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ON THE NUMBER OF BLOWING-UP SOLUTIONS TO A NONLINEAR ELLIPTIC EQUATION WITH CRITICAL GROWTH

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ABSTRACT. In this paper we estimate the number of solutions to

$$\begin{cases} -\Delta w + V(x)w = n(n-2)w^{(n+2)/(n-2)-\epsilon} & \text{in } \mathbf{R}^n \\ w > 0 & \text{in } \mathbf{R}^n \\ w \in \mathcal{D}^{1,2}(\mathbf{R}^n) \end{cases}$$

which blow up at a suitable critical point of the potential V as the parameter ϵ goes to zero.

1. Introduction. Let us consider the problem

$$(1) \qquad \begin{cases} -\Delta w + V(x)w = n(n-2)w^{(n+2)/(n-2)-\epsilon} & \text{in } \mathbf{R}^n \\ w > 0 & \text{in } \mathbf{R}^n \\ w \in \mathcal{D}^{1,2}(\mathbf{R}^n) \end{cases}$$

where $V : \mathbf{R}^n \to \mathbf{R}$ satisfies suitable conditions, $n \geq 3$ and $\epsilon > 0$ is a positive parameter. Here $\mathcal{D}^{1,2}(\mathbf{R}^n)$ is defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|u\|_{1,2} = (\int_{\mathbf{R}^n} |\nabla u|^2)^{1/2}$. It is a Hilbert space equipped with the inner product $(u,v)_{1,2} = \int_{\mathbf{R}^n} \nabla u \cdot \nabla v$. We refer the reader to the pioneering paper [**3**] on the critical Sobolev exponent.

In the critical case, i.e., $\epsilon = 0$, when $V \equiv 0$ on $\mathbf{R}^n$ it is well known (see [**1, 4, 15**]) that problem (1) has the family of solutions

$$U_{\delta,y}(x) = \delta^{-(n-2)/2}\, U\left(\frac{x-y}{\delta}\right), \quad x \in \mathbf{R}^n,$$

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where $\delta > 0$, $y \in \mathbf{R}^n$ and

$$U(x) = \frac{1}{(1+|x|^2)^{(n-2)/2}}.$$

In [**2**] the authors consider the case when V is not identically equal to zero and they prove the existence of a solution to (1), provided V satisfies some suitable conditions. The slightly subcritical case was firstly considered by Ding and Ni in [**5**], where the authors prove the existence of ground states solutions to (1), provided V belongs to a suitable class of potentials. Successively in [**14, 16**] it has been shown that the ground states solutions blow up at a global minimum point of the potential V. More recently in [**12**] and in [**10**] the authors consider a different class of potentials V, and they construct positive and sign changing solutions blowing up at one or more suitable critical points of V. Papers [**10, 12**] deal with both the slightly subcritical case and the slightly supercritical case, i.e., $\epsilon < 0$. In particular, in [**12**], the following existence result has been proved. Let V satisfy the following assumptions.

(i) $V \in L^{n/2}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$,

(ii) $V \in C^2(\mathbf{R}^n)$ and $\partial V/\partial x_i$, $\partial^2 V/(\partial x_i \partial x_j) \in L^\infty(\mathbf{R}^n)$, for any $i, = 1, 2, \dots, n$,

(iii) $V(x) > 0$ for any $x \in \mathbf{R}^n$,

(iv) $\|V\|_{L^{n/2}(\mathbf{R}^n)} \leq \mu_0$, see Lemma 2.1.

Theorem 1.1. *Let $n \geq 7$. Let y_0 be a "stable" critical point of the function V. Then there exists a family of solutions u_ϵ to* (1) *blowing up at the point y_0 as ϵ goes to zero. More precisely there exist $y_\epsilon \in \mathbf{R}^n$ and $\delta_\epsilon > 0$ with $y_\epsilon \to y_0$ and $\delta_\epsilon \to 0$ such that $u_\epsilon - U_{\delta_\epsilon, y_\epsilon} \to 0$ in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ as ϵ goes to zero.*

At this stage a natural question arises: how many solutions blowing up at y_0 do there exist?

In this paper we give an answer by following some ideas introduced by Grossi in [**8**]. In [**8**] the author studies the nonlinear Schrödinger equation

$$\begin{cases} -\epsilon^2 \Delta w + W(x) w = w^p & \text{in } \mathbf{R}^n \\ w > 0 & \text{in } \mathbf{R}^n \end{cases} \tag{2}$$

where $W : \mathbf{R}^n \to \mathbf{R}$ satisfies suitable conditions and $p \in (1, (n+2)/(n-2))$, and he establishes the exact number of single-peak solutions concentrating at a suitable critical point of the potential W as ϵ goes to zero. We would like to point out that, even if problems (2) and (1) have very different features, the results we get are very similar to the ones obtained in [**8**].

Let us mention our main results. Let y_0 be a fixed critical point of V. We assume the following assumption on V in a neighborhood of y_0.

(V_{y_0}) There exist $h_i : R^n \to R$ C^1-functions and $R_i : B(0,\rho) \to R$ and $\alpha_i \geq 1$ for $i = 1, \dots, n$, such that

(i) $\partial V / \partial x_i (y_0 + z) = h_i(z) + R_i(z)$ for $z \in B(0, \rho)$,

(ii) $|R_i(z)| \leq C |z|^{\beta_i}$ for $z \in B(0, \rho)$ with $\beta_i > \alpha_i$,

(iii) $h_i(tz) = t^{\alpha_i} h_i(z)$ for any $z \in \mathbf{R}^n$ and $t > 0$,

(iv) $h_i(z) = 0$ if and only if $z = 0$.

Moreover, assume

$$\overline{\alpha} := \max\{\alpha_i \mid i = 1, \dots, n\} < n - 4. \tag{3}$$

Therefore we can introduce the function $H_{y_0} : \mathbf{R}^n \to \mathbf{R}^n$:

$$(H_{y_0}(y))_i := \int_{\mathbf{R}^n} h_i(x+y)\, U^2(x)\, dx \quad \text{for} \quad i = 1, \dots, n. \tag{4}$$

First of all we prove the following nonexistence result, see also Example 3.3.

Theorem 1.2. *If $H_{y_0}(y) \neq 0$ for any $y \in \mathbf{R}^n$, then there is no solution to* (1) *blowing up and concentrating at y_0.*

Secondly we prove the following multiplicity result.

Let us introduce the set

$$Z_{y_0} := \left\{ y \in \mathbf{R}^N \mid y \text{ is a stable zero of } H_{y_0} \right\}. \tag{5}$$

We need also to assume the following technical condition.

$$\underline{\alpha} := \min \left\{ \alpha_i \mid i = 1, \dots, n \right\} < n - 5. \tag{6}$$

Theorem 1.3. *If $\# Z_{y_0} < \infty$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the number of the solutions of* (1) *blowing up and concentrating at y_0 is greater than or equal to $\# Z_{y_0}$.*

Finally we prove a uniqueness result, see also Examples 4.1 and 4.2. We need to assume the following further assumption on V:

(v) $x \cdot \nabla V(x) \in L^\infty(\mathbf{R}^n)$.

Theorem 1.4. *Let 0 be a regular value of H_{y_0}, i.e.,* $\det$ *Jac* $H_{y_0}(y) \neq 0$ *for any $y \in \mathbf{R}^n$ such that $H_{y_0}(y) = 0$. Then the number of solutions of* (1) *blowing up and concentrating at y_0 is equal to $\# Z_{y_0}$.*

We would like to quote the fact that computations in the critical case are more technical and delicate than in the subcritical one, because of both the decay of solutions, see Lemma 2.3, and the presence of the concentration parameter, see Lemma 3.5.

Finally let us make some comments about the supercritical case, i.e., $\epsilon < 0$. In [**12**] it was proved that if y_0 is a "stable" critical point of the function V with $V(y_0) < 0$, then there exists a family of solutions u_ϵ to (1) blowing up at y_0 as ϵ goes to 0. Also in this case one can ask how many solutions are generated by y_0. A partial answer was given in [**11**], where the authors consider a radial potential V with $V(0) < 0$. They construct infinitely many solutions blowing up at the origin as ϵ goes to zero, which resemble a super-position of spikes centered at the origin with different rates of concentration.

The paper is organized as follows. In Section 2 we recall the Liapunov-Schmidt procedure as performed in [**12**] and we prove a key result, see Lemma 2.3, about the decay of solutions. In Section 3 we prove

Theorem 1.2 and Theorem 1.3. In Section 4 we prove Theorem 1.4. In the Appendix we prove a technical result, see Lemma 3.5.

2. Preliminary remarks. First of all we rewrite problem (1) in a different way, see [**12**], namely

$$\begin{cases} u = i^*[f_\epsilon(u) - \delta^2 V_{\delta,y}\, u] \\ u \in X. \end{cases} \tag{7}$$

Here $i^* : L^{(2n)/(n+2)}(\mathbf{R}^n) \to \mathcal{D}^{1,2}(\mathbf{R}^n)$ is the adjoint operator of the immersion $i : \mathcal{D}^{1,2}(\mathbf{R}^n) \hookrightarrow L^{(2n)/(n-2)}(\mathbf{R}^n)$, i.e.,

$$i^*(u) = v \Longleftrightarrow (v, \varphi)_{1,2} = \int_{\mathbf{R}^n} u(x)\varphi(x)\, dx, \quad \forall\, \varphi \in \mathcal{D}^{1,2}(\mathbf{R}^n).$$

Moreover $f_\epsilon(s) = n(n-2)(s^+)^{(n+2)/(n-2)-\epsilon}$, $V_{\delta,y}(x) = V(\delta x + y)$ for some $\delta > 0$, $y \in \mathbf{R}^n$.

The Banach space $X = L^s(\mathbf{R}^n) \cap \mathcal{D}^{1,2}(\mathbf{R}^n)$ is equipped with the norm $\|u\|_X = \max\{\|u\|_{L^s(\mathbf{R}^n)}, \|u\|_{1,2}\}$. It is easy to verify that $u(x)$ is a solution of (7) if and only if $w(z) = \delta^{-2/(p-1-\epsilon)} u((z-y)/\delta)$, $p = (n+2)/(n-2)$, is a solution of (1).

We also point out the following result, see Lemma 2.3 and Lemma 2.4 of [**12**].

Lemma 2.1. *There exists a $\mu_0 > 0$ such that if $\|V\|_{L^{n/2}(\mathbf{R}^n)} \le \mu_0$, then the operator $-\Delta + V$ is coercive, i.e., there exists a $\delta > 0$ such that*

$$\int_{\mathbf{R}^n} (|\nabla u|^2 + V(x)u^2)\, dx \ge \delta \|u\|_{1,2}^2, \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbf{R}^n).$$

In particular, if $u \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ is a nontrivial solution of (7)*, then $u(x) > 0$ for all $x \in \mathbf{R}^n$.*

In order to solve (7) we use a well-known Ljapunov-Schmidt procedure, see, for example, [**12**]. More precisely, we look for a solution to (7) of the form $u^\epsilon_{\delta,y}(x) = U(x) + \phi^\epsilon_{\delta,y}(x)$, where the lower order term $\phi^\epsilon_{\delta,y}$ belongs to the space $K^\perp$ defined as follows:

$$K = \text{span}\,\{\psi_0, \psi_1, \dots, \psi_n\}$$

and

$$K^{\perp} = \{u \in X \mid (u, \varphi)_{1,2} = 0 \,\forall\, \varphi \in K\}.$$

Here the functions

$$\psi_0(x) := x \cdot \nabla U(x) + \frac{n-2}{2}\, U(x) = \frac{n-2}{2}\, \frac{1-|x|^2}{(1+|x|^2)^{n/2}}$$

and

$$\psi_i(x) := \frac{\partial U}{\partial x_i} = -\,(n-2)\, \frac{x_i}{(1+|x|^2)^{n/2}}, \quad i = 1, \dots, n,$$

are the solutions of the linearized problem, see [**13**, Lemma 4.2],

$$-\,\Delta\varphi = n(n+2)\, U^{4/(n-2)}\varphi \quad \text{in} \quad \mathbf{R}^n, \qquad \varphi \in \mathcal{D}^{1,2}(\mathbf{R}^n).$$

We introduce the projections $\Pi : X \to K$ and $\Pi^{\perp} : X \to K^{\perp}$. Therefore (7) turns out to be equivalent to the following system

$$\begin{cases} \Pi^{\perp}\left\{U + \phi - i^*\left[f_\epsilon(U+\phi) - \delta^2\, V_{\delta,y}(U+\phi)\right]\right\} = 0 \\ \Pi\,\left\{U + \phi - i^*\left[f_\epsilon(U+\phi) - \delta^2\, V_{\delta,y}(U+\phi)\right]\right\} = 0. \end{cases}$$

The following proposition allows us to solve the first equation in system (8) and to reduce problem (7) to a finite dimensional one, see [**12**, Proposition 3.1 and Lemma 1.15].

