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SOME EXISTENCE RESULTS FOR NONLINEAR INTEGRAL EQUATIONS VIA TOPOLOGICAL TRANSVERSALITY

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ABSTRACT. Existence results are established for nonlinear integral equations of Hammerstein and Urysohn type. The results complement and extend related work in the field. A principal feature of the paper is its rather easy proofs which are based on topological transversality theory rather than degree theory.

1. Introduction. In this paper we establish some existence results for nonlinear integral equations of Hammerstein and Urysohn type. The results and proofs complement and extend similar results in the literature; see, for example, [**6, 7**]. A principal feature of the paper is the rather easy proofs that are based on a more elementary topological structure than is usually used. We base our discussion on the topological transversality theory of A. Granas [**3**] rather than on the Leray-Schauder degree. The more elementary point of view of topological transversality has led to many new results about nonlinear differential systems; see [**2, 4, 5**] for an overview. However, the methods of topological transversality have not been used much (perhaps at all) in the treatment of nonlinear integral equations. We hope this discussion will stimulate further work.

The development of a topological degree, such as the Leray-Schauder degree, requires substantial and rather sophisticated preliminaries. In contrast, the results typically needed from topological transversality theory require nothing more demanding than Urysohn's lemma in a metric space and a few standard compactness arguments. Full proofs of the results summarized below may be found in [**4**, p. 14–15] or in [**1**, p. 57–60], which also contains further theory. In applications, some maps must be known to have nonzero degree or to be essential, in the case of topological transversality. In either approach, the Schauder

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fixed-point theorem plays a role. It is normally the deepest result used when the analysis is based on topological transversality.

The approach via topological transversality has the added advantage that it permits a cone formulation. This leads naturally to the existence of solutions in a cone, say the cone of nonnegative functions. Particular instances are noted in Section 3.

We are concerned primarily with the existence of solutions to nonlinear Urysohn equations

$$u(s) = f(s) + \int_0^1 k(s,t,u(t))\,dt, \quad 0 \le s \le 1. \tag{1.1}$$

The related Urysohn integral operator is

$$Ku(s) = \int_0^1 k(s,t,u(t))\,dt.$$

When a solution $u = u(t)$ is sought in a function space E, natural conditions are imposed on $k(s,t,u)$ which guarantee that $K : E \to E$ and, of course, $f \in E$ is assumed. Then (1.1) is equivalent to the nonlinear operator equation

$$u = Nu \tag*{(1.1)$'$}$$

where $N : E \to E$ is defined by $Nu = f + Ku$. Thus, (1.1) has a solution precisely when the operator N has a fixed point.

As noted above, we shall establish that N has a fixed point by means of topological transversality theory. We summarize the key elements of that theory next and refer to [**1**] for further elaboration and proofs.

A function is *compact* if it has relatively compact range. It is *completely continuous* if it maps bounded sets into relatively compact sets. By a map we mean a continuous function. Let C denote a convex subset of a normed linear space E. Let U be open in C, and denote by $\overline{U}$ and ∂U the closure and boundary of U in C. Denote by $K_{\partial U}(\overline{U}, C)$ the family of compact maps from $\overline{U}$ into C which are fixed point free on ∂U. A map $F \in K_{\partial U}(\overline{U}, C)$ is *essential* if every map in $K_{\partial U}(\overline{U}, C)$ which agrees with F on ∂U has a fixed point in U. Evidently every essential map has a fixed point. An easy application of the Schauder fixed point theorem establishes

Proposition 1.1. *Let $p_0 \in U$ be fixed. Then the constant map sending each point of $\overline{U}$ to p_0 is essential in $K_{\partial U}(\overline{U}, C)$.*

The next result permits the identification of other essential maps via homotopy. Two maps F and G in $K_{\partial U}(\overline{U}, C)$ are *homotopic* in $K_{\partial U}(\overline{U}, C)$ if there is a compact homotopy $H = H(u, \lambda) : \overline{U} \times [0,1] \to C$ such that $H_\lambda(u) = H(u, \lambda) : \overline{U} \to C$ belongs to $K_{\partial U}(\overline{U}, C)$ for each $\lambda \in [0,1]$, $F = H_0$, and $G = H_1$.

