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A MODIFIED DISCRETE SPECTRAL COLLOCATION METHOD FOR FIRST KIND INTEGRAL EQUATIONS WITH LOGARITHMIC KERNEL

J. SARANEN

ABSTRACT. Here we propose a modification of the discrete Galerkin method considered by Atkinson for Symm's equation with the logarithmic kernel. Our method has the computational complexity of the trigonometric collocation but it still retains the stability and the convergence properties of the original trigonometric Galerkin method. In particular, the method can be applied with any data having arbitrarily bad non-smoothness. Numerical experiments confirm our results.

1. Introduction. The essence of this paper consists of a simple remark on the discrete Galerkin method discussed by Atkinson in [**1**]. Atkinson considers a fully discrete approximation of the spectral Galerkin method [**3**] for Symm's integral equation with the logarithmic kernel. The method in [**1**] can also be viewed as a further discretization of the trigonometric collocation method. If the solution is smooth, this scheme retains the excellent convergenge properties of the trigonometric Galerkin method, but yields a low order convergence, or in the worst case is not at all applicable, if the solution (or equivalently the data) is not smooth. To be more precise, the minimal smoothness requirement is the continuity of the given right-hand side function.

Here we propose a slight modification of Atkinson's discretization. In our approach the given data is replaced by its L^2-projection onto the subspace of the trigonometric functions. We will show that this modification preserves all the convergence properties of the original trigonometric Galerkin method. In particular, it has optimal order convergence in the full range of the related Sobolev spaces. Moreover, the coefficient matrix remains same as in Atkinson's method. Thus, if the Fourier coefficients of the right-hand side are easy to calculate, the computational cost is very close to that of [**1**]. By using the well-known cosine transformation [**6, 7**] our modification carries also over

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to the case of an open arc, as it will be explained in Section 4. We have conducted several numerical experiments confirming our results. Some of those experiments are shown in Section 5. It is obvious that analogous modifications as presented here can be adapted in other situations as well; for a family of quadrature methods see the work [**5**] of Saranen and Sloan.

2. Preliminaries. Consider Symm's integral equation

$$-\frac{1}{\pi}\int_\Gamma \log|x-y|u_\Gamma(y)ds_y = f_\Gamma(x), \quad x\in\Gamma \tag{2.1}$$

on a smooth closed Jordan curve $\Gamma\subset\mathbf{R}^2$. We fix the 2π-periodic parametric representation $\theta\mapsto x(\theta):\mathbf{R}\to\Gamma$ of the curve such that $|x'(\theta)|>0,\ \theta\in\mathbf{R}$. With the notations

$$\begin{cases} u(\theta) := & u_\Gamma(x(\theta))|x'(\theta)| \\ f(\theta) := & f_\Gamma(x(\theta)) \end{cases} \tag{2.2}$$

equation (2.1) reads

$$(Lu)(\theta) = f(\theta), \quad \theta\in\mathbf{R} \tag{2.3}$$

where

$$(Lu)(\theta) = -\frac{1}{\pi}\int_{-\pi}^{\pi}\log|x(\theta)-x(\phi)|u(\phi)d\phi. \tag{2.4}$$

From now on we require the equation (2.3) be uniquely solvable (in appropriate function spaces). This property is guaranteed assuming that the capacity or the conformal radius $c(\Gamma)$ of Γ differs from one. Assuming $c(\Gamma)\neq 1$ the operator L defines an isomorphism $L:H^s\to H^{s+1},\ s\in\mathbf{R}$ between the Sobolev spaces

$$H^s = \left\{u \mid \|u\|_s := [|\hat{u}(0)|^2 + \sum_{k\neq 0}|k|^{2s}|\hat{u}(k)|^2]^{1/2} < \infty\right\} \tag{2.5}$$

of the 2π-periodic distributions u on $\mathbf{R}$. Here u has the Fourier series representation

$$u(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbf{Z}} \hat{u}(k) e^{ik\theta}, \qquad \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(\theta) e^{-ik\theta} d\theta. \tag{2.6}$$

Next we recall the discrete Galerkin method considered by Atkinson in [1]. Let

$$(u, v) = \int_{-\pi}^{\pi} u(\theta) \overline{v(\theta)} d\theta \tag{2.7}$$

be the duality between the spaces H^s, H^{-s} extending the H^0-inner product and let T^h be the space of 2π-periodic trigonometric functions u_h,

$$u_h(\theta) = \sum_{k \in \Lambda_n} c_k e^{ik\theta}, \quad c_k \in \mathbf{C}, \tag{2.8}$$

where $\Lambda_n = \{k \in \mathbf{Z} | -n/2 < k \leq n/2\}$, $h = 2\pi/n$, $n \in \mathbf{N}$. The operator L is decomposed as

$$L = A + B, \tag{2.9}$$

where the main part

$$(Au)(\theta) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \log |x_\rho(\theta) - x_\rho(\phi)| u(\phi) d\phi, \tag{2.10}$$

$x_\rho(\theta) = \rho e^{i\theta}$, $\rho = e^{-1/2}$, has the Fourier representation

$$(Au)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbf{Z}} \frac{\hat{u}(k)}{|\underline{k}|} e^{ik\theta} \tag{2.11}$$

with $|\underline{k}| = \max\{|k|, 1\}$, and where

$$(Bu)(\theta) = \int_{-\pi}^{\pi} b(\theta, \phi) u(\phi) d\phi \tag{2.12}$$

is an integral operator having the smooth kernel

$$b(\theta,\phi)=\begin{cases}-\frac{1}{\pi}\log\left|\dfrac{x(\theta)-x(\phi)}{x_\rho(\theta)-x_\rho(\phi)}\right|, & \theta\neq\phi \pmod{2\pi}\\ -\frac{1}{\pi}\log\left|\dfrac{x'(\theta)}{x'_\rho(\theta)}\right|, & \theta=\phi \pmod{2\pi}.\end{cases} \tag{2.13}$$