Proposition 2.2. *Let $n \geq 7$ and $s \in (n/(n-4), 2n/(n-2))$. There exist $\epsilon_0 > 0$ and $\delta_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, $\delta \in (0, \delta_0)$ and for any $y \in \mathbf{R}^n$ there exists a unique $\phi^{\epsilon}_{\delta,y} \in K^{\perp}$ such that*

$$\left\|\phi^{\epsilon}_{\delta,y}\right\|_X \leq C(\delta^2 + \epsilon)$$

and

$$\Pi^{\perp}\left\{U + \phi^{\epsilon}_{\delta,y} - i^*\left[f_\epsilon(U + \phi^{\epsilon}_{\delta,y}) - \delta^2\, V_{\delta,y}(U + \phi^{\epsilon}_{\delta,y})\right]\right\} = 0.$$

Moreover the map $(\delta, y) \to \phi^{\epsilon}_{\delta,y}$ is uniformly continuous.

According to Proposition 2.2, there exist real numbers $c_i(\epsilon, \delta, y)$ such that $u^\epsilon_{\delta,y} = U + \phi^\epsilon_{\delta,y}$ is a solution to

$$u^\epsilon_{\delta,y} - i^* \left[f_\epsilon(u^\epsilon_{\delta,y}) - \delta^2 \, V_{\delta,y} u^\epsilon_{\delta,y} \right] = \sum_{i=0}^{n} c_i(\epsilon\delta, y)\psi_i. \tag{9}$$

It is clear that, in order to solve the second equation in system (8), we need to find for ϵ small enough a parameter δ_ϵ and a point y_ϵ such that $c_i(\epsilon, \delta_\epsilon, y_\epsilon) = 0$ for any $i = 0, 1, \dots, n$.

At this aim the next result plays a crucial role in our analysis.

Lemma 2.3. *Let $u^\epsilon_{\delta,y} \in X$ be the solution to* (9)*. For any compact set K in $\mathbf{R}^n$, there exist $C > 0$, $\delta_0 > 0$ and $\epsilon_0 > 0$ such that, for any $y \in K$, $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$*

$$\left| u^\epsilon_{\delta,y}(x) \right| \le CU(x), \quad \forall \, x \in \mathbf{R}^n.$$

Proof.

Step 1. For any $G \subset \mathbf{R}^n$ compact, there exist $C(G) > 0$, $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for any $y \in K$, $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$

$$\left| u^\epsilon_{\delta,y}(x) \right| \le C(G), \quad \forall \, x \in G.$$

By contradiction we assume that there exist $\delta_m \to 0$, $\epsilon_m \to 0$, $y_m \to y_0$ and $x_m \to x_0$ such that $\left| u^{\epsilon_m}_{\delta_m,y_m}(x_m) \right| \to \infty$. We write for simplicity $u_m = u^{\epsilon_m}_{\delta_m,y_m}$ and we have that u_m solves the equation

$$\begin{aligned} -\Delta u_m + \delta_m^2 V_{\delta_m,y_m}(x) u_m = n(n-2) \left(u_m^+ \right)^{(4/(n-2))-\epsilon_m} u_m \\ - \sum_{i=0}^{n} c_i(\epsilon_m, \delta_m, y_m) \Delta\psi_i. \end{aligned}$$

Since $u_m \to U$ in $\mathcal{D}^{1,2}(\mathbf{R}^n)$, we get that for any $\eta > 0$ there exist $R > 0$ and $m_0 > 0$ such that, for $m \ge m_0$,

$$\int_{B(x_m,4R)} \left(u_m^+ \right)^{((4/(n-2))-\epsilon_m)n/2} \le \eta.$$

By Remark 2.4 we have that

$$\left\| \sum_{i=0}^{n} c_i(\epsilon_m, \delta_m, y_m)\Delta\psi_i \right\|_{\mathrm{L}^\infty(\mathbf{R}^n)} \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

By the first claim of Lemma 2.5, we obtain $(u_m^+)^{(4/(n-2))-\epsilon_m} \in L^{q/2}(B(x_m, 2R))$ with $q/2 = (2^*)^2/(2(2^* - 2 - \epsilon_m)) > n/2$.

Thus,

$$\left\| \left(u_m^+\right)^{(4/(n-2))-\epsilon_m} \right\|_{L^{(q/2)(B(x_m,2R))}} \leq c \left\| u_m \right\|_{L^{2^*}(B(x_m,4R))},$$

where c depends only on R^{-1} and n.

By the second claim of Lemma 2.5, we obtain

$$|u_m(x_m)| \leq \sup_{B(x_m,R)} |u_m(x)| \leq C_m \left(\int_{B(x_m,2R)} (1 + |u_m|^2) \right)^{1/2}.$$

By Remark 2.4 and the fact that $u_m \to U$ in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ we get that the sequences $\{(\int_{B(x_m,2R)}(1 + |u_m|^2))^{1/2}\}$ C_m are bounded. This gives a contradiction.

Step 2. There exist $R > 0$, $C > 0$, $\delta_0 > 0$ and $\epsilon_0 > 0$ such that, for any $y \in K$, $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$

$$\left|u_{\delta,y}^{\epsilon}(x)\right| \leq CU(x), \quad \forall\, x \in \mathbf{R}^n,\ |x| \geq R.$$

Let $w_{\delta,y}^{\epsilon} \in \mathcal{D}^{1,2}(\mathbf{R}^n)$ be the Kelvin transform of $u_{\delta,y}^{\epsilon}$, see, for example, [**6**]:

$$w_{\delta,y}^{\epsilon}(x) = \frac{1}{|x|^{n-2}}\, u_{\delta,y}^{\epsilon}\left(\frac{x}{|x|^2}\right).$$

We want to prove that there exist $r > 0$, $C > 0$, $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for any $y \in K$, $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$

$$|w_{\delta,y}^{\epsilon}(x)| \leq C, \quad \forall\, x \in \mathbf{R}^n,\ |x| \leq r.$$

We recall that $w_{\delta,y}^{\epsilon}$ satisfies the equation

$$-\Delta w_{\delta,y}^{\epsilon} + a_{\delta,y}^{\epsilon}(x)\, w_{\delta,y}^{\epsilon} = b_{\delta,y}^{\epsilon}(x)\, w_{\delta,y}^{\epsilon} + c_{\delta,y}^{\epsilon}(x),$$

where

$$a^\epsilon_{\delta,y}(x) = \frac{\delta^2}{|x|^4}\, V_{\delta,y}\left(\frac{x}{|x|^2}\right),$$
$$b^\epsilon_{\delta,y}(x) = n(n-2)\,\frac{1}{|x|^{\epsilon(n-2)}}\,(w^+)^{(4/(n-2))-\epsilon},$$
$$c^\epsilon_{\delta,y}(x) = -\,\frac{1}{|x|^{n+2}}\sum_{i=0}^{n} c_i(\epsilon,\delta,y)\Delta\psi_i\left(\frac{x}{|x|^2}\right).$$

It is possible to see that $(1/(|x|^{n+2}))\Delta\psi_i(x/|x|^2) \in L^\infty(B(0,1))$ and therefore by Remark 2.4 we have that $\|c^\epsilon_{\delta,y}\|_{L^\infty(B(0,1))}$ tends to zero for $\epsilon \to 0$ and $\delta \to 0$ uniformly with respect to $y \in K$. Moreover, since $u^\epsilon_{\delta,y}(x) \to U(x)$ in X for $\epsilon \to 0$ and $\delta \to 0$, we can prove that for any $\eta > 0$ there exist $0 < r_0 < 1$, $\delta_0 > 0$ and $\epsilon_0 > 0$ such that for any $y \in K$, $\delta \in (0,\delta_0)$ and $\epsilon \in (0,\epsilon_0)$

$$\|b^\epsilon_{\delta,y}\|_{L^{(n/2)}(B(0,4r_0))} < \eta.$$

Since $u^\epsilon_{\delta,y}(x) \to U(x)$ in X for $\epsilon \to 0$ and $\delta \to 0$, it is possible to see that $w^\epsilon_{\delta,y}(x) \to U(x)$ in X for $\epsilon \to 0$ and $\delta \to 0$. At this point by the first claim of Lemma 2.5 there exists a constant C_3 such that, for any $y \in K$, $\delta \in (0,\delta_0)$ and $\epsilon \in (0,\epsilon_0)$

$$\|w^\epsilon_{\delta,y}\|_{L^{((2^*)^2/2)(B(0,2r_0))}} \le C_3.$$

Then we can verify that

$$b^\epsilon_{\delta,y} \in L^{(n+1)/2}(B(0,2r_0)),$$

and there exists a constant C_4 such that, for any $y \in K$, $\delta \in (0,\delta_0)$ and $\epsilon \in (0,\epsilon_0)$

$$\left\|b^\epsilon_{\delta,y}\right\|_{L^{((n+1)/2)(B(0,2r_0))}} \le C_4 \|{w^\epsilon_{\delta,y}}^+\|^{q_\epsilon}_{L^{((2^*)^2/2)(B(0,2r_0))}},$$

with $q_\epsilon = (2(n+1)/(n-2)) - \epsilon((n+1)/2)$. Concluding, since $w^\epsilon_{\delta,y}(x) \to U(x)$ in X for $\epsilon \to 0$ and $\delta \to 0$, there exists a constant C_5 such that, for any $y \in K$, $\delta \in (0,\delta_0)$ and $\epsilon \in (0,\epsilon_0)$

$$\left\|b^\epsilon_{\delta,y}\right\|_{L^{((n+1)/2)(B(0,2r_0))}} \le C_5. \tag{10}$$

Now we can apply the second claim of Lemma 2.5 and, by (10), there exists a constant C_6 such that, for any $y \in K$, $\delta \in (0, \delta_0)$ and $\epsilon \in (0, \epsilon_0)$

$$\sup_{B(0,r_0)} |w^{\epsilon}_{\delta,y}(x)| \leq C_6.$$

So we get the claim. □

Remark 2.4. The constants $c_i(\epsilon, \delta, y)$ in equation (9) tend to 0 for $\epsilon \to 0$ and $\delta \to 0$ uniformly with respect to y in compact sets of $\mathbf{R}^n$, see Lemma 3.2 and Lemma 3.3 of [**12**].

For the sake of completeness, we recall the following well-known result, see [**6, 7, 9, 15**].

