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EXISTENCE AND UNIQUENESS FOR SPATIALLY INHOMOGENEOUS COAGULATION-CONDENSATION EQUATION WITH UNBOUNDED KERNELS

DONGHO CHAE AND PAVEL DUBOVSKIĬ

ABSTRACT. We prove the existence and uniqueness theorem for Smoluchovsky's model with condensation in three-dimensional space. Initial data are supposed to be small enough.

1. Introduction. We are concerned with the space-inhomogeneous Smoluchovsky's equation with condensation processes taken into account.

$$\begin{aligned}\frac{\partial}{\partial t}c(x,z,t)&+\frac{\partial}{\partial x}(r(x)c(x,z,t))+\mathrm{div}_z(v(x,z)c(x,z,t))\\ &=\frac{1}{2}\int_0^x K(x-y,y)c(x-y,z,t)c(y,z,t)\,dy\\ &\quad-c(x,z,t)\int_0^\infty K(x,y)c(y,z,t)\,dy;\\ &\quad x,t\in R^1_+=[0,\infty),\quad z\in R^3.\end{aligned}\tag{1.1}$$

It describes the time evolution of particles in disperse systems. Symmetric nonnegative on R^2_+ function $K(x,y)$ define intensity of merging of particles with masses x and y. Unknown function $c(x,z,t)$ is a distribution function for particles of the disperse system of mass $x\geq 0$ at time $t\in R_1^+=[0,\infty)$ at space point $z\in R^3$. The function $v(x,z)$ is a velocity of space transfer of particles; $r(x)$ is a scalar speed of growth of particles due to condensation of molecules (or, more generally, clusters) from outer medium, e.g., condensation of vapor on water drops in atmospheric clouds. In physically real situations we often have $r\sim x^\alpha$, $\alpha>0$, $0<x_0\leq x\leq \overline{x}$, where x_0 is a critical mass of a particle which splits regions of its stable and unstable state; $\overline{x}$ is a conventional

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boundary of satiation, after which the function $r(x)$ may be considered as bounded.

The equation (1.1) must be supplemented by an initial distribution

$$c(x,z,0)=c_{01}(x,z)\geq 0, \tag{1.2}$$

and a distribution function of condensation germs

$$c(0,z,t)=c_{02}(z,t). \tag{1.3}$$

Applications of (1.1) can be found in many problems including chemistry, e.g., reacting polymers, physics (aggregation of colloidal particles, growth of gas bubbles in solids), engineering (behavior of a fuel mixture in engines), astrophysics (formation of stars and planets) and meteorology (coagulation of drops in atmospheric clouds). Recent results for the space homogeneous case with $v\equiv 0$ and other references can be found in [**1, 13**]. The results in the spatially inhomogeneous case are much more poor than in the homogeneous one, as may be explained by the following reasons. The formal integration of (1.1) with weight x yields, the condensation r is assumed to be equal to zero,

$$\int_{-\infty}^{+\infty}\int_0^\infty xc(x,z,t)\,dx\,dz=\text{const}. \tag{1.4}$$

In the space homogeneous case we obtain the more valuable equality

$$\int_0^\infty xc(x,t)\,dx=\text{const}. \tag{1.5}$$

The correlations (1.4) and (1.5) express the mass conservation law. The large difference between the conservation laws (1.4) and (1.5) can be seen from the integral form of the problem (1.1), (1.2) (to simplify the exposition we take $r=0$ and $v=v(x)$):

$$\begin{aligned}c(x,z,t)&=c_0(x,z-v(x)t)\\&\quad+\int_0^t\bigg\{\frac{1}{2}\int_0^x K(x-y,y)c(x-y,z-v(x)(t-s),t)\\&\qquad c(y,z-v(x)(t-s),t)\,dy-c(x,z-v(x)(t-s),t)\\&\qquad\int_0^\infty K(x,y)c(y,z-v(x)(t-s),s)\,dy\bigg\}\,ds.\end{aligned} \tag{1.6}$$

To prove existence of the solution to (1.6) we usually build a sequence of solutions $c_n(x,z,t)$ of a regularized (more simple) problem. Such a sequence should converge to a function c. We do not discuss now in which topology this trend takes place. The main problem is to prove that the function constructed $c(x,z,t)$ is a solution to the original equation (1.6). Namely, we must demonstrate the possibility to pass to the limit as $n \to \infty$ in the equation (1.6) with c replaced by c_n. The most difficult stage is to show the admission to pass to the limit under sign of the integral over the infinite domain

$$\int_0^\infty K(x,y) c_n(y, z - v(x)(t-s), s)\, dy.$$

Let the coagulation kernel $K(x,y)$ be bounded. Then we ought to demonstrate the uniform smallness of the integral "tails"

$$\int_m^\infty c_n(y, z - v(x)(t-s), s)\, dy$$

for all $n \geq 1$. The value of m is taken sufficiently large. As long as we have the estimation like (1.5), then the problem can be solved by the following well-known trick:

$$\begin{aligned} \int_m^\infty c_n(y, z - v(x)(t-s), s)\, dy &\leq \frac{1}{m} \int_0^\infty y c_n(y, z - v(x)(t-s), s)\, dy \\ &\leq \frac{\text{const}}{m} \longrightarrow 0, \quad m \longrightarrow \infty. \end{aligned}$$

