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GENERALIZED CONDITIONAL YEH-WIENER INTEGRALS AND A WIENER INTEGRAL EQUATION

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ABSTRACT. Let $Q = [0,S] \times [0,T]$ and let $h \in L_2(Q)$. In this paper we evaluate conditional Yeh-Wiener integrals of the type

$$E\bigg[\exp\bigg\{\int_0^t\int_0^s \phi(\sigma,\tau,\int_0^\tau\int_0^\sigma h(u,v)\,dx(u,v))\,d\sigma\,d\tau\bigg\} \mid \int_0^t\int_0^s h(u,v)\,dx(u,v)=\xi\bigg].$$

The method we use to evaluate these conditional integrals is first to define a sample path-valued conditional Yeh-Wiener integral and show that it satisfies a Wiener integral equation. We next obtain a series solution to this Wiener integral equation which we then use to evaluate the above conditional Yeh-Wiener integral.

1. Introduction. For $Q = [0,S] \times [0,T]$, let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s,t)$ on Q such that $x(0,t) = x(s,0) = 0$ for every (s,t) in Q. Yeh [**10**] defined a Gaussian measure m_y on $C(Q)$ (later modified in [**11**]) such that as a stochastic process $\{x(s,t),(s,t) \in Q\}$ has mean $E[x(s,t)] \equiv \int_{C(Q)} x(s,t)m_y(dx) = 0$ and covariance $E[x(s,t)x(u,v)] = \min\{s,u\}\min\{t,v\}$. Let $C_w \equiv C[0,T]$ denote the standard Wiener space on $[0,T]$ with Wiener measure m_w. In [**12**], Yeh introduced the concept of the conditional Wiener integral of F given $X, E[F \mid X]$, and for the case $X(x) = x(T)$ obtained some very useful results including a Kac-Feynman integral equation.

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A very important class of functions in quantum mechanics are functions on $C[0,T]$ of the type

$$G(x) = \exp\left\{ \int_0^T \theta(s, x(s))\, ds \right\}$$

where $\theta : [0,T] \times \mathbf{R} \to \mathbf{C}$.

Yeh [**12**] gives a series expansion for the conditional Wiener integral

$$E\left[\exp\left\{ \int_0^t \theta(s, x(s))\, ds \right\} | x(t) = \xi \right]. \tag{1}$$

The corresponding problem for Yeh-Wiener space, namely finding

$$E\left[\exp\left\{ \int_0^t \int_0^s \phi(\sigma, \tau, x(\sigma, \tau))\, d\sigma\, d\tau \right\} \mid x(s,t) = \xi \right] \tag{2}$$

turned out to be a very difficult one. The standard approach is to find an integral equation involving the expression (2) and then solve the integral equation. Many attempted this approach without much success because the resulting integral equations were quite complicated and hence very difficult to solve; as an example, see [**2**, Theorem 2.1].

In [**8**], Park and Skoug used a different approach to solve the problem. They first defined a sample path-valued conditional Yeh-Wiener integral of the type

$$E\left[\exp\left\{ \int_0^t \int_0^s \phi(\sigma, \tau, x(\sigma, \tau))\, d\sigma\, d\tau \right\} \mid x(s, \cdot) = \psi(\cdot) \right], \tag{3}$$

which satisfies a Wiener integral equation similar to that of Cameron and Storvick [**1**]. The Wiener integral equation is then solved to evaluate (3), and then (2) is evaluated by integrating (3) appropriately.

The main purpose of this paper is to evaluate conditional Yeh-Wiener integrals of the type

$$\begin{aligned} E\Bigg[\exp\left\{ \int_0^t \int_0^s \phi\left(\sigma, \tau, \int_0^\tau \int_0^\sigma h(u,v)\, dx(u,v)\right) d\sigma\, d\tau \right\} \\ \mid \int_0^t \int_0^s h(u,v)\, dx(u,v) = \xi \Bigg], \end{aligned} \tag{4}$$

where $h \in L_2(Q)$.

It is clear that (2) is a special case of (4) with $h \equiv 1$ on Q. However, as we will see below, the evaluation of (4) is much more involved than the evaluation of (2). Also, the results in [**8**] follow from the results in this paper by simply choosing h to be identically one on Q.

