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# ON THE BACKWARD EULER METHOD FOR TIME DEPENDENT PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH NONSMOOTH INITIAL DATA

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ABSTRACT. In this paper the backward Euler method is applied for discretization in time for a time dependent parabolic integro-differential equation. A simple energy technique is used to derive almost optimal order error estimates when the initial function is only in $L^2$.

**1. Introduction.** In this paper we shall consider a time dependent parabolic integro-differential equation of the form

$$\begin{aligned} u_t + A(t)u &= \int_0^t B(t,s)u(s)\,ds \quad \text{in } \Omega \times J,\\ u &= 0 \quad \text{on } \partial\Omega \times J,\\ u(\cdot,0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^d$ with smooth boundary, $J$ denotes the interval $(0,T]$ with $T < \infty$, and $u(x,t)$ is a real-valued function in $\Omega \times J$ with $u_t = \partial u/\partial t$. We shall assume that $A(t)$ is a time dependent uniformly elliptic, second order self-adjoint linear partial differential operator in $\Omega$ and $B(t,s)$ is a second order partial differential operator with appropriately smooth coefficients.

Such problems and variants of them occur in several applications, such as in models for heat conduction in rigid materials with memory, the compression of poroviscoelastic media, reactor dynamics and epidemic phenomena in biology. For a detailed study, we refer to Yanik and Fairweather [**14**].

Let $H_0^1 = \{\phi \in H^1(\Omega) \mid \phi = 0 \text{ on } \partial\Omega\}$. Further, let $A(t;\cdot,\cdot)$ and $B(t,s;\cdot,\cdot)$ be the bilinear forms on $H_0^1 \times H_0^1$ corresponding to operators

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$A(t)$ and $B(t,s)$, respectively. The weak formulation of (1.1) is then defined as: Find $u : \overline{J} \to H_0^1$ such that

$$\begin{gathered}(u_t, \phi) + A(t; u, \phi) = \int_0^t B(t, s; u(s), \phi)\, ds, \\ \forall\, \phi \in H_0^1, \quad t \in J \\ u(0) = u_0.\end{gathered}$$

Here and below, we denote $(\cdot,\cdot)$ and $\|\cdot\|$ by the $L^2$ inner product and the induced norm on $L^2(\Omega)$.

For the purpose of Galerkin procedure, we assume that we are given a family $\{S_h\}$, $0 < h < 1$, of finite dimensional subspaces of $H_0^1$ such that

$$\begin{gathered}\inf_{\chi \in S_h} \{\|\phi - \chi\| + h\|\phi - \chi\|_1\} \le C h^r \|\phi\|_r, \\ \phi \in H^r \cap H_0^1, \quad r = 1, 2.\end{gathered}$$

The standard semi-discrete finite element approximation is then defined as a function $u_h : \overline{J} \to S_h$ such that

$$\begin{gathered}(u_{ht}, \chi) + A(t; u_h, \chi) = \int_0^t B(t, s; u_h(s), \chi)\, ds, \\ \forall\, \chi \in S_h, \quad t \in J, \\ u_h(0) = P_h u_0,\end{gathered} \tag{1.2}$$

where $P_h u_0$ is the $L^2$-projection of $u_0$ onto $S_h$.

In the present paper we shall discuss time discretization of (1.1) based on the backward Euler method. Let $k > 0$ be the time step and $t_n = nk$ with $T = Nk$. Further, let $U^n$ be the approximation of $u(t_n)$ and $\overline{\partial}_t U^n = k^{-1}(U^n - U^{n-1})$. Then the backward Euler scheme is to seek $U^n \in S_h$ such that, for $n = 1, 2, \dots, N$,

$$\begin{gathered}(\overline{\partial}_t U^n, \chi) + A(t_n; U^n, \chi) = k \sum_{j=0}^{n-1} B(t_n, t_j; U^j, \chi), \\ \forall\, \chi \in S_h, \\ U^0 = P_h u_0,\end{gathered} \tag{1.3}$$

where the integral term in (1.2) has been approximated by the rectangle rule

$$\int_0^{t_n} \phi(s)\, ds \approx k \sum_{j=0}^{n-1} \phi(t_j), \quad 0 < t_n \le T.$$

Below, we shall state the main result of this paper, whose proof will be carried out by energy arguments in Section 3.

**Theorem 1.1.** *Let $u$ be the exact solution of* (1.1) *and $U$ be the backward Euler approximation defined by* (1.3). *Then there exists a positive constant $C = C(T)$ such that, for $t_n \in (0, T]$,*

$$\|U^n - u(t_n)\| \le C t_n^{-1}\biggl(h^2 + k\biggl(1 + \biggl(\log\frac{1}{k}\biggr)\biggr)\biggr)\|u_0\|.$$

For our error analysis, we shall use the standard Sobolev space $H^m(\Omega)$, $m \in Z$ and its norm as $\|\cdot\|_m$. Let us define $\|\phi\|_{-j,h}$ as

$$\|\phi\|_{-j,h} = \sup_{\chi \in S_h} \frac{(\phi, \chi)}{\|\chi\|_j}, \quad j = 0, 1.$$

Throughout this paper $C$ denotes a generic positive constant independent of $h, k$ and any function involved and not necessarily the same at each occurrence.

The numerical solution of parabolic integro-differential equations was first studied by Douglas and Jones [**2**] using the finite difference method. Later, Yanik and Fairweather [**14**] presented fully discrete Galerkin finite element approximations to the solutions of a nonlinear parabolic integro-differential equation with $B$ at most of first order. For a more general parabolic integro-differential equation with $A$ independent of time, Sloan and Thomée [**10**] discussed the discretization in time with special attention paid to the storage requirements of the memory term.

Earlier, Thomée and Zhang [**12**] derived optimal $L^2$-error estimates for the semi-discrete Galerkin method applied to (1.1) with $A(t) = A$. The related fully discrete backward Euler scheme has been discussed by Thomée and Zhang [**13**], and optimal order error estimates are obtained through the semi-group theoretic approach when the given initial function is only in $L^2$. The method adopted also paid attention to the advantageous storage requirements of the memory term. Recently, for smooth initial data, Pani et al. [**7**] have also studied fully discrete numerical methods for (1.1) and obtained stability and optimal error estimates using energy arguments, and the methods considered there pay attention to the storage need during time-stepping. The semi-discrete Galerkin finite element approximation to (1.1) was presented by Pani and Sinha in [**6**], and optimal error estimates are derived using the parabolic duality argument and energy methods for rough initial data.

The related reference on finite element error analysis for parabolic equations with nonsmooth data can be found in Bramble et al. [**1**], Luskin and Rannacher [**5**], Huang and Thomée [**3,4**], Sammon [**8, 9**] and Thomée [**11**].

The layout of this paper is as follows. Section 2 contains some preliminary materials. Moreover, a stability result related to the semi-discrete solution $u_h$ is proved for our subsequent use. In Section 3 the backward Euler scheme for the discretization in time has been discussed. Finally, a proof of the main result, i.e., Theorem 1.1, is presented with the help of a series of lemmas.

**2. Preliminaries.** In this section we shall briefly review some basic results and stability estimates for our future use. For a proof, we refer to Huang and Thomée [**3**] and Pani et al. [**7**].

Let $T_h = T_h(t) : L^2 \to S_h$ be defined by

$$A(t; T_h\psi, \chi) = (\psi, \chi), \quad \forall \chi \in S_h.$$

We now recall some properties related to the solution operator $T_h$, namely, the operator $T_h$ is positive definite on $S_h$ and it approximates the exact solution operator $T = T(t) = A(t)^{-1}$ in the following sense

$$\|(T_h - T)\psi\| + h\|(T_h - T)\psi)\|_1 \le Ch^2\|\psi\|, \quad \psi \in L^2(\Omega). \tag{2.1}$$

Since $T_h$ is differentiable in time $t$, it is an easy exercise to show that

$$\|T_h'\psi\|_1 \le C\|T_h\psi\|_1 \le C\|\psi\|_{-1,h},$$

where $T_h'$ denotes the differentiation with respect to time $t$. We shall assume that the finite element mesh satisfies the quasi-uniformity condition. Then the following inverse estimate holds true for $S_h$, i.e., for $\chi \in S_h$,

$$\|\chi\|_1 \le Ch^{-1}\|\chi\|.$$

Let $\tilde{B}(\cdot,\cdot)$ be any bilinear form on $H_0^1 \times H_0^1$ associated with a second order partial differential operator. Then, using (2.1) and the inverse estimate, we have for $\psi, \chi \in S_h$,

$$\begin{aligned} |\tilde{B}(\psi, T_h\chi)| &\le |\tilde{B}(\psi, (T_h - T)\chi)| + |\tilde{B}(\psi, T_\chi)| \\ &\le C(\|\psi\|_1 h\|\chi\| + \|\psi\|\|\chi\|) \\ &\le C\|\psi\|\|\chi\|. \end{aligned} \tag{2.2}$$

