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SUPERCONVERGENCE OF MODIFIED PROJECTION METHOD FOR INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. Recently, Nair proposed a modification for the projection (Galerkin) method for the approximate solution of second kind integral equations and established conditions under which it was superconvergent (relative to the corresponding projection approximation). Its advantage over other superconvergent methods for such equations is its simplicity of implementation in requiring virtually no more computation than that implicit in the determination of the corresponding projection approximation. In this paper, we propose two variants of Nair's modification and establish conditions under which they are superconvergent.

1. Introduction. Using operator notation, second kind Fredholm integral equations take the form

$$x - Kx = y, \tag{1.1}$$

where the operator K is defined, in terms of its (known) kernel $k(s,t)$, by

$$(K\,v)(s) = \int_a^b k(s,t)v(t)dt, \quad a \le s \le b. \tag{1.2}$$

It is assumed that K is defined on a suitable Banach space X, that K is compact as a mapping from X into X, and that 1 is not an eigenvalue of K. Then, by Fredholm theory, (1.1) has a unique solution x in X for every y in X.

Approximate solutions are sought using a sequence of finite dimensional spaces X_n which, in practice, will be generated by polynomials or piecewise polynomials. Let $\boldsymbol{\pi}_n$ be a projection of X onto X_n. Then, assuming $(I - \boldsymbol{\pi}_n K)^{-1}$ exists, the *projection* (*Galerkin*) *approximation* x_n^{P} of x is defined by the equation

$$x_n^P - \boldsymbol{\pi}_n K x_n^{\mathrm{P}} = \boldsymbol{\pi}_n y.$$

The *iterated projection* (*or Sloan*) *approximation* $\tilde{x}_n^{\mathrm{P}} = x_n^S$ and the *Kantorovich approximation* x_n^K are defined by

$$x_n^{\mathrm{S}} = y + K\, x_n^{\mathrm{P}} \quad \text{and} \quad x_n^K = y + z_n^{\mathrm{P}},$$

respectively, where z_n^{P} is the projection solution of the equation $z - K\,z = K\,y$; i.e., z_n^{P} satisfies

$$z_n^{\mathrm{P}} - \boldsymbol{\pi}_n K\, z_n^{\mathrm{P}} = \boldsymbol{\pi}_n K\, y.$$

Recently, Nair [**7**] introduced the *modified projection approximation* x_n^M, defined by

$$x_n^M = x_n^{\mathrm{P}} + (I - \boldsymbol{\pi}_n)y,$$

and showed that it satisfies the relation

$$x - x_n^M = \boldsymbol{\pi}_n(x - x_n^{\mathrm{S}}) + (I - \boldsymbol{\pi}_n)(x - x_n^K). \tag{1.3}$$

Thus, if $\boldsymbol{\pi}_n$ is uniformly bounded, then (Theorem 2.1 in [**7**])

$$||x - x_n^M|| \le c \max\{||x - x_n^{\mathrm{S}}||, ||x - x_n^K||\}, \tag{1.4}$$

where c denotes (here and below) a generic constant.

In the standard superconvergence arguments (e.g., Sloan and Thomeé [**13**]), it is assumed that the projections are evaluated exactly. When numerical approximations are used, the proof of such results usually becomes technically more difficult. The advantage of Nair's [**7**] modification is that it is performed exactly using the information derived in constructing the projection approximation x_n^{P}. However, this practical advantage is achieved through a partial loss of full superconvergence, which is the topic of Nair [**7**] as well as of this paper.

It is known that, under certain smoothness assumptions on K and y, the Sloan approximation x_n^{S} converges to the exact solution x faster than x_n^{P}; i.e., x_n^{S} exhibits superconvergence (see Chandler [**1**], Chatelin and Lebbar [**2**], Graham *et al.* [**6**], Joe [**5**], and Sloan [**12**]). In addition, the speed of convergence of x_n^S also depends on the form of the projection method used (e.g., x_n^S does not always have the better convergence when collocation methods are applied).