2. Generalized sample path-valued conditional Yeh-Wiener integrals. For $h \in L_2(Q)$ define the stochastic integral z by

$$z(x) \equiv z(x;s,t) \equiv z_h(x;s,t) = \int_0^t \int_0^s h(u,v)\,dx(u,v) \tag{5}$$

for $x \in C(Q)$ and $(s,t) \in (Q)$. Then $z(x;\cdot,\cdot)$ is a Gaussian process with mean zero and covariance

$$E[z(x;s,t)z(x;s',t')] = \int_0^{t\wedge t'} \int_0^{s\wedge s'} h^2(u,v)\,du\,dv,$$

where $t \wedge t' = \min\{t,t'\}$.

Since the covariance function of $z(x;\cdot,\cdot)$ is continuous, we may assume that almost every sample path of $z(x;\cdot,\cdot)$ is in $C(Q)$.

For h in $L_2(Q)$, define $a(\cdot,\cdot)$ by

$$a(s,t) = \int_0^s h^2(u,t)\,du, \quad (s,t) \in Q. \tag{6}$$

We start by establishing two lemmas.

Lemma 1. *If $k \in L_\infty[0,T]$, then the stochastic integral $\int_0^t k(v)\,d_v z(x;s,v)$, $(s,t) \in Q$ exists for a.e. $x \in C(Q)$, and*

$$\int_0^t k(v) d_v z(x;s,v) = \int_0^t \int_0^s k(v)h(u,v)\,dx(u,v). \tag{i}$$

In particular,

$$\int_0^t \frac{a(s,v)}{a(S,v)} d_v z(x;S,v) = \int_0^t \int_0^S \frac{a(s,v)}{a(S,v)} h(u,v)\,dx(u,v), \quad 0 \le s \le S. \tag{ii}$$

Proof. The formula (i) follows from the definition of $z(x;s,t)$, while (ii) follows from the fact that $0 \le a(s,v)/a(S,v) \le 1$ for every $v \in [0,T]$ as $0 \le a(s,v) \le a(S,v)$. We tacitly use the convention $0/0 = 0$ when $a(S,v) = 0$. $\square$

Lemma 2. *For $h \in L_2(Q)$, assume that $F(z(x))$ is a Yeh-Wiener integrable function of x. Then, for almost every $\eta \in C[0,T]$, we have*

$$\begin{aligned}\text{(i)}\quad & E[F(z(x)) \mid z(x;S,\cdot) = \eta(\cdot)] \\ &= E\biggl[F\biggl(z(x;*,\cdot) - \int_0^{\cdot} \frac{a(*,v)}{a(S,v)} d_v z(x;S,v) \\ &\qquad + \int_0^{\cdot} \frac{a(*,v)}{a(S,v)}\, d\eta(v)\biggr)\biggr],\end{aligned}$$

and

$$\begin{aligned}\text{(ii)}\quad & E[F(z(x)) \mid z(x;S,T) = \xi] \\ &= E\biggl[F\biggl(z(x;*,\cdot) - \biggl(\int_0^{\cdot} a(*,v)\,dv \Big/ \int_0^T a(S,v)\,dv\biggr) \\ &\qquad \cdot [z(x;S,T) - \xi]\biggr)\biggr].\end{aligned}$$

Proof. (i) Under the conditioning $z(x;S,\cdot) = \eta(\cdot)$, we have

$$\int_0^t \frac{a(*,v)}{a(S,v)} d_v z(x;S,v) = \int_0^t \frac{a(*,v)}{a(S,v)}\, d\eta(v).$$

Thus, it is sufficient to show that $z(x;S,t)$ and $z(x;s,t) - \int_0^t (a(s,v)$

$a(S,v))\, d_v z(x;S,v)$ are independent for every $(s,t) \in Q$. Since they are Gaussian processes, we need only show that they are uncorrelated.