The operator B defines a continuous mapping $B: H^r\to H^s$ for all $r,s\in\mathbf{R}$. The discrete Galerkin method of [**1**] consists of a full discretization of the trigonometric Galerkin problem

$$u_h\in T^h : (Lu_h,v)=(f,v),\quad v\in T^h \tag{2.14}$$

such that the inner product $(\cdot,\cdot)$ is replaced by the discrete inner product

$$(u,v)_h := h\sum_{j\in\Lambda_n} u(\theta_j)\overline{v(\theta_j)}, \tag{2.15}$$

$\theta_j=jh$, and the operator L is replaced by the operator L_h,

$$(L_hu)(\theta)=(Au)(\theta)+(B_hu)(\theta), \tag{2.16}$$

with

$$(B_hu)(\theta):=h\sum_{j\in\Lambda_n} b(\theta,\theta_j)u(\theta_j). \tag{2.17}$$

Thus the method of Atkinson [**1**] becomes

$$u_h\in T^h : (L_hu_h,v)_h=(f,v)_h,\quad v\in T^h \tag{2.18}$$

which is equivalent to the trigonometric collocation problem

$$u_h\in T^h : (L_hu_h)(\theta_j)=f(\theta_j),\quad j\in\Lambda_n. \tag{2.19}$$

Due to the Fourier representation (2.11) it is inexpensive to set up the matrix equation for the unknowns c_k of u_h in (2.8). Moreover, if the function u is smooth, the convergence $u_h\to u$ is of high order; even exponential if u is analytic. This fact has also been observed by numerical experiments [**1**]. However, the method (2.19) uses pointwise values of the function f at the meshpoints θ_j. For the error analysis of (2.19) it is natural to assume that f is continuous; in fact the stronger condition $f \in C^4$ is imposed in [**1**].

In this paper we propose and analyze the method

$$u_h \in T^h : (L_h u_h, v)_h = (f, v), \quad v \in T^h, \tag{2.20}$$

which can be used for any given data f being a 2π-periodic distribution. Moreover, from the computational point of view the method (2.20) differs just slightly from (2.18); the left-hand side matrices coincide, only the right-hand side vector has to be calculated in a different manner. An equivalent form of (2.20), related to (2.19), reads

$$u_h \in T^h : (L_h u_h)(\theta_j) = (P_h f)(\theta_j), \quad j \in \Lambda_n \tag{2.21}$$

where $P_h f$ is the (extended) L^2-projection

$$(P_h f)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \Lambda_n} \hat{f}(k) e^{ik\theta} \tag{2.22}$$

of f onto the trigonometric subspace T^h. In the next section we will show that the method (2.20) is stable and has optimal order convergence in the all Sobolev spaces H^s, $s \in \mathbf{R}$.

3. Analysis. To analyze (2.20) let $Q_h : H^s \to T^h$, $s > 1/2$ be the trigonometric interpolation operator defined by the uniquely solvable equations

$$(Q_h u)(\theta_j) = u(\theta_j), \quad j \in \Lambda_n. \tag{3.1}$$

First we observe that the equations (2.20) can be given in an equivalent form as

$$u_h \in T^h : Q_h L_h u_h = P_h L u, \tag{3.2}$$

since for any solution u_h of (2.20) holds

$$(L_h u_h, v)_h = (Lu, v) = (P_h Lu, v) = (P_h Lu, v)_h, \quad v \in T^h \tag{3.3}$$

which coincides with (3.2). In the last equality above we have used the simple fact that

$$(w, v) = (w, v)_h \tag{3.4}$$

for all $w, v \in T^h$. In the further analysis we use some basic results concerning the accuracy of the projections Q_h and P_h;

Lemma 1. *The interpolation operator Q_h satisfies*

$$\|u - Q_h u\|_t \le c h^{s-t} \|u\|_s, \tag{3.5}$$

$0 \le t \le s,\ s > 1/2.$

Lemma 2. *The L^2-projection P_h satisfies*

$$\|u - P_h u\|_t \le c h^{s-t} \|u\|_s, \tag{3.6}$$

$t, s \in \mathbf{R},\ t \le s.$

For (3.5) see [4, Lemma 2.1]. The proof of (3.6) is even more elementary. Moreover, we need the following result of Saranen and Sloan.

Lemma 3. ([5, Lemma 5, Appendix]). *Let $s, t \in \mathbf{R}$ and $\tau > 0$. Then there exists a constant $c > 0$ such that*

$$\|(B - B_h) u_h\|_t \le c h^{\tau} \|u_h\|_s, \tag{3.7}$$

for all $u_h \in T^h$.

Now we are able to show the stability of our collocation-Galerkin method. In the proof we use the inverse estimate

$$\|u_h\|_s \le c h^{t-s} \|u_h\|_t, \quad u_h \in T^h \tag{3.8}$$

which is valid for all $t \le s$.

