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A CHEBYSHEV POLYNOMIAL COLLOCATION BIEM FOR MIXED BOUNDARY VALUE PROBLEMS ON NONSMOOTH BOUNDARIES

L. SCUDERI

ABSTRACT. We propose a Chebyshev polynomial collocation method to solve systems of boundary integral equations arising from a BIE formulation of the mixed Dirichlet-Neumann boundary value problem for the Laplace equation on a nonsmooth domain. In particular, we first improve the behavior of the solution near the corners by introducing a smoothing transformation and then we apply to the new system a collocation method using Chebyshev polynomial expansions as approximants and the zeros of Chebyshev polynomials as collocation nodes. We give a complete solvability and stability analysis of the transformed integral equations by using localization and Mellin techniques. The numerical results obtained show the efficiency of the method here proposed.

1. Introduction.

Several boundary value problems for an elliptic partial differential equation over a region Ω can be reformulated as equivalent integral equations over the boundary of Ω. Such a reformulation is called a boundary integral equation (BIE) and it may be used to solve Laplace's equation and many other elliptic equations, including the biharmonic equation, the Helmholtz equation, the equation of linear elasticity and the equation for Stokes' fluid flow.

In this paper we consider the numerical solution of a BIE reformulation of Laplace's equation in two dimensions. In particular, we consider the following mixed Dirichlet-Neumann boundary value problem for the

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Laplacian

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= \bar{f}_1 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \mathbf{n}} &= \bar{f}_2 \quad \text{on } \Gamma_N, \end{aligned} \tag{1.1}$$

in a simply connected region Ω with piecewise-smooth boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$; in (1.1) $\bar{f}_1$ and $\bar{f}_2$ are given on Γ_D and Γ_N respectively, and $\partial u/\partial \mathbf{n}$ denotes the derivative of u with respect to the outward normal vector $\mathbf{n}$.

For a BIE reformulation of (1.1) we use the single layer representation of the potential u, i.e.

$$u(P) = -\frac{1}{\pi} \int_\Gamma \log|P-Q| z(Q)\, dS_Q, \;\; P \in \Omega, \tag{1.2}$$

where $|P-Q|$ is the Euclidean distance between P and Q, dS_Q is the element of arc length and z is an unknown function called the "single layer" density. From the well-known "jump condition" for the normal derivative of the single layer potential at the boundary, we then have the following boundary integral equations

$$\begin{cases} -\frac{1}{\pi} \int_\Gamma \log|P-Q| z(Q)\, dS_Q = \bar{f}_1(P), & P \in \Gamma_D \\ z(P) - \frac{1}{\pi} \int_\Gamma \frac{\partial \log|P-Q|}{\partial \mathbf{n}_P} z(Q)\, dS_Q = \bar{f}_2(P), & P \in \Gamma_N \end{cases} \tag{1.3}$$

where the density function z is sought on Γ.

Nonsmooth domains, especially piecewise smooth domains, and mixed boundary conditions are very important in several problems arising from physics and engineering. Nevertheless, in the literature the case of smooth boundaries has been more extensively discussed, (see, for example, [**1**], [**29**] and the references given there). The nonsmooth case is significantly different from the smooth boundary one, both in the behavior of the solutions and in the properties of the integral operators, and therefore its numerical treatment is a rather delicate question.

The numerical methods commonly used in the case of boundaries with corners are product-integration, collocation and Galerkin using piecewise polynomial approximations based on properly graded meshes around each singularity (see, for instance, [**4**], [**5**] and, for a review, [**1**]). However, this approach is not quite satisfactory, because if on the one hand it allows to reach optimal rates of convergence, on the other hand it produces increasingly ill-conditioned linear systems to be solved as the local degree increases. Therefore, arbitrarily high orders of convergence, guaranteed by the theory, are not reached in the practical computations, unless pre-conditioning techniques are used. Moreover, in the literature there are only a few partial theoretical results regarding the use of graded meshes combined with collocation methods and BIE of first kind over polygons, (see [**6**]).

A recent different approach, solving BIE on planar curves with corners, consists in reformulating the integral equations in terms of a new unknown with better regularity properties than the original solution, see [**6–10, 15, 17, 18, 23, 31**]. Obviously this kind of approach is not new in the literature and it has been already used for solving other sorts of integral equations arising from various contexts, (see, for instance, [**16, 21, 24–28, 30**]). However, even if the transformed equations admit smoother solutions, which can be efficiently approximated by piecewise polynomials on uniform meshes or by global polynomials, they present some difficulties in the theoretical analysis of the equations themselves and of the corresponding numerical method, because of the complicated nature of the transformed integral kernels. A complete and successful analysis using localization and Mellin techniques has been recently given in [**6**], for the single layer equation, the so-called "Symm's equation," arising from a Dirichlet problem for the Laplace equation, and for a collocation method using splines of any order based on a uniform mesh to approximate the smooth transformed solution. Subsequently this analysis has been performed for other types of numerical methods solving the Symm equation, such as quadrature and qualocation, (with its discrete version) methods using splines, (see [**7**] and [**9, 17**]) and a collocation, (with its discrete version), method using trigonometric polynomials, (see [**10**]). Finally, in [**8**] a trigonometric collocation method has been proposed and analyzed also for solving the problem (1.1).