If y has less regularity, then it is preferable to use the Kantorovich approximation x_n^K rather than x_n^{P}, for in the Kantorovich method we approximate $z = K x$ rather than x itself (see Schock [**8**] and Sloan [**11**]). Moreover, the Kantorovich approximation, in general, has better global (uniform) convergence behaviour than the projection and Sloan approximations (see Schock [**9**]). In the light of these comments, x_n^M is likely to have better convergence rates than x_n^{P} under conditions favorable to x_n^{S} and x_n^K and, in addition, x_n^M requires (virtually) no more computation than x_n^{P} (see computational results in [**7**]). It should be noted, however, that x_n^M can never be better than x_n^K. This is an immediate consequence of the result that

$$(I - \boldsymbol{\pi}_n K)(x - x_n^K) = (I - \boldsymbol{\pi}_n) K x = (I - \boldsymbol{\pi}_n)(x - x_n^K),$$

since, using (1.3), it yields

$$(I - \boldsymbol{\pi}_n K)(x - x_n^K) = (I - \boldsymbol{\pi}_n)(x - x_n^M). \tag{1.5}$$

Thus, whenever $(I - \boldsymbol{\pi}_n K)^{-1}$ and $\boldsymbol{\pi}_n$ are uniformly bounded, we obtain, using (1.3), (1.4) and (1.5), that

$$c||x - x_n^K|| \le ||x - x_n^M|| \le c' \max\{||x - x_n^{\mathrm{S}}||, ||x - x_n^K||\},$$

where c' is a constant satisfying $||(I - \boldsymbol{\pi}_n K)^{-1}(I - \boldsymbol{\pi}_n)|| \le 1/c'$. Thus, whenever x_n^{S} has a better order of convergence than x_n^K, the order of convergence of x_n^M is exactly that of x_n^K. An example given by Sloan [**11**] can be used to illustrate this point. However, one iteration of x_n^K (i.e., the *iterated Kantorovich approximation*) $\tilde{x}_n^K = y + K x_n^K$ is best among all the former approximations (see Sloan [**11, 12**]). In [**13**], Sloan and Thomeé have shown that, in situations where the smoothing of powers of K is not optimal, partial superconvergence of x_n^{S} can be guaranteed under appropriate circumstances. They have also shown that the higher iterates of x_n^{P}, namely

$$x_n^{(k)} = y + K x_n^{(k-1)}, \qquad x_n^{(0)} = x_n^{\mathrm{P}}, \quad k = 1, 2, \ldots,$$

have improving rates of convergence provided the powers of K have appropriate smoothing properties.

In this paper, we introduce two new approximations which are variants of the modified approximation of Nair [**7**] and establish conditions for their superconvergence in terms of that for

$$\tilde{x}_n^K = y + K x_n^K \quad \text{and} \quad \tilde{x}_n^{\mathrm{S}} = y + K x_n^{\mathrm{S}}.$$

The two new approximations are

(1) $\hat{x}_n^M = y + z_n^M$,

(2) $\tilde{x}_n^M = y + K\, x_n^M$,

where

$$z_n^M = z_n^{\mathrm{P}} + (I - \boldsymbol{\pi}_n)K\, y$$

is the modified projection solution of $z - K\, z = K\, y$. The approximations $\hat{x}_n^M$ and $\tilde{x}_n^M$ can also be written as

$$\hat{x}_n^M = (y + z_n^{\mathrm{P}}) + (I - \boldsymbol{\pi}_n)K\, y = x_n^{\mathrm{K}} + (I - \boldsymbol{\pi}_n)K\, y, \tag{1.6}$$

$$\tilde{x}_n^M = y + K\,(x_n^{\mathrm{P}} + (I - \boldsymbol{\pi}_n)y) = x_n^{\mathrm{S}} + K\,(I - \boldsymbol{\pi}_n)y. \tag{1.7}$$

In fact, both of these approximations can be viewed as particular cases of

$$w_n = x_n + (K - K_n)y,$$

where x_n is the solution of

$$x_n - K_n x_n = y, \tag{1.8}$$

with $K_n = \boldsymbol{\pi}_n K$, for $\hat{x}_n^M$, and $K_n = K\,\boldsymbol{\pi}_n$ for $\tilde{x}_n^M$. Because of the form of expressions (1.6) and (1.7) for $\hat{x}_n^M$ and $\tilde{x}_n^M$, we shall refer to these approximations as the *modified Kantorovich approximation* and the *modified Sloan approximation*, respectively.