In our subsequent analysis, we shall also use the following properties related to the solution operator $T_n = A_h(t_n)^{-1} : S_h \to S_h$ where $A_h(t_n) : S_h \to S_h$ is defined by

$$(A_h(t_n)\psi, \chi) = A(t_n; \psi, \chi), \quad \psi, \chi \in S_h.$$

Suppose $\hat{T}_n = A(t_n)^{-1}$ to be the continuous analogue of $T_n = T_h(t_n)$. Then we have, see Pani et al. [**7**],

$$\|(T_n - \hat{T}_n)\psi\| + h\|(T_n - \hat{T}_n)\psi\|_1 \le Ch^2\|\psi\|, \quad \psi \in S_h.$$

Analogous to (2.2), we obtain

$$|\tilde{B}(\psi, T_n\chi)| \le C\|\psi\|\,\|\chi\|, \quad \psi, \chi \in S_h. \tag{2.3}$$

Moreover, $A(t_n; T_n\psi, \chi) = (\psi, \chi)$, $\psi, \chi \in S_h$, and hence,

$$(\overline{\partial}A)(t_n; T_{n-1}\psi, \chi) + A(t_n; (\overline{\partial}_n T_n)\psi, \chi) = 0, \tag{2.4}$$

where $(\overline{\partial}A)$ is the backward difference quotient with respect to the first variable $t$ at $t = t_n$. It is well known [**4**] that there exist positive generic constants $C_1$ and $C_2$ such that

$$C_{-1}\|\psi\|_{-1,h} \le \|T_n\psi\|_1 \le C_2\|\psi\|_{-1,h}. \tag{2.5}$$

Taking $\chi = (\overline{\partial}_t T_n)\psi$ in (2.4) and, using coercivity and boundedness of $A$, it is easy to obtain

$$\|(\overline{\partial}_t T_n)(\psi)\|_1 \le C\|T_{n-1}\psi\|_1 \le C\|\psi\|_{-1,h}. \tag{2.6}$$

Below we shall prove the following estimates for the semi-discrete solution $u_h$ satisfying (1.2) which will be of frequent use in our error analysis.

**Theorem 2.1.** *Let $u_h$ be the solution of* (1.2). *Then, for $u_0 \in L^2$, the following estimates*

(a) $\|u_h(t)\|^2 + \int_0^t \|u_h(s)\|_1^2 \, ds \le C\|u_0\|^2$,

(b) $t^2\|u_{ht}(t)\|^2 + \int_0^t s^2\|u_{hs}(s)\|_1^2 \, ds \le C\|u_0\|^2$,

(c) $\int_0^t s^2\|u_{hss}(s)\|_{-1,h}^2 \, ds \le C\|u_0\|^2$,

*and*

(d) $\int_0^t \|T_h u_{hss}(s)\|_{-1,h}^2 \, ds \le C\|u_0\|^2$

*hold.*

*Proof.* Setting $\chi = u_h$ in (1.2) and integrating the resulting equation from 0 to $t$, it is easy to obtain the estimate (a). To estimate (b), we first differentiate (1.2) with respect to time $t$ to have

$$\begin{aligned}(u_{htt}, \chi) + A(t; u_{ht}, \chi) = &-A_t(t; u_h, \chi) + B(t, t; u_h(t), \chi) \\ &+ \int_0^t B_t(t, s; u_h(s), \chi) \, ds.\end{aligned} \tag{2.7}$$

Choose $\chi = t^2 u_{ht}$ in the above equation and use the standard energy argument to prove (b). For the estimation of (c), use boundedness of $A$, $A_t$, $B$ and $B_t$ to obtain

$$\|u_{htt}\|_{-1,h} \le C\left(\|u_{ht}\|_1 + \|u_h\|_1 + \int_0^t \|u_h(s)\|_1 \, ds\right).$$

Applying estimates (a) and (b), it now follows that

$$\begin{aligned}&\int_0^t s^2\|u_{hss}(s)\|_{-1,h}^2 \, ds \\ &\qquad \le C\int_0^t s^2\left(\|u_{hs}\|_1^2 + \|u_h\|_1^2 + \int_0^s \|u_h(\tau)\|_1^2 \, d\tau\right) ds \\ &\qquad \le C\|u_0\|^2.\end{aligned}$$

Finally, for the estimation of (d), we take $\chi = T_h v_h$ for $v_h \in S_h$ in (2.7) and use the self-adjoint property of $T_h$ and (2.2) to have

$$|(T_h u_{htt}, v_h)| \le C\bigg( \|u_{ht}\|_{-1,h}\|v_h\|_1 + \|u_h\|\,\|v_h\| + \int_0^t \|u_h(s)\|\,ds\|v_h\|\bigg).$$

From (1.2), we obtain

$$\|u_{ht}\|_{-1,h} \le C\bigg(\|u_h\|_1 + \int_0^t \|u_h(s)\|_1\,ds\bigg).$$

Therefore,

$$\begin{aligned}\|T_h u_{htt}\|_{-1,h} \\ &\le C\bigg(\|u_h\| + \|u_h\|_1 + \int_0^t \|u_h(s)\|\,ds + \int_0^t \|u_h(s)\|_1\,ds\bigg),\end{aligned}$$

and hence,

$$\begin{aligned}\int_0^t \|T_h u_{hss}\|^2_{-1,h}\,ds &\le C\int_0^t \bigg(\|u_h(s)\|^2 + \|u_h(s)\|_1^2 \\ &\qquad + \int_0^s (\|u_h(\tau)\|^2 + \|u_h(\tau)\|_1^2)\,d\tau\bigg)\,ds \\ &\le C\|u_0\|^2.\end{aligned}$$

This now completes the proof. □

We shall also frequently use the discrete version of Gronwall's lemma which is stated as follows. For a proof, see Pani et al. [**7**, Lemma 2.3].

**Lemma 2.1.** *If* $\xi_n \ge 0$, $\alpha_n \ge \alpha_{n-1}$, $\beta_j \ge 0$ *and* $\xi_n \le \alpha_n + \sum_{j=0}^{n-1}\beta_j\xi_j$ *for* $n \ge 0$, *then* $\xi_n \le \alpha_n \exp(\sum_{j=0}^{n-1}\beta_j)$.

**3. Error analysis for backward Euler method.** In this section we shall be concerned with discretization in time by the backward Euler scheme given by (1.3) and derive almost optimal order error estimates in $L^2$ assuming $u_0 \in L^2$.

For the proof of Theorem 1.1, we split the error $U^n - u(t_n)$ as $(U^n - u_h^n) + (u_h^n - u(t_n))$ with $u_h^n = u_h(t_n)$. Since the estimate of $\|u_h^n - u(t_n)\|$ is known from Pani and Sinha [**6**, Theorem 4.1], it is enough to derive an estimate for $\|U^n - u_h^n\|$. Let $\eta^n = U^n - u_h^n$. From (1.2) and (1.3), we obtain an error equation in $\eta^n$ as

$$\begin{aligned} (\overline{\partial}_t \eta^n, \chi) + A(t_n; \eta^n, \chi) &= k \sum_{j=0}^{n-1} B(t_n; t_j; \eta^j, \chi) \\ &\quad + Q_B^n(u_h)(\chi) + (\tau^n, \chi), \\ \eta^0 &= 0, \end{aligned} \tag{3.1}$$

where $\tau^n = u_{ht}^n - \overline{\partial}_t u_h^n$ and $Q_B^n(u_h)(\chi) = k \sum_{j=0}^{n-1} B(t_n, t_j; u_h^j, \chi) - \int_0^{t_n} B(t_n, s; u_h(s), \chi)\, ds$.

In order to compute $\eta^n$, set $\eta^n = \sum_{i=1}^{2} \eta_i^n$ where $\eta_i^n$, $i = 1, 2$, are determined by

$$\begin{gathered} (\overline{\partial}_t \eta_1^n, \chi) + A(t_n; \eta_1^n; \chi) = k \sum_{j=0}^{n-1} B(t_n, t_j; \eta^j, \chi) + Q_B^n(u_h)(\chi), \\ \chi \in S_h, \quad n \geq 1, \\ \eta_1^0 = 0, \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} (\overline{\partial}_t \eta_2^n, \chi) + A(t_n; \eta_2^n, \chi) = (\tau^n, \chi), \\ \chi \in S_h, \quad n \geq 1, \\ \eta_2^0 = 0. \end{gathered} \tag{3.3}$$

For the estimation of $\eta_2^n$, we shall closely follow the analysis of Huang and Thomée [**3**].