This follows easily from the calculation

$$
\begin{aligned}
&E\left[z(x;S,t)\left\{z(x;s,t)-\int_0^t \frac{a(s,v)}{a(S,v)} d_v z(x;S,v)\right\}\right]\\
&= E\left[z(x;S,t)z(x;s,t) - z(x;S,t)\int_0^t\int_0^S \frac{a(s,v)}{a(S,v)} h(u,v)\,dx(u,v)\right]\\
&= \int_0^t\int_0^s h^2(u,v)\,du\,dv - \int_0^t\int_0^S \frac{h^2(u,v)a(s,v)}{a(S,v)}\,du\,dv\\
&= \int_0^t\int_0^s h^2(u,v)\,du\,dv - \int_0^t \frac{a(s,v)}{a(S,v)}\left(\int_0^S h^2(u,v)\,du\right)dv\\
&= \int_0^t a(s,v)\,dv - \int_0^t \frac{a(s,v)}{a(S,v)} a(S,v)\,dv = 0.
\end{aligned}
$$

(ii) Under the given conditioning, we have that $z(x;S,T)=\xi$ for almost every $x \in C(Q)$. Thus it suffices to show that the Gaussian processes $z(x;S,T)$ and $z(x;s,t) - (\int_0^t a(s,v)\,dv/\int_0^T a(S,v)\,dv)z(x;S,T)$ are independent for all $(s,t)\in Q$. To prove, simply proceed as in (i) above. $\square$

The following theorem plays an important role throughout this paper.

Theorem 1. *Let $h, a,$ and z be as in Lemma 2, and let $w(\cdot)$ be the standard Wiener process on $[0,T]$. Then*

(i) $E_w\left\{E\left[F(z(x)) \mid z(x;S,\cdot) = \int_0^{\cdot} \sqrt{a(S,v)}\,dw(v)\right]\right\} = E[F(z(x))]$

and

(ii) $E_w\{E[F(z(x)) \mid z(x;S,\cdot) = \eta(\cdot)]\} = E[F(z(x)) \mid z(x;S,T) = \xi]$,

where

$$
\begin{aligned}
(7)\quad \eta(\cdot) = &\int_0^{\cdot} \sqrt{a(S,v)}\,dw(v)\\
&- \left[\int_0^{\cdot} a(S,v)dv \Big/ \int_0^T a(S,v)\,dv\right]\left[\int_0^T \sqrt{a(S,v)}\,dw(v) - \xi\right].
\end{aligned}
$$

Proof. (i) Using formula (i) of Lemma 2, we have

$$(8)\quad E\Big[F(z(x)) \mid z(x;S,\cdot)=\int_0^{\cdot}\sqrt{a(S,v)}\,dw(v)\Big] \\ = E\Big[F\Big(z(x;*,\cdot)-\int_0^{\cdot}\frac{a(*,v)}{a(S,v)}\,d_vz(x;S,v)+\int_0^{\cdot}\frac{a(*,v)}{\sqrt{a(S,v)}}\,dw(v)\Big)\Big].$$

Let

$$(9)\quad Z(s,t)=z(x;s,t)-\int_0^t\frac{a(s,v)}{a(S,v)}\,d_vz(x;S,v)+\int_0^t\frac{a(s,v)}{\sqrt{a(S,v)}}\,dw(v).$$

Then $Z(\cdot,\cdot)$, as a process depending on the Yeh-Wiener process $x(\cdot,\cdot)$ and the Wiener process $w(\cdot)$, is a Gaussian process on Q with mean zero and covariance

$$E[Z(s,t)Z(s',t')]=\int_0^{t\wedge t'}\int_0^{s\wedge s'}h^2(u,v)\,du\,dv.$$

Thus, $Z(\cdot,\cdot)$ and $z(x;\cdot,\cdot)$ are equivalent processes, and hence, (i) follows from equation (8).

(ii) Using formula (i) of Lemma 2 with $\eta(\cdot)$ given by equation (7), it follows that

$$(10)\quad E_w\{E[F(z(x)) \mid z(x;S,\cdot)=\eta(\cdot)]\} \\ = E_w\Big\{E_x\Big[F\Big(z(x;*,\cdot)-\int_0^{\cdot}\frac{a(*,v)}{a(S,v)}d_vz(x;S,v) \\ +\int_0^{\cdot}\frac{a(*,v)}{\sqrt{a(S,v)}}\,dw(v) \\ -\Big[\int_0^{\cdot}a(*,v)\,dv\Big/\int_0^{T}a(S,v)\,dv\Big] \\ \cdot\Big[\int_0^T\sqrt{a(S,v)}dw(v)-\xi\Big]\Big)\Big]\Big\} \\ = E_w\Big\{E_x\Big[F\Big(Z(*,\cdot)-\Big[\int_0^{\cdot}a(*,v)\,dv\Big/\int_0^{T}a(S,v)\,dv\Big] \\ \cdot\Big[\int_0^T\sqrt{a(S,v)}dw(v)-\xi\Big]\Big)\Big]\Big\},$$

where $Z(\cdot,\cdot)$ is given by equation (9).