Following the spirit of these results, the author and Monegato have presented in [**21**] a simple smoothing change of variable and have shown in [**21, 28** and **23**] that this smoothing transformation combined with a collocation method based on orthogonal polynomial expansions of high degree leads to a highly competitive and quite general numerical approach. Indeed, in witness of the wideness of its applicability, we have applied it to some 1D weakly and Cauchy singular integral equations with nonsmooth input functions, (see [**21, 28**]) and very recently to the Symm integral equation in [**23**].

In the present paper we enlarge the applicability of the above-mentioned approach by applying it to solve the boundary integral equations (1.3). In particular, we introduce in (1.3) the parameterization of the curve Γ and then our smoothing change of variable, whose smoothing parameter is chosen according to the smoothness degree we ask for the final unknown function. Hence, we apply to the transformed system a collocation method using Chebyshev orthogonal expansions as approximants of the transformed density on each arc of the boundary, and the zeros of Chebyshev polynomials as collocation nodes. As we shall see in the next section, the role of these orthogonal polynomials is fundamental. They allow to understand the mapping properties of our main operators and they are needed to implement efficiently the numerical method.

Inspired by the technique developed in [**8**], we give a complete analysis of the stability and convergence properties of our Chebyshev collocation method, by using localization and Mellin techniques. As is usual in the analysis of approximation schemes for other Mellin convolution operators, stability is only proved by allowing the possibility that this method be modified slightly near the corners (see, for example, [**3, 6**–**13**, **15**–**17**, **23, 25**–**28**]). However, this modification seems not to be needed for stability in the practical computations. Indeed, the numerical results given in Section 4, have been obtained by applying the proposed collocation method without any modification. Moreover, the above modification has the disadvantage of affecting the theoretical order of convergence of the corresponding method. Indeed, when "finite section" approximations are introduced for proving the stability, then the convergence error is bounded in norm by two terms: one depends upon the global smoothness of the solution in the form of the Lagrangian interpolation error, the other stems from the modification and depends upon the behavior of the solution near and away from the corners. By using the optimal convergence of the Lagrangian interpolator based on some Jacobi interpolation nodes, the first term has an order of convergence which is twice that of the second term, provided that the boundary data $\bar{f}_1$ and $\bar{f}_2$ in (1.1) are sufficiently smooth. Therefore, in this last case if any modification is introduced and stability of the unmodified method is assumed, our method has an order of convergence which is twice the smoothness degree of the solution. Note that this property is another good consequence of the use of the orthogonal polynomials as approximants and the zeros of these polynomials as collocation nodes.

In the next section we give some preliminary results and definitions, which are used in Section 3, where we will prove the stability and convergence of the proposed numerical method. In Section 4 we test our method for two problems already considered in the literature; the numerical results show the efficiency of our approach.

2. Problem setting and preliminaries. In this section we reformulate the boundary integral equations (1.3) in terms of a system of two integral equations defined over the interval $(-1, 1)$, whose solution is smoother than the original one and is related to it by a known and simple connection. To this aim, we rewrite equations (1.3) as a 2×2 matrix integral equation system

$$
(2.1)\qquad \begin{cases} -(1/\pi)\int_{\Gamma_D} \log|P-Q| z_D(Q)\,dS_Q \\ -(1/\pi)\int_{\Gamma_N} \log|P-Q| z_N(Q)\,dS_Q = \bar{f}_1(P), \quad P \in \Gamma_D, \\ -(1/\pi)\int_{\Gamma_D} \frac{\partial \log|P-Q|}{\partial \mathbf{n}_P} z_D(Q)\,dS_Q + z_N(P) \\ -(1/\pi)\int_{\Gamma_N} \frac{\partial \log|P-Q|}{\partial \mathbf{n}_P} z_N(Q)\,dS_Q = \bar{f}_2(P), \quad P \in \Gamma_N, \end{cases}
$$

where $z_D := z|\Gamma_D$ and $z_N := z|\Gamma_N$ are sought on Γ_D and Γ_N, respectively.

In the sequel, for simplicity we will assume that Γ_D and Γ_N are smooth open arcs. Moreover, we will denote by P_i, $i = 0, 1$ the two interface points of the boundary Γ and by β_i, with $0 < \beta_i < 2\pi$, $i = 0, 1$, the interior angle of Γ at P_i.

It is known that the functions z_D and z_N have singularities at the corners of Γ, even if the boundary data $\bar{f}_1$ and $\bar{f}_2$ are smooth. Indeed, from [**4**] it follows that around P_i we have

$$
u(P) = c(\Theta)\, r^{(\pi/2\beta_i)} + \text{smoother terms}, \quad P \in \Omega,
$$

where (r, Θ) are the polar coordinates centered at P_i. Then, using (1.2) to define a potential not only in Ω but also in $\mathbb{R}^2 \setminus \bar{\Omega}$, by [**20**] the single layer z is the difference between the normal derivatives of u on Γ from inside and outside Γ. Therefore, near P_i, $i = 0, 1$, we get
(2.2)
$$z(P) = c\, r^{s_i} + \text{smoother terms}, \; s_i = \min\left\{\frac{\pi}{2\beta_i}, \frac{\pi}{2(2\pi - \beta_i)}\right\} - 1, \;\; P \in \Gamma.$$