The correlation (1.4) does not give us such convergence and we ought to seek other approaches. It is worthwhile to note that if the space velocity v does not depend on x, then the strong mass conservation law (1.5) holds and we obtain the desired trend. Namely this fact was used by Burobin in [**2**] where he considered the case $v(x) = \text{const}$ for $x \geq x_0$ which reduces the problem to the space uniform situation with the strong conservation law (1.5).

Other mathematical results for the spatially inhomogeneous equation are as follows. Dubovskiĭ [**4**] and Galkin [**10**] succeeded to prove existence and uniqueness theorems for bounded coagulation kernels. Unbounded coagulation kernels of special type were considered in [**11**]. The unique solvability of the problem with coagulation kernels of linear growth and particle fractionation taken into account was demonstrated in [**3**]. Influence of the condensation processes on the evolution of a coagulating system was studied for the space homogeneous case by Gajewski and Zaharias in [**6, 7**] and by Dubovskiĭ in [**5**]. Analysis of a spatially inhomogeneous coagulation model with condensation didn't undertake before. We prove the existence and uniqueness theorem for sufficiently small initial data with coagulation kernels admitting linear growth on infinity. These kernels include the considerable class of physically real ones.

2. Main result. Fix $T > 0$ and denote $\Omega_\lambda(T)$ the space of continuous functions in $R^1_+ \times R^3 \times [0,T]$ with the norm

$$\|c\|_\lambda = \sup_{0\le t\le T} \int_0^\infty \exp(\lambda x) \sup_{z\in R^3} |c(x,z,t)|\,dx.$$

We define $\Omega(T) = \cup_{\lambda>0}\Omega_\lambda(T)$. Let $\Omega^+_\lambda(T)$, $\Omega^+(T)$ be nonnegative cones in corresponding spaces.

Theorem 1. *Let the coagulation kernel K be a continuous, nonnegative and symmetric function, i.e., $K(x,y) = K(y,x) \ge 0$. Also let*

$$K(x,y) \le k(x+y) \quad \textit{where } k = \text{ const.}$$

Let the function $r(x)$ be nonnegative, bounded with its derivative and have continuous second derivative. Let the following inequality hold

$$\operatorname{div}_z v(x,z) + r'(x) \ge \delta > 0. \tag{2.1}$$

Suppose that functions v, c_{01}, c_{02} are continuous and, in addition, c_{01} and c_{02} are nonnegative. We impose the following conditions ensuring smallness of the initial data:

$$c_{01}(x,z) < A\exp(-ax), \quad a > 0; \tag{2.2}$$

$$\sup_{0\le t\le T} \exp(\delta t) c_{02}(z,t) < A. \tag{2.3}$$

Let $r(x) \geq 0$ *and*

$$R = \max\left\{\sup_{R^1_+} r(x), \sup_{R^1_+} |r'(x)|\right\} < \frac{\delta}{1+a}, \tag{2.4}$$

$$2\sqrt{\frac{kA}{\delta - R(1+a)}} + 2\frac{kAr(0)}{\delta(\delta - R(1+a))} < a. \tag{2.5}$$

Also let

$$c_{01}(0,z) = c_{02}(z,0), \quad z \in R^3.$$

Then there exists a continuous, differentiable along characteristics of the equation (1.1) *nonnegative solution* $c \in \Omega^+(T)$. *This solution is unique in* $\Omega(T)$ *the additional condition provided*

$$\begin{aligned} &\mathrm{div}_z v(x,z) + r'(x) \leq M(1+x), \\ &M = \mathrm{const}, \quad x \in R^1_+, \quad z \in R^3. \end{aligned}$$

First, we formulate an auxiliary result.

Lemma 2.1. *Let conditions of the theorem hold, and the coagulation kernel* K *has a compact support. Then there exists a unique solution* $c \in \Omega^+_a(T)$ *to the problem* (1.1)–(1.3).

The proof of Lemma 2.1 is based upon replacement of the integral with infinite upper limit in the main equation (1.1) to the integral over a compact domain. Due to this replacement the collision operator, which is expressed by the righthand side of the equation (1.1), maps $\Omega_a(T)$ into itself.