On the other hand, formula (ii) of Lemma 2 gives

$$(11) \quad E[F(z(x)) \mid z(x;S,T)=\xi]$$
$$= E\bigg\{F\bigg(z(x;*,\cdot) - \bigg[\int_0^{\cdot} a(*,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg][z(x;S,T)-\xi]\bigg)\bigg\}.$$

By checking the covariance, one can easily see that the two processes

$$Z(*,\cdot) - \bigg[\int_0^{\cdot} a(*,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg]\int_0^T \sqrt{a(S,v)}\,dw(v)$$

and

$$z(x;*,\cdot) - \bigg[\int_0^{\cdot} a(*,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg] z(x;S,T)$$

are equivalent processes. Thus, equation (ii) in Theorem 1 follows readily from equations (10) and (11). $\square$

3. Some examples. In this section we use the theory developed in Section 2 to evaluate the generalized sample path-valued conditional Yeh-Wiener integral of certain functions.

Example 1. Let $F(z(x)) = \int_Q z(x;s,t)\,ds\,dt$. Then, using Lemmas 1 and 2 and the Fubini theorem,

$$\begin{aligned} E\bigg(\int_Q & z(x;s,t)\,ds\,dt \mid z(x;S,\cdot)=\eta(\cdot)\bigg) \\ &= E\bigg(\int_Q \bigg[z(x;s,t) - \int_0^t\int_0^S \frac{a(s,v)}{a(S,v)} h(u,v)\,dx(u,v) \\ &\qquad + \int_0^t \frac{a(s,v)}{a(S,v)}\,d\eta(v)\bigg]\,ds\,dt\bigg) \\ &= \int_Q \int_0^t \frac{a(s,v)}{a(S,v)}\,d\eta(v)\,ds\,dt. \end{aligned}$$

In particular, if $h(s,t)\equiv 1$ on Q, then $a(s,t)=s$ on Q, and hence,

$$\int_Q\int_0^t \frac{a(s,v)}{a(S,v)}\,d\eta(v)\,ds\,dt = \frac{S}{2}\int_0^T \eta(t)\,dt$$

which agrees with the computation in [**8**].

Example 2. Let $F(z(x)) = \int_Q z^2(x;s,t)\,ds\,dt$. Then, proceeding as in Example 1 above, we obtain that

$$\begin{aligned} E\bigg(\int_Q z^2(x;s,t)\,ds\,dt \mid z(x;S,\cdot)=\eta(\cdot)\bigg) \\ = \int_Q \bigg[\int_0^t a(s,v)\,dv - \int_0^t \frac{a^2(s,v)}{a(S,v)}\,dv \\ + \bigg(\int_0^t \frac{a(s,v)}{a(S,v)}\,d\eta(v)\bigg)^2\bigg]\,ds\,dt. \end{aligned}$$

Our next example is somewhat more complicated, but it illustrates Lemma 2 and Theorem 1 quite nicely.

Example 3. Let $F(z(x)) = \exp\{\int_Q z(x;s,t)\,dx\,dt\}$. Then, using Lemmas 1 and 2 and the Fubini theorem,

$$\begin{aligned} J \equiv{} & E\bigg[\exp\bigg\{\int_Q z(x;s,t)\,ds\,dt\bigg\} \mid z(x;S,\cdot)=\eta(\cdot)\bigg] \\ ={} & E\bigg[\exp\bigg\{\int_Q \bigg(z(x;s,t) - \int_0^t\int_0^S \frac{a(s,v)}{a(S,v)}h(u,v)\,dx(u,v) \\ & + \int_0^t \frac{a(s,v)}{a(S,v)}\,d\eta(v)\bigg)\,ds\,dt\bigg\}\bigg] \\ ={} & E\bigg[\exp\bigg\{\int_Q (T-t)(S-s)h(s,t)\,dx(s,t) \\ & - \int_Q \frac{(T-t)}{a(S,t)}\bigg(\int_0^S a(s',t)\,ds'\bigg)h(s,t)\,dx(s,t) \\ & + \int_0^T \frac{(T-t)}{a(S,t)}\bigg(\int_0^S a(s,t)\,ds\bigg)\,d\eta(t)\bigg\}\bigg] \end{aligned} \tag{12}$$