Proposition 3.1. *The problem* (2.20) *has for any* $u \in H^s,\ s \in \mathbf{R}$ *a unique solution* $u_h \in T^h$ *for* $0 < h \le h_0$*, if* h_0 *is small enough* (h_0 *is independent on* u *and* s). *Moreover, there holds the stability estimate*

$$\|u_h\|_s \le c\|u\|_s, \quad s \in \mathbf{R}. \tag{3.9}$$

Proof. It is enough to show the estimate (3.9) *assuming* that $u_h \in T^h$ is a solution of (2.20). Then the *existence* of a unique solution follows, since (2.20) reduces to solution of a $n \times n$ system of equations, and by (3.9) any solution u_h with zero right-hand side vector vanishes. First we use the estimate

$$\|u_h\|_s \le c\|Lu_h\|_{s+1}, \tag{3.10}$$

with the decompositions

$$Q_h L u_h = P_h L u + Q_h(B - B_h)u_h, \tag{3.11}$$
$$(I - Q_h)Lu_h = (I - Q_h)Bu_h \tag{3.12}$$

which follow from (3.2) and (2.9), (2.11). By Lemma 2

$$\|P_h L u\|_{s+1} \le c\|Lu\|_{s+1} \le c\|u\|_s. \tag{3.13}$$

The inverse estimate (3.8) together with Lemma 1 and Lemma 3 yields

$$\begin{aligned} \|Q_h(B - B_h)u_h\|_{s+1} &\le ch^{-\max(0,s)}\|Q_h(B - B_h)u_h\|_1 \\ &\le ch^{-\max(0,s)}\|(B - B_h)u_h\|_1 \\ &\le ch^{\tau}\|u_h\|_s \end{aligned} \tag{3.14}$$

where $\tau > 0$. Moreover,

$$\|(I - Q_h)Bu_h\|_{s+1} \le ch\|Bu_h\|_{\max(1,s+2)} \le ch\|u_h\|_s. \tag{3.15}$$

Combining (3.10)–(3.15) we obtain the required estimate (3.9) if h is small enough. $\square$

Next we consider the convergence.

Theorem 3.1. *For the solution u_h of* (2.20) *holds the asymptotic error estimate*

$$\|u-u_h\|_t \le ch^{s-t}\|u\|_s \tag{3.16}$$

for all $t \le s$.

Proof. We start with

$$\|u-u_h\|_t \le c\|L(u-u_h)\|_{t+1} \tag{3.17}$$

and decompose $L(u-u_h)$ as

$$L(u-u_h) = P_hL(u-u_h) + (I-P_h)L(u-u_h). \tag{3.18}$$

Now, the equation (2.20) and the property $Q_hAu_h = P_hAu_h = Au_h$ give

$$P_hL(u-u_h) = (Q_h-I)B_hu_h - (P_h-I)B_hu_h + P_h(B_h-B)u_h, \tag{3.19}$$

$$(I-P_h)L(u-u_h) = (I-P_h)Lu - (I-P_h)Bu_h. \tag{3.20}$$

Lemma 3 together with the mapping property of B implies

$$\|B_hu_h\|_r \le c\|u_h\|_s, \quad u_h \in T^h \tag{3.21}$$

for all $r, s \in \mathbf{R}$. Taking $t+1 \ge 0$ and choosing r such that $r+1 > 1/2,\ r \ge s$ we thus obtain by Lemma 1

$$\|(Q_h-I)B_hu_h\|_{t+1} \le ch^{r-t}\|B_hu_h\|_{r+1} \le ch^{s-t}\|u_h\|_s. \tag{3.22}$$

If $t+1 \le 0$, we correspondingly get

$$\begin{aligned}\|(Q_h-I)B_hu_h\|_{t+1} &\le \|(Q_h-I)B_hu_h\|_0 \le ch^r\|B_hu_h\|_r \\ &\le ch^{s-t}\|u_h\|_s\end{aligned} \tag{3.23}$$

where r is chosen such that $r > 1/2,\ r \ge s-t$. Hence the estimate

$$\|(Q_h-I)B_hu_h\|_{t+1} \le ch^{s-t}\|u_h\|_s \tag{3.24}$$

holds for all t, s. Moreover, by Lemma 2, Lemma 3

(3.25)
$$\|(P_h - I)B_h u_h\|_{t+1} \le ch^{s-t}\|B_h u_h\|_{s+1} \le ch^{s-t}\|u_h\|_s,$$

(3.26)
$$\|P_h(B_h - B)u_h\|_{t+1} \le ch^{s-t}\|u_h\|_s.$$

Combining (3.19), (3.24)–(3.26) we get

(3.27)
$$\|P_h L(u - u_h)\|_{t+1} \le ch^{s-t}\|u_h\|_s.$$

Finally, by Lemma 2

(3.28)
$$\|(I - P_h)Lu\|_{t+1} \le ch^{s-t}\|Lu\|_{s+1} \le ch^{s-t}\|u\|_s,$$

(3.29)
$$\|(I - P_h)Bu_h\|_{t+1} \le ch^{s-t}\|Bu_h\|_{s+1} \le ch^{s-t}\|u_h\|_s.$$

From (3.17), (3.18) and (3.20) estimates (3.27)–(3.29) yield

(3.30)
$$\|u - u_h\|_t \le ch^{s-t}(\|u\|_s + \|u_h\|_s), \quad t \le s$$

which together with the stability result (3.9) proves our assertion (3.16). $\square$

In the above analysis we have assumed that the Fourier coefficients $\hat{f}(k),\ k \in \Lambda_n$ have been determined exactly. Since this is not always possible, we shortly discuss the case where the approximations $\tilde{f}(k)$ of $\hat{f}(k)$ are used. We assume that these approximations satisfy the accuracy