We approximate the original unbounded kernel by a sequence $\{K_n\}_{n=1}^\infty$ of kernels with compact supports. Each kernel from this sequence must satisfy the conditions of the theorem. Recalling Lemma 2.1 we get a sequence $\{c_n\}_{n=1}^\infty$ solutions of the problem (1.1)–(1.3) with kernels K_n and the same initial data c_{01} and c_{02}.

3. Main lemma. We make the change of variables

$$c_n(x,z,t) = (1-\tau)\hat{c}_n(x,z,\tau),$$
$$\tau = 1 - \exp(-\delta t), \quad n \geq 1.$$

Then the problem (1.1)–(1.3) takes the following form:

$$\begin{aligned}\delta\frac{\partial}{\partial\tau}\hat{c}_n(x,z,\tau) &+ (1-\tau)^{-1}(v(x,z), \nabla_z\hat{c}_n(x,z,\tau)) \\ &+ (1-\tau)^{-1}r(x)\frac{\partial}{\partial x}\hat{c}_n(x,z,\tau) \\ &= \frac{1}{2}\int_0^x K_n(x-y,y)\hat{c}_n(x-y,z,\tau)\hat{c}_n(y,z,\tau)\,dy \\ &- \hat{c}_n(x,z,\tau)\int_0^\infty K_n(x,y)\hat{c}_n(y,z,\tau)\,dy \\ &- [\mathrm{div}_z v(x,z) + r'(x) - \delta](1-\tau)^{-1}\hat{c}_n(x,z,\tau)\end{aligned} \tag{3.1}$$

with the initial and boundary conditions

$$\begin{aligned}\hat{c}_n(x,z,0) &= c_{01}(x,z), \\ \hat{c}_n(0,z,\tau) &= (1-\tau)^{-1}c_{02}(z,t).\end{aligned} \tag{3.2}$$

Lemma 3.1. *Let the conditions of Theorem 1 hold and the continuous function g be a solution of the equation*

$$\delta g_\tau(x,\tau) + \frac{r(x)}{1-\tau}g_x(x,\tau) = \frac{1}{2}kx\int_0^x g(x-y,\tau)g(y,\tau)\,dy, \tag{3.3}$$

$$g(x,0) = A\exp(-ax), \qquad g(0,\tau) = A. \tag{3.4}$$

Then

$$\begin{aligned}\hat{c}_n(x,z,\tau) &< g(x,\tau), \\ x \in R^1_+, \quad z \in R^3, &\quad \tau \in [0,1), \quad n \geq 1.\end{aligned} \tag{3.5}$$

Proof. Let a point (x_0, z_0, τ_0) be the first point where the functions $\hat{c}_n$ and g are equal:

$$\begin{aligned}\hat{c}_n(x_0,z_0,\tau_0) = g(x_0,\tau_0), &\qquad \hat{c}_n(x,z,\tau) < g(x,\tau), \\ 0 \leq \tau < \tau_0, \qquad 0 \leq x < x(\tau), &\qquad z = z(\tau).\end{aligned} \tag{3.6}$$

In (3.6) $x(\tau)$ and $z(\tau)$ mean the values on the characteristic passing through the point (x_0, z_0, τ_0) with $x(\tau_0) = x_0$, $z(\tau_0) = z_0$. Such a point, (x_0, z_0, τ_0), exists thanks to the continuity of $\hat{c}_n$, g, positivity of r and due to the expressions (2.2), (2.3) and (3.4). We integrate (3.1) and (3.3) along characteristics and obtain

$$
\begin{aligned}
(3.7)\quad \hat{c}_n(x_0, z_0, \tau_0) &\leq \frac{1}{2}\delta^{-1} \int_0^{\tau_0} \int_0^{x_0} K_n(x(s) - y, y) \\
&\qquad \cdot \hat{c}_n(x(s) - y, z(s), s)\hat{c}_n(y, z(s), s)\, dy\, ds \\
&< \frac{1}{2}\delta^{-1} k \int \int_0^{\tau_0} \int_0^{x_0} g(x(s) - y, s) g(y, s)\, dy\, ds \\
&= g(x_0, \tau_0).
\end{aligned}
$$

The inequality (3.7) yields the contradiction

$$\hat{c}_n(x_0, z_0, \tau_0) < g(x_0, \tau_0)$$

which proves Lemma 3.1. $\square$

By integrating (3.3) with the weight $\exp(\lambda x)$, we obtain

$$
\begin{aligned}
(3.8)\quad \delta H_\tau(\lambda, \tau) &- (1-\tau)^{-1} r(0) g(0, \tau) \\
&- (1-\tau)^{-1} \lambda \int_0^\infty \exp(\lambda x) r(x) g(x, \tau)\, dx \\
&- (1-\tau)^{-1} \int_0^\infty \exp(\lambda x) r'(x) g(x, \tau)\, dx \\
&= k H(\lambda, \tau) H_\lambda(\lambda, \tau),
\end{aligned}
$$

$$(3.9)\qquad H(\lambda, 0) = \frac{A}{a - \lambda}, \quad \lambda \in [0, a), \tau \in [0, 1).$$