$$
\begin{aligned}
&= \exp\bigg\{ \int_0^T \frac{(T-t)}{a(S,t)} \bigg(\int_0^S a(s,t)\,ds \bigg) \,d\eta(t) \bigg\} \\
&\cdot E\bigg[\exp\bigg\{ \int_Q (T-t)h(s,t)\bigg[(S-s) - \int_0^S \frac{a(s',t)}{a(S,t)}\,ds'\bigg] \,dx(s,t) \bigg] \\
&= \exp\{B/2\} \exp\bigg\{ \int_0^T \frac{(T-t)}{a(S,t)} \bigg(\int_0^S a(s,t)\,ds \bigg) \,d\eta(t)
\end{aligned}
$$

where

(13)

$$
\begin{aligned}
B &\equiv \int_Q \bigg\{ (T-t)h(s,t)\bigg[(S-s) - \int_0^S \frac{a(s',t)}{a(S,t)}\,ds'\bigg]\bigg\}^2 \,ds\,dt \\
&= \int_Q [(T-t)(S-s)h(s,t)]^2 \,ds\,dt - \int_0^T \frac{(T-t)^2}{a(S,t)} \bigg(\int_0^S a(s,t)\,ds \bigg)^2 dt \\
&= 2\int_Q (T-t)^2(S-s)a(s,t)\,ds\,dt - \int_0^T \frac{(T-t)^2}{a(S,t)} \bigg(\int_0^S a(s,t)\,ds \bigg)^2 dt.
\end{aligned}
$$

In particular, if $h(s,t) \equiv 1$ on Q, then $a(s,t) = s$ on Q, and hence the last expression in equation (12) becomes

$$
\exp\bigg\{ \frac{S^3T^3}{72} + \frac{S}{2} \int_0^T \eta(t)\,dt \bigg\}
$$

which agrees with the corresponding results in [**8**]. Furthermore, if we replace $\eta(\cdot)$ by $\int_0^{\cdot} \sqrt{a(S,v)}\,dw(v)$ in equation (12) and integrate in w over C_w, then

$$
\begin{aligned}
E_w[J] &= \exp\bigg\{ \frac{B}{2} \bigg\} E_w\bigg[\exp\bigg\{ \int_0^T \frac{(T-t)}{a(S,t)} \bigg(\int_0^S a(s,t)\,ds \bigg) \\
&\qquad\qquad \cdot \sqrt{a(S,t)}\,dw(t) \bigg\} \bigg] \\
&= \exp\bigg\{ \frac{1}{2} \int_Q [(T-t)(S-s)h(s,t)]^2 \,ds\,dt \bigg\} \\
&= \exp\bigg\{ \int_Q (T-t)^2(S-s)a(s,t)\,ds\,dt \bigg\},
\end{aligned}
$$

which agrees with $E[\exp\{\int_Q z(x;s,t)\,ds\,dt\}] = E[F(z(x))]$. Thus, we have verified directly that (i) of Theorem 1 holds for the function $F(z(x)) = \exp\{\int_Q z(x;s,t)\,ds\,dt\}$.

Next we will verify directly that (ii) of Theorem 1 also holds for this example. So let $\eta(\cdot)$ be given by (7). Then, integrating in w over C_w using (12), we obtain

(14)