In the next section, we obtain error estimates for $\hat{x}_n^M$ and $\tilde{x}_n^M$ in an abstract setting, while, in the final section, we examine their application to the integral equation (1.1).

2. Error bounds for $\hat{x}_n^M$ and $\tilde{x}_n^M$. We recall the basic assumption which we will invoke in the remainder of this paper. Let K denote a compact operator on a Banach space X and $\boldsymbol{\pi}_n$ a projection of X onto X_n. In addition, it is assumed that

(i) 1 is not an eigenvalue of K, and

(ii) for sufficiently large n, the $(I - \boldsymbol{\pi}_n K)^{-1}$ exist and are uniformly bounded.

Under these assumptions, each of the equations

$$u - K\, u = v$$

and

$$u - \boldsymbol{\pi}_n K\, u = v \tag{2.1}$$

has a unique solution in X for any given v in X. Thus, x_n^{P} and z_n^{P} exist uniquely as solutions of (2.1), with $v = \boldsymbol{\pi}_n y$ and $v = \boldsymbol{\pi}_n K\, y$, respectively. Thereby, the approximations

$$\hat{x}_n^M = y + z_n^M, \qquad z_n^M = z_n^{\mathrm{P}} + (I - \boldsymbol{\pi}_n) K\, y$$

and

$$\tilde{x}_n^M = y + K\, x_n^M, \qquad x_n^M = x_n^{\mathrm{P}} + (I - \boldsymbol{\pi}_n) y$$

are well defined, for sufficiently large n.

The error bounds to be given in Theorem 2.1 involve the iterated Kantorovich approximation

$$\tilde{x}_n^K = y + K\, x_n^K, \qquad x_n^K = y + z_n^{\mathrm{P}}$$

and the iterated Sloan approximation

$$\tilde{x}_n^{\mathrm{S}} = y + K\, x_n^{\mathrm{S}}, \qquad x_n^{\mathrm{S}} = y + K\, x_n^{\mathrm{P}}.$$

The following relations are easily derived from their definitions:

$$\begin{aligned} x - \tilde{x}_n^K &= K\,(x - x_n^K), \\ x - \tilde{x}_n^{\mathrm{S}} &= K\,(x - x_n^s) = K^{\,2}(x - x_n^{\mathrm{P}}), \\ K\,(I - \boldsymbol{\pi}_n) x &= (I - K\,\boldsymbol{\pi}_n)(x - x_n^S), \\ (I - \boldsymbol{\pi}_n) K\, x &= (I - \boldsymbol{\pi}_n K\,)(x - x_n^K), && (2.2) \\ K\,(I - \boldsymbol{\pi}_n) K\, x &= (I - K\,\boldsymbol{\pi}_n)(x - \tilde{x}_n^K). && (2.3) \end{aligned}$$

Theorem 2.1.

(a) *If the sequence $\boldsymbol{\pi}_n$ is uniformly bounded, then*

$$c_1 ||(I - \boldsymbol{\pi}_n) K^{\,2} x|| \le ||x - \hat{x}_n^M|| \le c_2 \max\left\{||x - \tilde{x}_n^K||, ||(I - \boldsymbol{\pi}_n) K^{\,2} x||\right\}$$

for some constants c_1 and c_2.

(b) *If the sequence* $(K\boldsymbol{\pi}_n)$ *is uniformly bounded, then*

$$||x-\tilde{x}_n^M|| \le c\max\left\{||K(I-\boldsymbol{\pi}_n)(x-x_n^{\mathrm{S}})||, ||x-\tilde{x}_n^K||, ||x-\tilde{x}_n^{\mathrm{S}}||\right\}$$

for some constant c.