Lemma 2.5. *Let $w \in H^1(\Omega)$ be a solution of equation*

$$-\Delta w + a(x)w = b(x)w + c(x) \tag{11}$$

where $c \in L^\infty(\Omega)$, $a, b \in L^{n/2}(\Omega)$ and $a(x) \geq 0$. There exists $\varepsilon_0 > 0$, depending only on n, such that if

$$\|b\|_{L^{(n/2)}(B(Q,2r))} < \varepsilon_0$$

for any $Q \in \mathbf{R}^n$, then

$$\|u\|_{L^{((2^*)^2/2)}(B(Q,r))} \leq C_1 \|u\|_{L^{2^*}(B(Q,2r))}, \tag{12}$$

where C_1 depends on n, $\|c\|_{L^\infty(\Omega)}$, r and is a bounded function of $\|c\|_{L^\infty(\Omega)}$.

Furthermore, if $b \in L^{q/2}(B(Q, 2r))$ with $q > n$, then

$$\sup_{B(Q,r)} |w(x)| \leq C_2 \left(\int_{B(Q,2r)} (1 + |w|^2) \right)^{1/2}, \tag{13}$$

where C_2 depends on n, $\|b\|_{L^{q/2}(\Omega)}$, $\|c\|_{L^\infty(\Omega)}$, r^{-1} and is a bounded function of $\|b\|_{L^{q/2}(\Omega)}$ and $\|c\|_{L^\infty(\Omega)}$.

3. A lower bound on the number of solutions. In this section we estimate the number of solutions to the second equation in system (8). More precisely, we estimate, for ϵ small enough, the numbers of points y_ϵ and the corresponding parameters δ_ϵ such that $c_i(\epsilon, \delta_\epsilon, y_\epsilon) = 0$ for any $i = 0, 1, \dots, n$, see (9).

Let us fix a critical point y_0 of V, and let us set

$$\delta^2 = \delta^2(\epsilon) = d_0\epsilon + \tilde{d}\epsilon^2, \text{ with } \tilde{d} \in \left[\eta, \frac{1}{\eta}\right] \text{ for some } \eta \in (0,1), \tag{14}$$

and

$$y = y_0 + \delta\tilde{y}, \text{ with } \tilde{y} \in K \text{ for some compact set } K \text{ in } \mathbf{R}^n. \tag{15}$$

Here

$$d_0 = -\frac{a_0}{V(y_0)} > 0$$

and

$$a_0 := n(n-2)\frac{\int_{\mathbf{R}^n} (\log U) U^p \psi_0}{\int_{\mathbf{R}^n} U\psi_0} < 0.$$

It is useful to point out that, with these choices, $\phi^\epsilon_{\delta,y} = \phi^\epsilon_{\tilde{d},\tilde{y}}$, see Proposition 2.2.

Let us make the following expansion.

Lemma 3.1. *Assume* (V_{y_0}) *with* $\overline{\alpha} < n-4$, *see* (3). *Then there holds*

$$\left(u^\epsilon_{\delta,y} - i^*\left[f_\epsilon\left(u^\epsilon_{\delta,y}\right) - \delta^2 V_{\delta,y}(x) u^\epsilon_{\delta,y}\right], \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2} = -\frac{1}{2}(d_0\epsilon)^{(3+\alpha_i)/2}\left(H_{y_0}(\tilde{y})\right)_i + o\big(\epsilon^{(3+\alpha_i)/2}\big). \tag{16}$$

Proof. It is easy to see that, for any $y \in \mathbf{R}^n$,

$$\left(u^\epsilon_{\delta,y} - i^*\left[f_\epsilon\left(u^\epsilon_{\delta,y}\right) - \delta^2 V_{\delta,y}(x) u^\epsilon_{\delta,y}\right], \frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2} = -\frac{\delta^3}{2}\int_{\mathbf{R}^n} \left.\frac{\partial V}{\partial z_i}\right|_{\delta x + y} \left(u^\epsilon_{\delta,y}\right)^2 dx.$$

By the hypotheses on the potential V, we obtain

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{\partial V}{\partial z_i}\bigg|_{y_0+\delta(x+\tilde{y})} \left(u^{\epsilon}_{\delta,y}\right)^2\, dx \\ &= \int_{B(-\tilde{y},(\rho/\delta))} \delta^{\alpha_i} h_i(x+\tilde{y}) \left(u^{\epsilon}_{\delta,y}\right)^2\, dx \\ &\quad + \int_{B(-\tilde{y},(\rho/\delta))} R_i(\delta(x+\tilde{y})) \left(u^{\epsilon}_{\delta,y}\right)^2\, dx \\ &\quad + \int_{\mathbf{R}^n\setminus B(-\tilde{y},(\rho/\delta))} \frac{\partial V}{\partial z_i}\bigg|_{y_0+\delta(x+\tilde{y})} \left(u^{\epsilon}_{\delta,y}\right)^2\, dx \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Now we can write

$$\begin{aligned}I_1 = \delta^{\alpha_i} &\int_{\mathbf{R}^n} h_i(x+\tilde{y})(U(x))^2\, dx \\ &- \delta^{\alpha_i} \int_{\mathbf{R}^n\setminus B(-\tilde{y},(\rho/\delta))} h_i(x+\tilde{y})(U(x))^2\, dx \\ &+ \delta^{\alpha_i} \int_{B(-\tilde{y},(\rho/\delta))} h_i(x+\tilde{y}) \left[2U\phi^{\epsilon}_{\delta,y} + \left(\phi^{\epsilon}_{\delta,y}\right)^2\right]\, dx.\end{aligned}$$

Since $\overline{\alpha} < n-4$, we have that

$$\begin{aligned}\int_{\mathbf{R}^n\setminus B(-\tilde{y},(\rho/\delta))} h_i(x+\tilde{y})(U(x))^2 &\le \int_{\mathbf{R}^n\setminus B(-\tilde{y},(\rho/\delta))} |x+\tilde{y}|^{\alpha_i}(U(x))^2 \\ &= o(1).\end{aligned}$$

By Lemma 2.3, choosing $\gamma > 0$ small enough, we have because $\overline{\alpha} < n-4$ that

$$\begin{aligned}\int_{B(-\tilde{y},(\rho/\delta))} h_i(x+\tilde{y}) \left[2U\phi^{\epsilon}_{\delta,y} + \left(\phi^{\epsilon}_{\delta,y}\right)^2\right]\, dx \\ &\le \int_{\mathbf{R}^n} |x+\tilde{y}|^{\alpha_i}(U(x))^{2-\gamma}|\phi|^{\gamma} \\ &\le \|\phi\|^{\gamma}_{\mathrm{L}^s}\left[\int_{\mathbf{R}^n} \left(|x+\tilde{y}|^{\alpha_i}(U(x))^{2-\gamma}\right)^{s/(s-\gamma)}\right]^{s-(\gamma/s)} \\ &= o(1).\end{aligned}$$

As regards the second integral, since $\overline{\alpha} < n-4$, we have

$$\begin{aligned} I_2 &= \int_{B(-\tilde{y},(\rho/\delta))} R_i(\delta(x+\tilde{y}))\left(u^{\epsilon}_{\delta,y}\right)^2\,dx \\ &\le C\delta^{\beta_i}\int_{B(-\tilde{y},(\rho/\delta))} |x+\tilde{y}|^{\beta_i}\,(U(x))^2\,dx \\ &= o(\delta^{\alpha_i}). \end{aligned}$$

Eventually

$$I_3 \le C\|\nabla V\|_{L^\infty(\mathbf{R}^n)}\int_{\mathbf{R}^n\setminus B(-\tilde{y},(r/\delta))}(U(x))^2\,dx = O(\delta^{n-4}),$$

and this completes the proof. □

By the previous result we deduce the following necessary condition.

Proposition 3.2. *Assume that* (V_{y_0}) *with* $\overline{\alpha} < n-4$, *see* (3). *Let* ϵ_k, $\tilde{d}_k$ *and* $\tilde{y}_k$ *be sequences such that* $\epsilon_k \to 0$, $\tilde{d}_k \to \tilde{d} > 0$ *and* $\tilde{y}_k \to \tilde{y}$. *If* $u_k := U + \phi^{\epsilon_k}_{\tilde{d}_k,\tilde{y}_k}$ *is a solution of* (7), *then* $H_{y_0}(\tilde{y}) = 0$.

Proof. If u_k is a solution of (7), by Lemma 3.1 we have

$$-\frac{1}{2}\,(d_0\epsilon_k)^{(3+\alpha_i)/2}\int_{\mathbf{R}^n} h_i(x+\tilde{y}_k)(U(x))^2\,dx + o\big(\epsilon_k^{(3+\alpha_i)/2}\big) = 0. \tag{17}$$

Since h_i is of class C^1 and homogeneous of degree α_i and $\overline{\alpha} < n-4$,

$$\begin{aligned} &\left|\int_{\mathbf{R}^n}\left[h_i(x+\tilde{y}_k) - h_i(x+\tilde{y})\right](U(x))^2\,dx\right| \\ &\qquad \le \int_{\mathbf{R}^n}|\nabla h_i(x+\tilde{y}_k+\theta(\tilde{y}-\tilde{y}_k))|\,|\tilde{y}-\tilde{y}_k|\,(U(x))^2\,dx \\ &\qquad \le C\,|\tilde{y}-\tilde{y}_k|\int_{\mathbf{R}^n}|x+\tilde{y}_k+\theta(\tilde{y}-\tilde{y}_k)|^{\alpha_i-1}\,(U(x))^2\,dx \\ &\qquad = O(|\tilde{y}-\tilde{y}_k|) \end{aligned}$$

for $k \to \infty$. By (17), we can conclude that

$$\int_{\mathbf{R}^n} h_i(x+\tilde{y})(U(x))^2\,dx + O(|\tilde{y}-\tilde{y}_k|) + o(1) = 0,$$

which completes the proof. □

Proof of Theorem 1.2. It follows from Proposition 3.2. □

The following nonexistence result holds, see also [**8**, Proposition 6.3].

Example 3.3. Let $V(x+y_0) \sim V(y_0)+x_1^3-x_1x_2x_3+x_3^3+\sum_{i=4}^n a_i x_i^{k_i}$ for $|x|$ small enough, where $a_i \in \mathbf{R}\setminus\{0\}$ and $k_i \in \mathbf{N}$. Then there is no solution to (1) blowing up and concentrating at y_0.

Proof. It holds

$$\begin{aligned}
(H_{y_0}(y))_1 &= 3Ay_1^2 + 3B - Ay_2y_3,\\
(H_{y_0}(y))_2 &= -\,Ay_1y_3\\
(H_{y_0}(y))_3 &= 3Ay_3^2 + 3B - Ay_1y_2,
\end{aligned}$$

where $A := \int_{\mathbf{R}^n} U^2(x)\,dx$ and $B := \int_{\mathbf{R}^n} x_1^2U^2(x)\,dx$, so that equation $H_{y_0}(y) = 0$ does not have any solutions. Therefore, the claim follows from Theorem 1.2. □

In the following we will prove the converse of Proposition 3.2.

Definition 3.4. We say that y is a stable zero of H_{y_0} if y is an isolated zero of H_{y_0} and there exists a neighborhood N of y such that $\deg\,(H_{y_0}, N, 0) \neq 0$.

Lemma 3.5. *Assume that* (V_{y_0}) *with* $\underline{\alpha} < n-5$, *see* (6).