(3.31)
$$|\hat{f}(k) - \tilde{f}(k)| \le ch^{\alpha}|\underline{k}|^{\beta}, \quad k \in \Lambda_n$$

with the numbers $\alpha > 1/2, \beta \in \mathbf{R}$. Define $\tilde{P}_h f \in T^h$,

(3.32)
$$(\tilde{P}_h f)(\theta) := \frac{1}{\sqrt{2\pi}} \sum_{k \in \Lambda_n} \tilde{f}(k) e^{ik\theta}.$$

The computed solution $\tilde{u}_h \in T^h$ satisfies

(3.33)
$$Q_h L_h \tilde{u}_h = \tilde{P}_h f,$$

which together with (3.2) yields

$$Q_h L_h(u_h - \tilde{u}_h) = P_h f - \tilde{P}_h f = P_h L L^{-1}(P_h f - \tilde{P}_h f). \tag{3.34}$$

By the stability (3.9) we thus obtain

$$\|u_h - \tilde{u}_h\|_t \le c\|L^{-1}(P_h f - \tilde{P}_h f)\|_t \le c\|P_h f - \tilde{P}_h f\|_{t+1}. \tag{3.35}$$

Now we deduce from (3.31)

$$\begin{aligned} \|P_h f - \tilde{P}_h f\|_{-\beta}^2 &= \sum_{k\in\Lambda_n} |\hat{f}(k) - \tilde{f}(k)|^2 |\underline{k}|^{-2\beta} \\ &\le c(f) h^{2\alpha-1}. \end{aligned} \tag{3.36}$$

Using (3.36) and the inverse estimates (3.8) we derive

$$\|P_h f - \tilde{P}_h f\|_r \le c(f) h^{\alpha - 1/2 - \max(\beta + r, 0)} \tag{3.37}$$

for any $r \in \mathbf{R}$. Combining (3.35), (3.37) with the error estimate (3.16) we have

$$\|u - \tilde{u}_h\|_t \le c\left(h^{s-t}\|u\|_s + c(f) h^{\alpha - 1/2 - \max(\beta + t + 1, 0)}\right). \tag{3.38}$$

In the last section we will have an example where α can chosen to be arbitrarily large. For such a case (3.38) still yields the optimal order convergence

$$\|u - \tilde{u}_h\|_t = O(h^{s-t}), \tag{3.39}$$

for $u \in H^s$.

Let us finally compare the result (3.16) with the convergence of the trigonometric collocation solution. Slightly modifying the analysis of Theorem 3.1 one obtains

Theorem 3.2. *Assume $u \in H^s$, $s > -1/2$. Then, for sufficiently small h there exists a unique solution u_h of* (2.19), *and the asymptotic error estimate*

$$\|u - u_h\|_t \le c h^{s-t}\|u\|_s \tag{3.40}$$

is valid for $-1 \le t \le s$.

4. Open arc. Here we consider the case of an open Jordan arc Γ. Assume that Γ has the infinitely differentiable representation $t \mapsto x(t)$, $-1 \le t \le 1$ such that $|x'(t)| > 0$ and that the endpoints of Γ don't coincide, i.e. $x(-1) \ne x(1)$. Using the cosine substitution $t = \cos\theta$, $0 \le \theta \le \pi$ and defining $a : [0, \pi] \to \Gamma$ by $a(\theta) = x(\cos\theta)$, we obtain from (2.1)

$$-\frac{1}{\pi}\int_0^\pi w(\varphi)\log|a(\theta) - a(\varphi)|d\varphi = g(\theta), \quad 0 \le \theta \le \pi \tag{4.1}$$

where

$$\begin{cases} w(\theta) := u_\Gamma(a(\theta))|x'(\cos\theta)|\,|\sin\theta|, \\ g(\theta) := f_\Gamma(a(\theta)). \end{cases} \tag{4.2}$$

Since these functions can be understood as 2π-periodic even functions on $\mathbf{R}$, equation (4.1) becomes

$$(L_e w)(\theta) := -\frac{1}{2\pi}\int_{-\pi}^{\pi} w(\varphi)\log|a(\theta) - a(\varphi)|d\varphi = g(\theta), \quad \theta \in \mathbf{R} \tag{4.3}$$

where L_e is decomposed as $L_e = A_e + B_e$ with

$$(A_e w)(\theta) := -\frac{1}{2\pi}\int_{-\pi}^{\pi} w(\varphi)\log[2e^{-1}|\cos\theta - \cos\varphi|]d\varphi, \tag{4.4}$$

$$(B_e w)(\theta) := \int_{-\pi}^{\pi} w(\varphi) b_e(\theta, \varphi)d\varphi, \tag{4.5}$$

where the kernel

$$b_e(\theta, \varphi) = \begin{cases} -\dfrac{1}{2\pi}\log\left|\dfrac{e}{2}\cdot\dfrac{a(\theta) - a(\varphi)}{\cos\theta - \cos\varphi}\right|, & \theta \ne -\varphi, \varphi \pmod{2\pi} \\ -\dfrac{1}{2\pi}\log\left|\dfrac{e}{2}\cdot x'(\cos\theta)\right|, & \theta = -\varphi, \varphi \pmod{2\pi} \end{cases} \tag{4.6}$$

is a smooth function on $\mathbf{R}\times\mathbf{R}$, and is 2π-periodic with respect to both variables.