Proof. Let K_n represent $\boldsymbol{\pi}_n K$ or $K\boldsymbol{\pi}_n$, and let x_n be the unique solution of (1.8). Then

$$(I-K_n)(x-x_n)=(K-K_n)x.$$

Denoting $w_n = x_n + (K-K_n)y$, we obtain

$$\begin{aligned} x-w_n &= (x-x_n)-(K-K_n)y \\ &= (x-x_n)-(K-K_n)(x-Kx) \\ &= (x-x_n)-(I-K_n)(x-x_n)+(K-K_n)Kx \\ &= K_n(x-x_n)+(K-K_n)Kx. \end{aligned} \tag{2.4}$$

Now, taking $K_n=\boldsymbol{\pi}_n K$, we obtain

$$x-\hat{x}_n^M = \boldsymbol{\pi}_n K(x-x_n^K)+(I-\boldsymbol{\pi}_n)K^2x.$$

In addition,

$$(I-\boldsymbol{\pi}_n)(x-\hat{x}_n^M)=(I-\boldsymbol{\pi}_n)K^2x.$$

Together these relations yield

$$||x-\hat{x}_n^M|| \le ||\boldsymbol{\pi}_n||\,||x-\tilde{x}_n^K||+||(I-\boldsymbol{\pi}_n)K^2x||$$

and

$$||(I-\boldsymbol{\pi}_n)K^2x|| \le (1+||\boldsymbol{\pi}_n||)||x-\hat{x}_n^M||.$$

Thus, the result in (a) follows by taking c_1 and c_2 such that

$$1+||\boldsymbol{\pi}_n|| \le 1/c_1 \quad \text{and} \quad c_2 = 1/c_1.$$

Now, taking $K_n = K\boldsymbol{\pi}_n$ in (1.8), we obtain, using (2.3),

$$\begin{aligned} x-\hat{x}_n^M &= K\boldsymbol{\pi}_n(x-x_n^{\mathrm{S}})+K(I-\boldsymbol{\pi}_n)K_x \\ &= K(x-x_n^s)-(K-K\boldsymbol{\pi}_n)(x-x_n^{\mathrm{S}})+K(I-\boldsymbol{\pi}_n)Kx \\ &= (x-\tilde{x}_n^{\mathrm{S}})-K(I-\boldsymbol{\pi}_n)(x-x_n^{\mathrm{S}})+(I-K\boldsymbol{\pi}_n)(x-\tilde{x}_n^K). \end{aligned}$$

Thus, the result in (b) follows on taking a c satisfying

$$3 + ||K\,\boldsymbol{\pi}_n|| \leq c. \ \square$$

Remark 2.2. It can be seen that the order of convergence of $||(I - \boldsymbol{\pi}_n)K^{\,2}x||$ to zero is the same as that of $||x - \hat{x}_n^K||$, where $\hat{x}_n^K$ denotes the second order Kantorovich approximation considered by Schock [**10**]:

$$\hat{x}_n^K = y + K\,y + \hat{z}_n, \qquad \hat{z}_n - \boldsymbol{\pi}_n K\,\hat{z}_n = \boldsymbol{\pi}_n K^{\,2} y.$$

Schock [**10**] has also characterized a class of operators K for which $\hat{x}_n^K$ is a better approximation of x than x_n^K.

We remark that the assumptions in Theorem 2.1 are satisfied if $||(I - \boldsymbol{\pi}_n)u|| \to 0$ as $n \to \infty$, for every u in X. However, this is a much stronger condition than is required. For example, in [**14**], Sloan and Burn considered the (Lagrange) interpolatory projection $\boldsymbol{\pi}_n$ with n Tchebychev's points as the nodes in $[-1,1]$ and showed that $||K\,(I - \boldsymbol{\pi}_n)u|| \to 0$ as $n \to \infty$ for every u in $X = C[-1,1]$, which implies that $(K\,\boldsymbol{\pi}_n)$ is uniformly bounded. In this case, $\boldsymbol{\pi}_n$ is not uniformly bounded. On the other hand, if we take X_n to be the space of piecewise polynomials of degree $\leq r-1$ with r Gaussian quadrature points placed on each subinterval as the nodes, then the interpolatory projections $\boldsymbol{\pi}_n$ onto X_n are known to be uniformly bounded (see Graham *et al.* [**6**]). But, $||(I - \boldsymbol{\pi}_n)u|| \to 0$ does not hold for all u in $X = C[-1,1]$.

The key point is that the assumptions used to obtain the results in Theorem 2.1 are minimal. In fact, Theorem 2.1 has been structured to give the interrelationships between the errors associated with the different approximations without considering whether convergence is guaranteed.