Then there holds

$$\begin{aligned}
(18)\quad &\left(u_{\delta,y}^{\epsilon} - i^*\left[f_\epsilon\left(u_{\delta,y}^{\epsilon}\right) - \delta^2\, V_{\delta,y}(x)u_{\delta,y}^{\epsilon}\right], \psi_0\right)_{1,2}\\
&\qquad = \epsilon^2\left[\tilde{d}\,V(y_0)\int_{\mathbf{R}^n} U\psi_0 + A + B\psi(\tilde{y})\right] + o\big(\epsilon^2\big)\,,
\end{aligned}$$

where

$$A = n(n-2)\int_{\mathbf{R}^n}(\log U)^2\,U^p\psi_0 - \frac{2n(n+2)}{n-2}\int_{\mathbf{R}^n}U^{p-2}\phi_0^2\,\psi_0 + n(n-2)\int_{\mathbf{R}^n}(1+p\log U)\,U^{p-1}\phi_0\,\psi_0 + d_0V(y_0)\int_{\mathbf{R}^n}\phi_0\,\psi_0,$$

$$\psi(\tilde{y}) = \frac{d_0^2}{2}\sum_{i,j=1}^{n}\frac{\partial^2 V}{\partial x_i\partial x_j}(y_0)\int_{\mathbf{R}^n}(x_i+\tilde{y}_i)(x_j+\tilde{y}_j)\,U\psi_0$$

and $B=1$ *if* $\underline{\alpha}>1$ *and* $B=0$ *if* $\underline{\alpha}=1$.

Proof. See Appendix. □

Proposition 3.6. *Assume that* (V_{y_0}) *with* $\overline{\alpha}<n-4$*, see* (3) *and* $\underline{\alpha}<n-5$*, see* (6)*. Let* $\tilde{y}\in Z_{y_0}$*, see* (5)*. Then there exists* $\epsilon_0>0$ *such that, for any* $\epsilon\in(0,\epsilon_0)$ *there exist* $\tilde{y}_\epsilon$ *and* $\tilde{d}_\epsilon$*, with* $\tilde{y}_\epsilon\to\tilde{y}$ *and* $\tilde{d}_\epsilon\to\tilde{d}>0$*, such that* $u_\epsilon := U+\phi^\epsilon_{\tilde{d}_\epsilon,\tilde{y}_\epsilon}$ *is a solution of* (7)*.*

Proof. Using (14) and (15), the problem reduces to find for ϵ small enough $\tilde{d}\in\mathbf{R}$ and $\tilde{y}\in\mathbf{R}^n$ such that the constants $c_i(\epsilon,\delta,y)$ for $i=0,1,\dots,n$ in (9) are zero, i.e.,

$$G^0_\epsilon(\tilde{d},\tilde{y}) = \frac{1}{\epsilon^2}\left(u^\epsilon_{\delta,y} - i^*\left[f_\epsilon\left(u^\epsilon_{\delta,y}\right) - \delta^2\,V_{\delta,y}(x)u^\epsilon_{\delta,y}\right],\psi_0\right)_{1,2} = 0,$$

$$G^i_\epsilon(\tilde{d},\tilde{y}) = -\frac{2}{(d_0\epsilon)^{(3+\alpha_i)/2}}\left(u^\epsilon_{\delta,y} - i^*\left[f_\epsilon\left(u^\epsilon_{\delta,y}\right) - \delta^2\,V_{\delta,y}(x)u^\epsilon_{\delta,y}\right],\frac{\partial u^\epsilon_{\delta,y}}{\partial x_i}\right)_{1,2} = 0,$$

for $i=1,2,\dots,n$. By Lemmas 3.5 and 3.1, we have that

$$G^0_\epsilon(\tilde{d},\tilde{y}) = \left(\tilde{d}\,V(y_0)\int_{\mathbf{R}^n}U\psi_0 + A + B\psi(\tilde{y})\right) + o(1),$$

$$G^i_\epsilon(\tilde{d},\tilde{y}) = \int_{\mathbf{R}^n}h_i(x+\tilde{y})(U(x))^2\,dx + o(1).$$

We apply Lemma 3.7. If $\tilde{y}_0$ is a stable zero of the vector field H_{y_0}, see (4), then $(\tilde{d}_0, \tilde{y}_0)$ is a stable zero of the vector field $G_0 = (G_0^0, G_0^1, \dots, G_0^n)$, where $\tilde{d}_0 = -(A + B\psi(\tilde{y}_0))/(V(y_0)\int_{\mathbf{R}^n} U\psi_0)$. Using the properties of Brouwer degree, for ϵ small enough there exist $\tilde{d}_\epsilon$ and $\tilde{y}_\epsilon$ such that

$$G_\epsilon(\tilde{d}_\epsilon, \tilde{y}_\epsilon) = \Big(G_\epsilon^0(\tilde{d}_\epsilon, \tilde{y}_\epsilon), G_\epsilon^1(\tilde{d}_\epsilon, \tilde{y}_\epsilon), \dots, G_\epsilon^n(\tilde{d}_\epsilon, \tilde{y}_\epsilon)\Big) = 0.$$

We recall that

$$G_\epsilon^0(\tilde{d}_\epsilon, \tilde{y}_\epsilon) = \frac{1}{\epsilon^2}\, c_0(\epsilon, \delta, y)$$

and, for $i = 1, 2, \dots, n$,

$$G_\epsilon^i(\tilde{d}_\epsilon, \tilde{y}_\epsilon) = -\frac{2}{(d_0\epsilon)^{(3+\alpha_i)/2}}\left[c_i(\epsilon, \delta, y) + o\left(\sum_{j=0}^n c_j(\epsilon, \delta, y)\right)\right].$$

Therefore $c_i(\epsilon, \delta, y) = 0$ for all $i = 0, 1, \dots, n$. That proves our claim. □

Proof of Theorem 1.3. We use Proposition 3.6, so it remains only to prove that two different stable zeros $\tilde{y}_0^1$, $\tilde{y}_0^2$ generate two different solutions. Let $u^{\epsilon,1}_{\delta_{\epsilon_1}, y_{\epsilon_1}}$, $u^{\epsilon,2}_{\delta_{\epsilon_2}, y_{\epsilon_2}}$ be the solutions of (7) generated respectively by $\tilde{y}_0^1$ and $\tilde{y}_0^2$. For $i = 1, 2$, let

$$w^{\epsilon,i}_{\delta_{\epsilon_i}, y_{\epsilon_i}}(z) = \delta_{\epsilon_i}^{-2/(p-1-\epsilon)}\, u^{\epsilon,i}_{\delta_{\epsilon_i}, y_{\epsilon_i}}\left(\frac{z - y_{\epsilon_i}}{\delta_{\epsilon_i}}\right)$$

be the corresponding solutions of (1), where $\delta_{\epsilon_i}^2 = \epsilon d_0 + \tilde{d}_{\epsilon_i}\epsilon^2$, $y_{\epsilon_i} = y_0 + \delta_{\epsilon_i}\tilde{y}_{\epsilon_i}$ and $\tilde{y}_{\epsilon_i} \to \tilde{y}_0^i$. It holds

$$\epsilon^{1/(p-1-\epsilon)}\left(d_0 + \tilde{d}_{\epsilon_1}\epsilon\right)^{1/(p-1-\epsilon)} w^{\epsilon,1}_{\delta_{\epsilon_1}, y_{\epsilon_1}}(y_{\epsilon_1}) = u^{\epsilon,1}_{\delta_{\epsilon_1}, y_{\epsilon_1}}(0) \to U(0),$$

$$\begin{aligned}\epsilon^{1/(p-1-\epsilon)}\left(d_0 + \tilde{d}_{\epsilon_2}\epsilon\right)^{1/(p-1-\epsilon)} & w^{\epsilon,2}_{\delta_{\epsilon_2}, y_{\epsilon_1}}(y_{\epsilon_1}) \\ &= u^{\epsilon,2}_{\delta_{\epsilon_2}, y_{\epsilon_2}}(\tilde{y}_{\epsilon_1} - \tilde{y}_{\epsilon_2}) \to U(\tilde{y}_0^1 - \tilde{y}_0^2) \neq U(0),\end{aligned}$$

because by standard regularity theory $u^{\epsilon,i}_{\delta_{\epsilon_i},y_{\epsilon_i}} \to U$ uniformly on compact sets of $\mathbf{R}^n$. Finally, we deduce that, for ϵ small enough, $w^{\epsilon,1}_{\delta_{\epsilon_1},y_{\epsilon_1}}(y_{\epsilon_1}) \neq w^{\epsilon,2}_{\delta_{\epsilon_2},y_{\epsilon_2}}(y_{\epsilon_1})$, which implies, using again the standard regularity theory, that $w^{\epsilon,1}_{\delta_{\epsilon_1},y_{\epsilon_1}} \neq w^{\epsilon,2}_{\delta_{\epsilon_2},y_{\epsilon_2}}$. That proves our claim. □

Lemma 3.7. *Let $\tilde{y}_0$ be a stable zero of the vector field H_{y_0}, $\gamma \in \mathbf{R}$, $\psi : \mathbf{R}^n \to \mathbf{R}$ be a continuous function and*

$$\tilde{d}_0 = \frac{-\gamma - \psi(\tilde{y}_0)}{V(y_0)\int_{\mathbf{R}^n} U\psi_0}.$$

Then $(\tilde{d}_0, \tilde{y}_0) \in \mathbf{R} \times \mathbf{R}^n$ is a stable zero of the following vector field $G : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$:

$$G(\tilde{d}, \tilde{y}) := \big(h(\tilde{d}) + \psi(\tilde{y}), H_{y_0}(\tilde{y})\big),$$

where

$$h(\tilde{d}) := V(y_0)\tilde{d}\int_{\mathbf{R}^n} U\psi_0 + \gamma.$$

Proof. Let $\mathcal{H} : [0,1] \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$ be the homotopy defined by

$$\mathcal{H}t, \tilde{d}, \tilde{y}) = \big(h(\tilde{d}) + t\psi(\tilde{y}), H_{y_0}(\tilde{y})\big).$$

It is easy to check that $\mathcal{H}(t, \tilde{d}, \tilde{y}) \neq 0$ for any $t \in [0,1]$ and for any $(\tilde{d}, \tilde{y}) \in \partial(I \times N)$, where I and N are neighborhoods of $\tilde{d}_0$ and $\tilde{y}_0$, respectively. By homotopy invariance of the degree, we get

$$\deg(G, I \times N, 0) = \deg(h, I, 0) \cdot \deg(H_{y_0}, N, 0) = \deg(H_{y_0}, N, 0). \qquad \square$$

4. Exact number of solutions.

Proof of Theorem 1.4. We apply Theorem 1.3, and it remains only to prove the following uniqueness result. Let $\tilde{y}$ be such that $H_{y_0}(\tilde{y}) = 0$. By contradiction, assume that for some sequence $\epsilon_j \to 0$

there exist two different solutions $w_{1,j}$ and $w_{2,j}$ to equation (1) such that $w_{i,j} \to U_{\delta_{i,j},y_{i,j}}$ in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ where

$$\begin{aligned} \delta_{i,j}^2 &= d_0\epsilon_j + \hat{d}_{i,j}\epsilon_j && \text{where} \quad d_j \to 0, \\ y_{i,j} &= y_0 + \delta_{i,j}\tilde{y} + \delta_{i,j}\hat{y}_{i,j} && \text{where} \quad \hat{y}_{i,j} \to 0. \end{aligned} \tag{19}$$

We set, for $i = 1, 2$,

$$u_{i,j}(x) = (d_0\epsilon_j)^{1/(p-1-\epsilon_j)}\, w_{i,j}\big((d_0\epsilon_j)^{1/2}\, x + y_0\big). \tag{20}$$