Thus z_D and z_N have this behavior near the corners P_i. To smooth these irregularities, we introduce a smoothing parameterization $\alpha(t)$, which improves the behavior of the unknown function z by incorporating the Jacobian of the transformation. Indeed, the new unknown function will be $z(\alpha(t))|\alpha'(t)|$, whose smoothness degree at the corners depends upon a smoothing parameter: the larger its value, the smoother the transformed density. More precisely, we first introduce in (2.1) a piecewise-smooth parameterization $\bar{\alpha} : [0, 2] \to \Gamma$ such that on each smooth arc $|\bar{\alpha}'|$ is bounded above and below by positive constants, and
$$\bar{\alpha} : \begin{cases} \bar{t} \in [0, 1] \longrightarrow \bar{\alpha}(\bar{t}) \in \Gamma_D, \\ \bar{t} \in [1, 2] \longrightarrow \bar{\alpha}(\bar{t}) \in \Gamma_N. \end{cases}$$

We then consider the smoothing change of variable $\bar{t} = \gamma(t)$, with

$$\gamma(t) = \frac{\int_0^t x^{q-1}(1-x)^{q-1}\, dx}{\int_0^1 x^{q-1}(1-x)^{q-1}\, dx}, \quad t \in [0, 1], \; q > 1. \tag{2.3}$$

This transformation has been already defined and used in other contexts (see [**21**–**23**]) and it revealed itself to be very efficient and, especially, simple to handle. Note that γ maps $(0, 1)$ onto $(0, 1)$ and satisfies the conditions $\gamma^{(k)}(0) = \gamma^{(k)}(1) = 0$, $k = 1, \cdots, q-1$, $q > 1$. Through the parameter q we control the smoothness degree of the final unknown function.

Thus, we define the following "smoothing parameterization"

$$\alpha(t) = \begin{cases} \alpha^{(1)}(t) := \bar{\alpha}(\gamma(t+1)) \in \Gamma_D & t \in [-1, 0], \\ \alpha^{(2)}(t) := \bar{\alpha}(\gamma(t) + 1) \in \Gamma_N & t \in [0, 1]. \end{cases} \tag{2.4}$$

We explicitly note that the interface points of Γ are given by $P_0 = \alpha^{(1)}(-1) = \alpha^{(2)}(1)$ and $P_1 = \alpha^{(1)}(0) = \alpha^{(2)}(0)$. As we will show, besides the smoothing parameterization, another crucial device in the following analysis is the introduction of the weight functions $\omega_1(s) := 1/\sqrt{-s(1+s)}$ and $\omega_2(s) := 1/\sqrt{s(1-s)}$ in the integrals on $[-1,0]$ and $[0,1]$, respectively. Therefore, substituting $P = \alpha(t)$ and $Q = \alpha(s)$ into (2.1), introducing the above weight functions and multiplying the second equation of the system by $|\alpha^{(2)'}(t)|\omega_2^{-1}(t)$, we have

$$(2.5)\quad \begin{cases} -1/\pi \int_{-1}^{0} \log\left|\alpha^{(1)}(t) - \alpha^{(1)}(s)\right| \bar{z}_1(s)\omega_1(s)\,ds \\ -(1/\pi)\int_0^1 \log\left|\alpha^{(1)}(t) - \alpha^{(2)}(s)\right|\bar{z}_2(s)\omega_2(s)\,ds \\ \quad = \bar{f}_1(\alpha^{(1)}(t)), & t \in [-1,0], \\ -1/\pi \int_{-1}^{0} \omega_2^{-1}(t)\,k_1(t,s)\,\bar{z}_1(s)\omega_1(s)\,ds + \bar{z}_2(t) \\ -1/\pi \int_0^1 \omega_2^{-1}(t)\,k_2(t,s)\,\bar{z}_2(s)\omega_2(s)\,ds \\ \quad = \bar{f}_2(\alpha^{(2)}(t))\left|\alpha^{(2)'}(t)\right|\omega_2^{-1}(t), & t \in [0,1], \end{cases}$$

where

$$(2.6)\quad \begin{aligned} \bar{z}_1(s) &:= z_D(\alpha^{(1)}(s))|\alpha^{(1)'}(s)|\,\omega_1^{-1}(s), \\ \bar{z}_2(s) &:= z_N(\alpha^{(2)}(s))|\alpha^{(2)'}(s)|\,\omega_2^{-1}(s), \end{aligned}$$

and

(2.7)

$$k_1(t,s) := \frac{\alpha_2^{(2)'}(t)[\alpha_1^{(2)}(t) - \alpha_1^{(1)}(s)] - \alpha_1^{(2)'}(t)[\alpha_2^{(2)}(t) - \alpha_2^{(1)}(s)]}{[\alpha_1^{(2)}(t) - \alpha_1^{(1)}(s)]^2 + [\alpha_2^{(2)}(t) - \alpha_2^{(1)}(s)]^2},$$

$$k_2(t,s) := \frac{\alpha_2^{(2)'}(t)[\alpha_1^{(2)}(t) - \alpha_1^{(2)}(s)] - \alpha_1^{(2)'}(t)[\alpha_2^{(2)}(t) - \alpha_2^{(2)}(s)]}{[\alpha_1^{(2)}(t) - \alpha_1^{(2)}(s)]^2 + [\alpha_2^{(2)}(t) - \alpha_2^{(2)}(s)]^2},$$

being $\alpha^{(i)}(t) = (\alpha_1^{(i)}(t), \alpha_2^{(i)}(t))$, $i = 1,2$. Note that in (2.6) the weight functions ω_1 and ω_2 do not spoil the smoothness degree of the solutions $\bar{z}_1$ and $\bar{z}_2$ because the Jacobian $|\alpha^{(i)'}|$, $i = 1,2$, precisely absorbs the irregularities at the endpoints of the domain of integration.