$$
\begin{aligned}
K \equiv E_w\bigg[E\bigg(&\exp\bigg\{\int_Q z(x;s,t)\,ds\,dt\bigg\} \mid z(x;S,\cdot)=\eta(\cdot)\bigg)\bigg] \\
&= \exp\bigg\{\frac{B}{2}\bigg\}E_w\bigg(\exp\bigg\{\int_0^T \frac{(T-t)}{a(S,t)}\bigg(\int_0^S a(s,t)\,ds\bigg)\sqrt{a(S,t)}\,dw(t) \\
&\quad - \bigg[\int_0^T \frac{(T-t)}{a(S,t)}\bigg(\int_0^S a(s,t)\,ds\bigg)a(S,t)\,dt \Big/ \int_0^T a(S,v)\,dv\bigg] \\
&\quad \cdot \bigg[\int_0^T \sqrt{a(S,v)}\,dw(v)-\xi\bigg]\bigg\}\bigg) \\
&= \exp\bigg\{\frac{B}{2}+\xi\int_0^T (T-t)\int_0^S a(s,t)\,ds\,dt \Big/ \int_0^T a(S,v)\,dv\bigg\} \\
&\quad \cdot E_w\bigg(\exp\bigg\{\int_0^T \bigg[\frac{(T-t)}{\sqrt{a(S,t)}}\int_0^S a(s,t)\,ds \\
&\quad - \sqrt{a(S,t)}\int_0^T (T-v)\int_0^S a(s,v)\,ds\,dv \Big/ \int_0^T a(S,v)\,dv\bigg]\,dw(t)\bigg\}\bigg) \\
&= \exp\bigg\{\frac{B}{2}+\xi\int_0^T (T-t)\int_0^S a(s,t)\,ds\,dt \Big/ \int_0^T a(S,v)\,dv+\frac{C}{2}\bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
C &= \int_0^T \bigg[\frac{(T-t)}{\sqrt{a(S,t)}}\int_0^S a(s,t)\,ds \\
&\qquad - \sqrt{a(S,t)}\int_0^T (T-v)\int_0^S a(s,v)\,ds\,dv \Big/ \int_0^T a(S,v)\,dv\bigg]^2 dt \\
&= \int_0^T \frac{(T-t)^2}{a(S,t)}\bigg[\int_0^S a(s,t)\,ds\bigg]^2 dt \\
&\qquad - \bigg\{\int_Q (T-t)a(s,t)\,ds\,dt\bigg\}^2 \Big/ \int_0^T a(S,v)\,dv.
\end{aligned}
$$

On the other hand, using Lemma 1, we obtain

$$
E\bigg[\exp\bigg\{\int_Q z(x;s,t)\,ds\,dt\bigg\} \mid z(x;S,T)=\xi\bigg] \tag{15}
$$

$$
\begin{aligned}
&= E\bigg[\exp\bigg\{\int_Q \bigg[z(x;s,t) \\
&\quad - \bigg(\int_0^t a(s,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg)(z(x;S,T)-\xi)\bigg]\,ds\,dt\bigg\}\bigg] \\
&= \exp\bigg\{\xi \int_Q \bigg(\int_0^t a(s,v)\,dv\bigg)\,ds\,dt \Big/ \int_0^T a(S,v)\,dv\bigg\} \\
&\cdot E\bigg(\exp\bigg\{\int_Q \bigg[(T-t)(S-s) \\
&\quad - \int_Q (T-v)a(u,v)\,du\,dv \Big/ \int_0^T a(S,v)\,dv\bigg] h(s,t)\,dx(s,t)\bigg\}\bigg) \\
&= \exp\bigg\{\xi \int_0^T (T-t)\int_0^S a(s,v)\,ds\,dt \Big/ \int_0^T a(S,v)\,dv\bigg\} \\
&\cdot \exp\bigg\{\frac{1}{2}\int_Q \bigg[(T-t)(S-s) \\
&\quad - \int_Q (T-v)a(u,v)\,du\,dv \Big/ \int_0^T a(S,v)\,dv\bigg]^2 h^2(s,t)\,ds\,dt\bigg\} \\
&= \exp\bigg\{\xi \int_0^T (T-t)\int_0^S a(s,t)\,ds\,dt \Big/ \int_0^T a(S,v)\,dv\bigg\} \\
&\cdot \exp\bigg(\frac{1}{2}\bigg\{\int_Q [(T-t)(S-s)h(s,t)]^2\,ds\,dt \\
&\quad - \bigg[\int_Q (T-v)a(u,v)\,du\,dv\bigg]^2 \Big/ \int_0^T a(S,v)\,dv\bigg\}\bigg) \\
&= K
\end{aligned}
$$

by equation (14), and so (ii) of Theorem 1 holds for $F(z(x)) = \exp\{\int_Q z(x;s,t)\,ds\,dt\}$.