They are solutions to equation

$$-\Delta u_{i,j} + d_0\epsilon_j\, V_{(d_0\epsilon_j)^{1/2},y_0}(x)\, u_{i,j} = u_{i,j}^{p-\epsilon_j}, \tag{21}$$

with the property $u_{i,j}(x) \to U(x-\tilde{y})$ in $\mathcal{D}^{1,2}(\mathbf{R}^n)$. By (20) and by Lemma 2.5 we deduce that, for some positive constant c,

$$0 \le u_{i,j}(x) \le c\, \frac{1}{\big(1+|x-\tilde{y}|^2\big)^{(n-2)/2}}, \quad \forall\, x \in \mathbf{R}^n. \tag{22}$$

We set

$$v_j(x) := \frac{u_{1,j}(x) - u_{2,j}(x)}{\|u_{1,j} - u_{2,j}\|_{\mathcal{D}^{1,2}}}. \tag{23}$$

Then we have

$$-\Delta v_j + d_0\epsilon_j\, V_{(d_0\epsilon_j)^{1/2},y_0}(x) v_j = \rho_j(x) v_j, \tag{24}$$

where

$$\rho_j(x) := (p-\epsilon_j)\int_0^1 [t\, u_{1,j}(x) + (1-t)\, u_{2,j}(x)]^{p-1-\epsilon_j}\, dt. \tag{25}$$

By (22) we deduce that

$$0 \le \rho_j(x) \le c\, \frac{1}{\big(1+|x-\tilde{y}|^2\big)^2}, \quad \forall\, x \in \mathbf{R}^n. \tag{26}$$

Up to a subsequence, we can assume that $v_j \to v$ weakly in $\mathcal{D}^{1,2}(\mathbf{R}^n)$ and almost everywhere in $\mathbf{R}^n$. Moreover, by (24) we deduce that v is a solution to

$$-\Delta v(x) = p\, U^{p-1}(x-\tilde{y})v(x), \quad x \in \mathbf{R}^n. \tag{27}$$

Then there exist real numbers $\gamma_0, \gamma_1, \dots, \gamma_j$ such that

$$v(x) = \sum_{k=1}^{n} \gamma_j \frac{\partial U}{\partial x_k}(x-\tilde{y}) + \gamma_0 \frac{1-|x-\tilde{y}|^2}{(1+|x-\tilde{y}|^2)^{n/2}}. \tag{28}$$

First of all we verify that $\gamma_0 = 0$. We multiply (21) by $x \cdot \nabla u_{i,j} + ((n-2)/2)u_{i,j}$, and we get for $i = 1, 2$,

$$\begin{aligned}
0 &= (n-1)\int_{\mathbf{R}^n} |\nabla u_{i,j}|^2(x)\, dx \\
&\quad - d_0\epsilon_j \int_{\mathbf{R}^n} u_{i,j}^2(x)\left(V_{(d_0\epsilon_j)^{1/2},y_0}(x) + \frac{1}{2}\, x\cdot\nabla V_{(d_0\epsilon_j)^{1/2},y_0}(x)\right) dx \\
&\quad - \left(\frac{n-2}{2} - \frac{n}{p+1-\epsilon_j}\right)\int_{\mathbf{R}^n} u_{i,j}^{p+1-\epsilon_j}(x)\, dx,
\end{aligned}$$

and then

$$\begin{aligned}
0 &= (n-1)\int_{\mathbf{R}^n} \nabla v_j \nabla(u_{1,j}+u_{2,j}) \\
&\quad - d_0\epsilon_j \int_{\mathbf{R}^n} v_j(u_{1,j}+u_{2,j})\left(V_{(d_0\epsilon_j)^{1/2},y_0} + \frac{1}{2}\, x\cdot\nabla V_{(d_0\epsilon_j)^{1/2},y_0}\right) \\
&\quad - \left(\frac{n-2}{2} - \frac{n}{p+1-\epsilon_j}\right)\int_{\mathbf{R}^n} \left(u_{1,j}^{p+1-\epsilon_j} - u_{2,j}^{p+1-\epsilon_j}\right) \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{29}$$

By Hölder's inequality and (22), we get

$$\begin{aligned}
|I_2| &\le d_0\epsilon_j \|v_j\|_{\mathrm{L}^{(2n)/(n-2)}} \|u_{1,j}+u_{2,j}\|_{\mathrm{L}^{(2n)/(n+2)}} \left(\|V + x\cdot\nabla V\|_{\mathrm{L}^\infty}\right) \\
&= o(1)
\end{aligned} \tag{30}$$

and also

$$\begin{aligned}
|I_3| &= \left(\frac{n-2}{2} - \frac{n}{p+1-\epsilon_j}\right)\int_{\mathbf{R}^n} v_j(x)\hat{\rho}_j(x) \\
&\le \left(\frac{n-2}{2} - \frac{n}{p+1-\epsilon_j}\right)\|v_j\|_{\mathrm{L}^{(2n)/(n-2)}} \|\hat{\rho}_j\|_{\mathrm{L}^{(2n)/(n+2)}} = o(1),
\end{aligned} \tag{31}$$

where, because of (22),

$$
\begin{aligned}
0 \le \hat\rho_j(x) &= (p+1-\epsilon_j)\int_0^1 \left(t\,u_{1,j} + (1-t)\,u_{2,j}\right)^{p-\epsilon_j}\,dt\\
&\le c\,\frac{1}{\left(1+|x-\tilde y|^2\right)^{(n+2)/2}}.
\end{aligned}
$$

Since $v_j \to v$ weakly in $\mathcal{D}^{1,2}$ and $u_{i,j} \to U(x-\tilde y)$ strongly in $\mathcal{D}^{1,2}$, by (28), (29), (30) and (31) we deduce

$$
\begin{aligned}
0 &= \int_{\mathbf{R}^n} \nabla v(x) \nabla U(x-\tilde y)\,dx = \int_{\mathbf{R}^n} v(x)\,U^p(x-\tilde y)\,dx\\
&= \gamma_0 \int_{\mathbf{R}^n} \psi_0(x)\,U^p(x)\,dx,
\end{aligned}
$$

which implies $\gamma_0 = 0$.

In the following we will show that, if the determinant of the Jacobian matrix of H_{y_0} is different from zero, then $\gamma_1 = \cdots = \gamma_n = 0$.

We multiply (21) by $\partial u_{i,j}/\partial x_k$ and we get for $i = 1, 2$,

$$
0 = \int_{\mathbf{R}^n} u_{i,j}^2(x)\,\frac{\partial V}{\partial x_k}\left((d_0\epsilon_j)^{1/2}x + y_0\right)dx.
$$

Then, using also assumption (V_{y_0}), we get

$$
\begin{aligned}
0 &= \int_{\mathbf{R}^n} v_j(x)\big(u_{1,j}(x)+u_{2,j}(x)\big)\,\frac{\partial V}{\partial x_k}\left((d_0\epsilon_j)^{1/2}x+y_0\right)dx\\
&= (d_0\epsilon_j)^{\alpha_k/2} \int_{\{|x|\le R(d_0\epsilon_j)^{-1/2}\}} v_j(x)\big(u_{1,j}(x)+u_{2,j}(x)\big)h_k(x)\,dx\\
&\quad + \int_{\{|x|\le R(d_0\epsilon_j)^{-1/2}\}} v_j(x)\big(u_{1,j}(x)+u_{2,j}(x)\big)R_k(x)\,dx\\
&\quad + \int_{\{|x|\ge R(d_0\epsilon_j)^{-1/2}\}} v_j(x)\big(u_{1,j}(x)+u_{2,j}(x)\big)\\
&\qquad\qquad \times \frac{\partial V}{\partial x_k}\left((d_0\epsilon_j)^{1/2}x+y_0\right)dx\\
&= (d_0\epsilon_j)^{\alpha_k/2}\left[2\int_{\mathbf{R}^n} v(x)\,U(x-\tilde y)h_k(x)\,dx + o(1)\right].
\end{aligned}
$$

Therefore, we have for any $h = 1, \dots, n$

$$0 = \int_{\mathbf{R}^n} v(x)\, U(x-\tilde{y}) h_k(x)\, dx = \sum_{l=1}^{n} \gamma_l \int_{\mathbf{R}^n} \frac{\partial U}{\partial x_l}\,(x-\tilde{y})\, U(x-\tilde{y}) h_k(x)\, dx,$$

which implies that $\gamma_1 = \cdots = \gamma_n = 0$.

Let us prove that a contradiction arises. We multiply (24) by v_j and, taking into account that $\|v_j\|_{\mathcal{D}^{1,2}} = 1$ and also that $V > 0$ in $\mathbf{R}^n$, we get

$$\begin{aligned} 1 &= \int_{\mathbf{R}^n} |\nabla v_j|^2(x)\, dx \\ &= -\, d_0\epsilon_j \int_{\mathbf{R}^n} V_{(d_0\epsilon_j)^{1/2}, y_0}(x) v_j^2(x)\, dx + \int_{\mathbf{R}^n} \rho_j(x) v_j^2(x)\, dx \\ &\le \int_{\mathbf{R}^n} \rho_j(x) v_j^2(x)\, dx \\ &= \int_{\{|x| \le R\}} \rho_j(x) v_j^2(x)\, dx + \int_{\{|x|\ge R\}} \rho_j(x) v_j^2(x)\, dx, \end{aligned} \tag{32}$$

for some $R > 0$. By (26) and Hölder's inequality, we deduce that there exists an $R > 0$ such that, for any j,

$$\int_{\{|x|\ge R\}} \rho_j(x) v_j^2(x)\, dx \le \frac{1}{2}. \tag{33}$$

Moreover, since v_j solves equation (24) and it is bounded in $\mathcal{D}^{1,2}(\mathbf{R}^n)$, by standard regularity theory we deduce that $v_j \to 0$ uniformly on compact sets of $\mathbf{R}^n$, and so

$$\lim_j \int_{\{|x|\le R\}} \rho_j(x) v_j^2(x)\, dx = 0. \tag{34}$$

Finally by (32), (33) and (34) a contradiction arises. $\square$

The following uniqueness result holds, see [**8**, Corollary 6.4].

Example 4.1. Assume that y_0 is a nondegenerate critical point of V. Then there exists exactly one solution of (1) blowing up and concentrating at y_0.

We give also the following example.

Example 4.2. Let $V(x+y_0) \sim V(y_0)+ax_1^4+x_2^4-bx_1^2x_2^2+x_3^2+\cdots+x_n^2$ for $|x|$ small enough, with $a \in (0,1)$ and $b \in (6a,6)$. Then there exist exactly three solutions of (1) blowing up and concentrating at y_0. In particular, if $a \in (0,1/9)$ and $b \in (6a, 2\sqrt{a})$, there exist exactly three solutions of (1) blowing up and concentrating at y_0, which is a local minimum point of V.

Proof. It is easy to check that

$$\begin{aligned}
(H_{y_0}(y))_1 &= 4a\,Ay_1^3 + 2B(6a-b)y_1 - 2b\,Ay_1\,y_2^2,\\
(H_{y_0}(y))_2 &= 4Ay_2^3 + 2B(6-b)y_2 - 2b\,Ay_1^2\,y_2\\
(H_{y_0}(y))_i &= 2Ay_i \quad \text{if} \quad i=3,\dots,n,
\end{aligned}$$

where $A := \int_{\mathbf{R}^n} U^2(x)\,dx$ and $B := \int_{\mathbf{R}^n} x_1^2\,U^2(x)\,dx$. Since $a<1$ and $b \in (2a,2)$, a straightforward computation shows that $Z_{y_0} = \{0, (\zeta,0), (-\zeta,0)\}$, where ζ and $-\zeta$ solve the equation $2aA\zeta^2 = B(b-2a)$. Finally, it is not difficult to prove that

$$\det Jac\,H_{y_0}(0) = 4(2A)^{n-2}B^2\,\frac{b^2-4ab-12a}{a} \neq 0$$

and

$$\det Jac\,H_{y_0}\big((\pm\zeta,0)\big) = 4(2A)^{n-2}B^2(6a-b)(6-b) \neq 0.$$

Therefore the claim follows from Theorem 1.4. $\square$

Appendix

We recall the following result, see [**12**, Lemma 2.2 and Remark 2.9].