Then, by simple linear changes of variables mapping $[0,1]$ and $[-1,0]$ onto $[-1,1]$ and setting

$$(2.8)\quad \omega(x) := \frac{1}{\sqrt{1-x^2}},$$

equations (2.5) become

$$
\left\{\begin{array}{l} V_{11}z_1(x)+V_{12}z_2(x)=f_1(x) \\ K_{21}z_1(x)+(I+K_{22})z_2(x)=f_2(x) \end{array}\right. \qquad x\in[-1,1] \tag{2.9}
$$

where
(2.10)

$$
\begin{aligned}
V_{11}z_1(x) &:= -\frac{1}{\pi}\int_{-1}^{1}\log\left|\alpha^{(1)}\left(\frac{x-1}{2}\right)-\alpha^{(1)}\left(\frac{y-1}{2}\right)\right| z_1(y)\omega(y)\,dy,\\
V_{12}z_2(x) &:= -\frac{1}{\pi}\int_{-1}^{1}\log\left|\alpha^{(1)}\left(\frac{x-1}{2}\right)-\alpha^{(2)}\left(\frac{y+1}{2}\right)\right| z_2(y)\omega(y)\,dy,\\
K_{21}z_1(x) &:= -\frac{1}{2\pi}\int_{-1}^{1}\omega^{-1}(x)\,k_1\left(\frac{x+1}{2},\frac{y-1}{2}\right)z_1(y)\omega(y)\,dy,\\
K_{22}z_2(x) &:= -\frac{1}{2\pi}\int_{-1}^{1}\omega^{-1}(x)\,k_2\left(\frac{x+1}{2},\frac{y+1}{2}\right)z_2(y)\omega(y)\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
z_1(y) &:= \bar{z}_1\left(\frac{y-1}{2}\right), \qquad z_2(y) := \bar{z}_2\left(\frac{y+1}{2}\right),\\
f_1(x) &:= \bar{f}_1\left(\alpha^{(1)}\left(\frac{x-1}{2}\right)\right),\\
f_2(x) &:= \bar{f}_2\left(\alpha^{(2)}\left(\frac{x+1}{2}\right)\right)\left|\alpha^{(2)\prime}\left(\frac{x+1}{2}\right)\right|\frac{\omega^{-1}(x)}{2}.
\end{aligned} \tag{2.11}
$$

Equivalently, we can rewrite system (2.9) as follows

$$
\mathbf{B}\mathbf{z} := \begin{pmatrix} V_{11} & V_{12} \\ K_{21} & I+K_{22} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =: \mathbf{f}. \tag{2.12}
$$

Now, let us introduce the operator

$$
Av(x) := -\frac{1}{\pi}\int_{-1}^{1}\log|x-y|v(y)\omega(y)\,dy, \tag{2.13}
$$

since both in the theoretical analysis and in the numerical treatment it is convenient to rewrite V_{11} as $(V_{11}-A)+A$ and the matrix $\mathbf{B}$ as

$$
\mathbf{B} = \begin{pmatrix} A & V_{12} \\ 0 & I \end{pmatrix} + \begin{pmatrix} V_{11}-A & 0 \\ K_{21} & K_{22} \end{pmatrix} =: \mathbf{A}+\mathbf{K}. \tag{2.14}
$$

Indeed, as we shall see shortly, in the subsequent analysis it is fundamental to deal with the operator A since its mapping properties (see (2.16) and (2.17)) are essential. This is the main reason why we have been forced to introduce the Chebyshev weight function in each integral in (2.5).

We now introduce the function spaces which are the natural setting for examining (2.12) and its numerical resolution. Let L^2_ω be the Hilbert space of all square integrable functions on $(-1,1)$ with respect to the weight $\omega(x)$ given by (2.8), endowed with the scalar product $(u,v)_\omega := (1/\pi)\int_{-1}^{1} u(x)\overline{v(x)}\omega(x)\,dx$ and the norm $||u||_\omega = \sqrt{(u,u)_\omega}$. When $\omega(x)\equiv 1$ the corresponding space is L^2.

Let p_m be the classical Chebyshev orthogonal polynomial of the first kind and of degree m, normalized with respect to the scalar product $(\cdot,\cdot)_\omega$ and with positive leading coefficient. Furthermore, for a real number $r\geq 0$, we define the subspace $L^2_{\omega,r} := L^2_{\omega,r}(-1,1)$ of L^2_ω as follows

$$L^2_{\omega,r} = \{u \in L^2_\omega : ||u||_{\omega,r} < \infty\},$$

where

$$||u||_{\omega,r} := \sqrt{(u,u)_{\omega,r}}, \quad (u,v)_{\omega,r} := \sum_{m=0}^{\infty}(1+m)^{2r}(u,p_m)_\omega\overline{(v,p_m)}_\omega.$$

$L^2_{\omega,r}$ is still a Hilbert space and $L^2_{\omega,0}\equiv L^2_\omega$. Similarly, we define the spaces $L^2_{\omega^{-1}}$ and $L^2_{\omega^{-1},r}$. In [**2**] it has been proved that the "weighted Sobolev-like space" $L^2_{\omega,r}$ has properties analogous to those of the classical Sobolev space on the unit circle. Here we report some of those results, which will be used in the sequel:

(i) $L^2_{\omega,r}\subseteq L^2_{\omega,t}$ and $||u||_{\omega,t}\leq ||u||_{\omega,r}$ for all $0\leq t\leq r$;

(ii) after denoting by D^k the operator of differentiation of k-th order and setting $\varphi(x) := \sqrt{1-x^2}\equiv\omega^{-1}(x)$, the following norms

$$(2.15)\qquad ||u||_{\omega,r} \;\sim\; \sum_{k=0}^{r}||\varphi^k D^k u||_\omega \sim\; ||u||_\omega + ||\varphi^r D^r u||_\omega$$

are equivalent ([**2**, pages 196–197]) for $x\in[-1,1]$ and an integer $r\geq 0$.