Topological Transversality Theorem 1.2. *Let F and G be homotopic maps in $K_{\partial U}(\overline{U}, C)$. Then F is essential if and only if G is essential.*

The following nonlinear alternative, a variant of the classical Leray-Schauder alternative, is often useful in establishing that a particular operator has a fixed point. It is an immediate consequence of the topological transversality theorem and Proposition 1.1.

Theorem 1.3. *Let $N : \overline{U} \to C$ be a compact map, $p_0 \in U$, and $N_\lambda(u) = N(u, \lambda) : \overline{U} \times [0,1] \to C$ a compact map with $N_1 = N$ and N_0 the constant map to p_0. Then either*

(1) *N has a fixed point in $\overline{U}$; or*

(2) *there exists $\lambda \in (0,1)$ such that N_λ has a fixed point in ∂U.*

In typical applications of this nonlinear alternative, a priori bounds are established for solutions u to $N_\lambda u = u$. Then U is chosen so that possibility (2) in Theorem 1.3 cannot occur and, hence, N has a fixed point.

2. Preliminaries. Consider the nonlinear Urysohn equation

$$u(s) = f(s) + \int_0^1 k(s, t, u(t))\, dt, \quad 0 \le s \le 1. \tag{2.1}$$

For the moment assume $k(s, t, u)$ is continuous on $[0,1] \times [0,1] \times \mathbf{R}^d$ with values in $\mathbf{R}^d$ for some $d \ge 1$. Likewise, assume $f(s)$ is continuous from $[0,1]$ to $\mathbf{R}^d$. Thus, (2.1) is a scalar equation when $d = 1$ and a system of such equations when $d > 1$. As usual, $C[0,1]$ is the Banach space of continuous functions $u : [0,1] \to \mathbf{R}^d$ equipped with the maximum norm

$$|u|_0 = \max\{|u(t)| : t \in [0,1]\}.$$

We express (2.1) in operator form by

$$u = f + Ku$$

where

$$K : C[0,1] \to C[0,1]$$

is the integral operator

$$Ku(s) = \int_0^1 k(s,t,u(t))\,dt. \tag{2.2}$$

Lemma 2.1. *Let $k(s,t,u)$ be continuous on $[0,1]\times[0,1]\times\mathbf{R}^d$ into $\mathbf{R}^d$. Then $K : C[0,1] \to C[0,1]$ is continuous and completely continuous.*

Proof. The continuity of K follows easily from the uniform continuity of k on compact sets. The complete continuity of K is a direct consequence of the Arzela-Ascoli theorem and the aforementioned uniform continuity. $\square$

Next we relax the hypotheses on $k(s,t,u)$ but still wish to retain the conclusions in Lemma 2.1. To begin with, let m be a positive integer and $1 \le p \le \infty$. A function $g : [0,1]\times\mathbf{R}^m \to \mathbf{R}^d$ is an *L^p-Carathéodory function* provided: If $g = g(t,z)$, then

(a) the map $z \to g(t,z)$ is continuous for almost all t in $[0,1]$,

(b) the map $t \to g(t,z)$ is measurable for all z in $\mathbf{R}^m$,

(c) for each $r > 0$, there exists $h_r \in L^p[0,1]$ such that $|z| \le r$ implies $|g(t,z)| \le h_r(t)$ for almost all t in $[0,1]$.

Sometimes we simply call g a Carathéodory function.

Now let $k : [0,1] \times [0,1] \times \mathbf{R}^d \to \mathbf{R}^d$. We make the following assumptions (H1–H3) on k.

H1. For each $s \in [0,1]$, the function $k_s : [0,1] \times \mathbf{R}^d \to \mathbf{R}^d$ given by $k_s(t,u) = k(s,t,u)$ is an L^1-Carathéodory function.