Lemma 6.1. *Let $s > (n/(n-2))$. If $u \in L^{(2n)/(n+2)}(\mathbf{R}^n) \cap L^{(ns)/(n+2s)}(\mathbf{R}^n)$, then $i^*(u) \in L^s(\mathbf{R}^n) \cap \mathcal{D}^{1,2}(\mathbf{R}^n)$ and*

$$\|i^*(u)\|_X \leq C(n,s)\left(\|u\|_{L^{(ns)/(n+2s)}} + \|u\|_{L^{(2n)/(n+2)}}\right).$$

The following technical lemmas provide useful estimates.

Lemma 6.2. *There exist $\epsilon_0 > 0$ and a constant $C > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$, there hold*

$$\begin{aligned} \|f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\|_{L^{(2n)/(n+2)}(\mathbf{R}^n)} &= o(\epsilon), \\ \|f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\|_{L^{(sn)/(n+2s)}(\mathbf{R}^n)} &= o(\epsilon), \end{aligned} \tag{35}$$

for any $s > 1$ and

$$\|f'_\epsilon(U) - f'_0(U)\|_{L^{(n/2)}(\mathbf{R}^n)} \le C\epsilon. \tag{36}$$

Proof. By the mean value theorem we get, for any $x \in \mathbf{R}^n$,

$$\begin{aligned} f_\epsilon(U)(x) - f_0(U)(x) = &- \epsilon\, n(n-2)(\log U(x))(U(x))^p \\ &- \epsilon\, n(n-2)(\log U(x))(U(x))^p \left[(U(x))^{-\theta_x \epsilon} - 1\right], \end{aligned}$$

for some $\theta_x \in (0,1)$. Estimate (35) follows since $(\log U)U^p \in L^t(\mathbf{R}^n)$ for all $t > (n/(n+2))$ and, since $|(U(x))^{-\theta_x\epsilon} - 1| \le c|x|^{\varepsilon_1}$ for ε_1 small enough, $(\log U(x))(U(x))^p \left[(U(x))^{-\theta_x\epsilon} - 1\right] \in L^t(\mathbf{R}^n)$ for all $t > (n/(n+2)) + \varepsilon_2$ for ε_2 small enough.

By the mean value theorem we get for any $x \in \mathbf{R}^n$

$$\begin{aligned} &f'_\epsilon(U)(x) - f'_0(U)(x) \\ &\quad = - \epsilon\, n(n-2) \left[U^{p-1-\theta_x\epsilon}(x) + (p - \theta_x\epsilon)(\log U(x))\, U^{p-1-\theta_x\epsilon}(x)\right], \end{aligned}$$

for some $\theta_x \in (0,1)$. Estimate (35) follows since

$$U^{p-1-\theta_x\epsilon} + (\log U)\, U^{p-1-\theta_x\epsilon} \in L^t(\mathbf{R}^n) \quad \text{for all} \quad t \ge \frac{n}{2}. \qquad \square$$

Lemma 6.3. *There exist $\epsilon_0 > 0$ and a constant $C > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and for any $\phi \in \mathcal{D}^{1,2}(\mathbf{R}^n)$*

$$|f_\epsilon(U + \phi) - f_\epsilon(U) - f'_\epsilon(U)\phi| \le C|\phi|^{p-\epsilon}. \tag{37}$$

Proof. It is enough to point out that there exist $\epsilon_0 > 0$ and a constant $C > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$ and for any $x_1, x_2 \in \mathbf{R}$,

$$|f_\epsilon(x_1) - f_\epsilon(x_2) - f'_\epsilon(x_2)(x_1 - x_2)| \leq C(x_1 - x_2)^{p-\epsilon}. \qquad \square$$

Lemma 6.4. *Let $\tilde{d}$ and $\tilde{y}$ be fixed as in* (14) *and* (15)*. Then we have*

$$\phi^\epsilon_{\delta,y} = \epsilon \left(\phi_0 + \widetilde{\phi}^\epsilon_{\tilde{d},\tilde{y}}\right),$$

where $\phi_0 \in K^\perp$ is the unique solution of

$$-\Delta\phi_0 - f'_0(U)\phi_0 = -\, n(n-2)(\log U)\, U^{(n+2)/(n-2)} - d_0 V(y_0)\, U. \tag{38}$$

Moreover,

$$\left\|\widetilde{\phi}^\epsilon_{\tilde{d},\tilde{y}}\right\|_X \longrightarrow 0 \quad \textit{as } \epsilon \textit{ goes to zero.}$$

Proof. By Proposition 2.2 we have

$$\begin{aligned}\Pi\left\{\phi^\epsilon_{\delta,y} - i^*\left[f'_0(U)\phi^\epsilon_{\delta,y})\right]\right\} = \Pi\left\{i^*\left[f_\epsilon(U + \phi^\epsilon_{\delta,y}) - f_0(U) - f'_0(U)\phi^\epsilon_{\delta,y}\right.\right.\\ \left.\left. - \delta^2\, V_{\delta,y}(U + \phi^\epsilon_{\delta,y})\right]\right\}.\end{aligned}$$

For simplicity we write $\phi = \epsilon\big(\phi_0 + \widetilde{\phi}\big)$ instead of $\phi^\epsilon_{\delta,y} = \epsilon\big(\phi_0 + \widetilde{\phi}^\epsilon_{\tilde{d},\tilde{y}}\big)$. Substituting, we obtain

$$\begin{aligned}&\epsilon\,\Pi\left\{[\mathrm{Id} - i^* f'_0(U)]\big(\phi_0 + \widetilde{\phi}\big)\right\}\\ &\quad = \Pi i^*\left[f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\right] + \Pi i^*\left[f_\epsilon(U) - f_0(U)\right]\\ &\qquad + \Pi i^*\left\{\left[f'_\epsilon(U) - f'_0(U)\right]\phi\right\} - \epsilon\,(d_0 + \tilde{d}\epsilon)\Pi i^*\left[V_{\delta,y}(U+\phi)\right].\end{aligned} \tag{39}$$

Now by Lemma 6.1, (37) and using interpolation, we get

$$\begin{aligned}&\|i^*\left[f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\right]\|_{1,2}\\ &\qquad \leq C\|f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\|_{L^{(2n)/(n+2)}(\mathbf{R}^n)}\\ &\qquad \leq C\||\phi|^{p-\epsilon}\|_{L^{(2n)/(n+2)}(\mathbf{R}^n)} \leq C\|\phi\|^{p-\epsilon}_X,\end{aligned} \tag{40}$$

because $((2n)/(n+2))(p-\epsilon) \in (s,(2n)/(n-2))$ for ϵ small enough, and also

$$\begin{aligned} &\|i^*\left[f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\right]\|_{L^s(\mathbf{R}^n)} \\ &\qquad\qquad \leq C\|f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\|_{L^{(sn)/(n+2s)}(\mathbf{R}^n)} \\ &\qquad\qquad \leq C\||\phi|^{p-\epsilon}\|_{L^{(sn)/(n+2s)}(\mathbf{R}^n)} \leq C\|\phi\|_X^{p-\epsilon}, \end{aligned} \tag{41}$$

because $((sn)/(n+2s))(p-\epsilon) \in (s,(2n)/(n-2))$. Therefore, by (40) and (41) we deduce that

$$\|i^*\left[f_\epsilon(U+\phi) - f_\epsilon(U) - f'_\epsilon(U)\phi\right]\|_X \leq C\|\phi\|_X^{p-\epsilon}. \tag{42}$$

By Lemma 6.1 and (35) we get

$$\begin{aligned} &\|i^*\left[f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\right]\|_{1,2} \\ &\qquad \leq C\,\|f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\|_{L^{(2n)/(n+2)}(\mathbf{R}^n)} \\ &\qquad = o(\epsilon), \end{aligned} \tag{43}$$

$$\begin{aligned} &\|i^*\left[f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\right]\|_{L^s(\mathbf{R}^n)} \\ &\qquad \leq C\,\|f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\|_{L^{(sn)/(n+2s)}(\mathbf{R}^n)} \\ &\qquad = o(\epsilon). \end{aligned} \tag{44}$$

Therefore, by (43) and (44) we deduce that

$$\|i^*\left[f_\epsilon(U) - f_0(U) + \epsilon\, n(n-2)(\log U)\, U^p\right]\|_X = o(\epsilon). \tag{45}$$

By Lemma 6.1 and (36) we get

(46)

$$\begin{aligned} \|i^*\left[(f'_\epsilon(U) - f'_0(U))\,\phi\right]\|_{1,2} &\leq C\,\|(f'_\epsilon(U) - f'_0(U))\,\phi\|_{L^{(2n)/(n+2)}(\mathbf{R}^n)} \\ &\leq C\epsilon\,\|\phi\|_{L^{(2n)/(n-2)}(\mathbf{R}^n)} \end{aligned}$$

and also

(47)

$$\begin{aligned} \|i^*\left[(f'_\epsilon(U) - f'_0(U))\,\phi\right]\|_{L^s(\mathbf{R}^n)} &\leq C\,\|(f'_\epsilon(U) - f'_0(U))\,\phi\|_{L^{(sn)/(n+2s)}(\mathbf{R}^n)} \\ &\leq C\epsilon\,\|\phi\|_{L^s(\mathbf{R}^n)}. \end{aligned}$$

Therefore, by (46) and (47) we deduce that

$$\|i^*\left[\left(f'_\epsilon(U) - f'_0(U)\right)\phi\right]\|_X \leq C\epsilon\|\phi\|_X. \tag{48}$$

Finally, by the fact that $U \in L^{(2n)/(n+2)}(\mathbf{R}^n)$, $U \in L^{(sn)/(9n+2s)}(\mathbf{R}^n)$ for $s > (n/(n-4))$, $U(x)|x|^2 \in L^{(2n)/(n+2)}(\mathbf{R}^n)$, $U(x)|x|^2 \in L^{(sn)/(n+2s)}(\mathbf{R}^n)$ for $s > (n/(n-4))$, and since

$$\begin{aligned} &V\left(y_0 + \delta\left(x + \tilde{y}\right)\right) \\ &\qquad = V(y_0) + \frac{1}{2}\,\delta^2 \sum_{i,j=1}^{n} \left.\frac{\partial^2 V}{\partial x_i \partial x_j}\right|_{y_0+\theta\delta(x+\tilde{y})} (x+\tilde{y})_i\,(x+\tilde{y})_j \end{aligned}$$

with $\theta \in (0,1)$, we get

$$\left\| i^*\left[\big(d_0 + \tilde{d}\epsilon\big)V\big(y_0 + \delta(x+\tilde{y})\big)(U+\phi) - d_0 V(y_0)\,U\right]\right\|_X = o(1). \tag{49}$$

Concluding, by (39), (42), (45), (48) and (49),

$$\begin{aligned} \phi_0 - i^* f'_0(U)\phi_0 &= i^*\left[n(n-2)(\log U)\,U^p - d_0 V(y_0)\,U\right], \\ \widetilde{\phi} - i^* f'_0(U)\widetilde{\phi} &= \gamma_\epsilon, \end{aligned}$$

where γ_ϵ depends on $\tilde{d}$, $\tilde{y}$, ϕ, but $\|\gamma_\epsilon\|_X = o(1)$. Here we used the fact that

$$i^*\left[-\,n(n-2)(\log U)\,U^p - d_0 V(y_0)\,U\right] \in K^\perp. \qquad \square$$

Proof of Lemma 3.5. We point out that Lemma 6.4 plays a crucial role in the proof. By summing and subtracting, using definition of d_0, we can write

$$\begin{aligned} &\left(u^\epsilon_{\delta,y} - i^*\left[f_\epsilon\left(u^\epsilon_{\delta,y}\right) - \delta^2\,V_{\delta,y}u^\epsilon_{\delta,y}\right], \psi_0\right)_{1,2} \\ &\qquad = \int_{\mathbf{R}^n}\left[f_0(U) - f_\epsilon(U) - \epsilon\,n(n-2)(\log U)\,U^p\right]\psi_0 \\ &\qquad\quad - \int_{\mathbf{R}^n}\left[f_\epsilon\left(U + \phi^\epsilon_{\delta,y}\right) - f_\epsilon(U) - f'_\epsilon(U)\phi^\epsilon_{\delta,y}\right]\psi_0 \\ &\qquad\quad - \int_{\mathbf{R}^n}\left[f'_\epsilon(U) - f'_0(U)\right]\phi^\epsilon_{\delta,y}\,\psi_0 + \int_{\mathbf{R}^n}\left[\delta^2\,V_{\delta,y} - \epsilon d_0 V(y_0)\right]U\psi_0 \\ &\qquad\quad + \delta^2\int_{\mathbf{R}^n} V_{\delta,y}\phi^\epsilon_{\delta,y}\,\psi_0 := I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Let us write for simplicity in this proof $\phi^{\epsilon}_{\delta,y} = \phi = \epsilon\bigl(\phi_0 + \widetilde{\phi}\bigr)$.