Now we recall some results on the operator A, which play a fundamental role in the analysis that follows. First we recall the well-known special property of A (see, for instance, [**30**]):

$$Ap_m(x) = \begin{cases} \ln 2\, p_0(x) & m = 0, \\ (1/m)\, p_m(x) & m \geq 1. \end{cases} \quad x \in [-1, 1] \tag{2.16}$$

This latter expression implies that the operator

$$A : L^2_{\omega,r} \longrightarrow L^2_{\omega,r+1}, \quad r \geq 0, \tag{2.17}$$

is an isomorphism between the two spaces with inverse

$$A^{-1} = H_{\omega^{-1}} D + I_\omega, \tag{2.18}$$

where

$$H_{\omega^{-1}} v(x) := -\frac{1}{\pi} \oint_{-1}^{1} \omega^{-1}(y) \frac{v(y)}{y-x}\, dy, \quad Dv(x) := \frac{dv}{dx} \tag{2.19}$$

(the symbol $\oint$ denotes an integral defined in the principal value sense), and

$$I_\omega v(x) := \frac{1}{\pi \ln 2} \int_{-1}^{1} \omega(x) v(x)\, dx, \tag{2.20}$$

(see [**23**, Lemma 2]). Moreover,

$$H_{\omega^{-1}} : L^2_{\omega^{-1},r} \longrightarrow L^2_{\omega,r}, \quad D : L^2_{\omega,r+1} \longrightarrow L^2_{\omega^{-1},r}, \quad r \geq 0, \tag{2.21}$$

are bounded operators, while

$$I_\omega : L^2_{\omega,r+1} \longrightarrow L^2_{\omega,r}, \quad r \geq 0, \tag{2.22}$$

is bounded and compact.

To proceed further, as in [**23, 24, 28**] in order to use a Mellin technique similar to that in [**8**], we shift the space setting from L^2_ω, $L^2_{\omega^{-1}}$ to L^2, by employing the following correspondences:

$$\begin{aligned} v \in L^2_\omega &\Longleftrightarrow \tilde{v} := \omega^{1/2} v \in L^2 \text{ and } \ ||\tilde{v}||_{L^2} = ||v||_\omega, \\ w \in L^2_{\omega^{-1}} &\Longleftrightarrow \tilde{w} := \omega^{-1/2} w \in L^2 \text{ and } ||\tilde{w}||_{L^2} = ||w||_{\omega^{-1}}. \end{aligned} \tag{2.23}$$

Taking into account their mapping properties, in the L^2-setting the operators A, $H_{\omega^{-1}}$ and D will be defined as

$$\begin{aligned} \tilde{A} &:= \omega^{1/2} A \omega^{-1/2}, & \tilde{A}\tilde{v} &= \omega^{1/2} A v, \\ \widetilde{H}_{\omega^{-1}} &:= \omega^{1/2} H_{\omega^{-1}} \omega^{1/2}, & \widetilde{H}_{\omega^{-1}}\tilde{w} &= \omega^{1/2} H_{\omega^{-1}} w, \\ \widetilde{D} &:= \omega^{-1/2} D \omega^{-1/2}, & \widetilde{D}\tilde{v} &= \omega^{-1/2} D v. \end{aligned} \tag{2.24}$$

We then set

$$\begin{aligned} \tilde{V}_{1i} &:= \omega^{1/2} V_{1i} \omega^{-1/2}, \tilde{V}_{1i}\tilde{v} = \omega^{1/2} V_{1i} v, \\ \tilde{K}_{2i} &:= \omega^{1/2} K_{2i} \omega^{-1/2}, \tilde{K}_{2i}\tilde{v} = \omega^{1/2} K_{2i} v, \end{aligned} \tag{2.25}$$

where V_{1i} and K_{2i}, $i = 1, 2$, are given by (2.10).

Finally, we define the following normed space

$$\widetilde{L}^2_1 := \{\tilde{v} = \omega^{1/2} v \in L^2 \,:\, ||\tilde{v}||_{\widetilde{L}^2_1} := ||\tilde{v}||_{L^2} + ||\widetilde{D}\tilde{v}||_{L^2} < \infty\}, \tag{2.26}$$

with $\widetilde{D}$ given by (2.24); since the following relation

$$||v||_{\omega,1} \sim \; ||v||_\omega + ||\varphi D v||_\omega = ||\tilde{v}||_{L^2} + ||\widetilde{D}\tilde{v}||_{L^2} = ||\tilde{v}||_{\widetilde{L}^2_1}, \quad \tilde{v} = \omega^{1/2} v,$$

holds, we then have that for any $v \in L^2_{\omega,1}$, $\tilde{v} = \omega^{1/2} v \in \widetilde{L}^2_1$ and vice versa.