Then, as Carathéodory showed, (a) and (b) imply that $k_s(t, u(t)) = k(s, t, u(t))$ is measurable for any measurable function $u(t)$. In view of (c), $k_s(t, u(t))$ is integrable when $u \in C[0, 1]$. Thus, $Ku(s) \in \mathbf{R}^d$ for each $u \in C[0, 1]$ and s in $[0, 1]$. Next, let $u_n \to u$ in $C[0, 1]$. Then there exists $r > 0$ such that $|u_n|_0 \le r$ and $|u|_0 \le r$. By H1, there is a function $h_{s,r} \in L^1[0, 1]$ such that $|k_s(t, u_n(t))| = |k(s, t, u_n(t))| \le h_{s,r}(t)$ for almost all t in $[0, 1]$. Also, from H1, for each s in $[0, 1]$, $k(s, t, u_n(t)) \to k(s, t, u(t))$ for almost all t in $[0, 1]$. Lebesgue's dominated convergence theorem implies that $Ku_n(s) \to Ku(s)$ pointwise on $[0, 1]$. To guarantee that this convergence is uniform and, hence, that $Ku \in C[0, 1]$ and K is continuous, we assume:

H2. For each $r > 0$ and s in $[0, 1]$,

$$\int_0^1 \sup_{|u| \le r} |k(s', t, u) - k(s, t, u)| \, dt \to 0 \quad \text{as } s' \to s.$$

With $u_n \to u$ in $C[0, 1]$ and $r > 0$ as above, it follows that

$$|Ku_n(s') - Ku_n(s)| \le \int_0^1 \sup_{|u| \le r} |k(s', t, u) - k(s, t, u)| \, dt.$$

Consequently, $\{Ku_n\}$ is equicontinuous at s for each s in $[0, 1]$, and, hence, uniformly equicontinuous on $[0, 1]$. Since Ku_n also converges pointwise to Ku on $[0, 1]$, it follows that the convergence is uniform. Then $Ku \in C[0, 1]$, $Ku_n \to Ku$ in $C[0, 1]$ and $K : C[0, 1] \to C[0, 1]$ is continuous.

The reasoning following H2 also shows that KB is equicontinuous for each bounded set B in $C[0, 1]$. KB will also be bounded if we assume:

H3. For each $r > 0$,

$$\sup_{s \in [0,1]} \int_0^1 \sup_{|u| \le r} |k(s, t, u)| \, dt < \infty.$$

Consequently, if $k(s, t, u)$ satisfies H1–H3, then $K : C[0, 1] \to C[0, 1]$ is continuous and completely continuous. In regard to H3, note that, by H1, given $r > 0$ and s in $[0, 1]$ there is an $h_{s,r} \in L^1[0, 1]$ such that

$$|k(s, t, u)| \le h_{s,r}(t) \quad \text{for a.e. } t \text{ in } [0, 1]$$

and $|u| \leq r$. We call $k(s,t,u)$ *L^1-Carathéodory uniformly in s* if in H1

$$\sup_{s\in[0,1]} \int_0^1 h_{s,r}(t)\,dt < \infty.$$

H3 follows easily from this assumption. We summarize this discussion in

Theorem 2.2. *Let $k : [0,1] \times [0,1] \times \mathbf{R}^d \to \mathbf{R}^d$ satisfy* H1–H3 *or alternatively satisfy* H1, H2 *and be L^1-Carathéodory uniformly in s. Then the integral operator K in* (2.2) *maps $C[0,1]$ into itself and is continuous and completely continuous.*

Remark. The reasoning above applies when $[0,1]$ is replaced by any compact set in some Euclidean space.

Hammerstein integral operators provide important special cases of the Urysohn operator (2.2). In the Hammerstein case, $k(s,t,u) = l(s,t)g(t,u)$ where $l(s,t) = l(t,s)$ is a symmetric kernel. We drop the symmetry requirement on $l(s,t)$ and formulate the following result.