(I) By the Taylor formula, we have

$$I_1 = \epsilon^2\, n(n-2) \int_{\mathbf{R}^n} (\log U)^2\, U^p\, \psi_0 + o\bigl(\epsilon^2\bigr). \tag{50}$$

(II) The integral

$$I_2 = -\,n(n-2) \int_{\mathbf{R}^n} \Bigl\{ \Bigl[(U+\phi)^+ \Bigr]^{p-\epsilon} - U^{p-\epsilon} - (p-\epsilon) U^{p-1-\epsilon} \phi \Bigr\}\, \psi_0$$

can be divided into two integrals: one on the set $D_\epsilon = \{x \in \mathbf{R}^n \mid U(x) + \phi(x) > 0\}$ and the other on the complement $\mathbf{R}^n \setminus D_\epsilon$. We prove that

$$\begin{aligned}
&\int_{D_\epsilon} \Bigl\{ \Bigl[(U+\phi)^+ \Bigr]^{p-\epsilon} - U^{p-\epsilon} - (p-\epsilon)\, U^{p-1-\epsilon} \phi \Bigr\}\, \psi_0 \\
&\qquad = \epsilon^2\, \frac{p(p-1)}{2} \int_{\mathbf{R}^n} U^{p-2}\, \phi_0^2\, \psi_0 + o\bigl(\epsilon^2\bigr), \\
&\qquad\qquad \times \int_{\mathbf{R}^n \setminus D_\epsilon} \bigl[-\,U^{p-\epsilon} - (p-\epsilon)\, U^{p-1-\epsilon} \phi \bigr]\, \psi_0 = o\bigl(\epsilon^2\bigr).
\end{aligned}$$

As regards the second equation, since, for any $x \in \mathbf{R}^n \setminus D_\epsilon$, we have $U(x) \le |\phi(x)|$, and since $\psi_0(x) \le C U(x)$ for some $C > 0$, we obtain

$$\begin{aligned}
\left| \int_{\mathbf{R}^n \setminus D_\epsilon} \bigl[-\,U^{p-\epsilon} - (p-\epsilon)\, U^{p-1-\epsilon} \phi \bigr]\, \psi_0 \right| &\le C' \int_{\mathbf{R}^n} |\phi(x)|^{p+1-\epsilon} \\
&\le C'' \|\phi\|_X^{p+1-\epsilon}
\end{aligned}$$

with C', C'' positive constants. This, using Lemma 6.4, completes the evaluation, as $\max\{s, 2\} < p + 1 - \epsilon < 2^*$ for ϵ sufficiently small.

For the first equation, we evaluate

$$\begin{aligned}
&\int_{D_\epsilon}\big\{(U+\phi)^{p-\epsilon}-U^{p-\epsilon}-(p-\epsilon)\,U^{p-1-\epsilon}\phi\big\}\,\psi_0-\epsilon^2\,\frac{p(p-1)}{2}\int_{\mathbf{R}^n}U^{p-2}\phi_0^2\,\psi_0\\
&\quad+\int_{D_\epsilon}\bigg\{(U+\phi)^{p-\epsilon}-U^{p-\epsilon}-(p-\epsilon)U^{p-1-\epsilon}\phi\\
&\qquad\qquad\qquad\qquad-\frac{(p-\epsilon)(p-1-\epsilon)}{2}\,U^{p-2-\epsilon}\phi^2\bigg\}\psi_0\\
&\quad+\int_{D_\epsilon}\bigg\{\frac{(p-\epsilon)(p-1-\epsilon)}{2}\,U^{p-2-\epsilon}\phi^2-\epsilon^2\,\frac{p(p-1)}{2}\,U^{p-2}\phi_0^2\bigg\}\,\psi_0\\
&\qquad\qquad-\epsilon^2\,\frac{p(p-1)}{2}\int_{\mathbf{R}^n\setminus D_\epsilon}U^{p-2}\phi_0^2\psi_0=A_1+A_2+A_3.
\end{aligned}$$

Since there exists a constant $C>0$ such that

$$\left|(1+t)^{p-\epsilon}-1-(p-\epsilon)\,t-\frac{1}{2}\,(p-\epsilon)(p-1-\epsilon)\,t^2\right|\le C|t|^3,$$

for any $t>-1$, we have

$$\begin{aligned}
|A_1|&=\bigg|\int_{D_\epsilon}U^{p-\epsilon}\bigg\{\left(1+\frac{\phi}{U}\right)^{p-\epsilon}-1-(p-\epsilon)\frac{\phi}{U}\\
&\qquad\qquad\qquad-\frac{(p-\epsilon)(p-1-\epsilon)}{2}\left(\frac{\phi}{U}\right)^2\bigg\}\,\psi_0\bigg|\\
&\le C\int_{D_\epsilon}U^{p-\epsilon}\left|\frac{\phi}{U}\right|^3|\psi_0|=C\int_{D_\epsilon}\frac{|\phi|^3|\psi_0|}{U^{3-p+\epsilon}}.
\end{aligned}$$

Choosing a constant $0<\gamma<1$ and using Lemma 2.3 and the fact that $|\psi_0(x)|\le CU(x)$ for any $x\in\mathbf{R}^n$ and for some constant $C>0$, we obtain

$$\begin{aligned}
|A_1|&\le C_1\int_{\mathbf{R}^n}\frac{|\phi|^3}{U^{2-p+\epsilon}}=C_1\int_{\mathbf{R}^n}|\phi|^{2+\gamma}\,\frac{|\phi|^{1-\gamma}}{U^{2-p+\epsilon}}\\
&\le C_2\int_{\mathbf{R}^n}|\phi|^{2+\gamma}\,U^{p-1-\epsilon-\gamma}\le C_3\|\phi\|^{2+\gamma}_{L^{(2n)/(n-2)}(\mathbf{R}^n)}=O\big(\epsilon^{2+\gamma}\big),
\end{aligned}$$

provided that ϵ and γ are sufficiently small. Let us write

$$\begin{aligned} A_2 = &\frac{1}{2}\int_{D_\epsilon}[(p-\epsilon)(p-1-\epsilon)-p(p-1)]\,U^{p-2-\epsilon}\phi^2\,\psi_0 \\ &+\frac{1}{2}\int_{D_\epsilon}p(p-1)\left(U^{p-2-\epsilon}-U^{p-2}\right)\phi^2\,\psi_0 \\ &+\frac{1}{2}\int_{D_\epsilon}p(p-1)\,U^{p-2}\left(\phi^2-\epsilon^2\phi_0^2\right)\psi_0\,. \end{aligned}$$

By the Taylor formula and Lemma 6.4, one immediately finds that there exists $0<\theta<1$ such that

$$\begin{aligned} &\left|\int_{D_\epsilon}[(p-\epsilon)(p-1-\epsilon)-p(p-1)]\,U^{p-2-\epsilon}\phi^2\,\psi_0\right| \\ &\qquad\qquad \le C\epsilon\|\phi\|^2_{L^{(2n)/(n-2)}(\mathbf{R}^n)}=O\big(\epsilon^3\big)\,, \end{aligned}$$

$$\begin{aligned} \left|\int_{D_\epsilon}\left(U^{p-2-\epsilon}-U^{p-2}\right)\phi^2\,\psi_0\right| &= \epsilon\left|\int_{D_\epsilon}(-\log U)\,U^{p-2-\theta\epsilon}\phi^2\,\psi_0\right| \\ &\le C\epsilon\|\phi\|^2_{L^{(2n)/(n-2)}(\mathbf{R}^n)}=O\big(\epsilon^3\big)\,, \end{aligned}$$

$$\begin{aligned} \left|\int_{D_\epsilon}U^{p-2}\big(\phi^2-\epsilon^2\phi_0^2\big)\psi_0\right| &= \epsilon^2\left|\int_{D_\epsilon}U^{p-2}\big(2\phi_0\widetilde{\phi}+\widetilde{\phi}^2\big)\psi_0\right| \\ &\le C\epsilon^2\left(\|\widetilde{\phi}\|_{L^{(2n)/(n-2)}(\mathbf{R}^n)}+\|\widetilde{\phi}\|^2_{L^{(2n)/(n-2)}(\mathbf{R}^n)}\right) \\ &= o\big(\epsilon^2\big)\,. \end{aligned}$$

Finally,

$$A_3=-\,\epsilon^2\,\frac{p(p-1)}{2}\int_{\mathbf{R}^n\setminus D_\epsilon}U^{p-2}\phi_0^2\,\psi_0=o\big(\epsilon^2\big)\,,$$

because the measure of the set $\mathbf{R}^n\setminus D_\epsilon$ tends to zero for $\epsilon\to 0$. So

$$I_2=-\,\frac{2n(n+2)}{n-2}\int_{\mathbf{R}^n}U^{p-2}\phi_0^2\,\psi_0+o\big(\epsilon^2\big)\,. \tag{51}$$

(III) By the Taylor formula we have, for some $\theta = \theta(x,\epsilon)$, $0 < \theta < 1$,

(52)
$$f'_\epsilon(U) - f'_0(U) = n(n-2)\Big\{ -\epsilon\, U^{p-1}(1 + p\,\log U) + e\,\frac{1}{2}\,\epsilon^2\, U^{p-1-\theta\epsilon}\log U\,[2 + (p-\theta\epsilon)\log U]\Big\}.$$

By (52) and Lemma 6.4 we conclude that

$$I_3 = \epsilon^2\, n(n-2)\int_{\mathbf{R}^n}(1 + p\,\log U)\,U^{p-1}\phi_0\,\psi_0 + o\big(\epsilon^2\big)\,; \tag{53}$$

in fact, $(1 + p\log U)\,U^{p-1}\psi_0$, $U^{p-1-\theta\epsilon}\log U[2 + (p-\theta\epsilon)\log U]\psi_0 \in L^{(2n)/(n+2)}(\mathbf{R}^n)$.