Therefore, from (2.17), (2.21) for $r = 0$ and (2.24), in particular we have that

$$\tilde{A} : L^2 \longrightarrow \widetilde{L}^2_1, \qquad \widetilde{D} : \widetilde{L}^2_1 \longrightarrow L^2 \tag{2.27}$$

are bounded.

In the L^2-setting equation (2.12) becomes

$$\tilde{\mathbf{B}} \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \quad \tilde{z}_i, \tilde{f}_i \in L^2, \quad i = 1, 2, \tag{2.28}$$

where

$$\tilde{\mathbf{B}} = \begin{pmatrix} \tilde{A} & \tilde{V}_{12} \\ 0 & I \end{pmatrix} + \begin{pmatrix} \tilde{V}_{11} - \tilde{A} & 0 \\ \tilde{K}_{21} & \tilde{K}_{22} \end{pmatrix} =: \tilde{\mathbf{A}} + \tilde{\mathbf{K}} \tag{2.29}$$

and $\tilde{z}_i = \omega^{1/2} z_i$, $\tilde{f}_i = \omega^{1/2} f_i$, $i = 1, 2$. We explicitly note that the operator $\tilde{K}_{22} : L^2 \to L^2$ is compact since it has a continuous kernel. Moreover, by using a localization and Mellin technique (see Lemma 5.2, Remark 5.1 and (B1) in the Appendix) we are able to show that the operators

$$\tilde{V}_{12} : L^2 \longrightarrow \widetilde{L}^2_1, \qquad \tilde{K}_{21} : L^2 \longrightarrow L^2 \tag{2.30}$$

are bounded. Since the technique used to prove these results is similar to that used in [**8**] and the proofs are quite technical, these latters are reported in the Appendix. For the convenience of the reader we have summarized there also the basic properties of the Mellin operators, thus making our exposition self-contained. For the proof of our assertions we cannot refer directly to [**8**], because there are some relevant differences with that paper. For instance, the authors of [**8**] have considered a periodic setting, where a periodic Sobolev space is associated with each arc of the polygon. Here, we study our transformed system in the L^2-space; moreover, we deal with weighted operators.

From (2.30) it is now immediate to show that the operator

$$\tilde{\mathbf{A}} : L^2 \times L^2 \longrightarrow \widetilde{L}^2_1 \times L^2, \tag{2.31}$$

defined in (2.29), is an isomorphism with inverse given by

$$\tilde{\mathbf{A}}^{-1} = \begin{pmatrix} \tilde{A}^{-1} & -\tilde{A}^{-1}\tilde{V}_{12} \\ 0 & I \end{pmatrix}. \tag{2.32}$$

Therefore, we can set

$$\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{B}} = \mathbf{I} + \tilde{\mathbf{M}}, \tag{2.33}$$

where

$$\tilde{\mathbf{M}} := \tilde{\mathbf{A}}^{\mathbf{-1}}\tilde{\mathbf{K}} = \begin{pmatrix} \tilde{A}^{-1} & -\tilde{A}^{-1}\tilde{V}_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{V}_{11} - \tilde{A} & 0 \\ \tilde{K}_{21} & \tilde{K}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{M} & \widetilde{E} \\ \tilde{K}_{21} & \tilde{K}_{22} \end{pmatrix},$$

$$\tilde{M} = \tilde{A}^{-1}(\tilde{V}_{11} - \tilde{A}) - \tilde{A}^{-1}\tilde{V}_{12}\tilde{K}_{21}, \qquad \widetilde{E} = -\tilde{A}^{-1}\tilde{V}_{12}\tilde{K}_{22}. \tag{2.34}$$

From Lemma 5.2, Remark 5.3 and (B1) in the Appendix, it follows that

$$\tilde{M} : L^2 \longrightarrow L^2 \tag{2.35}$$

is bounded; further, $I + \tilde{M}$ is a Fredholm operator of index zero on L^2, (see Theorem 5.4 in the Appendix). Using this latter result, here we will prove the following

Theorem 2.1. *Assuming that equations* (1.3) *with* $\bar{f}_i \equiv 0$, $i = 1, 2$ *have in* $L^p(\Gamma)$ *a unique solution* $z \equiv 0$ *for any* $p > 1$, *for* $q \geq 1$ *the operator*

$$\tilde{\mathbf{B}} : L^2 \times L^2 \longrightarrow \widetilde{L}_1^2 \times L^2$$

has a bounded inverse.

Proof. From the above-mentioned Theorem 5.4 and recalling the compactness of $\widetilde{E}$ and $\tilde{K}_{22}$, we have that $\det(\mathbf{I} + \tilde{\mathbf{M}})$ is a Fredholm operator with index 0 in L^2 and, hence, $\mathbf{I} + \tilde{\mathbf{M}}$ is a Fredholm operator in $L^2 \times L^2$. Thus, from (2.31) it follows that $\tilde{\mathbf{B}} = \tilde{\mathbf{A}}(\mathbf{I} + \tilde{\mathbf{M}})$ is a Fredholm operator of index zero from $L^2 \times L^2$ into $\widetilde{L}_1^2 \times L^2$. Therefore, under the assumption of the theorem and proceeding as in the proof of [**6**, Theorem 2], see also [**8**, Corollary 5.1], it can be proved that $\tilde{\mathbf{B}}\tilde{\mathbf{z}} = \mathbf{0}$ with $\tilde{\mathbf{z}} \in L^2 \times L^2$ implies $\tilde{\mathbf{z}} = \mathbf{0}$. This last conclusion proves the theorem. $\square$

Throughout the paper we will denote by c a generic constant and by $\widetilde{E}$ the generic compact operator in L^2; they may take different values at its different occurrences.