In (3.8) and (3.9) we have used the notation

$$H(\lambda, \tau) = \int_0^\infty \exp(\lambda x) g(x, \tau)\, dx.$$

Taking (2.4) into account, we obtain from (3.8):

$$(3.10)\qquad \delta H_\tau - k H H_\lambda \leq \frac{A r(0)}{1 - \tau} + \frac{1 + a}{1 - \tau} R H.$$

To find an estimate for the function $H(\lambda, \tau)$ we need the following lemma.

Lemma 3.2. *A solution of the differential inequality* (3.10) *with the initial condition* (3.9) *obeys for some* $0 < \tilde{\lambda} < a$ *the following correlation*

$$H(\lambda, \tau) < F(\lambda, \tau), \quad \tau \in [0, 1), \quad 0 \le \lambda \le \tilde{\lambda},$$

where the function F *is defined as a solution to the majorant equation*

$$\delta F_\tau(\lambda, \tau) - kF(\lambda, \tau)F_\lambda(\lambda, \tau) = \frac{Ar(0)}{1-\tau} + \frac{1+a}{1-\tau} RF(\lambda, \tau), \tag{3.11}$$

$$F(\lambda, 0) = \frac{D}{a - \lambda}, \quad \lambda \in [0, \tilde{\lambda}], \quad \tau \in [0, 1], \tag{3.12}$$

$$D > A. \tag{3.13}$$

Proof. We shall prove by contradiction. Consider the family of characteristics of the problem (3.11), (3.12). Define

$$Q(\lambda_0, \tau_0) = \{(\lambda, \tau) : 0 \le \tau \le \tau_0,\ 0 \le \lambda < \lambda(\tau)\},$$

where $\lambda(\tau)$ is a value of λ on the characteristic curve $\Gamma(\lambda_0, \tau_0)$ which goes through the point (λ_0, τ_0). In addition, we suppose $0 < \lambda(0) < a$. We choose a point (λ_0, τ_0) such that

$$F(\lambda_0, \tau_0) = H(\lambda_0, \tau_0), \quad \text{but } H(\lambda, \tau) < F(\lambda, \tau)$$
$$\text{if } (\lambda, \tau) \in Q(\lambda_0, \tau_0).$$

We point out that $\tau_0 > 0$ because (3.13) holds. Let us consider the characteristic curve $\Gamma'(\lambda_0, \tau_0)$ of the problem (3.10), (3.9) with the inclusion $\Gamma'(\lambda_0, \tau_0) \in Q(\lambda_0, \tau_0)$ taken into account. Then

$$\begin{aligned}
&H(\lambda_0, t_0) \\
&\quad \le H(\lambda_0^2, 0) + \int_{\Gamma'(\lambda_0, \tau_0)} \left\{ \frac{R(1+a)}{1-\tau} H(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} d\tau \\
&\quad < H(\lambda_0^1, 0) + \int_{\Gamma(\lambda_0, \tau_0)} d\tau \left\{ \frac{R(1+a)}{1-\tau} H(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} \\
&\quad < F(\lambda_0^1, 0) + \int_{\Gamma(\lambda_0, \tau_0)} d\tau \left\{ \frac{R(1+a)}{1-\tau} F(\lambda(\tau), \tau) + \frac{Ar(0)}{1-\tau} \right\} \\
&\quad = F(\lambda_0, \tau_0).
\end{aligned}$$

In the last expression λ_0^1 and λ_0^2 are beginnings of the characteristic curves $\Gamma(\lambda_0, \tau_0)$ and $\Gamma'(\lambda_0, \tau_0)$, and $\lambda_0^1 > \lambda_0^2$. We have used as well that $\lambda < a$ and the function H increases in λ. Finally, $H(\lambda_0, \tau_0) < F(\lambda_0, \tau_0)$. This contradiction with the hypothesis $F(\lambda_0, \tau_0) = H(\lambda_0, \tau_0)$ proves Lemma 3.2. $\square$

Let us consider properties of the function $F(\lambda, \tau)$. We make in (3.11), (3.12) the change of variables

$$F(\lambda, \tau) = (1-\tau)^{-\eta} L(\lambda, \tau), \qquad \lambda \in [0, \tilde{\lambda}], \tau \in [0, 1). \tag{3.14}$$

Hence,

$$\delta L_\tau(\lambda, \tau) - k(1-\tau)^{-\eta} L(\lambda, \tau) L_\lambda(\lambda, \tau) = Ar(0)(1-\tau)^{\eta-1}, \tag{3.15}$$

$$L(\lambda, 0) = \frac{D}{a-\lambda}, \quad \lambda \in [0, \tilde{\lambda}], \tau \in [0, 1). \tag{3.16}$$

