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SMOOTH POINTS OF ESSENTIALLY BOUNDED VECTOR FUNCTION SPACES

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ABSTRACT. We characterize the smooth points of $L_\infty(X)$, where X is any normed space.

1. Introduction. Let X be a normed space and $x, y \in X$. The one-sided derivatives at $x \neq 0$ in the direction $y \neq 0$ are

$$D_X^{\pm}(x,y) = \lim_{h\to 0^{\pm}} \frac{\|x+hy\| - \|x\|}{h}.$$

Both limits always exist and, if they have the same value, we write $D_X(x,y) = D_X^+(x,y) = D_X^-(x,y)$. It is easy to see that this is equivalent to saying: For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < h < \delta$ implies $\|x+hy\| + \|x-hy\| < 2\|x\| + \varepsilon h$.

We say that $x \neq 0$ is *smooth*, if $D(x,y)$ exists, for every $y \in S_X$, where S_X denotes the unit sphere of X, or equivalently, if there is a unique norm-one functional $x^* \in X^*$, the topological dual of X, such that $x^*(x) = \|x\|$ [**1**, page 179]. Since $D_X(tx,y) = D_X(x,y)$ for $t > 0$, we can restrict our attention to the smooth points of S_X.

Deeb and Khalil [**3**] have characterized the smooth points of the Lebesgue-Bochner spaces $L_p(I,X)$, $1 \leq p < \infty$, when I has finite measure and X has a separable dual. Cerda, Hudzik and Mastylo [**2**] characterize the smooth points of the Köthe-Bochner space $E(X)$, if X is real with separable dual, E is order continuous, and the norm of E^* is strictly monotonic. In this paper we characterize the smooth points of $L_\infty(X)$. In contrast to the $L_p(I,X)$, $1 \leq p < \infty$, it is worth noticing that the smoothness of $x \in L_\infty(X)$ does not imply the smoothness of $x(t) \in X$ for almost every $t \in T$.

Let (T, Σ, μ) be a complete, positive measure space and X a normed space. The function $x : T \to X$ is said to be *simple* if there

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exist $T_1, \dots, T_n \in \Sigma$, disjoint, and $x_1, \dots, x_n \in X$ such that $x = \sum_{i=1}^n x_i \chi_{T_i}$, where χ_{T_i} is the characteristic function of T_i. The function $x : T \to X$ is defined as *measurable* if, for every finite measurable set F, there exists a sequence of simple functions $\{s_n\}_{n\in\mathbf{N}}$ such that $x\chi_F = \lim_{n\to\infty} s_n$ almost everywhere [**4**]. The set of measurable functions is a linear space.

A measurable set A is called an *atom* if $\mu(A) > 0$ and, whenever B is a measurable subset of A, we have either $\mu(B) = 0$ or $\mu(A\backslash B) = 0$.

We use $L_\infty(X)$ to denote the space of measurable equivalence classes of functions $x : T \to X$ such that ess $\sup_{t\in T}\{\|x(t)\|_X\} < \infty$, where ess sup denotes the essential supremum, i.e.,

$$\operatorname*{ess\,sup}_{t\in T}\{\|x(t)\|_X\} = \inf\{c : \mu\{t \in T : \|x(t)\|_X > c\} = 0\}.$$

It is a normed space, normed by $\|x\| = \text{ess}\ \sup_{t\in T}\{\|x(t)\|_X\}$.

If $X = \mathbf{R}$, we write $L_\infty(X) = L_\infty$. To avoid confusion, from now on we shall use $\|\cdot\|$ for the norm in $L_\infty(X)$ and $\|\cdot\|_X$ for the norm in X.

We collect the following easy results in a lemma.

Lemma 1. (i) *If the function $x : T \to X$ is measurable and A is a finite-measure atom, then $x\chi_A$ is a constant function on A. If $X = \mathbf{R}$, the assumption "finite measure" can be removed.*

(ii) *Let $x \in S_{L_\infty}$, A be an atom and $\|x\chi_{T\backslash A}\| < 1$. Then $|x(t)| = 1$ for almost every $t \in A$.*

(iii) *Let $x, y \in L_\infty$, $x \geq 0$ and $y \geq 0$. If A is an atom, then $\|(x+y)\chi_A\| = \|x\chi_A\| + \|y\chi_A\|$.*

2. Smooth points in L_∞ and $L_\infty(X)$.

We begin with the scalar case.

Theorem 2. *Let $x \in S_{L_\infty}$ and $A = \{t \in T : |x(t)| = 1\}$. Then x is smooth if and only if A is an atom and $\|x\chi_{T\backslash A}\| < 1$.*

Proof. Suppose either A is non-atom with $\mu(A) > 0$ or $\|x\chi_{T\backslash A}\| = 1$.