Theorem 2.3. *Let $k(s,t,u) = l(s,t)g(t,u)$ and assume*:

A. *$l_s(t) = l(s,t) \in L^p[0,1]$ for each $s \in [0,1]$.*

B. *The map $s \mapsto l_s$ is continuous from $[0,1]$ to $L^p[0,1]$.*

C. *$g(t,u)$ is L^q-Carathéodory where $1/p + 1/q = 1$.*

Then $k(s,t,u)$ satisfies H1–H3.

Proof. Clearly $k_s(t,u) = l_s(t)g(t,u)$ satisfies the continuity and measurability conditions required by H1. Given $r > 0$, there exists $h_r(t) \in L^q[0,1]$ such that $|g(t,u)| \leq h_r(t)$ for almost all $t \in [0,1]$ and all $|u| \leq r$. Thus, $|k_s(t,u)| \leq |l_s(t)|h_r(t)$ for almost all $t \in [0,1]$ and all $|u| \leq r$ and the right member is in $L^1[0,1]$. So H1 holds.

Next,

$$\begin{aligned}\int_0^1 \sup_{|u|\le r} &|l(s',t)g(t,u)-l(s,t)g(t,u)|\,dt\\ &= \int_0^1 |l(s',t)-l(s,t)|\sup_{|u|\le r}|g(t,u)|\,dt\\ &\le \int_0^1 |l(s',t)-l(s,t)|h_r(t)\,dt\\ &\le ||l_{s'}-l_s||_p||h_r||_q \to 0 \quad \text{as } s'\to s\end{aligned}$$

by B. Thus, H2 holds. Likewise, by B

$$\begin{aligned}\sup_{s\in[0,1]}\int_0^1 \sup_{|u|\le r}|l(s,t)g(t,u)|\,dt &\le \sup_{s\in[0,1]}||l_s||_p||h_r||_q\\ &= ||h_r||_q \max_{s\in[0,1]}||l_s||_p,\end{aligned}$$

and H3 holds. □

3. Existence results. Consider the Urysohn equation

$$u(s)=f(s)+\int_0^1 k(s,t,u(t))\,dt \tag{3.1}$$

and the related family of problems

$$u(s)=f(s)+\lambda\int_0^1 k(s,t,u(t))\,dt \tag{3.2}$$

for $\lambda\in(0,1)$. Throughout this section assume that $f\in C[0,1]$ and k satisfies H1–H3 in Section 2. In case $k(s,t,u)=l(s,t)g(t,u)$, we assume that l and g satisfy A, B and C in Section 2. Then $k=lg$ satisfies H1–H3 by Theorem 2.3. The assumptions above imply that the corresponding Urysohn integral operator

$$Ku(s)=\int_0^1 k(s,t,u(t))\,dt \tag{3.3}$$

maps $C[0,1]$ into itself and that K is continuous and completely continuous. It follows that

$$N : C[0,1] \times [0,1] \to C[0,1], \qquad N(u,\lambda) = f + \lambda K u,$$

is continuous and completely continuous. Evidently, (3.2) is equivalent to $u = N(u,\lambda)$. The restriction of $N(u,\lambda)$ to $\overline{U} \times [0,1]$ is a continuous, compact map for any bounded subset U of $C[0,1]$.

We use these observations and the nonlinear alternative (Theorem 1.3) to establish, very easily, several existence theorems for Urysohn and Hammerstein integral equations (and systems of such equations). By a solution of (3.1) we mean a continuous function u that satisfies (3.1).

Theorem 3.1. *Let k be bounded. Then* (3.1) *has a solution.*

Proof. Let $|k(s,t,u)| \le \tilde{M}$. Then $|N(u,\lambda)| < M = |f|_0 + \tilde{M} + 1$. Apply the nonlinear alternative with $C = C[0,1]$, $U = \{u \in C : |u|_0 < M\}$, and $p_0 = f$. Alternative (2) is impossible by the choice of U so $N(u,1)$ has a fixed point; equivalently, (3.1) has a solution. $\square$

Example. $u(s) = \cos t + \int_0^1 e^{st} \sin(t^2 e^{u(t)})\, dt$ has a solution.