3. Superconvergence of $\hat{x}_n^M$ and $\tilde{x}_n^M$. In this section, we apply the results of §2 to the integral equation (1.1). We now assume that the Banach space $X = L_p = L_p[a,b], 1 \leq p \leq \infty$, and take X_n as the space $S_{r,n}$ of all piecewise polynomials of degree $\leq r-1$ defined on the mesh

$$\Delta_n := \{a = t_0 < t_1 < \cdots < t_n = b\}$$

with no continuity requirement at the nodes. Let $\boldsymbol{\pi}_n$ be the projection of L_p onto $S_{r,n}$ defined by

$$\langle \boldsymbol{\pi}_n v, u_n \rangle = \langle v, u_n \rangle, \quad \text{for all } v \in L_p, \quad u_n \in S_{r,n},$$

where $\langle .,. \rangle$ denotes the L_2-inner product. We further assume that the sequence of meshes Δ_n is quasiuniform (i.e., $h_i \geq ch$, where $h = \max_{1 \leq i \leq n} h_i), h_i = t_i - t_{i-1}$. As a consequence, $\boldsymbol{\pi}_n$ is uniformly bounded (see Werschulz [**15**, Lemma 3.1]). It also follows that

$$||(I - \boldsymbol{\pi}_n)v|| = ||(I - \boldsymbol{\pi}_n)(v - u_n)||, \quad \text{for all } u_n \in S_{r,n},$$

and hence

$$||(I - \boldsymbol{\pi}_n)v|| \leq c \inf_{u_n \in S_{r,n}} ||v - u_n||, \tag{3.1}$$

where $c \geq 1 + ||\boldsymbol{\pi}_n||$, for all n. Here $||v||$ denotes the L_p-norm of the function v.

For non-negative integers l, we let W_p^l denote the Sobolev space of functions defined on $[a,b]$ and equipped with the norm

$$||v||_{l,p} = \max_{0 \leq j \leq l} ||v^{(j)}||_p,$$

where $v^{(j)}$ is the j-th distributional derivative of v. In order to describe the smoothing properties of K , we consider, as in [**13**], a class $\mathcal{K}_{p,m,l}$ of operators $\mathbf{K}$ which maps L_p into W_p^m and satisfies

$$||K\,v||_{m,p} \leq c||v||_{-l,p}, \quad \vartheta \in L_p,$$

for some constant c. Here, the negative Sobolev norm $||\cdot||_{-l,p}$ is defined by

$$||v||_{-l,p} := \sup_{u \in W_q^l} \frac{|\langle v, u \rangle|}{||u||_{l,q}}, \quad v \in L_p, \tag{3.2}$$

with $1/p + 1/q = 1$. We recall the following well- known result on the approximation properties of $S_{r,n}$ (see de Vore [**3**] or Demko [**4**]).

Proposition 3.1. *If $\vartheta \in W_p^l$, then, for each $n \geq 0$,*

$$d_p(v, s_{r,n}) = \inf_{u_n \in S_{r,n}} ||v - u_n|| \leq ch^\nu ||v||_{l,p},$$

where $\nu = \min\{l, r\}, 1 \leq p \leq \infty$, and c is a constant.

From this result, using the compactness of K and uniform boundedness of the $\boldsymbol{\pi}_n$, one gets $||(I - \boldsymbol{\pi}_n)K|| \to 0$ as $n \to \infty$, so that, for n sufficiently large, $(I - \boldsymbol{\pi}_n K)^{-1}$ exists that is uniformly bounded. Here, and in what follows, $||A||$ denotes the norm of the operator A induced by the norm of the L_p space. Unless otherwise specified, $||v||$ denotes the L_p-norm of the function v.

Throughout the remainder of this paper, we shall impose the following smoothness conditions on K:

(1) K^*, the adjoint of K, maps W_p^l into itself for every non-negative integer $l \leq r$, where $1/p + 1/q = 1$;

(2) K is of class of $\mathcal{K}_{p,0,l_1}$ and $\mathcal{K}_{p,m_1,0}$ for some non-negative integers $l_1, m_1 \leq r$; and

(3) K^2 is of class $\mathcal{K}_{p,0,l_2}$ and $\mathcal{K}_{p,m_2,0}$ for some non-negative integers $l_2, m_2 \leq r$.