We then prove the existence of $P, Q \in \Sigma$ such that

$$P \cap Q = \varnothing, \quad \mu(P) > 0, \quad \mu(Q) > 0 \quad \text{and} \quad \|x\chi_P\| = \|x\chi_Q\| = 1. \tag{2.1}$$

If A is non-atom with $\mu(A) > 0$, then obviously (2.1) holds. Let $\|x\chi_{T\backslash A}\| = 1$. Take $0 < r_1 < r_2 < \cdots < 1$ with $\lim_{n\to\infty} r_n = 1$. Define $A_n = \{t \in T\backslash A : r_{n-1} < |x(t)| \le r_n\}$. We claim that there exists a subsequence $(A_{n_k})_{k\in\mathbf{N}}$ with $\mu(A_{n_k}) > 0$ for every $k = 1, 2, \dots$. Otherwise, we may suppose that $\mu(A_n) = 0$ for every $n \in \mathbf{N}$; thus, $\mu(\cup_{n\in\mathbf{N}} A_n) = \mu\{t \in T\backslash A : r_1 < |x(t)|\} = 0$. Therefore, we have the contradiction $\|x\chi_{T\backslash A}\| \le r_1 < 1$. Now it is easy to check that $P = \cup_{k\,\mathrm{even}} A_{n_k}$ and $Q = \cup_{k\,\mathrm{odd}} A_{n_k}$ satisfy (1).

Let $T_+ = \{t \in T : x(t) \ge 0\}$, $T_- = \{t \in T : x(t) < 0\}$ and $y = \chi_{P\cap T_+} - \chi_{P\cap T_-}$.

For every $h > 0$, we have $|x(t) + hy(t)| = |x(t)| + h$, if $t \in P$ and $|x(t) - hy(t)| = |x(t)|$, if $t \in Q$. Thus $1 + h \ge \|x + hy\| \ge \|(x + hy)\chi_P\| = \|(|x| + h)\chi_P\| = 1 + h$ and $\|x - hy\| = \|x\chi_Q\| = 1$. Therefore, $D^+_{L_\infty}(x, y) = 1$ and $D^-_{L_\infty}(x, y) = 0$.

Conversely, let A be an atom, $\|x_{\chi T\backslash A}\| = r < 1$ and $y \in S_{L_\infty}$. We prove that $D^+_{L_\infty}(x, y) = D^-_{L_\infty}(x, y)$. If $0 \le h \le (1 - r)/2$, then for almost every $t' \in T\backslash A$ and $t \in A$, we have by Lemma 1 (ii)

$$\begin{aligned} |x(t') \pm hy(t')| &\le r + h|y(t')| \le r + h \le 1 - h \\ &\le 1 - h\|y\chi_A\| \le 1 - h|y(t)| \\ &= |x(t)| - h|y(t)| \le |x(t) \pm hy(t)|. \end{aligned} \tag{2.2}$$

Therefore $\|x \pm hy\| = \|(x \pm hy)\chi_A\|$. Set $B = \{t \in A : \operatorname{sgn} x(t) = \operatorname{sgn} y(t)\}$, where sgn denotes the sign function. Then $B \in \Sigma$ and

$$|x(t) \pm hy(t)| = (1 \pm h|y(t)|)\chi_B(t) + (1 \mp h|y(t)|)\chi_{A\backslash B}(t), \quad \text{for a.e. } t \in A.$$

If $\mu(B) > 0$, then $\mu(A\backslash B) = 0$. So $\|(x + hy)\chi_A\| = \|(x + hy)\chi_B\| = 1 + h\|y\chi_B\|$ and $\|(x - hy)\chi_A\| = \|(x - hy)\chi_B\| = 1 - h\|y\chi_B\|$. Then we have $D^+_{L_\infty}(x, y) = \|y\chi_B\| = D^-_{L_\infty}(x, y)$. If $\mu(B) = 0$, then $\mu(A\backslash B) > 0$ and we obtain $D^+_{L_\infty}(x, y) = -\|y\chi_{A\backslash B}\| = D^-_{L_\infty}(x, y)$. $\square$

Now the vectorial case.