In the expression (3.14) the notation $\eta = R(1+a)\delta^{-1}$ is introduced. The characteristic equation of the problem (3.15), (3.16) has the form

$$\frac{d\lambda}{d\tau} = -k\delta^{-1}(1-\tau)^{-\eta} L(\lambda, \tau). \tag{3.17}$$

As far as on each characteristic

$$L(\lambda, \tau) = \frac{D}{a-\lambda_0} + \int_0^\tau Ar(0)\delta^{-1}(1-\tau)^{\eta-1}\, d\tau,$$

then from (3.17) we obtain

$$\frac{d\lambda}{d\tau} = -k\delta^{-1}(1-\tau)^{-\eta}\left(\frac{D}{a-\lambda_0} - Ar(0)\delta^{-1}\eta^{-1}[(1-\tau)^\eta - 1]\right).$$

Hence,

$$\begin{aligned}\lambda(\tau) = {} & \frac{Akr(0)\tau}{\delta^2\eta} + \lambda_0 \\ & - \frac{k(1-(1-\tau)^{1-\eta})}{\delta(1-\eta)}\left[\frac{Ar(0)}{\delta\eta} + \frac{D}{a-\lambda_0}\right].\end{aligned} \tag{3.18}$$

Let us ascertain whether the characteristics (3.18) with starting points λ_0^1 and λ_0^2 can intersect. If they intersect then

$$\lambda_0^1 - \lambda_0^2 = \frac{2kD}{\delta(1-\eta)}\left[1-(1-\tau)^{1-\eta}\right]\frac{\lambda_0^1-\lambda_0^2}{(a-\lambda_0^1)(a-\lambda_0^2)},$$

whence

$$1-(1-\tau)^{1-\eta} = \delta k^{-1} D^{-1}(a-\lambda_0^1)(a-\lambda_0^2)(1-\eta).$$

Consequently, the characteristic curves of the problem (3.15), (3.16) have no intersection, if

$$D < \frac{\delta a^2}{k}\left(1 - R\frac{1+a}{\delta}\right) \tag{3.19}$$

and the initial conditions are sufficiently small:

$$0 < \lambda_0^1, \lambda_0^2 < a - \sqrt{\frac{kD}{\delta - R(1+a)}}. \tag{3.20}$$

The inequality (3.19) brings us the correlation

$$R < \frac{\delta}{1+a}. \tag{3.21}$$

We should reveal now when the problem (3.15), (3.16) has smooth solution for small $\lambda > 0$ for all $\tau \in [0,1)$. This condition holds if characteristics have no intersection and $\lambda(1) > 0$. On $\tau = 1$ we obtain from (3.18)

$$\lambda_0 - \frac{kAr(0)}{\delta^2(1-\eta)} - \frac{kD}{\delta(1-\eta)(a-\lambda_0)} > 0.$$

Hence,

$$\frac{kAr(0)}{2\delta^2(1-\eta)} + \frac{a}{2} - C < \lambda_0 < \frac{kAr(0)}{2\delta^2(1-\eta)} + \frac{a}{2} + C, \tag{3.22}$$

where

$$C = \sqrt{\left(\frac{kAr(0)}{2\delta^2(1-\eta)} + \frac{a}{2}\right)^2 - \frac{k}{\delta(1-\eta)}(Aar(0)\delta^{-1} + D)}. \tag{3.23}$$

To obtain suitable $\lambda_0 > 0$ we ought to have concordance between (3.20) and (3.22). Hence, the following correlation has to take place

$$\frac{kAr(0)}{2\delta^2(1-\eta)} + \frac{a}{2} - C < a - \sqrt{\frac{kA}{\delta(1-\eta)}}. \tag{3.24}$$

We have changed D in (3.20) into A thanks to (3.13). Hence, (3.24) holds, provided

$$a > 2\sqrt{\frac{kA}{\delta(1-\eta)}} + \frac{kAr(0)}{\delta^2(1-\eta)}. \tag{3.25}$$

It is easy to see that the subradicand expression in (3.23) is positive if the stronger inequality than (3.25) is true:

$$a > 2\sqrt{\frac{kA}{\delta(1-\eta)}} + 2\frac{kAr(0)}{\delta^2(1-\eta)}. \tag{3.26}$$

The inequality (3.26) ensures the correctness of (3.20) and (3.24) and holds thanks to the condition (2.5) of Theorem 1. Consequently, for $0 \le \tau \le 1$ and small enough $\lambda > 0$, there exists a continuous function $F(\lambda, \tau)$, and the supremum $\sup_{0\le\tau\le 1} F(\lambda,\tau)$ covers the integrals

$$\sup_{t\in[0,\infty)} \int_0^\infty \exp(\lambda x) \sup_{z\in R^3} c_n(x,z,t)\, dx$$

uniformly with respect to $n \ge 1$ for sufficiently small $\lambda > 0$. Consequently, we have proved the following lemma.