Now the equation (4.3) will be considered in the Sobolev spaces H_e^s of 2π-periodic even functions on $\mathbf{R}$;

$$H_e^s = \{w \in H^s \mid \widehat{w}(-k) = \widehat{w}(k),\ k \in \mathbf{Z}\}. \tag{4.7}$$

Function $w \in H_e^s$ has also the cosine expansion

$$w(\theta) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{2}\widehat{w}(0) + \sum_{k=1}^{\infty}\widehat{w}(k)\cos k\theta\right], \tag{4.8}$$

$$\widehat{w}(k) = \sqrt{\frac{2}{\pi}}\int_0^{\pi} w(\theta)\cos k\theta d\theta \tag{4.9}$$

and the norm is given by

$$\|w\|_s = \left(|\widehat{w}(0)|^2 + \sum_{k=1}^{\infty}|k|^{2s}|\widehat{w}(k)|^2\right)^{1/2}. \tag{4.10}$$

Since the capacity of Γ is assumed to be different from one, the operator $L_e : H_e^s \to H_e^{s+1}$ is an isomorphism for all $s \in \mathbf{R}$, [**7**]. Moreover, the operator A_e has the cosine series representation

$$(A_e w)(\theta) = \sqrt{\frac{2}{\pi}}\left[\frac{1}{2}\widehat{w}(0) + \sum_{k=1}^{\infty}\frac{\widehat{w}(k)}{|k|}\cos k\theta\right] \tag{4.11}$$

and $B_e : H_e^r \to H_e^s$ is continuous for all $r, s \in \mathbf{R}$.

For the numerical solution of the equation (4.1) we proceed quite analogously with the treatment of a closed curve. However, since the functions w and g are even functions, it is natural to impose the same condition also on the approximate solution. For this purpose we assume that $n = 2m+1$ is an odd integer. Then we have the following properties. The set $\Lambda_n = \{k \in \mathbf{Z} \mid -m \le k \le m\}$ and the mesh $\{\theta_j \mid j \in \Lambda_n\} = \{jh \mid -m \le j \le m\}$ are symmetric with respect to the origin. This implies that the interpolation and the L^2-projection of even functions are trigonometric even functions. More precisely, letting

$$T_e^h = \{w_h \in T^h \mid w_h(\theta) = w_h(-\theta),\ \theta \in \mathbf{R}\} \tag{4.12}$$

be the space of the trigonometric even functions, we have $Q_h w \in T_e^h$, if $w \in H_e^s,\ s > 1/2$ and $P_h w \in T_e^h$ for $w \in H_e^s,\ s \in \mathbf{R}$. Note that the space T_e^h has only $m+1$ free parameters in contrast to the full space T^h with the dimension $n = 2m+1$. The functions w_h in T_e^h are given by the pure cosine representation,

$$w_h(\theta) = {\sum_{k=0}^{m}}' a_k \cos k\theta, \quad a_k \in \mathbf{C}, \tag{4.13}$$

where the notation $\sum'$ means that the first term of the sum is to be halved.

Now we define the discretized operator $L_{e,h}$ as

$$L_{e,h} w = A_e w + B_{e,h} w, \quad w \in H_e^s,\ s > 1/2 \tag{4.14}$$

with

$$\begin{aligned}(B_{e,h} w)(\theta) &= h \sum_{j \in \Lambda_n} b_e(\theta, \theta_j) w(\theta_j) \\ &= 2h {\sum_{j=0}^{m}}' b_e(\theta, \theta_j) w(\theta_j).\end{aligned} \tag{4.15}$$

Analogously to (2.18) and (2.20) we consider the discretized methods for (4.1);

$$w_h \in T_e^h : (L_{e,h} w_h, v)_h = (g, v)_h, \quad v \in T_e^h, \tag{4.16}$$

$$w_h \in T_e^h : (L_{e,h} w_h, v)_h = (g, v), \quad v \in T_e^h. \tag{4.17}$$

For functions $w, v \in H_e^s,\ s > 1/2$ the discrete inner product $(w, v)_h$ takes the form

$$(w, v)_h = h \sum_{j \in \Lambda_n} w(\theta_j) \overline{v(\theta_j)} = 2h {\sum_{j=0}^{m}}' w(\theta_j) \overline{v(\theta_j)}. \tag{4.18}$$

By the unique solvability of the cosine interpolation in T_e^h at the points $\{\theta_j\}_0^m$, we find that (4.16) is equivalent to

$$w_h \in T_e^h : (L_{e,h} w_h)(\theta_j) = g(\theta_j), \quad 0 \le j \le m, \tag{4.19}$$

and (4.17) is equivalent to

$$w_h \in T_e^h : (L_{e,h} w_h)(\theta_j) = (P_h g)(\theta_j), \quad 0 \le j \le m. \tag{4.20}$$

Thus, in these problems we have $m+1$ equations for the unknowns a_k, $0 \le k \le m$ of the representation (4.13). Now, Lemmas 1–3 of Section 3 are applicable for the analysis of problems (4.19) and (4.20); especially (3.7) holds when B, B_h are replaced by $B_e, B_{e,h}$, and T_e^h replaces T^h. Repeating the analysis of Section 3, we obtain

Theorem 4.1. *For any $w \in H_e^s$, $s \in \mathbf{R}$ the problem (4.20) has a unique solution $w_h \in T_e^h$ for $0 < h \le h_0$, if h_0 is small enough, and there holds the asymptotic error estimate*

$$\|w - w_h\|_t \le ch^{s-t} \|w\|_s \tag{4.21}$$

for all $t \le s$.