**Lemma 3.1.** *Let $\eta_2^n$ be a solution of* (3.3). *Then, for $n = 1, 2, \ldots, N$,*

$$t_n^2 \|\eta_2^n\|^2 + k \sum_{j=1}^{n} t_j^2 \|\eta_2^j\|_1^2 \leq C k^2 \|u_0\|^2.$$

*Proof.* Set $\tilde\eta_2^n = t_n\eta_2^n$ and $\tilde\tau^n = t_n\tau^n$. Multiply (3.3) by $t_n$ to have

$$(\overline{\partial}_t\tilde\eta_2^n,\chi) + A(t_n;\tilde\eta_2^n,\chi) = (\tilde\tau^n,\chi) + (\eta_2^{n-1},\chi). \tag{3.4}$$

Taking $\chi = \tilde\eta_2^n$ in (3.4) and using coercivity of $A$, we obtain

$$\frac{1}{2}\overline{\partial}_t\|\tilde\eta_2^n\|^2 + c\|\tilde\eta_2^n\|_1^2 \le \|\tilde\tau^n\|_{-1,h}\|\tilde\eta_2^n\|_1 + \|\eta_2^{n-1}\|_{-1,h}\|\tilde\eta_2^n\|_1.$$

Sum $n$ from 1 to $m$ to have

$$\begin{aligned} t_m^2\|\eta_2^m\|^2 + k\sum_{n=1}^m t_n^2\|\eta_2^n\|_1^2 \le Ck\sum_{n=1}^m t_n^2\|\tau^n\|_{-1,h}^2& \\ + Ck\sum_{n=1}^{m-1}\|\eta_2^n\|_{-1,h}^2&. \end{aligned}\tag{3.5}$$

To estimate the first term on the righthand side, we note that

$$\tau^n = \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})u_{hss}(s)\,ds,$$

and, hence,

$$\|\tau^n\|_{-1,h}^2 \le \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2\|u_{h_{ss}}(s)\|_{-1,h}^2\,ds.$$

Since $t_n(s-t_{n-1}) \le sk$ for $s\in[t_{n-1},t_n]$, we obtain, using Theorem 2.1,

$$\begin{aligned} k\sum_{n=1}^m t_n^2\|\tau^n\|_{-1,h}^2 &\le Ck^2\int_0^{t_n} s^2\|u_{h_{ss}}(s)\|_{-1,h}^2\,ds \\ &\le Ck^2\|u_0\|^2. \end{aligned}\tag{3.6}$$

Next, to estimate the second term on the righthand side of (3.5), we proceed as follows.

Using the property of $T_n = T_h(t_n)$, first write error equation (3.3) in the form

$$T_n\overline{\partial}_t\eta_2^n + \eta_2^n = T_n\tau^n.$$

Now rewrite the above equation as

$$\overline{\partial}_t(T_n\eta_2^n) + \eta_2^n = T_n\tau^n + (\overline{\partial}_t T_n)\eta_2^{n-1} = F_n + (\overline{\partial}_t T_n)\eta_2^{n-1},$$

where $F_n = T_n\tau^n$. Taking the inner product with $T_n\eta_2^n$ and using (2.5)–(2.6), we find that

$$\begin{aligned}\frac{1}{2}\overline{\partial}_t(\|T_n\eta_2^n\|^2) + c\|\eta_2^n\|_{-1,h}^2 &\le \|F_n\|_{-1,h}\|T_n\eta_2^n\|_1 \\ &\quad + \|(\overline{\partial}_t T_n)\eta_2^{n-1}\|_1\|T_n\eta_2^n\|_{-1,h} \\ &\le C\|F_n\|_{-1,h}\|\eta_2^n\|_{-1,h} \\ &\quad + \|\eta_2^{n-1}\|_{-1,h}\|T_n\eta_2^n\|.\end{aligned}$$

Using Young's inequality, it follows that

$$\begin{aligned}\frac{1}{2}\overline{\partial}_t(\|T_n\eta_2^n\|^2) + c\|\eta_2^n\|_{-1,h}^2 &\le \frac{c}{4}(\|\eta_2^n\|_{-1,h}^2 + \|\eta_2^{n-1}\|_{-1,h}^2) \\ &\quad + C\|F_n\|_{-1,h}^2 + C\|T_n\eta_2^n\|^2.\end{aligned}$$

Sum $n$ from 1 to $m$ to have

$$\|T_m\eta_2^m\|^2 + ck\sum_{n=1}^m \|\eta_2^n\|_{-1,h}^2 \le Ck\sum_{n=1}^m \|F_n\|_{-1,h}^2 + Ck\sum_{n=1}^m \|T_n\eta_2^n\|^2.$$

An application of the discrete Gronwall's lemma, Lemma 2.1, leads to

$$\|T_m\eta_2^m\|^2 + ck\sum_{n=1}^m \|\eta_2^n\|_{-1,h}^2 \le Ck\sum_{n=1}^m \|F_n\|_{-1,h}^2. \tag{3.7}$$

To estimate the term on the righthand side of (3.7), we note that

$$\begin{aligned}\|F_n\|_{-1,h}^2 &= \|T_n\tau^n\|_{-1,h} \\ &\le \frac{1}{k}\int_{t_{n-1}}^{t_n}(s-t_{n-1})^2\|T_n u_{hss}(s)\|_{-1,h}^2\,ds.\end{aligned}$$

For $\tilde{s} \in (t_{n-1}, t_n)$ use the mean value theorem and properties of $T_h$ to obtain

$$\begin{aligned}\|T_n u_{hss}(s)\|_{-1,h} &\le \|T_h(s)u_{hss}(s)\|_{-1,h} + k\|T_h'(\tilde{s})u_{hss}(s)\|_{-1,h} \\ &\le \|T_h(s)u_{hss}(s)\|_{-1,h} + Ck\|T_h'(\tilde{s})u_{hss}(s)\|_1 \\ &\le \|T_h(s)u_{hss}(s)\|_{-1,h} + Ck\|u_{hss}(s)\|_{-1,h}.\end{aligned}$$

Therefore, again using Theorem 2.1, we obtain

$$
\begin{aligned}
k\sum_{n=1}^{m}\|F_n\|_{-1,h}^2 &\le Ck^2\int_0^{t_m}\|T_h(s)u_{hss}(s)\|_{-1,h}^2\,ds \\
&\quad + Ck^2\int_0^{t_m} s^2\|u_{hss}(s)\|_{-1,h}^2\,ds \\
&\le Ck^2\|u_0\|^2.
\end{aligned}
\tag{3.8}
$$

Combine (3.5)–(3.8) to obtain the desired estimate. This completes the proof. □

To achieve a bound for $\eta^n$, it remains to obtain an estimate for $\eta_1^n$. Below we shall derive this using a series of lemmas.

Let $Q_A^m(u_h)(\chi) = -k\sum_{n=1}^m A(t_n; u_h^n, \chi) + \int_0^{t_m} A(s; u_h(s), \chi)\,ds$, and

$$
\begin{aligned}
\overline{Q}_B^m(u_h)(\chi) = k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1} B(t_n, t_j; u_h^j, \chi) \\
- \int_0^{t_m}\int_0^s B(s,\tau; u_h(\tau), \chi)\,d\tau\,ds
\end{aligned}
$$

be the quadrature error when we apply the right rectangle rule. In the following lemma, we shall derive some estimates related to the above quadrature errors for our future use.

**Lemma 3.2.** *With $Q_A^n$, $Q_B^n$ and $\overline{Q}_B^m$ defined as above, there is a positive constant $C$ such that, for $\chi \in S_h$,*

$$
\begin{aligned}
|Q_A^n(u_h)(T_n\chi)| + |Q_B^n(u_h)(T_n\chi)| + |\overline{Q}_B^m(u_h)(T_m\chi)| \\
\le Ck\left(1 + \log\frac{1}{k}\right)\|u_0\|\|\chi\|.
\end{aligned}
$$

*Proof.* Using the right rectangle rule we note that

$$\begin{aligned} Q_A^n(u_h)(T_n\chi) = & -kA(t_1, u_h^1, T_n\chi) \\ & + \int_0^{t_1} A(s, u_h(s), T_n\chi)\, ds \\ & + \sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} (t_{j-1} - s) \\ & \qquad \cdot [A(s, u_{hs}(s), T_n\chi) + A_s(s, u_h(s), T_n\chi)]\, ds. \end{aligned}$$

Now apply (2.3) to have

$$\begin{aligned} |Q_A^n(u_h)(T_n\chi)| \le & \; Ck\|u_h(t_1)\|\|\chi\| \\ & + C\int_0^{t_1} \|u_h(s)\|\|\chi\|\, ds \\ & + Ck\sum_{j=2}^{n} \int_{t_{j-1}}^{t_j} (\|u_{hs}(s)\| + \|u_h(s)\|)\, ds\|\chi\|. \end{aligned}$$