4. Evaluation of conditional Yeh-Wiener integrals. Let $\theta(s,t,u)$ be a bounded continuous function on $Q \times \mathbf{R}$, and let

$$\theta(\sigma, z(x;\sigma,\cdot)) = \int_0^T \phi(\sigma,\tau,z(x;\sigma,\tau))\,d\tau.$$

Then

$$F(s,z(x)) \equiv \exp\bigg\{\int_0^s \int_0^T \phi(\sigma,\tau,z(x;\sigma,\tau))\,d\tau\,d\sigma\bigg\}$$
$$= \exp\bigg\{\int_0^s \theta(\sigma,z(x;\sigma,\cdot))\,d\sigma\bigg\}.$$

Since $\partial F(s,z(x))/\partial s = \theta(s,z(x;s,\cdot))F(s,z(x))$, by integrating over $[0,s]$, $0 < s \le S$, it follows that

$$F(s,z(x)) - 1 = \int_0^s \theta(\sigma,z(x;\sigma,\cdot))F(\sigma,z(x))\,d\sigma. \tag{16}$$

If we take the sample path-valued conditional expectation under the conditioning $z(x;s,\cdot) = \psi(\cdot)$ and then use the Fubini theorem, (16) yields

$$\begin{aligned}(17)\quad & E[F(s,z(x)) \mid z(x;s,\cdot) = \psi(\cdot)] \\ &= 1 + \int_0^s E[\theta(\sigma,z(x;\sigma,\cdot))F(\sigma,z(x)) \mid z(x;s,\cdot) = \psi(\cdot)]\,dx.\end{aligned}$$

Using formula (i) of Lemma 2, for $0 < \sigma \le s \le S$, equation (17) yields

$$\begin{aligned}(18)\quad & E[\theta(\sigma,z(x;\sigma,\cdot))F(\sigma,z(x)) \mid z(x;s,\cdot) = \psi(\cdot)] \\ &= E\bigg[\theta\bigg(\sigma, z(x,\sigma,\cdot) - \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)} d_v z(x;s,v) + \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\,d\psi(v)\bigg) \\ &\qquad \cdot \exp\bigg\{\int_0^{\sigma} \theta(u, z(x;u,\cdot) - \int_0^{\cdot} \frac{a(u,v)}{a(\sigma,v)} d_v z(x;\sigma,v) \\ &\qquad\qquad + \int_0^{\cdot} \frac{a(u,v)}{a(\sigma,v)} d_v\bigg[z(x;\sigma,v) - \int_0^{v} \frac{a(\sigma,v')}{a(s,v')} d_{v'} z(x;s,v')\bigg] \\ &\qquad\qquad + \int_0^{\cdot} \frac{a(u,v)}{a(s,v)}\,d\psi(v)\bigg)\,du\bigg\}\bigg].\end{aligned}$$

We next observe that, for $0 \le u \le \sigma \le s \le S$, $z(x;u,\cdot) - \int_0^{\cdot}(a(u,v)/a(\sigma,v))\,d_v z(x;\sigma,v)$ and $z(x;\sigma,\cdot) - \int_0^{\cdot}(a(\sigma,v)/a(s,v))d_v z(x;s,v)$ are independent processes, and that $z(x;\sigma,\cdot) - \int_0^{\cdot}(a(\sigma,v)/a(s,v))d_v z(x;s,v)$

is equivalent to $\int_0^{\cdot} \sqrt{a(\sigma,v)[1-a(\sigma,v)/a(s,v)]}\, dw(v)$ for fixed σ and s. Thus, we may rewrite (18) in the form

$$
\begin{aligned}
(19)\quad & E[\theta(\sigma, z(x;\sigma,\cdot))F(\sigma, z(x)) \mid z(x;s,\cdot)=\psi(\cdot)] \\
&= E_w\bigg\{\theta\bigg(\sigma, \int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) + \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg) \\
&\cdot E_x\bigg[\exp\bigg\{\int_0^{\sigma} \theta\bigg(u, z(x;u,\cdot) - \int_0^{\cdot} \frac{a(u,v)}{a(\sigma,v)} d_v z(x;\sigma,v) \\
&\quad + \int_0^{\cdot} \frac{a(u,v)}{a(\sigma,v)} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) + \int_0^{\cdot} \frac{a(u,v)}{a(s,v)}\, d\psi(v)\bigg) du\bigg\}\bigg]\bigg\} \\
&= E_w\bigg\{\theta\bigg(\sigma, \int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) + \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg) \\
&\cdot E_x\bigg[\exp\bigg\{\int_0^{\sigma} \theta(u, z(x;u,\cdot))\, du\bigg\} \mid z(x;\sigma,\cdot) \\
&= \int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) + \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg]\bigg\}.
\end{aligned}
$$