3. Stability and convergence for a collocation method. In this section we will study the stability and the convergence of a collocation method we propose for solving equation (2.12).

Introducing the projector

$$\mathbf{P}_n = \begin{pmatrix} P_n & 0 \\ 0 & P_n \end{pmatrix}, \tag{3.1}$$

where P_n denotes the Lagrange interpolation projector associated with the zeros of the n-th Chebyshev polynomial $p_n(x)$, we consider the following collocation method

$$\mathbf{P}_n(\mathbf{A}+\mathbf{K})\mathbf{z}_n = \mathbf{P}_n\mathbf{f}, \quad \mathbf{z}_n \in \mathbb{P}_n \times \mathbb{P}_n, \tag{3.2}$$

where $\mathbb{P}_n$ is the set of all algebraic polynomials of degree less than or equal to $n-1$. Recalling (2.29) in the L^2-setting, method (3.2) becomes

$$\tilde{\mathbf{P}}_n(\tilde{\mathbf{A}}+\tilde{\mathbf{K}})\tilde{\mathbf{z}}_n = \tilde{\mathbf{P}}_n\tilde{\mathbf{f}}, \quad \tilde{\mathbf{z}}_n = \omega^{1/2}\mathbf{z}_n,$$

where

$$\tilde{\mathbf{P}}_n = \begin{pmatrix} \tilde{P}_n & 0 \\ 0 & \tilde{P}_n \end{pmatrix}, \quad \tilde{P}_n = \omega^{1/2}P_n\omega^{-1/2}.$$

To prove the stability of our collocation method, we introduce the following truncation operator

$$T^r v(x) = \begin{cases} v(x), & x \in (-1+r, 1-r), \\ 0, & x \in (-1,-1+r) \cup (1-r, 1), \end{cases} \tag{3.3}$$

for $0 < r < 1/2$, and we set

$$\widetilde{T}^r := \omega^{1/2}T^r\omega^{-1/2}. \tag{3.4}$$

Note that for $\tilde{v} \in L^2$ it is $\widetilde{T}^r\tilde{v} = T^r\tilde{v}$.

We are only able to prove the stability by allowing the possibility that the method be slightly modified near the corners. Therefore, we will prove the stability of the following collocation method

$$\tilde{\mathbf{P}}_n(\tilde{\mathbf{A}}+\tilde{\mathbf{K}}^{\frac{i^\star}{n}})\tilde{\mathbf{z}}_n^\star = \tilde{\mathbf{P}}_n\tilde{\mathbf{f}}, \quad \tilde{\mathbf{z}}_n^\star = \omega^{1/2}\mathbf{z}_n^\star \in \omega^{1/2}\mathbb{P}_n \times \omega^{1/2}\mathbb{P}_n, \tag{3.5}$$

with

$$\tilde{\mathbf{K}}^{\frac{i^\star}{n}} = \begin{pmatrix} (\tilde{V}_{11}-\tilde{A})\widetilde{T}^{\frac{i^\star}{n}} & 0 \\ \tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}} & \tilde{K}_{22} \end{pmatrix},$$

for any fixed integer $i^\star$ and n sufficiently large. However, the modification seems not to be needed in the practical computations, as the numerical results confirm. Moreover, as we shall see at the end of this section, assuming that our method is stable without any modification, we have derived for it an order of convergence which is twice that of the modified method.

To prove the stability of (3.5), we rewrite it as an equivalent non-standard projection method for the equation

$$(\mathbf{I}+\tilde{\mathbf{M}})\tilde{\mathbf{z}}=\tilde{\mathbf{e}},\quad \tilde{\mathbf{M}}=\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{K}},\ \tilde{\mathbf{e}}=\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{f}}, \tag{3.6}$$

(see (2.33)). To this aim, for any $\tilde{\mathbf{z}}\in L^2\times\widetilde{L}_1^2$, let $\widetilde{\mathbf{S}}_n\tilde{\mathbf{z}}$ solve the collocation system

$$\tilde{\mathbf{P}}_n\tilde{\mathbf{A}}\widetilde{\mathbf{S}}_n\tilde{\mathbf{z}}=\tilde{\mathbf{P}}_n\tilde{\mathbf{A}}\tilde{\mathbf{z}}. \tag{3.7}$$