By Theorem 2.1, we obtain

$$\begin{aligned} |Q_A^n(u_h)(T_n\chi)| & \le Ck\bigg(1 + \sum_{j=2}^{n} \log\frac{t_j}{t_{j-1}}\bigg)\|u_0\|\|\chi\| \\ & \le Ck\bigg(1 + \log\frac{1}{k}\bigg)\|u_0\|\|\chi\|. \end{aligned}$$

Next, using the left rectangle rule, we rewrite

$$\begin{aligned} Q_B^n(u_h)(T_n\chi) = & \; kB(t_n, 0; u_h(0), T_n\chi) \\ & - \int_0^{t_1} B(t_n, s; u_h(s), T_n\chi)\, ds \\ & + \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (s - t_{j+1})[B(t_n; s, u_{hs}(s), T_n\chi) \\ & \qquad\qquad + B_s(t_n, s; u_h(s), T_n\chi)]\, ds. \end{aligned}$$

A similar argument as above shows that

$$\begin{aligned} |Q_B^n(u_h)(T_n\chi)| & \le Ck\bigg(1 + \sum_{j=1}^{n-1} \log\frac{t_{j+1}}{t_j}\bigg)\|u_0\|\|\chi\| \\ & \le Ck\bigg(1 + \log\frac{1}{k}\bigg)\|u_0\|\|\chi\|. \end{aligned}$$

Finally, to estimate $\overline{Q}_B^m$, we now split

$$
\begin{aligned}
\overline{Q}_B^m = k\sum_{n=1}^{m}\Bigg[k\sum_{j=0}^{n-1}B(t_n,t_j;u_h^j,T_m\chi)& \\
-\int_0^{t_n}B(t_n,s;u_h(s),T_m\chi)\,ds\Bigg]& \\
+\Bigg[k\sum_{n=1}^{m}\int_0^{t_n}B(t_n,s;u_h(s),T_m\chi)\,ds& \\
-\int_0^{t_m}\int_0^{s}B(s,\tau;u_h(\tau),T_m\chi)\,d\tau\,ds\Bigg]& \\
=\overline{Q}_{1,B}^m(u_h)(T_m\chi)+\overline{Q}_{2,B}^m(u_h)(T_m\chi).&
\end{aligned}
\tag{3.9}
$$

Note that

$$\overline{Q}_{1,B}^m = k\sum_{n=1}^{m}Q_B^n(u_h)(T_m\chi),$$

and hence, using the estimate of $Q_B^n$ (replacing $T_n\chi$ by $T_m\chi$) we have

$$|\overline{Q}_{1,B}^m(u_h)(T_m\chi)| \le Ck\left(1+\log\frac{1}{k}\right)\|u_0\|\|\chi\|. \tag{3.10}$$

For the second term on the right of (3.9), it now follows that

$$
\begin{aligned}
\overline{Q}_{2,B}^m(u_h)(T_m\chi) &= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}(s-t_{n-1})s \\
&\qquad\cdot\frac{\partial}{\partial s}\left[\int_0^s B(s,\tau;u_h(\tau),T_m\chi)\,d\tau\right]ds \\
&= \sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}(s-t_{n-1}) \\
&\qquad\cdot\Bigg[B(s,s;u_h(s),T_m\chi) \\
&\qquad\qquad+\int_0^s B_s(s,\tau;u_h(\tau),T_m\chi)\,d\tau\Bigg]\,ds.
\end{aligned}
$$

Again, a use of (2.3) yields

$$
\begin{aligned}
|\overline{Q}_{2,B}^{m}(u_h)(T_m\chi)| &\le Ck\sum_{n=1}^{m}\int_{t_{n-1}}^{t_n}[\|u_h(s)\|\|\chi\| \\
&\qquad + \int_0^s \|u_h(s)\|\|\chi\|\,d\tau]\,ds \\
&\le Ck^2\sum_{n=1}^{m}\|u_0\|\|\chi\| \\
&\le Ck\|u_0\|\|\chi\|.
\end{aligned} \tag{3.11}
$$

Now combine (3.9)–(3.11) to estimate the third term. This completes the proof. □

**Lemma 3.3.** *There is a positive constant $C$ such that the following estimate holds for $n = 1, 2, \ldots, N$,*

$$
\begin{aligned}
k\sum_{j=1}^{n} t_j\|T_j\overline{\partial}_t\eta_1^j\|_1^2 + t_n\|\eta_1^n\|^2 &\le C\Big[k^2\Big(1+\log\frac{1}{k}\Big)^2\|u_0\|^2 + t_{n-1}\|\hat{\eta}^{n-1}\|^2\Big] \\
&\quad + C\Big[k\sum_{j=1}^{n-1}\|\hat{\eta}^j\|^2 + k\sum_{j=1}^{n}\|\eta_1^j\|^2 \\
&\qquad + k\sum_{j=1}^{n-1} t_j^2\|\eta^j\|^2\Big],
\end{aligned}
$$

*where $\hat{\eta}^n = k\sum_{j=0}^{n}\eta^j$.*

*Proof.* Choose $\chi = t_n\overline{\partial}_t(T_n\eta_1^n)$ in (3.2) to have

$$
\begin{aligned}
&t_n(\overline{\partial}_t\eta_1^n, \overline{\partial}_t(T_n\eta_1^n)) + t_nA(t_n;\eta_1^n, \overline{\partial}_t(T_n\eta_1^n)) \\
&\qquad = k\sum_{j=0}^{n-1} t_nB(t_n;t_j;\eta^j, \overline{\partial}_t(T_n\eta_1^n)) + t_nQ_B^n(u_h)(\overline{\partial}_t(T_n\eta_1^n)).
\end{aligned}
$$

Note that

$$
\begin{aligned}
t_n(\overline{\partial}_t\eta_1^n, \overline{\partial}_t(T_n\eta_1^n)) &= t_n(\overline{\partial}_t\eta_1^n, T_n\overline{\partial}_t\eta_1^n) \\
&\quad + t_n(\overline{\partial}_t\eta_1^n, (\overline{\partial}_t T_n)\eta_1^{n-1}), \\
t_n A(t_n; \eta_1^n, \overline{\partial}_t(T_n\eta_1^n)) &\geq \frac{1}{2}\overline{\partial}_t[t_n\|\eta_1^n\|^2] - \frac{1}{2}\|\eta_1^{n-1}\|^2 \\
&\quad - t_n(\overline{\partial}A)(t_n; \eta_1^n, T_{n-1}\eta_1^{n-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
t_n Q_B^n(u_h)(\overline{\partial}_t(T_n\eta_1^n)) &= \overline{\partial}_t[t_n Q_B^n(u_h)(T_n\eta_1^n)] \\
&\quad - Q_B^n(u_h)(T_{n-1}\eta_1^{n-1}) \\
&\quad - t_{n-1}\overline{\partial}_t(Q_B^n(u_h))(T_{n-1}\eta_1^{n-1}).
\end{aligned}
$$

For $n = 1$, it is easy to obtain

$$
kt_1\|T_1\overline{\partial}_t\eta_1^1\|_1^2 + t_1\|\eta_1^1\|^2 \leq Ck^2\|u_0\|^2 + Ckt_1^2\|\eta^1\|^2. \tag{3.12}
$$

We now sum $n$ from 2 to $m$ to have

$$
\begin{aligned}
&k\sum_{n=2}^{m} t_n(\overline{\partial}_t\eta_1^n, T_n\overline{\partial}_t\eta_1^n) + \frac{1}{2}t_m\|\eta_1^m\|^2 \\
&\qquad \leq \frac{1}{2}k\sum_{n=2}^{m}\|\eta_1^{n-1}\|^2 \\
&\qquad\quad + \left|k\sum_{n=2}^{m}[-t_n(\overline{\partial}_t\eta_1^n, (\overline{\partial}_t T_n)\eta_1^{n-1}) + t_n(\overline{\partial}A)(t_n; \eta_1^n, T_{n-1}\eta_1^{n-1})]\right| \\
&\qquad\quad + \left|k\sum_{n=2}^{m} t_n B(t_n, t_{n-1}; \hat{\eta}^{n-1}, \overline{\partial}_t(T_n\eta_1^n))\right| \\
&\qquad\quad + \left| - k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1} t_n(\overline{\partial}_2 B)(t_n, t_j; \hat{\eta}^{j-1}, \overline{\partial}_t(T_n\eta_1^n))\right| \\
&\qquad\quad + [|t_m Q_B^m(u_h)(T_m\eta_1^m) - t_1 Q_B^1(u_h)(T_1\eta_1^1)|] \\
&\qquad\quad + \left| - k\sum_{n=2}^{m} Q_B^n(u_h)(T_{n-1}\eta_1^{n-1})\right| \\
&\qquad\quad + \left| - k\sum_{n=2}^{m} t_{n-1}\overline{\partial}_t(Q_B^n(u_h))(T_{n-1}\eta_1^{n-1})\right| + \frac{1}{2}t_1\|\eta_1^1\|^2
\end{aligned}
$$

$$= |I_1| + |I_2| + |I_3| + |I_4| + |I_5| + |I_6| + |I_7| + \frac{1}{2} t_1 \|\eta_1^1\|^2.$$

