, $N \geq 4$, Ann. Mat. Pura Appl. **147** (1987), 267–301.

4. ———, *The maximal rank conjecture for non-special curves in* $\mathbf{P}^n$, Math. Z. **196** (1987), 355–367.

5. A. Bertram, *An application of general Kodaira vanishing to embedded projective varieties*, e-preprint alg-geom/9707001.

6. J. Rathmann, *An infinitesimal approach to a conjecture of Eisenbud and Harris*, unpublished preprint.

7. P. Vermeire, *On the regularity of powers of ideal sheaves*, Compositio Math. **131** (2002), 161–172.

8. J. Wahl, *On the cohomology of the square of an ideal sheaf*, J. Algebraic Geom. **6** (1997), 481–511.

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