Theorem 3. *Let $x \in S_{L_\infty(X)}$ and $A = \{t \in T : \|x(t)\|_X = 1\}$. Then x is smooth if and only if A is an atom, $\|x\chi_{T\setminus A}\| < 1$ and, for every $y \in S_{L_\infty(X)}$, there exists $D_X(x(t), y(t))$ uniformly in $\{(x(t), y(t)),$ for a.e. $t \in A\}$.*

Proof. Assume that A is an atom and $\|x\chi_{T\setminus A}\| = r < 1$. Changing $|\cdot|$ to $\|\cdot\|_X$ in (2.2), we obtain

$$\|x(t') \pm hy(t')\|_X \le \|x(t) \pm hy(t)\|_X, \quad \text{for a.e. } t' \in T\setminus A, \quad t \in A,$$

whenever $0 \le h \le (1-r)/2$. Therefore $\|x \pm hy\| = \|(x \pm hy)\chi_A\|$. Since A is an atom and the functions $\|x(\cdot) \pm h(\cdot)\|_X$ are positive, by Lemma 1 we have

$$\begin{aligned}\|x + hy\| + \|x - hy\| &= \|(x+hy)\chi_A\| + \|(x-hy)\chi_A\| \\ &= \operatorname*{ess\,sup}_{t\in T}\{\|(x(\cdot) + hy(\cdot))\chi_A(\cdot)\|_X\} \\ &\quad + \operatorname*{ess\,sup}_{t\in T}\{\|x(\cdot) - hy(\cdot))\chi_A(\cdot)\|_X\} \\ &= \operatorname*{ess\,sup}_{t\in T}\{\|(x(\cdot) + hy(\cdot))\chi_A(\cdot)\|_X \\ &\qquad + \|x(\cdot) - hy(\cdot))\chi_A(\cdot)\|_X\}.\end{aligned}$$

Thus the existence of $D_{L_\infty(X)}(x, y)$ is equivalent to the existence of $D_X(x(t), y(t))$ uniformly in $\{(x(t), y(t)),$ for a.e. $t \in A\}$.

Conversely, suppose that $x \in S_{L_\infty(X)}$ is smooth and write $Z = \{t \in T : x(t) = 0\}$. Let $u(\cdot) \in S_{L_\infty}$ and take $w \in S_X$. The function $y(t) = (u(t)x(t)/\|x(t)\|_X)\chi_{T\setminus Z}(t) + u(t)w\chi_Z(t)$ belongs to $y \in S_{L_\infty(X)}$. Moreover, for $h \ge 0$,

$$\begin{aligned}\|x \pm hy\| &= \left\| \frac{x(\cdot)}{\|x(\cdot)\|_X}(\|x(\cdot)\|_X \pm hu(\cdot))\chi_{T\setminus Z}(\cdot) \pm hu(\cdot)w\chi_Z(\cdot)\right\|_{L_\infty} \\ &= \|(\|x(\cdot)\|_X \pm hu(\cdot))\|_{L_\infty}.\end{aligned}$$

Hence the existence of $D_{L_\infty(X)}(x, y)$ implies the existence of $D_{L_\infty}(\|x(\cdot)\|_X, u(\cdot))$. By Theorem 2, A is an atom and $\|(\|x(\cdot)\|_X)\chi_{T\setminus A}\|_{L_\infty} < 1$. Moreover, as we have already proved, the existence of $D_{L_\infty(X)}(x, y)$ is equivalent to the existence of $D_X(x(t), y(t))$ uniformly in $\{(x(t), y(t)),$ for a.e. $t \in A\}$. $\square$

If μ is σ-finite, every atom has finite measure, and then each function $x \in L_\infty(X)$ is a constant on the atom. Consequently, we obtain

Corollary 4. *Let μ be σ-finite. Then $x \in S_{L_\infty(X)}$ is smooth if and only if $A = \{t \in T : \|x(t)\|_X = 1\}$ is an atom, $\|x\chi_{T\setminus A}\| < 1$ and $x(t)$ is smooth for almost every $t \in A$.*

When (T, Σ, μ) is a discrete measure space, one has $L_\infty = l_\infty$ and ess sup = sup. If $\{X_i\}_{i\in I}$ is a family of normed spaces, the space of functions $x : I \to \cup_{i\in I} X_i$, such that $x_i \in X_i$ for each $i \in I$ and $(\|x_i\|_i) \in l_\infty$ is a normed space endowed with the norm $\|x\| = \sup_{i\in I} \|x_i\|_i$. We denote it by $l_\infty(X_i)$. Since, in this case, each element of I is an atom of measure one, we get as a consequence of Theorem 2 and Corollary 4:

Corollary 5. (i) *$x \in S_{l_\infty}$ is smooth if and only if there exists $j \in I$ such that $\sup_{i\neq j} |x_i| < 1$.*

(ii) *$x \in S_{l_\infty(X_i)}$ is smooth if and only if there exists $j \in I$ such that $\sup_{i\neq j} \|x_i\|_i < 1$ and $x_j \in X_j$ is smooth.*

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