For later reference, we state the following result. The proof follows from Theorem 1 of Sloan and Thomeé [**13**], on using the fact that K is of class $\mathcal{K}_{p,O,l}$ if K^* maps L_p into W_q^l [**13**, p. 9].

Proposition 3.2. *Let $v \in L_p$ and u_n^P be the projection solution of the equation $u - K u = v$. Then*

$$||u - u_n^P||_{-l,p} \leq ch^l ||u - u_n^P||_p,$$

where $l \in \{l_1, l_2\}$, and $v \in L_p$.

We are now in a position to state and prove

Theorem 3.3. *Under the above smoothness assumptions on K, it follows that*

(i) $||x - x_n^K|| \le ch^{m_1}||x||$,
(ii) $||x - x_n^{\mathrm{S}}|| \le ch^{l_1}||x - x_n^{\mathrm{P}}||$,
(iii) $||x - \tilde{x}_n^K|| \le ch^{l_1+m_1}||x||$,
(iv) $||x - \tilde{x}_n^{\mathrm{S}}|| \le ch^{l_2}||x - x_n^{\mathrm{P}}||$,

where c denotes a generic constant.

Proof. (i). From the relations (2.2) and (3.1), we have

$$\begin{aligned}||x - x_n^K|| &\le ||(I - \boldsymbol{\pi}_n K)^{-1}||\, ||(I - \boldsymbol{\pi}_n)K x|| \\ &\le ||(I - \boldsymbol{\pi}_n K)^{-1}||(1 + ||\boldsymbol{\pi}_n||)d_p(K x, S_{r,n}).\end{aligned}$$

Since K is of class $\mathcal{K}_{p,m_1,0}$, we have

$$K x \in W_p^{m_1}, \qquad ||K x||_{m_1,p} \le c||x||.$$

Hence, by Proposition 3.1, we obtain

$$d_p(K x, S_{r,n}) \le ch^{m_1}||K x||_{m_1,p} \le ch^{m_1}||x||,$$

which yields the required result.

(ii). Since K is of class $\mathcal{K}_{p,0,l_1}$,

$$||x - x_n^{\mathrm{S}}|| = ||K(x - x_n^{\mathrm{P}})|| \le c||x - x_n^{\mathrm{P}}||_{-l_1,p}.$$

Applying Proposition 3.2, we obtain the required result.

(iii) Since K is of class $\mathcal{K}_{p,0,l_1}$,

$$||x - \tilde{x}_n^K|| = ||K(x - x_n^K)|| = ||K(z - z_n^{\mathrm{P}})|| \le c||z - z_n^{\mathrm{P}}||_{-l_1,p},$$

where $z = K x$ and z_n^{P} is the projection approximation of the solution of $z - K z = K y$. Now, by Proposition 3.3 and (i), we obtain the required result,

$$||x - \tilde{x}_n^K|| \le ch^{l_1}||z - z_n^{\mathrm{P}}|| = ch^{l_1}||x - x_n^K|| \le ch^{l_1+m_1}||x||.$$

(iv). Since K^2 is of class $\mathcal{K}_{p,0,l_2}$, we have, on using Proposition 3.2, that

$$||x - \tilde{x}_n^{\mathrm{S}}|| = ||K^2(x - x_n^{\mathrm{P}})|| \le c||x - x_n^{\mathrm{P}}||_{-l_2,p} \le ch^{l_2}||x - x_n^{\mathrm{P}}||,$$

which is the required result. □

Next, we state and prove the main theorem of this section.

Theorem 3.4. *Under the above smoothness assumptions on K, it follows that*

(i) $||x - \hat{x}_n^M|| \le c \max\{h^{l_1+m_1}, h^{m_2}\}$,

(ii) $||x - \tilde{x}_n^M|| \le c \max\{h^{l_1+m_1}, h^{l_2}||x - x_n^{\mathrm{P}}||, h^{l_1+m_1}||x - x_n^{\mathrm{P}}||\}$.

Proof. Initially, we show that

(a) $||(I - \boldsymbol{\pi}_n)K^2 x|| \le ch^{m_2}||x||$ and

(b) $||K(I - \boldsymbol{\pi}_n)(x - x_n^{\mathrm{S}}) \le ch^{l_1+m_1}||x - x_n^{\mathrm{P}}||$.