For $I_1$ and $I_2$, apply (2.3), (2.5) and (2.6) to obtain

$$\begin{aligned}
|I_1| + |I_2| \le& \frac{1}{2} k \sum_{n=1}^{m-1} \|\eta_1^n\|^2 \\
&+ Ck \sum_{n=2}^{m} [t_n \|\overline{\partial}_t \eta_1^n\|_{-1,h} \|(\overline{\partial}_t T_n)\eta_1^{n-1}\|_1 + t_n \|\eta_1^n\| \|\eta_1^{n-1}\|] \\
\le& \frac{1}{2} k \sum_{n=1}^{m-1} \|\eta_1^n\|^2 \\
&+ Ck \sum_{n=2}^{m} [t_n \|T_n \overline{\partial}_t \eta_1^n\|_1 \|\eta_1^{n-1}\| + t_n \|\eta_1^n\| \|\eta_1^{n-1}\|],
\end{aligned}$$

and hence,

$$|I_1| + |I_2| \le Ck \sum_{n=1}^{m} \|\eta_1^n\|^2 + \frac{1}{2} k \sum_{n=2}^{m} t_n \|T_n \overline{\partial}_t \eta_1^n\|_1^2.$$

To estimate $I_3$, we first rewrite it as

$$\begin{aligned}
I_3 =& k \sum_{n=2}^{m} \overline{\partial}_t [t_n B(t_n, t_{n-1}; \hat{\eta}^{n-1}, T_n \eta_1^n)] \\
&- k \sum_{n=2}^{m} B(t_n, t_{n-1}; \hat{\eta}^{n-1}, T_{n-1} \eta_1^{n-1}) \\
&- k \sum_{n=2}^{m} t_{n-1} B(t_n, t_{n-1}; \eta^{n-1}, T_{n-1} \eta_1^{n-1}) \\
&- k \sum_{n=2}^{m} t_{n-1} (\overline{\partial}_1 B)(t_n, t_{n-1}; \hat{\eta}^{n-2}, T_{n-1} \eta_1^{n-1}) \\
&- k \sum_{n=2}^{m} t_{n-1} (\overline{\partial}_2 B)(t_{n-1}, t_{n-1}; \hat{\eta}^{n-2}, T_{n-1} \eta_1^{n-1}),
\end{aligned}$$

where $\overline{\partial}_1 B$ and $\overline{\partial}_2 B$ are the difference quotients of $B$ with respect to the first and second arguments, respectively. The first term on the righthand side of $I_3$ can be written as $t_m B(t_m, t_{m-1}; \hat{\eta}^{m-1}, T_m \eta_1^m)$.

Now applying (2.3) to all the terms in $I_3$ and, since $t_m = t_{m-1} + k$, we have

$$\begin{aligned} |I_3| \leq C(t_{m-1}+k)\|\hat{\eta}^{m-1}\|^2 \\ + Ck\sum_{n=1}^{m-1}\|\hat{\eta}^n\|^2 + Ck\sum_{n=1}^{m-1}\|\eta_1^n\|^2 \\ + Ck\sum_{n=1}^{m-1}t_n^2\|\eta^n\|^2 + \frac{1}{8}t_m\|\eta_1^m\|^2. \end{aligned}$$

For $I_4$, let us rewrite it as

$$\begin{aligned} I_4 &= -k^2\sum_{j=1}^{m-1}\sum_{n=j+1}^{m} t_n(\overline{\partial}_2 B)(t_n, t_j; \hat{\eta}^{j-1}, \overline{\partial}_t(T_n\eta_1^n)) \\ &= -k^2\sum_{j=1}^{m-1}\sum_{n=j+1}^{m} \overline{\partial}_t[t_n(\overline{\partial}_2 B)(t_n, t_j; \hat{\eta}^{j-1}, T_n\eta_1^n)] \\ &\quad + k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}[(\overline{\partial}_2 B)(t_n, t_j; \hat{\eta}^{j-1}, T_{n-1}\eta_1^{n-1}) \\ &\qquad\qquad + t_{n-1}(\overline{\partial}_{21} B)(t_n, t_j; \hat{\eta}^{j-1}, T_{n-1}\eta_1^{n-1})] \\ &= -k\sum_{j=1}^{m-1} t_m(\overline{\partial}_2 B)(t_m, t_j; \hat{\eta}^{j-1}, T_m\eta_1^m) \\ &\quad + k\sum_{j=1}^{m-1} t_j(\overline{\partial}_2 B)(t_j, t_j; \hat{\eta}^{j-1}, T_j\eta_1^j) \\ &\quad + k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}[(\overline{\partial}_2 B)(t_n, t_j; \hat{\eta}^{j-1}, T_{n-1}\eta_1^{n-1}) \\ &\qquad\qquad + t_{n-1}(\overline{\partial}_{21} B)(t_n, t_j; \hat{\eta}^{j-1}, T_{n-1}\eta_1^{n-1})], \end{aligned}$$

whence $\overline{\partial}_{21}B$ is the difference quotient of $\overline{\partial}_2 B$ with respect to the first argument, and hence,

$$
\begin{aligned}
|I_4| \leq Ck \sum_{j=1}^{m-1} t_m \|\hat{\eta}^{j-1}\| \|\eta_1^m\| \\
+ Ck \sum_{j=1}^{m-1} t_j \|\hat{\eta}^{j-1}\| \|\eta_1^j\| \\
+ Ck^2 \sum_{n=2}^{m} \sum_{j=1}^{n-1} \|\hat{\eta}^{j-1}\| \|\eta_1^{n-1}\| \\
+ Ck^2 \sum_{n=2}^{m} \sum_{j=1}^{n-1} t_{n-1} \|\hat{\eta}^{j-1}\| \|\eta_1^{n-1}\|.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, it now follows that

$$
\begin{aligned}
|I_4| \leq Ck \sum_{j=1}^{m-1} \|\hat{\eta}^j\|^2 + Ck \sum_{j=1}^{m-1} \|\eta_1^j\|^2 \\
+ Ck^2 \sum_{n=2}^{m} \sum_{j=1}^{n-1} \|\hat{\eta}^j\|^2 + \frac{1}{8} t_m \|\eta_1^m\|^2.
\end{aligned}
$$

For $I_5$ and $I_6$, use Lemma 3.2 to obtain

$$
\begin{aligned}
|I_5| + |I_6| \leq Ck^2 \left(1 + \log \frac{1}{k}\right)^2 \|u_0\|^2 \\
+ Ck \sum_{n=1}^{m-1} \|\eta_1^n\|^2 \\
+ \frac{1}{4}(t_m \|\eta_1^m\|^2 + t_1 \|\eta_1^1\|^2).
\end{aligned}
$$

Finally, for $I_7$, rewrite it as

$$
\begin{aligned}
I_7 &= -\sum_{n=2}^{m} t_{n-1} \Big[ kB(t_n, t_{n-1}; u_h^{n-1}, T_{n-1}\eta_1^{n-1}) \\
&\qquad - \int_{t_{n-1}}^{t_n} B(t_n, s; u_h(s), T_{n-1}\eta_1^{n-1})\, ds \Big] \\
&\quad - k \sum_{n=2}^{m} t_{n-1} Q_{\bar{\partial}_1 B}^{n-1}(T_{n-1}\eta_1^{n-1}) \\
&= I_7^1 + I_7^2,
\end{aligned}
$$

where $\overline{\partial}_1 B$ is the difference quotient of $B$ with respect to the first argument. Using (2.3), we have

$$\begin{aligned}
|I_7^1| &\le C\sum_{n=2}^{m} t_{n-1}\int_{t_{n-1}}^{t_n}\left|(s-t_n)\frac{\partial}{\partial s}[B(t_n,s;u_h(s),T_{n-1}\eta_1^{n-1})]\right| ds\\
&\le k\sum_{n=2}^{m}\int_{t_{n-1}}^{t_n} s(|B(t_n,s;u_{hs}(s),T_{n-1}\eta_1^{n-1})|\\
&\qquad\qquad + |B_s(t_n,s;u_h(s),T_{n-1}\eta_1^{n-1})|)\, ds\\
&\le Ck\sum_{n=2}^{m}\int_{t_{n-1}}^{t_n} s(\|u_{hs}(s)\|+\|u_h(s)\|)\, ds\|\eta_1^{n-1}\|.
\end{aligned}$$

Again, use Theorem 2.1 to obtain

$$|I_7^1| \le Ck^2\|u_0\|\sum_{n=2}^{m}\|\eta_1^{n-1}\| \le Ck^2\|u_0\|^2 + Ck\sum_{n=1}^{m-1}\|\eta_1^n\|^2.$$

For $I_7^2$, Lemma 3.2 can be easily modified to have

$$|I_7^2| \le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2 + Ck\sum_{n=1}^{m-1}\|\eta_1^n\|^2.$$

Combining the above estimates, we obtain the required estimate using (3.12), and this completes the proof. □

Note that the righthand side of the estimate $t_n\|\eta^n\|$ in the previous lemma involves terms containing $\hat{\eta}^n$. Therefore, in the following lemma we shall obtain some estimates related to $\hat{\eta}^n$.