If we set

$$
(20) \qquad G(s,\psi(\cdot)) \equiv E[F(s,z(x)) \mid z(x;s,\cdot)=\psi(\cdot)],
$$

then substituting equation (19) into (17) we see that $G(s,\psi(\cdot))$ satisfies the Wiener integral equation

(21)

$$
\begin{aligned}
G(s,\psi(\cdot)) = 1 + \int_0^s E_w\bigg[\theta\bigg(\sigma, &\int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) \\
&+ \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg) \\
&\cdot G\bigg(\sigma, \int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) \\
&+ \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg)\bigg]\, d\sigma.
\end{aligned}
$$

This Wiener integral equation is very similar to the Cameron-Storvick integral equation [**1**, Theorem 2], and thus (21) has a series solution

$$G(s,\psi(\cdot)) = \sum_{k=0}^{\infty} H_k(s,\psi(\cdot)), \tag{22}$$

where the sequence $\{H_k\}$ is given inductively by $H_0(s,\psi(\cdot)) \equiv 1$, and

$$\begin{aligned} H_{k+1}(s,\psi(\cdot)) = \int_0^s E_w\bigg[\theta\bigg(\sigma, &\int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) \\ &+ \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg) \\ &\cdot H_k\bigg(\sigma, \int_0^{\cdot} \sqrt{a(\sigma,v)\bigg[1-\frac{a(\sigma,v)}{a(s,v)}\bigg]}\, dw(v) \\ &+ \int_0^{\cdot} \frac{a(\sigma,v)}{a(s,v)}\, d\psi(v)\bigg)\bigg]\, d\sigma. \end{aligned}$$

If $\theta(s,\xi)$ is bounded, say $|\theta(s,\xi)| \le M$ on $[0,S]\times\mathbf{R}$, then by induction it follows that

$$|H_{k+1}(s,\psi(\cdot))| \le \frac{(Ms)^k}{k!} \le \frac{(MS)^k}{k!} \quad \text{for } k = 0,1,2,\dots,$$

and hence,

$$\sum_{k=0}^{\infty} |H_k(s,\psi(\cdot)| \le \exp\{MS\},$$

and so the series solution (22) converges uniformly on $[0,S]$. Thus, $G(s,\psi(\cdot))$ given by (22) is a bounded continuous solution of (21), and the fact that such a solution is unique can be shown in the usual way.

We are now ready to evaluate

$$I \equiv E\bigg[\exp\bigg\{\int_Q \phi(\sigma,\tau,z(x;\sigma,\tau))\, d\sigma\, d\tau\bigg\} \mid z(x;S,T) = \xi\bigg] \tag{23}$$

under the assumption that ϕ is bounded and continuous. Applying Theorem 1, equations (20) and (22), we have that

$$
\begin{aligned}
I &= E_w\bigg\{E\bigg[F(S,z(x)) \mid z(x;S,\cdot) = \int_0^{\cdot} \sqrt{a(S,v)}\,dw(v) \\
&\qquad\qquad - \bigg(\int_0^{\cdot} a(S,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg) \\
&\qquad\qquad \cdot \bigg(\int_0^T \sqrt{a(S,v)}\,dw(v) - \xi\bigg)\bigg]\bigg\} \\
&= E_w\bigg\{G\bigg(S, \int_0^{\cdot} \sqrt{a(S,v)}\,dw(v) \\
&\qquad\qquad - \bigg(\int_0^{\cdot} a(S,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg) \\
&\qquad\qquad \cdot \bigg(\int_0^T \sqrt{a(S,v)}\,dw(v) - \xi\bigg)\bigg)\bigg\} \\
&= \sum_{k=0}^{\infty} E_w\bigg[H_k\bigg(S, \int_0^{\cdot} \sqrt{a(S,v)}\,dw(v) \\
&\qquad\qquad - \bigg(\int_0^{\cdot} a(S,v)\,dv \Big/ \int_0^T a(S,v)\,dv\bigg) \\
&\qquad\qquad \cdot \bigg(\int_0^T \sqrt{a(S,v)}\,dw(v) - \xi\bigg)\bigg)\bigg].
\end{aligned}
\tag{24}
$$

Now each Wiener integral in the summand can be expressed in terms of Lebesgue integrals in the usual way, as Cameron and Storvick demonstrated in [**1**].

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