The required results (i) and (ii) then follow from Theorems 2.1 and 3.3.

Using Proposition 3.1 and the fact that K^2 is of class $\mathcal{K}_{p,m_2,0}$, we obtain

$$||(I - \boldsymbol{\pi}_n)K^2 x|| \le ch^{m_2}||K^2 x||_{l_2,p} \le ch^{m_2}||x||,$$

which proves (a). Since K is of class $\mathcal{K}_{p,0,l_1}$,

$$\begin{aligned} ||K(I - \boldsymbol{\pi}_n)(x - x_n^{\mathrm{S}})|| &= ||K(I - \boldsymbol{\pi}_n)\mathbf{K}(x - x_n^{\mathrm{P}})|| \\ &\le c||(I - \boldsymbol{\pi}_n)K(x - x_n^{\mathrm{P}})||_{-l_1,p}. \end{aligned}$$

It now follows from (3.2) (the definition of the negative norm $||\cdot||_{-l_1,p}$), that it is sufficient to prove that

$$|\langle (I - \boldsymbol{\pi}_n)K(x - x_n^{\mathrm{P}}), v\rangle| \le ch^{l_1+m_1}||x - x_n^{\mathrm{P}}||\,||v||_{l_1,q}$$

for every $v \in W_q^{l_1}$. Note that

$$\begin{aligned} \langle (I - \boldsymbol{\pi}_n)\mathbf{K}(x - x_n^{\mathrm{P}}), v\rangle &= \langle (I - \boldsymbol{\pi}_n)K(x - x_n^{\mathrm{P}}), v - u_n\rangle \\ &\le ||(I - \boldsymbol{\pi}_n)K(x - x_n^{\mathrm{P}})||_{0,p}||v - u_n||_{o,q}, \end{aligned}$$

for every $u_n \in S_{r,n}$. Hence,

$$\begin{aligned}|\langle (I - \boldsymbol{\pi}_n)K\,(x - x_n^{\mathrm{P}}), v\rangle| &\le ||(I - \boldsymbol{\pi}_n)K\,(x - x_n^{\mathrm{P}})||d_q(v, S_{r,n})\\ &\le c\; d_p(K\,(x - x_n^{\mathrm{P}}), S_{r,n})d_q(v, S_{r,n})\\ &\le ch^{l_1+m_1}||K\,(x - x_n^{\mathrm{P}})||_{m_1,p}||v||_{l_1,p}\\ &\le ch^{l_1+m_1}||x - x_n^{\mathrm{P}}||\;||v||_{l_1,q}.\end{aligned}$$

Here we have used the relation (3.1), Proposition 3.1, and the fact that K is of class $\mathcal{K}_{p,m_1,0}$. □

If the non-homogeneous part y in (1.1) belongs to W_p^ρ, for some non-negative integer ρ, then we find that

$$x = y + K\,x \in W_p^\beta, \quad \beta = \min\{\rho, m_1\}.$$

Thus,

$$\begin{aligned}||(I - \boldsymbol{\pi}_n)x|| &\le (1 + ||\boldsymbol{\pi}_n||)d_p(x, S_{r,n})\\ &\le ch^\beta||x||_{\beta,p},\end{aligned}$$

so that

$$||x - x_n^{\mathrm{P}}|| = ||(I - \boldsymbol{\pi}_n K\,)^{-1}(I - \boldsymbol{\pi}_n)x|| \le ch^\beta||x||_{\beta,p}.$$

Now, as a corollary to the above two theorems we can state

Theorem 3.5. *In addition to the above smoothness conditions on K, let $y \in W_p^\rho$ for some non-negative integer ρ. Then, for a generic constant c depending on x, we obtain the following inequalities*:

(i) $||x - x_n^{\mathrm{P}}|| \le ch^\beta, \quad \beta = \min\{\rho, m_1\}$,

(ii) $||x - x_n^K|| \le ch^{m_1}$,

(iii) $||x - x_n^{\mathrm{S}}|| \le ch^{l_1+\beta}$,

(iv) $||x - \tilde{x}_n^K|| \le ch^{l_1+m_1}$,

(v) $||x - \tilde{x}_n^{\mathrm{S}}|| \le ch^{l_2+\beta}$,

(vi) $||x - \hat{x}_n^M|| \le c\max\{h^{l_1+m_1}, h^{m_2}\}$,

(vii) $||x - \tilde{x}_n^M|| \le c\max\{h^{l_1+m_1}, h^{l_2+\beta}\}$.