A slight modification of the proof permits us to show there is a nonnegative solution, when it is reasonable to expect one.

Theorem 3.2. *Let $f \ge 0$ and k be nonnegative and bounded for all $s,t \in [0,1]$ and $u \ge 0$. Then* (3.1) *has a nonnegative solution.*

Proof. Replace C in the previous proof by $C = \{u \in C[0,1] : u \ge 0\}$ which is a convex subset of $C[0,1]$. Then $N(u,\lambda) : \overline{U} \times [0,1] \to C$ and the nonlinear alternative applies as before. $\square$

Example.

$$u(s) = t^2 + \int_0^1 \frac{e^{-su(t)}}{1 + u(t)^2}\, dt$$

has a solution $u(t) \ge 0$. Note that $k(s,t,u)$ is not bounded for all real u.

Remark. Theorem 3.2 can also be deduced from Theorem 3.1 by means of the auxiliary kernel $\tilde{k}(s,t,u) = k(s,t,|u|)$.

Theorem 3.3. *Assume* $|k(s,t,u)| \le \varphi(|u|)$ *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a nondecreasing, Borel function such that* $L = \overline{\lim}_{x\to\infty} \varphi(x)/x < 1$. *Then* (3.1) *has a solution.*

Proof. Suppose $u = N(u,\lambda)$ for some $u \in C[0,1]$ and $\lambda \in (0,1)$. Then

$$|u(s)| \le |f|_0 + \int_0^1 \varphi(|u(t)|)\,dt \le |f|_0 + \varphi(|u|_0)$$

because φ is nondecreasing. Thus,

$$1 \le |f|_0/|u|_0 + \varphi(|u|_0)/|u|_0$$

provided $u \ne 0$. It follows that there exists a constant $M < \infty$ and independent of λ in (0,1) such that $|u|_0 < M$ for any u that satisfies $u = N(u,\lambda)$. Otherwise, there would exist $u_n = N(u_n,\lambda_n)$ with $|u_n|_0 \to \infty$ as $n \to \infty$ and the displayed inequality would yield the contradiction $1 \le L$. Thus, $u = N(u,\lambda)$ implies $|u|_0 \le M$. Now apply the nonlinear alternative just as in Theorem 3.1 to obtain a solution to (3.1). □

Remark. If $[0,1]$ is replaced by a compact set D in $\mathbf{R}^d$, the condition $L < 1$ is replaced by $L|D| < 1$ where $|D|$ is the volume of D in $\mathbf{R}^d$.

Theorem 3.3 applies, in particular, to kernels with sublinear growth in u because $\varphi(x) = \alpha x^\beta$ with $\alpha > 0$ and $0 < \beta < 1$ has $L = 0$. The theorem also covers mildly linear growth in u described by $\varphi(x) = \alpha x$ for $0 \le \alpha < 1$, and oscillatory behavior with $\varphi(x) = \alpha x|\sin x|$, $0 \le \alpha < 1$. A slight modification in the proof of Theorem 3.3 allows a growth rate that includes an integrable singularity.

Theorem 3.4. *Assume* $|k(s,t,u)| \le \beta(s,t)\varphi(|u|)$ *where for each* $s \in [0,1]$, $\beta_s = \beta(s,t) \in L^1[0,1]$, *and* φ *and* L *are as in Theorem* 3.3. *Let* $b = \sup\{||\beta_s||_1 : s \in [0,1]\}$. *Then* (3.1) *has a solution if* $bL < 1$.

Next we turn to nonlinear Hammerstein equations.