(IV) There holds

$$\begin{aligned} I_4 &= \int_{\mathbf{R}^n}\big[\delta^2\, V_{\delta,y} - \epsilon d_0 V(y_0)\big]\,U\psi_0 \\ &= +\,\epsilon^2\tilde{d}\,V(y_0)\int_{\mathbf{R}^n}U\psi_0 + \delta^2\int_{\mathbf{R}^n}[V_{\delta,y}(x) - V(y_0)]\,U\psi_0. \end{aligned}$$

We consider for the moment the case $\underline{\alpha} > 1$. By assumption (V_{y_0}) on the potential V, the first integral gives

$$\begin{aligned} &\epsilon d_0\int_{\mathbf{R}^n}[V(y_0 + \delta(x+\tilde{y})) - V(y_0)]\,U\psi_0 \\ &\quad = \epsilon\delta d_0\int_{\mathbf{R}^n}\nabla V(y_0 + \theta\delta(x+\tilde{y}))\cdot(x+\tilde{y})\,U\psi_0 \\ &\quad = \epsilon\delta d_0\int_{\{x\in\mathbf{R}^n\,|\,\theta\delta|x+\tilde{y}|<\rho\}}\sum_{i=1}^n h_i(\theta\delta(x+\tilde{y}))(x_i+\tilde{y}_i)\,U\psi_0 \\ &\qquad + \epsilon\delta d_0\int_{\{x\in\mathbf{R}^n\,|\,\theta\delta|x+\tilde{y}|<\rho\}}\sum_{i=1}^n R_i(\theta\delta(x+\tilde{y}))(x_i+\tilde{y}_i)\,U\psi_0 \\ &\qquad + \epsilon\delta d_0\int_{\{x\in\mathbf{R}^n\,|\,\theta\delta|x+\tilde{y}|\geq\rho\}}\nabla V(y_0 + \theta\delta(x+\tilde{y}))\cdot(x+\tilde{y})\,U\psi_0 \\ &\quad = B_1 + B_2 + B_3, \end{aligned}$$

with $0 < \theta < 1$. There exist constants $C, C' > 0$ such that if $\underline{\alpha} < n-5$,

$$|B_1| = \epsilon\delta d_0 \left| \int_{\{x\in\mathbf{R}^n \,|\, \theta\delta|x+\tilde{y}|<\rho\}} \sum_{i=1}^{n} (\theta\delta|x+\tilde{y}|)^{\alpha_i} \right.$$
$$\left. \times\, h_i\left(\frac{x+\tilde{y}}{|x+\tilde{y}|}\right)(x_i+\tilde{y}_i)U\psi_0 \right|$$
$$\le \epsilon\delta^{1+\alpha}\, C \sum_{i=1}^{n} \int_{\mathbf{R}^n} |x+\tilde{y}|^{\alpha_i+1} U|\psi_0|$$
$$\le \epsilon^{(3+\alpha)/2}\, C' = o\big(\epsilon^2\big)\,.$$

Analogously $|B_2| = o(\epsilon^2)$. We observe now that the following set inclusions hold

$$\begin{aligned}\{x\in\mathbf{R}^n \mid \theta\delta|x+\tilde{y}|\ge\rho\} &\subset \{x\in\mathbf{R}^n \mid \delta|x+\tilde{y}|\ge\rho\}\\ &\subset \{x\in\mathbf{R}^n \mid \delta|x|\ge\rho'\}\end{aligned} \tag{55}$$

for some $\rho' > 0$. Therefore, we have

$$|B_3| \le \epsilon\delta d_0 \|\nabla V\|_{L^\infty(\mathbf{R}^n)} \int_{\{x\in\mathbf{R}^n \,|\, \delta|x|\ge\rho'\}} |x+\tilde{y}|U|\psi_0| = O\big(\epsilon^{(n-2)/2}\big)\,.$$

Concluding, if $\underline{\alpha} > 1$, we have

$$I_4 = \epsilon^2\tilde{d}\,V(y_0)\int_{\mathbf{R}^n} U\psi_0 + o\big(\epsilon^2\big)\,. \tag{56}$$

Let us consider now the case $\underline{\alpha} = 1$. Then, for $z \in B(0,\rho)$, it holds $V(y_0+z) = V(y_0) + \mathcal{H}(z) + \sigma(z)$, where $\mathcal{H}$ is homogenous of degree 2 and $|\sigma(z)| \le c|z|^\gamma$ with $2 < \gamma$ where $\gamma :=$

$\min\{\beta_i+1,\ \alpha_i+1 \mid i=1,\dots,n,\ \alpha_i\neq 1\}$. Then we get

$$
\begin{aligned}
&\epsilon d_0 \int_{\mathbf{R}^n} [V(y_0+\delta(x+\tilde y)) - V(y_0)]\, U\psi_0 \\
&= \epsilon d_0 \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|<\rho\}} [V(y_0+\delta(x+\tilde y)) - V(y_0)]\, U\psi_0 \\
&\quad + \epsilon d_0 \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|\geq\rho\}} [V(y_0+\delta(x+\tilde y)) - V(y_0)]\, U\psi_0 \\
&= \epsilon\delta^2 \frac{d_0}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(y_0) \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|<\rho\}} (x_i+\tilde y_i)(x_j+\tilde y_j)\, U\psi_0 \\
&\quad + \epsilon d_0 \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|<\rho\}} \sigma\left(\delta(x+\tilde y)\right) U\psi_0 \\
&\quad + \epsilon d_0 \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|\geq\rho\}} [V(y_0+\delta(x+\tilde y)) - V(y_0)]\, U\psi_0 \\
&= D_1 + D_2 + D_3
\end{aligned}
$$

We have

$$
\begin{aligned}
D_1 &= \epsilon\delta^2 \frac{d_0}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(y_0) \int_{\mathbf{R}^n} (x_i+\tilde y_i)(x_j+\tilde y_j)\, U\psi_0 \\
&\quad - \epsilon\delta^2 \frac{d_0}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(y_0) \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|\geq\rho\}} (x_i+\tilde y_i)(x_j+\tilde y_j)\, U\psi_0 \\
&= \epsilon^2 \frac{d_0^2}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}(y_0) \int_{\mathbf{R}^n} (x_i+\tilde y_i)(x_j+\tilde y_j) U\psi_0 + o\big(\epsilon^2\big)\,.
\end{aligned}
$$

Moreover, the second term gives

$$
|D_2| \leq \epsilon d_0 \int_{\{x\in\mathbf{R}^n \,|\, \delta|x+\tilde y|<\rho\}} |\delta(x+\tilde y)|^{\gamma}\, U|\psi_0|\,.
$$

Now, since $\gamma>2$, we can write $\gamma = 2+\gamma_1+\gamma_2$ with $\gamma_1,\gamma_2>0$ and we obtain

$$
|D_2| \leq \epsilon\delta^{2+\gamma_1}\, d_0 \int_{\mathbf{R}^n} |x+\tilde y|^{2+\gamma_1}\, \rho^{\gamma_2}\, U|\psi_0| \leq C\epsilon^{2+(\gamma_1/2)} = o\big(\epsilon^2\big)\,,
$$

provided γ_1 is small enough. Finally, D_3 is $o(\epsilon^2)$, analogously to the previous B_3. Finally, if $\underline{\alpha} = 1$, we have

$$I_4 = \epsilon^2 \tilde{d}\, V(y_0) \int_{\mathbf{R}^n} U\psi_0 + \epsilon^2 \frac{d_0^2}{2} \sum_{i,j=1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_j}(y_0) \int_{\mathbf{R}^n} (x_i + \tilde{y}_i)(x_j + \tilde{y}_j)\, U\psi_0 + o\big(\epsilon^2\big)\,. \tag{57}$$

(V) We have

$$\begin{aligned} I_5 &= \delta^2 \int_{\mathbf{R}^n} V(y_0 + \delta(x+\tilde{y}))\phi\,\psi_0 \\ &= \delta^2\, V(y_0) \int_{\mathbf{R}^n} \phi\,\psi_0 + \delta^4 \int_{\mathbf{R}^n} \sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}\Big|_{y_0+\theta(x+\tilde{y})} (x+\tilde{y})_i (x+\tilde{y})_j \phi\,\psi_0 \\ &= \epsilon^2 d_0 V(y_0) \int_{\mathbf{R}^n} \phi\,\psi_0 + o\big(\epsilon^2\big)\,. \end{aligned}$$

Indeed, by Lemma 6.4,

$$\delta^2\, V(y_0) \int_{\mathbf{R}^n} \phi\,\psi_0 = \epsilon^2 d_0 V(y_0) \int_{\mathbf{R}^n} \phi_0\,\psi_0 + o\big(\epsilon^2\big)\,.$$

Moreover, using the inequality $|\phi(x)| \leq cU(x)$ proved in Lemma 2.3, given s for γ small enough, we have

$$\begin{aligned} &\left| \int_{\mathbf{R}^n} \sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}\Big|_{y_0+\theta(x+\tilde{y})} (x+\tilde{y})_i (x+\tilde{y})_j \phi\,\psi_0 \right| \\ &\leq c_1 \max_{i,j} \left\| \frac{\partial^2 V}{\partial x_i \partial x_j} \right\|_{\mathrm{L}^\infty} \int_{\mathbf{R}^n} |\phi\|^\gamma\, U^{1-\gamma} |\psi_0| \left(|x|^2 + |\tilde{y}|^2\right) \\ &\leq c_2 \|\phi\|_{\mathrm{L}^s}^\gamma \left(\||x|^2\, U^{1-\gamma}\psi_0\|_{\mathrm{L}^{s/(s-\gamma)}} + \|U^{1-\gamma}\psi_0\|_{\mathrm{L}^{s/(9s-\gamma)}}\right) = O\big(\epsilon^\gamma\big)\,. \end{aligned} \tag{58}$$

By (50), (51), (53), (56), (57) and (58) we obtain our claim. ◻

REFERENCES

1. T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), 573–598.

2. V. Benci and G. Cerami, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ *in* $\mathbf{R}^N$, J. Funct. Anal. **88** (1990), 91–117.

3. H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

4. L.A. Caffarelli, B. Gidas and J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), 271–297.

5. W.Y. Ding and W.M. Ni, *On the existence of positive entire solutions of a semilinear elliptic equation*, Arch. Rational Mech. Anal. **91** (1986), 283–308.

6. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1998.

7. E. Giusti, *Equazioni ellittiche del secondo ordine*, Quaderni U.M.I., Pitagora Editrice, Bologna, 1978.

8. M. Grossi, *On the number of single-peak solutions of the nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 261–280.

9. Z.-C. Han, *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **8** (1991), 159–174.

10. N. Hirano, A.M. Micheletti and A. Pistoia, *Existence of sign changing solutions for some critical problems on* $\mathbb{R}^N$, Commun. Pure Appl. Anal. **4** (2005), 143–164.

11. A.M. Micheletti, M. Musso and A. Pistoia, *Super-position of spikes for a slightly super-critical elliptic equation in* $\mathbf{R}^N$, Discrete Contin. Dyn. Syst. **12** (2005), 747–760.

12. A.M. Micheletti and A. Pistoia, *Existence of blowing-up solutions for a slightly subcritical or a slightly supercritical non-linear elliptic equation on* $\mathbf{R}^n$, Nonlinear Anal. **52** (2003), 173–195.

13. W.-M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), 247–281.

14. X.B. Pan and X. Wang, *Blow-up behaviour of ground states of semilinear elliptic equations in* $\mathbf{R}^N$ *involving critical exponents*, J. Differential Equations **99** (1992), 78–107.

15. G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.

16. X. Wang, *On location of blow-up of ground states of semilinear elliptic equations in* $\mathbf{R}^N$ *involving critical exponents*, J. Differential Equations **127** (1996), 148–173.

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