Theorem 4.2. *Assume $w \in H_e^s$, $s > -1/2$. Then, for sufficiently small h, there exists a unique solution $w_h \in T_e^h$ of (4.19) with the estimate*

$$\|w - w_h\|_t \le ch^{s-t} \|w\|_s, \tag{4.22}$$

$-1 \le t \le s$.

5. Examples. In this section we apply our results to solution of the Dirichlet type boundary value problem

$$\begin{cases} \Delta\Phi = 0 & \text{in } \Omega, \\ \Phi = f_\Gamma & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbf{R}^2$ is a smooth bounded domain. Assuming that the conformal radius of Γ differs from 1 we can represent the solution Φ by the single layer representation

$$\Phi(x) = -\frac{1}{\pi} \int_\Gamma \log|x-y| u_\Gamma(y) ds_y, \quad x \in \Omega \tag{5.2}$$

which leads to solution of Symm's integral equation (2.1). Using the previous notations (2.2) we obtain

$$\Phi(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \log|x - x(\varphi)|u(\varphi)d\varphi. \tag{5.3}$$

Having found, from (2.20), the approximation $u_h \in T^h$ of the density u, we define the corresponding approximation

$$\Phi_h(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \log|x - x(\varphi)|u_h(\varphi)d\varphi \tag{5.4}$$

of the potential. Finally, we calculate the numerical approximation $\Phi_h^h(x)$ of $\Phi_h(x)$ by using the composite trapezoidal rule such that

$$\Phi_h^h(x) = -\frac{h}{\pi}\sum_{j\in\Lambda_n} \log|x - x(\theta_j)|u_h(\theta_j). \tag{5.5}$$

We estimate the asymptotic accuracy of $\Phi_h^h(x)$ as an approximation of $\Phi(x)$ as follows. Let $u \in H^s$ be the solution of (2.3) for some fixed index $s \in \mathbf{R}$. First we have by Theorem 3.2

$$\begin{aligned} |\Phi(x) - \Phi_h(x)| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} \log|x - x(\varphi)|(u_h(\varphi) - u(\varphi))d\varphi\right| \\ &\le c\|u - u_h\|_t \le ch^{s-t}\|u\|_s \end{aligned} \tag{5.6}$$

for $t \in \mathbf{R}$, $t \le s$. Secondly, from [5, Lemma 4 in Appendix] and (3.9) follows

$$|\Phi_h(x) - \Phi_h^h(x)| \le ch^\tau\|u_h\|_s \le ch^\tau\|u\|_s \tag{5.7}$$

for any $\tau > 0$. Since $t \le s$ in (5.6) can be chosen freely, we obtain by (5.6), (5.7)

$$|\Phi(x) - \Phi_h^h(x)| \le ch^\tau\|u\|_s \tag{5.8}$$

for all $\tau > 0$. Thus, the asymptotic accurary of the numerical potential $\Phi_h^h(x)$ is of an arbitrary high order in powers of h, independent of the smoothness of the function u.

We have carried out several numerical experiments testing the convergence of the potential outside of the boundary. Below we present the computed results in three different cases, where the given boundary function is piecewise continuous, Dirac's distribution at a boundary point, or has a logarithmic singularity. All these experiments confirm the excellent convergence obtained in this paper. In the computations we have used an even number n of the discretization points $\theta_j,\ j \in \Lambda_n$ such that $n = 4, 8, 16, 32, 64, 128, 256$. The calculations were carried out by using the double precision accuracy. In the most experiments, it turned out that the maximal accuracy limited by the double precision is achieved already with relative small values of n, such as $n = 128$ or even 64. It was also observed that having reached this limit, the computed solution begins to lose accuracy when the number n increases. In the following examples we give the absolute error $|\Phi(x) - \Phi_h^h(x)|$ of the potential and the experimental convergence rate

$$\text{ecr} = \ln\left(\frac{|\Phi(x) - \Phi_{2h}^{2h}(x)|}{|\Phi(x) - \Phi_h^h(x)|}\right) \Big/ \ln 2. \tag{5.9}$$

Example 1. Consider the case where the given boundary function f_Γ is piecewise constant. The numerical experiments are shown for the circle $\Gamma = \{x : |x| = 2\}$, and the function $f(\theta)$ is defined by

$$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi. \end{cases} \tag{5.10}$$

The corresponding potential $\Phi(x)$ (also known as the harmonic measure of the arc $\{x \in \Gamma : \mathrm{Im} x > 0\}$) has the representation

$$\Phi(x) = \frac{1}{2} + \frac{1}{\pi}[\arg(2 + x) - \arg(2 - x)], \quad x \in \Omega \tag{5.11}$$

where the argument function is chosen such that $|\arg(2 \pm x)| < \pi/2,\ x \in \Omega$. The convergence of the potential was tested at several interior points $x \in \Omega$. Below, in Table 1, we illustrate the results for the choices $x_1 = (0,1)$, $x_2 = (1,1)$ and $x_3 = (1.8, 0.1)$ giving there the absolute errors $|\delta_h \Phi(x_i)|$ and the experimental convergence rates. It is clearly seen that the experimental convergence rate increases unboundedly until the double precision accuracy is achieved. Moreover, the convergence becomes slower when x approaches a discontinuity at the boundary.

In Table 2 we give the corresponding results when applying the trigonometric collocation with $f(0) = f(\pi) = 1/2$ at the discontinuity points. There are no convergence results available for this method (with a discontinuous right-hand side). However, we observe quadratic convergence. The same phenomenon was observed when applying the collocation at the midpoints of the mesh.