Theorem 3.5. *Assume that* $k(s,t,u) = l(s,t)g(t,u)$ *where* $l(s,t)$ *is a nonnegative, symmetric, positive definite kernel with smallest eigenvalue* λ_1*, that* A *holds with* $p = 2$*, and* $|g(t,u)| \le \alpha(t) + c(t)|u|$ *for* $\alpha, c \in C[0,1]$*. Then* (3.1) *has a solution if* $|c|_0 < \lambda_1$*.*

Proof. Recall that g and l satisfy A, B, and C in Section 2 according to our standing assumptions. In particular, the integral operator L with kernel l maps $C[0,1]$ into itself. Let $\langle \cdot, \cdot \rangle$ be the usual inner product on $L_2[0,1]$. Then $u = N(u,\lambda)$ implies that

$$\begin{aligned} |u(s)| &\le |f(t)| + \int_0^1 l(s,t)\alpha(t)\,dt + \int_0^1 l(s,t)c(t)|u(t)|\,dt \\ |u(s)| &\le |f|_0 + |L\alpha|_0 + |c|_0 \int_0^1 l(s,t)|u(t)|\,dt. \end{aligned} \tag{3.4}$$

Multiply by $|u(s)|$ and integrate with respect to s to obtain

$$\begin{aligned} ||u||_2^2 &\le (|f|_0 + |L\alpha|_0)||u||_1 + |c|_0 \langle L|u|, |u|\rangle, \\ ||u||_2^2 &\le (|f|_0 + |L\alpha|_0)||u||_2 + |c|_0 \frac{||u||_2^2}{\lambda_1}, \\ ||u||^2 &\le \frac{|f|_0 + |L\alpha|_0}{1 - |c|_0/\lambda_1} \equiv M_1, \end{aligned}$$

an a priori bound in $L_2[0,1]$. Return to (3.4) and apply the Schwarz inequality to get

$$|u(s)| \le |f|_0 + |L\alpha|_0 + |c|_0 ||l_s||_2 ||u||_2.$$

By B, $||l_s||_2$ is bounded for s in $[0,1]$ and, since $||u||_2 \le M_1$, we infer the existence of a constant M independent of λ in $(0,1)$ such that $|u|_0 < M$. Now, existence of a solution to (3.1) follows exactly as in Theorem 3.1. □

Remark. Just as in passing from Theorem 3.1 to Theorem 3.2, we can obtain a nonnegative solution u to (3.1) if we assume that $f(t) \ge 0$ and $0 \le g(t,u) \le \alpha(t) + c(t)u$ for $0 \le t \le 1$ and $u \ge 0$ in Theorem 3.5.

We conclude with a more subtle result.

Theorem 3.6. *Assume* $k(s,t,u) = l(s,t)g(t,u)$ *satisfies*:

(i) *The function* $g(t,u)/u \geq 0$ *for* $u \neq 0$*, and there are constants* $0 < \alpha < \beta$*,* $\gamma \geq 0$*, and* $M > 0$ *such that*

$$\alpha|u|^\gamma \leq g(t,u)/u \leq \beta|u|^\gamma$$

for $0 \leq t \leq 1$ *and* $|u| > M$.

(ii) $g(t,0) = 0$ *and* $g_u(t,u)$ *is an* L^1*-Carathéodory function.*

(iii) $l(s,t)$ *is symmetric, nonpositive definite, and* $p = \gamma + 2$ *in condition* A *of Theorem* 2.3.

Then (3.1) *has a solution.*

Remark. The conditions A, B, C with $p = \gamma + 2$ hold in particular when $l(s,t)$ is continuous and $g(t,u)$ is continuously differentiable, which also implies $g_u(t,u)$ is L^1-Carathéodory.