Lemma 3.3 (Main). *Let the conditions of Theorem* 1 *hold. Then there exist positive constants* $\tilde{\lambda}$ *and* E *such that*

$$\sup_{0\le\lambda\le\tilde{\lambda}} \|c_n\|_\lambda \le E < \infty, \quad n \ge 1.$$

4. Proof of existence.

Lemma 4.1. *Let the conditions of the theorem hold. Then the sequence* $\{c_n\}_{n=1}^{\infty}$ *is uniformly bounded and equicontinuous on each compact in* $R^1_+ \times R^3 \times [0,T]$.

The proof is similar to the proof of the analogous lemma in [**8**].

By standard diagonal process we choose from the sequence $\{c_n\}_{n=1}^{\infty}$ a subsequence $\{c_{n'}\}_{n'=1}^{\infty}$, which converges on each compact to a continuous function $c \geq 0$. Such a subsequence exists due to Lemma 4.1. As the corollary of Lemma 3.3 we have as well

$$\sup_{0\leq\lambda\leq\tilde{\lambda}} \|c\|_\lambda \leq E.$$

Lemma 3.3 allows us to pass to limit in equation (1.1) written in the integral form. Actually, this lemma ensures that "tails" of integrals $\int_m^\infty$ tend to zero uniformly with respect to n as $m\to\infty$. Consequently, the function c satisfies the integral equation

$$\begin{aligned}c(x,z,t) &= c_{01}(x(0),z(0)) \\ &\quad + \int_{\Gamma(x,z,t)} \left\{ \frac{1}{2}\int_0^{x(s)} K(x(s)-y,y)c(x(s) \right. \\ &\qquad\qquad - y, z(s), s)c(y,z(s),s)\,dy \\ &\quad - [r'(x(s)) + \mathrm{div}_z v(x(s),z(s))]c(x(s),z(s),s) \\ &\quad \left. - c(x(s),z(s),s)\int_0^\infty K(x(s),y)c(y,z(s),s)\,dy \right\} ds,\end{aligned} \tag{4.1}$$

where $\Gamma(x,z,t)$ is the part $s \leq t$ of the characteristic going through the point (x,z,t). In (4.1) we assume that the characteristic begins at the coordinate axis $t=0$. As long as the characteristic begins at the coordinate axis $x=0$ then the expression (4.1) will be unsufficiently changed. Direct differentiation (4.1) persuades us that the function c has continuous derivatives along characteristics. Existence of a solution $c \in \Omega^+(T)$ has been proved.

5. Proof of uniqueness. We prove uniqueness by contradiction. Suppose that there are two solutions to the problem (1.1)–(1.3) $c_1, c_2 \in$

$\Omega(T)$. We make the substitution

$$c_i = (1+x)^{-1} d_i(x), \quad i = 1, 2,$$

and denote

$$u(x,t) = \sup_{z \in R^3} |d_1(x,z,t) - d_2(x,z,t)|,$$
$$\psi(x,t) = \sup_{z \in R^3} |d_1(x,z,t) + d_2(x,z,t)|.$$

From the equation (1.1) written in the integral form, we obtain

(5.2)

$$\begin{aligned} u(x_0,t_0) \le \int_{\Gamma(x_0,t_0)} \Bigg\{ \frac{1}{2} k \int_0^{x(s)} & (1 + x(x_0,t_0,s)) \\ & \cdot u(x(x_0,t_0,s) - y, s)\psi(y,s)\,dy \\ + k(1 + x(x_0,t_0,s)) & u(x(x_0,t_0,s),s) \int_0^\infty \psi(y,s)\,dy \\ + k(1 + x(x_0,t_0,s)) & \psi(x(x_0,t_0,s),s) \int_0^\infty u(y,s)\,dy \\ + M(1+x) & u(x(x_0,t_0,s),s) \Bigg\}\,ds, \quad x_0 \in R^1_+,\ 0 \le t_0 \le T. \end{aligned}$$