TABLE 1. Piecewise continuous data. Collocation-Galerkin. Circle, $r = 2$. Absolute error. Experimental convergence rate. Points $x_1 = (0,1)$, $x_2 = (1,1)$, $x_3 = (1.8, 0.1)$.

n	$\lvert\delta_h \Phi(x_1)\rvert$	ecr	$\lvert\delta_h \Phi(x_2)\rvert$	ecr	$\lvert\delta_h \Phi(x_3)\rvert$	ecr
4	$0.429 \cdot 10^{-1}$		$0.560 \cdot 10^{-1}$		0.311	
8	$0.690 \cdot 10^{-2}$	2.64	$0.385 \cdot 10^{-1}$	0.54	0.162	0.94
16	$0.230 \cdot 10^{-3}$	4.91	$0.531 \cdot 10^{-2}$	2.86	$0.941 \cdot 10^{-1}$	0.76
32	$0.467 \cdot 10^{-6}$	8.94	$0.176 \cdot 10^{-3}$	4.92	$0.399 \cdot 10^{-1}$	1.24
64	$0.364 \cdot 10^{-11}$	16.97	$0.354 \cdot 10^{-6}$	8.96	$0.416 \cdot 10^{-2}$	3.26
128	$0.634 \cdot 10^{-16}$	15.81	$0.274 \cdot 10^{-11}$	16.92	$0.748 \cdot 10^{-4}$	5.80

TABLE 2. Piecewise continuous data. Collocation. Circle, $r = 2$. Absolute error. Experimental convergence rate. Points $x_1 = (0,1)$, $x_2 = (1,1)$, $x_3 = (1.8, 0.1)$.

n	$\lvert\delta_h \Phi(x_1)\rvert$	ecr	$\lvert\delta_h \Phi(x_2)\rvert$	ecr	$\lvert\delta_h \Phi(x_3)\rvert$	ecr
4	$0.321 \cdot 10^{-1}$		0.111		0.314	
8	$0.784 \cdot 10^{-2}$	2.04	$0.144 \cdot 10^{-1}$	2.94	0.168	0.91
16	$0.195 \cdot 10^{-2}$	2.01	$0.422 \cdot 10^{-2}$	1.77	0.104	0.69
32	$0.490 \cdot 10^{-3}$	1.99	$0.106 \cdot 10^{-2}$	2.00	$0.550 \cdot 10^{-1}$	0.92
64	$0.123 \cdot 10^{-3}$	2.00	$0.265 \cdot 10^{-3}$	1.99	$0.168 \cdot 10^{-1}$	1.71
128	$0.307 \cdot 10^{-4}$	2.00	$0.664 \cdot 10^{-4}$	2.00	$0.391 \cdot 10^{-2}$	2.10

Example 2. Here we consider numerical determination of the Poisson kernel $P(x,y),\ x\in\Omega,\ y\in\Gamma$, which for all $y\in\Gamma$ is a harmonic function on Ω such that for given f on Γ (f continuous) the function

$$\Phi(x) := \int_\Gamma P(x,y)f(y)ds_y \tag{5.12}$$

satisfies the boundary value problem (5.1). In the case of the disc $\Omega = \{x|\ |x|<r\}$ the Poisson kernel is given by

$$P(x,y) = \frac{r^2-|x|^2}{2\pi r|x-y|^2}. \tag{5.13}$$

For more general domains the existence and construction of the kernel $P(x,y)$ can be derived, e.g., by means of the conformal mappings. Formally, taking the Dirac distribution $f=\delta_{x_0},\ x_0\in\Gamma$ in (5.12) we find that the corresponding potential $\Phi(x)=P(x,x_0)$ satisfies the boundary value problem

$$\begin{cases} \Delta_x P(x,x_0) = 0 & \text{in } \Omega \\ P(x,x_0)|_\Gamma = \delta_{x_0} & \text{on } \Gamma. \end{cases} \tag{5.14}$$

Our trigonometric collocation-Galerkin method provides an effective solution of (5.14). Representing the solution as

$$P(x,x_0) = -\frac{1}{\pi}\int_\Gamma u_{\Gamma,x_0}(y)\ln|x-y|ds_y, \quad x\in\Omega \tag{5.15}$$

the density function u_{Γ,x_0} satisfies

$$-\frac{1}{\pi}\int_\Gamma u_{\Gamma,x_0}(y)\ln|x-y|ds_y = \delta_{x_0}(x), \quad x\in\Gamma. \tag{5.16}$$

For the computations we have chosen the function $f(\theta)$ in (2.4) to be the 2π-periodic Dirac's distribution corresponding the point $\theta_0=0$. Then the potential is, $\Phi(x)=P(x,x_0)|x'(\theta_0)|, x_0=x(\theta_0)$. In the Table 3 we give the absolute errors and the experimental convergence rates at the points $x_1=(0,1),\ x_2=(1,1),\ x_3=(1.8,0.1)$ for the circle with the radius $r=2$. The singularity is located at the point

$x_0 = (2, 0)$. Similar phenomena occur as in the previous example. In particular the accuracy decreases and the convergence slows down when the singularity is approached.