With $\hat{\eta}^n = k\sum_{j=0}^{n}\eta^j$, clearly $\overline{\partial}_t\hat{\eta}^n = \eta^n$ and $\hat{\eta}^0 = 0$. Multiply (1.3) by $k$ and then sum with respect to $n$ from 1 to $m$ with $1 \le n \le m \le N$ to have

$$\begin{aligned}
(U^m,\chi) + k\sum_{n=1}^{m} A(t_n;U^n,\chi)\\
= k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1} B(t_n,t_j;U^j,\chi) + (P_h u_0,\chi).
\end{aligned} \tag{3.13}$$

Integrate (1.2) from 0 to $t$ to obtain

$$(3.14)\quad (u_h(t),\chi)+\int_0^t A(s;u_h(s),\chi)\,ds = (P_h u_0,\chi)+\int_0^t\int_0^s B(s,\tau;u_h(\tau),\chi)\,d\tau\,ds.$$

Using (3.14) at $t=t_m$ and (3.13), we find that

$$(\overline{\partial}_t\hat{\eta}^m,\chi)+k\sum_{n=1}^{m}A(t_n;\eta^n,\chi)=k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}B(t_n,t_j;\eta^j,\chi)+Q_A^m(u_h)(\chi) + \overline{Q}_B^m(u_h)(\chi).$$

Since $k\sum_{n=1}^m A(t_n;\eta^n,\chi) = A(t_m;\hat{\eta}^m,\chi)-k\sum_{n=1}^m(\overline{\partial}A)(t_n;\hat{\eta}^{n-1},\chi)$, where $(\overline{\partial}A)(t_n;\cdot,\cdot) = k^{-1}[A(t_n;\cdot,\cdot)-A(t_{n-1};\cdot,\cdot)]$ is the backward difference quotient of $A(t,\cdot,\cdot)$ with respect to the first variable at $t=t_n$, we obtain

$$(3.15)\quad \begin{aligned}(\overline{\partial}_t\hat{\eta}^m,\chi)+A(t_m;\hat{\eta}^m,\chi) &= k\sum_{n=1}^{m}(\overline{\partial}A)(t_n;\hat{\eta}^{n-1},\chi)\\ &\quad + k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1}B(t_n,t_j;\eta^j,\chi)\\ &\quad + Q_A^m(u_h)(\chi)+\overline{Q}_B^m(u_h)(\chi).\end{aligned}$$

**Lemma 3.4.** *With $\hat{\eta}^n$ given as above, we have*

$$\|T_n\hat{\eta}^n\|_1^2+k\sum_{j=1}^{n}\|\hat{\eta}^j\|^2\le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2.$$

*Proof.* Choose $\chi = T_m\hat{\eta}^m$ in (3.15) to obtain

$$(3.16)\quad \begin{aligned}(\overline{\partial}_t\hat{\eta}^m, T_m\hat{\eta}^m) + A(t_m;\hat{\eta}^m, T_m\hat{\eta}^m) &\\ = k\sum_{n=1}^{m}(\overline{\partial}A)(t_n, \hat{\eta}^{n-1}, T_m\hat{\eta}^m)&\\ + k^2\sum_{n=1}^{m}\sum_{j=0}^{n-1} B(t_n, t_j;\eta^j, T_m\hat{\eta}^m)&\\ + Q_A^m(u_h)(T_m\hat{\eta}^m)&\\ + \overline{Q}_B^m(u_h)(T_m\hat{\eta}^m).&\end{aligned}$$

For $m = 1$, it follows that

$$\frac{1}{k}(\hat{\eta}^1, T_1\hat{\eta}^1) + \|\hat{\eta}^1\|^2 = Q_A^1(u_h)(T_1\hat{\eta}^1) + \overline{Q}_B^1(u_h)(T_1\hat{\eta}^1).$$

Applying (2.3) to the terms appearing on the right of the above equation, we obtain

$$(3.17)\qquad \|T_1\hat{\eta}^1\|_1^2 + k\|\hat{\eta}^1\|^2 \le Ck^2\|u_0\|^2.$$

We first note that

$$\begin{aligned}2(\overline{\partial}_t\hat{\eta}^m, T_m\hat{\eta}^m) = \overline{\partial}_t[(\hat{\eta}^m, T_m\hat{\eta}^m)]&\\ + k(\overline{\partial}_t\hat{\eta}^m, T_m\overline{\partial}_t\hat{\eta}^m)&\\ - (\hat{\eta}^{m-1}, (\overline{\partial}_t T_m)\hat{\eta}^{m-1})&\end{aligned}$$

and for $m \ge 2$,

$$\begin{aligned}k^2\sum_{n=2}^{m}\sum_{j=0}^{n-1} B(t_n, t_j;\eta^j, T_m\hat{\eta}^m) &= k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1} B(t_n, t_j;\overline{\partial}_t\hat{\eta}^j, T_m\hat{\eta}^m)\\ &= k\sum_{n=2}^{m} B(t_n, t_{n-1};\hat{\eta}^{n-1}, T_m\hat{\eta}^m)\\ &\quad - k^2\sum_{n=2}^{m}\sum_{j=1}^{n-1}(\overline{\partial}_2 B)(t_n, t_j;\hat{\eta}^{j-1}, T_m\hat{\eta}^m),\end{aligned}$$

where $\overline{\partial}_2 B$ is the backward difference quotient of $B$ with respect to the second argument. Here we have also used summation by parts.

Sum (3.16) with respect to $m$ from 2 to $l$ with $m \le l \le N$ and use (2.3) to have

$$
\begin{aligned}
(\hat\eta^l, T_l\hat\eta^l) + 2k\sum_{m=2}^{l}\|\hat\eta^m\|^2 \le |(\hat\eta^1, T_1\hat\eta^1)| \\
+ C\Bigg[ k^2\sum_{m=2}^{l}\sum_{n=1}^{m-1}\|\hat\eta^n\|\|\hat\eta^m\| \\
+ k\sum_{m=2}^{l}\|\hat\eta^{m-1}\|_{-1,h}\|(\overline{\partial}_t T_m)\hat\eta^{m-1}\|_1 \\
+ k^3\sum_{m=2}^{l}\sum_{n=2}^{m}\sum_{j=1}^{n-1}\|\hat\eta^{j-1}\|\|\hat\eta^m\| \\
+ k\sum_{m=2}^{l}|Q_A^m(u_h)(T_m\hat\eta^m)| \\
+ k\sum_{m=2}^{l}|\overline{Q}_B^m(u_h)(T_m\hat\eta^m)|\Bigg].
\end{aligned}
$$

For the third term on the righthand side, we shall use (2.5)–(2.6) and apply Lemma 3.2 for the last two terms. Then use Young's inequality to obtain

$$
\begin{aligned}
\|T_l\hat\eta^l\|_1^2 + k\sum_{m=2}^{l}\|\hat\eta^m\|^2 \le C\|T_1\hat\eta^1\|_1^2 \\
+ Ck^2\left(1 + \log\frac{1}{k}\right)^2\|u_0\|^2 \\
+ Ck^2\sum_{m=2}^{l}\sum_{j=1}^{m-1}\|\hat\eta^j\|^2 \\
+ Ck\sum_{m=1}^{l-1}\|T_m\hat\eta^m\|_1^2.
\end{aligned}
$$

With the help of (3.17), we have

$$\|T_l\hat{\eta}^l\|_1^2 + k\sum_{m=1}^{l}\|\hat{\eta}^m\|^2 \le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2 + Ck^2\sum_{m=1}^{l-1}\sum_{j=1}^{m}\|\hat{\eta}^j\|^2 + Ck\sum_{m=1}^{l-1}\|T_m\hat{\eta}^m\|_1^2.$$

Apply the discrete Gronwall's lemma to complete the rest of the proof.
□

**Lemma 3.5.** *With $\hat{\eta}^n$ as above, the following estimate*

$$k\sum_{j=1}^{n}\|T_j\overline{\partial}_t\hat{\eta}^j\|_1^2 + \|\hat{\eta}^n\|^2 \le Ck\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2$$