Here $\Gamma(x_0,t_0)$ is an orthogonal projection of the curve $\Gamma(x_0,z,t_0)$ on the plane (x,t); $x(x_0,t_0,s)$ is a current value of the variable x on the curve $\Gamma(x_0,t_0)$ which depends on the parameter $s \le t_0$. We should point out that $\Gamma(x,t)$ is an integral curve of the equation

$$dx/dt = r(x). \tag{3.5}$$

Lemma 5.1. *Let a nonnegative function $u(x,t)$ be a solution to the integral inequality (5.2). Then there exists such a continuous differentiable (by both arguments) function $f(x,t)$ that*

$$u(x,t) \le f(x,t), \quad x \in R^1_+,\ 0 \le t \le T, \tag{5.4}$$

$$f(x,0) \equiv f(0,t) \equiv 0 \tag{5.5}$$

and the function f satisfies the following differential inequality

$$
\begin{aligned}
\frac{\partial}{\partial t} f(x,t) + r(x)\frac{\partial}{\partial x} f(x,t) \le{}& \frac{1}{2}k(1+x)\int_0^x f(x-y,t)\psi(y,t)\,dy \\
&+ k(1+x)f(x,t)\int_0^\infty \psi(y,t)\,dy \\
&+ k(1+x)\psi(x,t)\int_0^\infty f(y,t)\,dy \\
&+ M(1+x)f(x,t).
\end{aligned} \tag{5.6}
$$

Proof. Let us denote the righthand side of the inequality (5.2) as $f(x_0,t_0)$ and the contents of the braces $\{.\}$ under the main integral in (5.2) as $w(x(s),s)$. Then for derivatives of the function f we obtain

$$
\frac{\partial f(x_0,t_0)}{\partial t_0} = w(x_0,t_0) + \int_0^{t_0} w'_x(x(x_0,t_0,s),s)x'_{t_0}(x_0,t_0,s)\,ds, \tag{5.7}
$$

$$
\frac{\partial f(x_0,t_0)}{\partial x_0} = \int_0^{t_0} w'_x(x(x_0,t_0,s),s)x'_{x_0}(x_0,t_0,s)\,ds. \tag{5.8}
$$

Our nearest goal is to estimate the derivative $x'_{t_0}(x_0,t_0,s)$. Integration of the equation (5.3) yields

$$
t_0 - s = Q(x_0) - Q(x(x_0,t_0,s)),
$$

where $Q(x)$ is the primitive function to $[r(x)]^{-1}$. Hence,

$$
x(x_0,t_0,s) = Q^{-1}(Q(x_0) - t_0 + s). \tag{5.9}
$$

Here the function Q^{-1} is the inverse function to Q. Utilizing the rule for differentiating of the inverse function and taking into account $Q'(x) = [r(x)]^{-1}$, we obtain

$$
x'_{t_0}(x_0,t_0,s) = -r(x(x_0,t_0,s)). \tag{5.10}
$$

Similarly from (5.9) we conclude

$$
x'_{x_0}(x_0,t_0,s) = \frac{r(x(x_0,t_0,s))}{r(x_0)}. \tag{5.11}
$$

By substituting the expressions (5.10) and (5.11) into (5.7), (5.8) and utilizing the inequality (5.2) and definition of function f, we establish (5.4). This proves Lemma 5.1.

Lemma 5.2. *Let $v(\lambda, t)$ be a real continuous function having continuous partial derivatives v_λ and $v_{\lambda\lambda}$ on*

$$D = \{0 \le \lambda \le \lambda_0, \quad 0 \le t \le T\}.$$

Assume that $\alpha(\lambda)$, $\beta(\lambda,t)$, $\gamma_1(\lambda,t)$, $\gamma_2(\lambda,t)$, $\theta_1(\lambda,t)$ and $\theta_2(\lambda,t)$ are real and continuous on D, have their continuous partial derivatives in λ and the functions $v, v_\lambda, \beta, \gamma_1, \gamma_2$ are nonnegative. Suppose that the following inequalities hold on D:

$$v(\lambda,t) \le \alpha(\lambda) + \int_0^t [\beta(\lambda,s)v_\lambda(\lambda,s) + \gamma_1(\lambda,s)v(\lambda,s) + \theta_1(\lambda,s)]\,ds, \tag{5.12}$$

$$v_\lambda(\lambda,t) \le \alpha_\lambda(\lambda) + \int_0^t \left\{ \frac{\partial}{\partial\lambda}\,[\beta(\lambda,s)v_\lambda(\lambda,s) + \gamma_1(\lambda,s)v(\lambda,s) + \theta_1(\lambda,s)] + \gamma_2(\lambda,s)v(\lambda,s) + \theta_2(\lambda,s) \right\} ds. \tag{5.13}$$

Let

$$c_0 = \sup_{0\le\lambda\le\lambda_0} \alpha, \qquad c_1 = \sup_D \beta,$$
$$c_2 = \sup_D(\beta\gamma_2 + \gamma_1), \qquad c_3 = \sup_D(\beta\theta_2 + \theta_1).$$

Then

$$v(\lambda,t) \le c_0 \exp(c_2 t) + \frac{c_3}{c_2}(\exp(c_2 t) - 1)$$

in any region $R \subset D$:

$$R = \{(\lambda,t) : 0 \le t \le t' < T'; \lambda_1 - c_1 t \le \lambda \le \lambda_0 - c_1 t, 0 < \lambda_1 < \lambda_0\}$$

where $T' = \min\{(\lambda_1/c_1),\ T\}$.