Example 3. In this example the right-hand side function $f(\theta)$ in (2.3) has a logarithmic singularity at the point $\theta_0 = 0$, such that

$$f(\theta) = -\frac{1}{\pi}\cos\theta \cdot \ln|x(\theta) - x(\theta_0)|. \tag{5.17}$$

In the case of a circle the Fourier coefficients of $f(\theta)$ can be given explicitly, but for the general smooth curves we need to use numerical integration. To compute the approximations $\tilde{f}(k)$ of $\hat{f}(k)$ we proceed as follows. Abbreviating

$$g(\theta) := -\frac{1}{\pi}\ln|x(\theta) - x(\theta_0)| \tag{5.18}$$

we have

$$\hat{f}(k) = \frac{1}{2}(\hat{g}(k-1) + \hat{g}(k+1)). \tag{5.19}$$

Now, by using (2.11)–(2.13) we obtain
(5.20)

$$\begin{aligned}
\hat{g}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\Big(-\frac{1}{\pi}\ln|x(\theta)-x(\theta_0)|\Big)e^{-ik\theta}d\theta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\Big(-\frac{1}{\pi}\ln|x_\rho(\theta)-x_\rho(\theta_0)|\Big)e^{-ik\theta}d\theta + \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} b(\theta)e^{-ik\theta}d\theta \\
&= \frac{1}{2\pi|\underline{k}|} + \hat{b}(k),
\end{aligned}$$

with the smooth function $b(\theta) := b(\theta, \theta_0)$. We determine the approximation $\tilde{f}(k)$ using (5.19), (5.20) such that the Fourier coefficient $\hat{b}(k)$ is replaced by the composite trapezoidal approximation $\tilde{b}(k)$;

$$\tilde{b}(k) := \frac{\sqrt{2\pi}}{n}\sum_{j\in\Lambda_n} b(\theta_j)e^{-ik\theta_j}. \tag{5.21}$$

It is easy to see that the approximation $\tilde{f}(k)$ satisfies the estimate (3.31) with $\beta = 0$ for any $\alpha > 0$. Hence we obtain the optimal order convergence (3.39) for the boundary and thus the estimate

$$|\Phi(x) - \tilde{\Phi}_h^h(x)| \le ch^\tau, \tag{5.22}$$

$c = c(u, \tau, x)$, for any $\tau > 0$. Here the approximation $\tilde{\Phi}_h^h(x)$ is determined by the formula (5.5) replacing $u_h(\theta_j)$ by $\tilde{u}_h(\theta_j)$.

The computed results are shown in the Table 4 for the ellipse $\Gamma = \{x(\theta) = (6\cos\theta, 2\sin\theta)\}$ at the points $x_1 = (-5, 0)$, $x_2 = (0, 1)$ and $x_3 = (5, 0)$. Now the exact potential is not known. Therefore we have replaced the value $\Phi(x_i)$ by the value $\Phi_h^h(x_i)$, $n = 256$, in calculation of the absolute error $|\delta_h \Phi(x_i)|$ and the experimental convergence rate.

TABLE 3. Dirac's distribution. Circle, $r = 2$.
Absolute error. Experimental convergence rate.
Points $x_1 = (0, 1)$, $x_2 = (1, 1)$, $x_3 = (1.8, 0.1)$.

n	$\lvert\delta_h \Phi(x_1)\rvert$	ecr	$\lvert\delta_h \Phi(x_2)\rvert$	ecr	$\lvert\delta_h \Phi(x_3)\rvert$	ecr
4	0.134		0.885		13.18	
8	$0.136 \cdot 10^{-1}$	3.30	0.159	2.47	9.62	0.45
16	$0.534 \cdot 10^{-3}$	4.67	$0.201 \cdot 10^{-1}$	2.98	5.52	0.80
32	$0.113 \cdot 10^{-5}$	8.89	$0.802 \cdot 10^{-3}$	4.65	1.93	1.52
64	$0.890 \cdot 10{-}11$	16.94	$0.173 \cdot 10^{-5}$	8.85	0.252	2.94
128	$0.347 \cdot 10^{-12}$	4.70	$0.135 \cdot 10^{-10}$	16.97	$0.523 \cdot 10^{-2}$	5.59
256	$0.413 \cdot 10^{-12}$	-0.251	$0.158 \cdot 10^{-11}$	3.09	$0.343 \cdot 10^{-5}$	10.57

TABLE 4. Logarithmic singularity. Ellipse, half axis 6 and 2.
Absolute error. Experimental convergence rate.
Points $x_1 = (-5, 0)$, $x_2 = (0, 1)$, $x_3 = (5.0, 0.0)$.

n	$\lvert\delta_h \Phi(x_1)\rvert$	ecr	$\lvert\delta_h \Phi(x_2)\rvert$	ecr	$\lvert\delta_h \Phi(x_3)\rvert$	ecr
4	$0.139 \cdot 10^{-1}$		0.135		$0.985 \cdot 10^{-1}$	
8	$0.138 \cdot 10^{-1}$	$0.10 \cdot 10^{-3}$	$0.318 \cdot 10^{-1}$	2.09	$0.136 \cdot 10^{-1}$	2.85
16	$0.201 \cdot 10^{-3}$	6.11	$0.335 \cdot 10^{-2}$	3.24	$0.143 \cdot 10^{-3}$	6.58
32	$0.659 \cdot 10^{-6}$	8.25	$0.104 \cdot 10^{-3}$	5.01	$0.110 \cdot 10^{-5}$	7.02
64	$0.140 \cdot 10^{-10}$	15.52	$0.221 \cdot 10^{-6}$	8.88	$0.692 \cdot 10^{-11}$	17.27

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Department of Mathematics, University of Oulu, 90570 Oulu, Finland