*holds.*

*Proof.* Take $\chi = \overline{\partial}_t(T_m\hat{\eta}^m)$ in (3.15) to obtain

$$\begin{aligned}(\overline{\partial}_t\hat{\eta}^m, \overline{\partial}_t(T_m\hat{\eta}^m)) + A(t_m;\hat{\eta}^m, \overline{\partial}_t(T_m\hat{\eta}^m)) &= k\sum_{n=1}^{m}(\overline{\partial}A)(t_n, \hat{\eta}^{n-1}, \overline{\partial}_t(T_m\hat{\eta}^m)) \\ &\quad + k\sum_{n=1}^{m}\sum_{j=0}^{n-1}B(t_n, t_j;\eta^j, \overline{\partial}_t(T_m\hat{\eta}^m)) \\ &\quad + Q_A^m(u_h)(\overline{\partial}_t(T_m\hat{\eta}^m)) \\ &\quad + \overline{Q}_B^m(u_h)(\overline{\partial}_t(T_m\hat{\eta}^m)).\end{aligned} \tag{3.18}$$

Note that $(\overline{\partial}_t\hat{\eta}^m, \overline{\partial}_t(T_m\hat{\eta}^m)) = (\overline{\partial}_t\hat{\eta}^m, T_m\overline{\partial}_t\hat{\eta}^m) + (\overline{\partial}_t\hat{\eta}^m, (\overline{\partial}_tT_m)\hat{\eta}^{m-1})$, and

$$A(t_m;\hat{\eta}^m, \overline{\partial}_t(T_m\hat{\eta}^m)) \ge \frac{1}{2}\overline{\partial}_t\|\hat{\eta}^m\|^2 - (\overline{\partial}A)(t_m;\hat{\eta}^m, T_{m-1}\hat{\eta}^{m-1}).$$

For $m=1$, use (2.3) and Young's inequality to obtain

$$k\|T_1\overline{\partial}_t\hat{\eta}^1\|_1^2+\|\hat{\eta}^1\|^2\le Ck^2\|u_0\|^2. \tag{3.19}$$

For $m\ge 2$, sum (3.18) with respect to $m$ from 2 to $l$ to obtain

$$\begin{aligned}
k\sum_{m=2}^{l}\|T_m\overline{\partial}_t\hat{\eta}^m\|_1^2+\frac{1}{2}\|\hat{\eta}^l\|^2
&\le \frac{1}{2}\|\hat{\eta}^1\|^2+\bigg|k\sum_{m=2}^{l}[-(\overline{\partial}_t\hat{\eta}^m,(\overline{\partial}_tT_m)\hat{\eta}^{m-1})\\
&\qquad\qquad +(\overline{\partial}A)(t_m;\hat{\eta}^m,T_{m-1}\hat{\eta}^{m-1})]\bigg|\\
&\quad+\bigg|k^2\sum_{m=2}^{l}\sum_{n=1}^{m}(\overline{\partial}A)(t_n;\hat{\eta}^{n-1},\overline{\partial}_t(T_m\hat{\eta}^m))\bigg|\\
&\quad+\bigg|k^2\sum_{m=2}^{l}\sum_{n=1}^{m}B(t_n,t_{n-1};\hat{\eta}^{n-1},\overline{\partial}_t(T_m\hat{\eta}^m))\bigg|\\
&\quad+\bigg|-k^3\sum_{m=2}^{l}\sum_{n=2}^{m}\sum_{j=1}^{n-1}(\overline{\partial}_2B)(t_n,t_j;\hat{\eta}^{j-1},\overline{\partial}_t(T_m\hat{\eta}^m))\bigg|\\
&\quad+\bigg|k\sum_{m=2}^{l}Q_A^m(u_h)(\overline{\partial}_t(T_m\hat{\eta}^m))\bigg|\\
&\quad+\bigg|k\sum_{m=2}^{l}\overline{Q}_B^m(u_h)(\overline{\partial}_t(T_m\hat{\eta}^m))\bigg|\\
&=|I_1|+|I_2|+|I_3|+|I_4|+|I_5|+|I_6|+|I_7|.
\end{aligned}$$

In view of (3.19), $I_1$ is bounded by the term on the right of (3.19). From (2.3), (2.5) and (2.6) we have, for $I_2$,

$$\begin{aligned}
|I_2|&\le Ck\sum_{m=2}^{l}[\|\overline{\partial}_t\hat{\eta}^m\|_{-1,h}\|(\overline{\partial}_tT_m)\hat{\eta}^{m-1}\|_1+\|\hat{\eta}^m\|\|\hat{\eta}^{m-1}\|]\\
&\le Ck\sum_{m=2}^{l}[\|T_m\overline{\partial}_t\hat{\eta}^m\|_1\|\hat{\eta}^{m-1}\|+\|\hat{\eta}^m\|\|\hat{\eta}^{m-1}\|]\\
&\le \varepsilon k\sum_{m=2}^{l}\|T_m\overline{\partial}_t\hat{\eta}^m\|_1^2+C(\varepsilon)k\sum_{m=1}^{l}\|\hat{\eta}^m\|^2.
\end{aligned}$$

To estimate $I_3$, we observe that

$$\begin{aligned}
I_3 &= k^2 \sum_{n=1}^{l} \sum_{m=n+1}^{l} (\overline{\partial} A)(t_n; \hat{\eta}^{n-1}, \overline{\partial}_t (T_m \hat{\eta}^m)) \\
&= k \sum_{n=1}^{l} (\overline{\partial} A)(t_n; \hat{\eta}^{n-1}, T_1 \hat{\eta}^l) \\
&\quad - k \sum_{n=1}^{l} (\overline{\partial} A)(t_n; \hat{\eta}^{n-1}, T_n \hat{\eta}^n)
\end{aligned}$$

and, hence, an application of (2.3) yields

$$|I_3| \le C(\varepsilon) k \sum_{n=1}^{l-1} \|\hat{\eta}^n\|^2 + Ck \sum_{n=1}^{l} \|\hat{\eta}^n\|^2 + \varepsilon \|\hat{\eta}^l\|^2.$$

Similarly, we have for $I_4$ and $I_5$

$$|I_4| + |I_5| \le C(\varepsilon) k \sum_{n=1}^{l-1} \|\hat{\eta}^n\|^2 + Ck \sum_{n=1}^{l} \|\hat{\eta}^n\|^2 + \varepsilon \|\hat{\eta}^l\|^2.$$

For $I_6$, we find that

$$\begin{aligned}
I_6 &= k \sum_{m=2}^{l} \overline{\partial}_t [Q_A^m(u_h)(T_m \hat{\eta}^m)] \\
&\quad - k \sum_{m=2}^{l} \overline{\partial}_t (Q_A^m(u_h))(T_{m-1} \hat{\eta}^{m-1}) \\
&= [Q_A^l(u_h)(T_l \hat{\eta}^l) - Q_A^1(u_h)(T_1 \hat{\eta}^1)] \\
&\quad - k \sum_{m=2}^{l} \overline{\partial}_t (Q_A^m(u_h))(T_{m-1} \hat{\eta}^{m-1}) \\
&= I_6^1 + I_6^2.
\end{aligned}$$

From Lemma 3.2 and (3.19), we have

$$\begin{aligned}
|I_6^1| &\le Ck \left(1 + \log \frac{1}{k}\right) \|u_0\| \|\hat{\eta}^l\| + Ck \|u_0\| \|\hat{\eta}^1\| \\
&\le C(\varepsilon) k^2 \left(1 + \log \frac{1}{k}\right)^2 \|u_0\|^2 + \varepsilon \|\hat{\eta}^l\|^2.
\end{aligned}$$

Note that

$$I_6^2 = k\sum_{m=2}^{l}\left[k^{-1}\left(kA(t_m; u_h^m, T_{m-1}\hat{\eta}^{m-1}) - \int_{t_{m-1}}^{t_m} A(s; u_h(s), T_{m-1}\hat{\eta}^{m-1})\,ds\right)\right].$$

Apply (2.3) to obtain

$$|I_6^2| \le Ck\sum_{m=2}^{l}\frac{1}{t_{m-1}}\int_{t_{m-1}}^{t_m} s(\|u_{hs}(s)\| + \|u_h(s)\|)\,ds\|\hat{\eta}^{m-1}\|.$$

By Theorem 2.1, it now follows that

$$\begin{aligned}|I_6^2| &\le Ck^{1/2}\|u_0\|\left(\sum_{m=2}^{l}\frac{k^2}{t_{m-1}^2}\right)^{1/2}\left(k\sum_{m=2}^{l}\|\hat{\eta}^{m-1}\|^2\right)^{1/2}\\ &\le Ck\|u_0\|^2 + Ck\sum_{m=1}^{l-1}\|\hat{\eta}^m\|^2.\end{aligned}$$