Proof. Let us denote $w(\lambda, t)$ the righthand side of the inequality (5.12). By differentiating in t and λ, we obtain from (5.12) and (5.13):

$$\begin{aligned} w_t &= \beta v_\lambda + \gamma_1 v + \theta_1 \\ &\leq \beta(w_\lambda + \gamma_2 v + \theta_2) + \gamma_1 v + \theta_1 \\ &\leq \beta w_\lambda + (\beta\gamma_2 + \gamma_1) w + (\beta\theta_2 + \theta_1). \end{aligned} \tag{5.14}$$

Hence, for the derivative along characteristic $d\lambda/dt = -\beta$ we obtain from (5.14)

$$\frac{dw}{dt} \leq (\beta\gamma_2 + \gamma_1) w + (\beta\theta_2 + \theta_1). \tag{5.15}$$

Let us denote

$$u(t) = \bar{c_0} \exp(c_2 t) + (\bar{c_3}/c_2)(\exp(c_2 t) - 1)$$

with $\bar{c_0} > c_0$, $\bar{c_3} > c_3$. Obviously, $u(0) > w(\lambda, 0)$ for all $\lambda \in [0, \lambda_0]$. Let $(\hat{\lambda}, \hat{t})$ be the first point on a characteristic curve where $w = u$. Then in the point $(\hat{\lambda}, \hat{t})$

$$\frac{d(u - w)}{dt} \leq 0$$

and, consequently,

$$w_t - \beta w_\lambda \geq u_t = c_2 w + \bar{c_3}. \tag{5.16}$$

Comparing (5.16) with (5.15) we obtain the contradiction

$$(\beta\gamma_2 + \gamma_1) w + (\beta\theta_2 + \theta_1) > c_2 w + c_3 \geq (\beta\gamma_2 + \gamma_1) w + (\beta\theta_2 + \theta_1).$$

This proves Lemma 5.2. It is worth pointing out that the majorant function $u(t)$ can decrease in t if $c_3 < 0$ and $c_0 c_2 + c_3 < 0$. We should emphasize also that Lemma 5.2 was formulated without proof in [**9**] on $\theta_1 = \theta_2 = \gamma_2 = 0$ and additional assumptions on nonnegativity of functions in (5.12), (5.13).

Proof of Theorem 1. Let us integrate (5.6) in x with weight $\exp(\lambda x)$ and $x \exp(\lambda x)$, $0 \leq \lambda \leq \tilde{\lambda}$. Then for

$$\begin{aligned} G(\lambda, t) &= \int_0^\infty \exp(\lambda x) f(x, t)\, dx, \\ \Psi(\lambda, t) &= \int_0^\infty \exp(\lambda x) \psi(x, t)\, dx \end{aligned}$$

we obtain the following correlations with (5.5) taken into account

$$\begin{aligned} G_t(\lambda,t) &\le \left(\frac{3}{2}k\Psi(\lambda,t)+M\right)G_\lambda(\lambda,t) \\ &\qquad + \left[\frac{5}{2}k(\Psi+\Psi_\lambda)+R(2+\tilde{\lambda})+M\right]G(\lambda,t), \\ G_{\lambda t}(\lambda,t) &\le \frac{\partial}{\partial\lambda}\biggl(\left(\frac{3}{2}k\Psi(\lambda,t)+M\right)G_\lambda(\lambda,t) \\ &\qquad\qquad + \left[\frac{5}{2}k(\Psi+\Psi_\lambda)+R(2+\tilde{\lambda})+M\right]G(\lambda,t)\biggr). \end{aligned}$$

Thanks to (5.5) and Lemma 5.2 we obtain

$$\int_0^\infty \exp(\lambda x) f(x,t) = 0 \tag{5.17}$$

in the region R defined in Lemma 5.2. Since $f(x,t)$ is continuous, $f(x,t)=0$ for $0\le t\le t'$, $0\le x<\infty$. Consequently, the integral (5.17) is equal to zero not only in R but for all $0\le\lambda\le\tilde{\lambda}$, $0\le t\le t'$. Applying the same reasonings to the interval $[t',2t']$ we conclude that $f(x,t)=0$ for $0\le t\le 2t'$, $0\le x<\infty$, and, continuing this process, we establish that $f(x,t)\equiv 0$. Utilizing (5.4) completes the proof of Theorem 1. □

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Department of Mathematics, Seoul National University, Seoul, 151-742, Korea

Institute of Numerical Mathematics, Russian Academy of Sciences, Lenunski prospecr 32A, 117334, Moscow, Russia