Proof. Define

$$a(t,u) = \begin{cases} g(t,u)/u, & u \neq 0,\ 0 \leq t \leq 1 \\ g_u(t,0), & u = 0,\ 0 \leq t \leq 1 \end{cases}.$$

It is easy to check that $a(t,u) \geq 0$ is an L^1-Carathéodory function, a fact used several times below. Fix $u \in C[0,1]$ such that $u = N(u,\lambda)$. Let $b(t) = a(t,u(t))$. Then

$$u(s) = f(s) + \lambda \int_0^1 l(s,t)b(t)u(t)\,dt.$$

Multiply by $b(s)u(s)$, integrate with respect to s, and recall that $l(s,t)$ is nonpositive definite to obtain

$$\int_0^1 b(s)u(s)^2\,ds = \int_0^1 b(s)f(s)u(s)\,ds + \lambda\langle Lbu, bu\rangle,$$
$$\int_0^1 b(s)u(s)^2\,ds \leq \int_0^1 \sqrt{b(s)}f(s)\sqrt{b(s)}u(s)\,ds.$$

By the Schwarz inequality
(3.5)
$$\int_0^1 b(s)u(s)^2\,ds \le \left(\int_0^1 b(s)f(s)^2\,ds\right)^{1/2}\left(\int_0^1 b(s)u(s)^2\,ds\right)^{1/2},$$
$$\int_0^1 b(s)u(s)^2\,ds \le \int_0^1 b(s)f(s)^2\,ds.$$

Let $I=\{t\in[0,1]:|u(t)|>M\}$. Then, from (i),

$$\int_0^1 b(s)u(s)^2\,ds \ge \int_I b(s)u(s)^2\,ds \ge \alpha\int_I |u(s)|^{\gamma+2}\,ds, \tag{3.6}$$

and

$$\int_0^1 b(s)f(s)^2\,ds \le |f|_0^2\int_{I^c} b(s)\,ds + \beta\int_I |u(s)|^{\gamma}f(s)^2\,ds. \tag{3.7}$$

Since g_u is L^1-Carathéodory, there exists $h_M(t)\in L^1[0,1]$ such that

$$|b(s)| = |g_u(s,\vartheta_s u(s))| \le h_M(s) \quad \text{a.e. } s\in I^c, \tag{3.8}$$

where $0\le\vartheta_s<1$ is determined using the mean value theorem. It follows that there is a constant M_1 (independent of λ) such that

$$\begin{aligned}\int_0^1 b(s)f(s)^2\,ds &\le M_1+\beta\int_I |u(s)|^{\gamma}f(s)^2\,ds,\\ &\le M_1+\beta\left(\int_I |u(s)|^{\gamma p}\,ds\right)^{1/p}\left(\int_I f(s)^{2q}\,ds\right)^{1/q},\end{aligned}$$

for any $p\ge1$ and $1/p+1/q=1$. If $\gamma>0$, choose p such that $\gamma p=\gamma+2$, i.e., $p=(\gamma+2)/\gamma>1$. Then

$$\int_0^1 b(s)f(s)^2\,ds \le M_1+\beta||f||_{2q}^2\left(\int_I |u(s)|^{\gamma+2}\,ds\right)^{1/p}. \tag{3.9}$$

From (3.5), (3.6), and (3.9) there are constants M_1 and M_2 such that

$$\alpha\int_I |u(s)|^{\gamma+2}\,ds \le M_1+\beta M_2\left(\int_I |u(s)|^{\gamma+2}\,ds\right)^{1/p}$$

for $p = (\gamma + 2)/\gamma > 1$. Since $\alpha \int_{I^c} |u(s)|^{\gamma+2}\, ds \leq \alpha M^{\gamma+2}$ there is a constant M_3 such that

$$\alpha \int_0^1 |u(s)|^{\gamma+2}\, ds \leq M_3 + \beta M_2 \biggl(\int_0^1 |u(s)|^{\gamma+2}\, ds \biggr)^{1/p}$$

for $p = 1 + 2/\gamma$. It follows that

$$\int_0^1 |u(s)|^{\gamma+2}\, ds \leq M_4 \tag{3.10}$$

for some constant M_4 independent of λ. Now consider $\gamma = 0$. When $\gamma = 0$, (3.7) and (3.8) imply that

$$\int_0^1 b(s) f(s)^2\, ds \leq |f|_0^2 \int_0^1 h_M(s)\, ds + \beta \int_I f(s)^2\, ds = \tilde{M}$$

a bound independent of λ. Now (3.5), (3.6) and this bound yields

$$\alpha \int_I |u(s)|^2\, ds \leq \tilde{M},$$

and, hence,

$$\alpha \int_0^1 |u(s)|^2\, ds \leq \tilde{M} + \alpha M^2$$

which is a bound of the form (3.10) when $\gamma = 0$. Thus, (3.10) holds for $\gamma \geq 0$.