Finally, for $I_7$, we use summation by parts to obtain

$$\begin{aligned}I_7 &= [\overline{Q}_B^l(u_h)(T_l\hat{\eta}^l) - \overline{Q}_B^1(u_h)(T_1\hat{\eta}^1)]\\ &\quad - k\sum_{m=2}^{l}\overline{\partial}_t(\overline{Q}_B^m(u_h))(T_{m-1}\hat{\eta}^{m-1})\\ &= I_7^1 + I_7^2.\end{aligned}$$

Using Lemma 3.2 and (3.19), it now yields

$$|I_7^1| \le C(\varepsilon)k^2\left(1 + \log\frac{1}{k}\right)^2\|u_0\|^2 + \varepsilon\|\hat{\eta}^l\|^2.$$

To estimate $I_7^2$, we rewrite it as

$$\begin{aligned}
I_7^2 = -\Big[&k^2 \sum_{m=2}^{l} \sum_{j=0}^{m-1} B(t_m, t_j; u_h^j, T_{m-1}\hat{\eta}^{m-1}) \\
&- k \sum_{m=2}^{l} \int_0^{t_m} B(t_m, s; u_h(s), T_{m-1}\hat{\eta}^{m-1})\, ds\Big] \\
- \Big[&k \sum_{m=2}^{l} \int_0^{t_m} B(t_m, s; u_h(s), T_{m-1}\hat{\eta}^{m-1})\, ds \\
&- \sum_{m=2}^{l} \int_{t_{m-1}}^{t_m} \int_0^{s} B(s, \tau; u_h(\tau), T_{m-1}\hat{\eta}^{m-1})\, d\tau\, ds\Big] \\
= I_7^{21} &+ I_7^{22}.
\end{aligned}$$

For $I_7^{21}$, apply Lemma 3.2 to obtain

$$\begin{aligned}
|I_7^{21}| &\le k \sum_{m=2}^{l} |Q_B^m(u_h)(T_{m-1}\hat{\eta}^{m-1})| \\
&\le Ck^2 \left(1 + \log \frac{1}{k}\right)^2 \|u_o\|^2 + Ck \sum_{m=1}^{l-1} \|\hat{\eta}^m\|^2
\end{aligned}$$

To estimate $I_7^{22}$, we note that

$$\begin{aligned}
|I_7^{22}| &= \Big| - \sum_{m=2}^{l} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) \frac{\partial}{\partial s} \\
&\qquad \times \left( \int_0^s B(s, \tau; u_h(\tau), T_{m-1}\hat{\eta}^{m-1})\, d\tau \right) ds \Big| \\
&\le k \sum_{m=2}^{l} \int_{t_{m-1}}^{t_m} (|B(s, s; u_h(s), T_{m-1}\hat{\eta}^{m-1})| \\
&\qquad\qquad + \int_0^s |B_s(s, \tau; u_h(\tau), T_{m-1}\hat{\eta}^{m-1})|\, d\tau)\, ds,
\end{aligned}$$

and, hence, using the property (2.3) and Theorem 2.1, we obtain

$$\begin{aligned} |I_7^{22}| &\le Ck \sum_{m=2}^{l} \int_{t_{m-1}}^{t_m} (\|u_h(s)\| \\ &\qquad + \int_0^s \|u_h(\tau)\|\, d\tau)\, ds \|\hat{\eta}^{m-1}\| \\ &\le Ck^2 \|u_0\|^2 + Ck \sum_{m=1}^{l-1} \|\hat{\eta}^m\|^2. \end{aligned}$$

Combining all the above estimates and choosing $\varepsilon$ appropriately, we arrive at

$$k \sum_{m=2}^{l} \|T_m \overline{\partial}_t \hat{\eta}^m\|_1^2 + \|\hat{\eta}^l\|^2 \le Ck \left(1 + \log \frac{1}{k}\right)^2 \|u_0\|^2 + Ck \sum_{m=1}^{l} \|\hat{\eta}^m\|^2.$$

Adding $k\|T_1 \overline{\partial}_t \hat{\eta}^1\|_1^2$ to both sides of the above inequality and making use of (3.19) and Lemma 3.4, we now complete the rest of the proof. $\square$

**Lemma 3.6.** *With $\hat{\eta}^n$ as above, the following estimate*

$$k \sum_{j=1}^{n} t_j \|T_j \overline{\partial}_t \hat{\eta}^j\|_1^2 + t_n \|\hat{\eta}^n\|^2 \le Ck^2 \left(1 + \log \frac{1}{k}\right)^2 \|u_0\|^2$$

*holds.*

*Proof.* Setting $\chi = t_m \overline{\partial}_t (T_m \hat{\eta}^m)$ in (3.15) and repeating the arguments of Lemma 3.5, we obtain the required estimates. For the sake of clarity, we present below a short proof.

Note that, except for the term $I_6$, all other terms in the previous lemma, i.e. Lemma 3.3, are bounded by $Ck^2(1+\log(1/k))^2\|u_0\|^2$, and, hence, we shall only estimate $I_6$. Write $I_6$ for the present case as

$$I_6 = k \sum_{m=2}^{l} t_m Q_A^m(u_h)(\overline{\partial}_t(T_m \hat{\eta}^m))$$

$$\begin{aligned}
&= k\sum_{m=2}^{l} \overline{\partial}_t[t_m Q_A^m(u_h)(T_m\hat{\eta}^m)] \\
&\quad - k\sum_{m=2}^{l} t_{m-1}\overline{\partial}_t[Q_A^m(u_h)](T_{m-1}\hat{\eta}^{m-1}) \\
&\quad - k\sum_{m=2}^{l} Q_A^m(u_h)(T_{m-1}\hat{\eta}^{m-1}) \\
&= [t_l Q_A^l(u_h)(T_l\hat{\eta}^l) - t_1 Q_A^1(u_h)(T_1\hat{\eta}^1)] \\
&\quad - k\sum_{m=2}^{l} t_{m-1}\overline{\partial}_t[Q_A^m(u_h)](T_{m-1}\hat{\eta}^{m-1}) \\
&\quad - k\sum_{m=2}^{l} Q_A^m(u_h)(T_{m-1}\hat{\eta}^{m-1}) \\
&= I_6^1 + I_6^2 + I_6^3.
\end{aligned}$$

Using Lemmas 3.2 and 3.4, the terms $I_6^1$ and $I_6^3$ are bounded as desired. To estimate $I_6^2$, we find that

$$\begin{aligned}
|I_6^2| &\le k\sum_{m=2}^{l} t_{m-1}\int_{t_{m-1}}^{t_m}\left|\frac{\partial}{\partial s}[A(;u_h(s), T_{m-1}\hat{\eta}^{m-1}]\right| ds \\
&\le Ck\sum_{m=2}^{l}\int_{t_{m-1}}^{t_m} s(\|u_{hs}(s)\| + \|u_h(s)\|)\, ds\|\hat{\eta}^{m-1}\|.
\end{aligned}$$

Apply Theorem 2.1 and Lemma 3.4 to obtain

$$\begin{aligned}
|I_6^2| &\le Ck\sum_{m=2}^{l}\int_{t_{m-1}}^{t_m}\|u_0\|\, ds\|\hat{\eta}^{m-1}\| \\
&\le Ck^2\|u_0\|^2 + Ck\sum_{m=1}^{l-1}\|\hat{\eta}^m\|^2 \\
&\le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2,
\end{aligned}$$

and this completes the proof. $\square$

Finally we obtain the following estimate for $\eta_1^n$.

**Lemma 3.7.** *Let $\eta_1^n$ be a solution of* (3.12). *Then there is a constant $C$ independent of $k$ such that*

$$\|T_1\eta_1^n\|_1^2 + k\sum_{j=1}^{n}\|\eta_1^j\|^2 \le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2.$$

*Proof.* The proof will follow along the lines of that of Lemma 3.4 taking $\chi = T_n\eta_1^n$ in (3.12). We, therefore, omit the details. □

*Proof of Theorem* 1.1. We write $U^n - u(t_n)$ as $U^n - u(t_n) = \eta^n + e(t_n)$. From Pani and Sinha [**6**, Theorem 4.1], we have

$$\|e(t_n)\| \le Ch^2t_n^{-1}\|u_0\|.$$

Since the estimate for $\eta_2^n$ can be derived from Lemma 3.1, it is sufficient to derive an estimate for $\|\eta_1^n\|$. Now, use of Lemmas 3.4–3.7 in Lemma 3.3 yields

$$t_n\|\eta_1^n\|^2 \le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2 + Ck\sum_{j=1}^{n-1}t_n^2\|\eta^n\|^2.$$

Altogether, we obtain

$$t_n^2\|\eta^n\|^2 \le Ck^2\left(1+\log\frac{1}{k}\right)^2\|u_0\|^2 + Ck\sum_{j=1}^{n-1}t_j^2\|\eta^j\|^2.$$

Now apply the discrete Gronwall lemma and then triangle inequality to complete the proof. □

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