Remark 3.6. From Theorem 3.5, we find that if $m_2 \geq l_1 + m_1$ (respectively $l_2 + \beta \geq l_1 + m_1$), then $\hat{x}_n^M$ (respectively $\tilde{x}_n^M$) has a better order of accuracy than x_n^{P}, x_n^K, and x_n^{S} and has the same order of accuracy as $\tilde{x}_n^K$.

Thus, if y and K are not so smooth, but K^2 is, then $\hat{x}_n^M$ and $\tilde{x}_n^M$ have better order of convergence than x_n^K and x_n^{S}, and they do not require as much computation as for $\tilde{x}_n^K$ and $\tilde{x}_n^{\mathrm{S}}$. We also remark that if $X = L_\infty, y \in C^\rho, k \in C^{l_1}$ with $\rho < l_1 = m_1$, then the results in Theorem 3.5, (i)–(iv), are the same as those in Sloan [**11**, (4.4)–(4.7)].

Example 3.7. Consider the following example of Sloan and Thomeé [**13**], where $X = L_2[0, 2\pi]$ and

$$(K\, v)(s) = \lambda \int_0^{2\pi} \log |\Big(\frac{s-t}{2}\Big)| v(s)\, ds,$$

assuming λ is not an eigenvalue of K. They show that

(i) $K = K^*$ is of class $\mathcal{K}_{p,0,1}$ and $\mathcal{K}_{p,1,0}$, and

(ii) K^i is of class $\mathcal{K}_{p,m,l}$ for $i \geq m + l$.

Thus we have $l_1 = 1 = m_1$ and $l_2 = 2 = m_2 = l_1 + m_1$.

Computational Remarks 3.8. The following expressions show explicitly the quantities to be computed:

$$\begin{aligned}
\tilde{x}_n^K &= y + K\, x_n^K = y + K\, y + K\, z_n^{\mathrm{P}}, \\
\tilde{x}_n^{\mathrm{S}} &= y + K\, x_n^{\mathrm{S}} = y + K\, y + K^2 x_n^{\mathrm{P}}, \\
\hat{x}_n^M &= y + z_n^M = y + K\, y + (z_n^{\mathrm{P}} - \boldsymbol{\pi}_n K\, y), \\
\tilde{x}_n^M &= y + K\, x_n^M = y + K\, y + K\, (x_n^{\mathrm{P}} - \boldsymbol{\pi}_n y).
\end{aligned}$$

We note that $\boldsymbol{\pi}_n K\, y$ is required to compute z_n^{P}, and $\boldsymbol{\pi}_n y$ is needed to compute x_n^{P}. Thus $\hat{x}_n^M$ and $\tilde{x}_n^M$ require less computations than $\tilde{x}_n^K$ and $\tilde{x}_n^S$.

The following table shows the operations to be performed in obtaining $\tilde{x}_n^K, \tilde{x}_n^{\mathrm{S}}, \hat{x}_n^M$ and $\tilde{x}_n^M$ except for the common expression $y + K\, y$:

Method	Solve	Compute
$\tilde{x}_n^K$	$u_n - \boldsymbol{\pi}_n K\, u_n = \boldsymbol{\pi}_n K\, y$	$K\, u_n$
$\tilde{x}_n^{\mathrm{S}}$	$v_n - \boldsymbol{\pi}_n K\, v_n = \boldsymbol{\pi}_n y$	$K^{\,2} v_n$
$\hat{x}_n^M$	$u_n - \boldsymbol{\pi}_n K\, u_n = \boldsymbol{\pi}_n K\, y$	$u_n - \boldsymbol{\pi}_n K\, y$
$\tilde{x}_n^M$	$v_n - \boldsymbol{\pi}_n K\, v_n = \boldsymbol{\pi}_n y$	$K\,(v_n - \boldsymbol{\pi}_n y)$

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