Finally, to obtain an a priori bound in the maximum norm, return to (3.2) to find

$$\begin{aligned} |u(s)| &\leq |f|_0 + \int_0^1 |l(s,t)||g(t,u(t))|\, dt \\ &\leq |f|_0 + \biggl(\int_0^1 |l(s,t))^p\, dt \biggr)^{1/p} \biggl(\int_0^1 |g(t,u(t))|^q\, dt \biggr)^{1/q} \end{aligned}$$

with $p = \gamma + 2$ as in (iii) and, therefore, $q = (\gamma+2)/(\gamma+1)$. By B there is a constant M_s such that $||l_s||_p \leq M_5$ for all s in $[0,1]$. Consequently,

$$|u(s)| \leq |f|_0 + M_5 \biggl(\int_0^1 |g(t,u(t))|^q\, dt \biggr)^{1/q} \tag{3.11}$$

where $q = (\gamma+2)/(\gamma+1)$. Since g is an L^q-Carathéodory function, there exists $h_M(t) \in L_q[0,1]$ such that

$$|g(t,u(t))| \le h_M(t) \quad \text{a.e. } t \in I^c,$$

while by (i),

$$|g(t,u(t))| \le \beta |u(t)|^{\gamma+1} \quad \text{for } t \in I.$$

These estimates imply that

$$\int_0^1 |g(t,u(t))|^q \, dt \le \int_0^1 h_M(t)^q \, dt + \int_0^1 \beta^q |u(t)|^{\gamma+2} \, dt,$$

where we have used $q(\gamma+1) = \gamma+2$. Finally, (3.10), (3.11) and this estimate yields a constant M_6 such that $|u(s)| \le M_6$ for all $s \in [0,1]$. □

As mentioned in the introduction, the interval $[0,1]$ can be replaced by any compact set in $\mathbf{R}^d$ and the same reasoning establishes the corresponding theorems. In a final example, we apply Theorem 3.6 to a compact domain $D \subset \mathbf{R}^d$ for $d=2$ and $d=3$. We assume D has a smooth boundary ∂D.

Example. Consider the nonlinear Dirichlet problem

$$\begin{cases} \Delta u = g(x,u), & x \in D \\ u = \varphi(x), & x \in \partial D \end{cases} \tag{3.12}$$

with smooth boundary data $\varphi(x)$ and $g \in C^1(D \times \mathbf{R})$. If f is the harmonic function in D with boundary values φ and $l(x,y)$ is the Green's function for the Laplacian with zero boundary data, then (3.12) is equivalent to the Hammerstein equation

$$u(x) = f(x) + \int_D l(x,y) g(y,u(y)) \, dy. \tag{3.13}$$

The Green's function is symmetric and negative definite. If $d=2$, $l(x,y)$ has a logarithmic singularity when $y=x$ and, hence, $l_x \in L^p(D)$ for any $p \ge 1$. If $g(t,u)$ satisfies (i) and (ii) in Theorem 3.6 for some $\gamma > 0$, then $l(x,y)$ satisfies (iii) for $p = \gamma+2$. Therefore, (3.13) and, hence, (3.12) has a solution. If $d = 3$, $l(x, y)$ has a singularity of the form $|x - y|^{-1}$ and $l_x \in L^p(D)$ only for $p \leq 2$. In this case Theorem 3.6 only applies when $\gamma = 0$, in which case $p = 2$ in (iii).

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