Since $\tilde{P}_n$ commutes with $\tilde{A}$ on $\omega^{1/2}\mathbb{P}_n$, see [**23**] or use (2.16), the unique solution of (3.7) is

$$\widetilde{\mathbf{S}}_n\tilde{\mathbf{z}}=\begin{pmatrix}\tilde{A}^{-1} & -\tilde{A}^{-1}\tilde{P}_n\tilde{V}_{12}\\ 0 & I\end{pmatrix}\tilde{\mathbf{P}}_n\begin{pmatrix}\tilde{A} & \tilde{V}_{12}\\ 0 & I\end{pmatrix}\tilde{\mathbf{z}};$$

hence we have

$$\widetilde{\mathbf{S}}_n\tilde{\mathbf{z}}=\begin{pmatrix}\widetilde{S}_n & \widetilde{Q}_n\\ 0 & \tilde{P}_n\end{pmatrix}\tilde{\mathbf{z}}, \tag{3.8}$$

where $\widetilde{S}_n=\tilde{A}^{-1}\tilde{P}_n\tilde{A}$, $\widetilde{Q}_n=\tilde{A}^{-1}\tilde{P}_n\tilde{V}_{12}(I-\tilde{P}_n)$. It is easily seen that equation (3.5) is equivalent to the following one

$$(\mathbf{I}+\widetilde{\mathbf{S}}_n\tilde{\mathbf{M}}^{\frac{i^\star}{n}})\tilde{\mathbf{z}}_n^\star=\widetilde{\mathbf{S}}_n\tilde{\mathbf{e}},\quad \tilde{\mathbf{z}}_n^\star\in\omega^{1/2}\mathbb{P}_n\times\omega^{1/2}\mathbb{P}_n, \tag{3.9}$$

where

$$\tilde{\mathbf{M}}^{\frac{i^\star}{n}}=\tilde{\mathbf{A}}^{-1}\tilde{\mathbf{K}}^{\frac{i^\star}{n}}=\begin{pmatrix}\tilde{M}\widetilde{T}^{\frac{i^\star}{n}} & \widetilde{E}\\ \tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}} & \tilde{K}_{22}\end{pmatrix}. \tag{3.10}$$

The operator $\widetilde{\mathbf{S}}_n$ satisfies the following error estimate:

Lemma 3.1. *For any* $\tilde{\mathbf{z}}=\omega^{1/2}\mathbf{z}$ *with* $\mathbf{z}\in L^2_{\omega,l}\times L^2_{\omega,l}$, $l\ge 1$, *the bound*

$$||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}\le\frac{c}{n^l}||\mathbf{z}||_{L^2_{\omega,l}\times L^2_{\omega,l}} \tag{3.11}$$

holds with a positive constant c independent of n and $\mathbf{z}$.

Proof. Definition (3.8) of $\widetilde{\mathbf{S}}_n$ implies

$$||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} \leq ||(I-\widetilde{S}_n)\tilde{z}_1||_{L^2} + ||(I-\tilde{P}_n)\tilde{z}_2||_{L^2} + ||\widetilde{Q}_n\tilde{z}_2||_{L^2}. \tag{3.12}$$

Then, from [**23**, Lemma 3] we have

$$||(I-\widetilde{S}_n)\tilde{z}_1||_{L^2} \leq \frac{c}{n^l}||z_1||_{\omega,l}, \quad l \geq 1, \tag{3.13}$$

and from

$$||(I-P_n)z||_{\omega,s} \leq \frac{c}{n^{l-s}}||z||_{\omega,l}, \quad l \geq s \geq 0,\ l > 1/2, \tag{3.14}$$

(see [**2**, Theorem 3.4] or [**23**, equation (3.8)]), we derive

$$||(I-\tilde{P}_n)\tilde{z}_2||_{L^2} = ||(I-P_n)z_2||_{\omega} \leq \frac{c}{n^l}||z_2||_{\omega,l}, \quad l \geq 1. \tag{3.15}$$

Finally, taking into account the boundedness of the operators $\tilde{A}^{-1}: \widetilde{L}^2_1 \to L^2$, $\tilde{V}_{12}: L^2 \to \widetilde{L}^2_1$ (see (2.27) and (2.30)) and of $\tilde{P}_n: \widetilde{L}^2_1 \to \widetilde{L}^2_1$ (for this latter combine (3.14) with (2.26)), from (3.15) it follows

$$||\widetilde{Q}_n\tilde{z}_2||_{L^2} = ||\tilde{A}^{-1}\tilde{P}_n\tilde{V}_{12}(I-\tilde{P}_n)\tilde{z}_2||_{L^2} \leq ||(I-\tilde{P}_u)\tilde{z}_2||_{L^2} \leq \frac{c}{n^l}||z_2||_{\omega,l}, \quad l \geq 1. \tag{3.16}$$

Finally, combining (3.13) and (3.15), (3.16) with (3.12) we obtain (3.11). □

The following lemma is crucial for the stability of (3.9).

Lemma 3.2. *For $q \geq 2$ and each $\varepsilon > 0$, there exists $i^\star \geq 1$ independent of n such that the bound*

$$||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{z}}||_{L^2\times L^2} \leq \varepsilon\,||\tilde{\mathbf{z}}||_{L^2\times L^2}, \quad \tilde{\mathbf{z}} \in L^2 \times L^2,\ \tilde{\mathbf{z}} = \omega^{1/2}\mathbf{z},$$

holds for all n sufficiently large.

Proof. From (3.8), (3.10) and (3.16) we derive

$$
\begin{aligned}
||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{z}}||_{L^2\times L^2} \le [||(I-\widetilde{S}_n)\tilde{M}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \\
+ c\,||(I-\tilde{P}_n)\tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \\
+ ||(I-\widetilde{S}_n)\widetilde{E}\tilde{z}_2||_{L^2} \\
+ c\,||(I-\tilde{P}_n)\tilde{K}_{22}\tilde{z}_2||_{L^2}].
\end{aligned}
\tag{3.17}
$$

Since $\widetilde{S}_n$ converges strongly to I on L^2 and $\tilde{K}_{22}$ is bounded from L^2 to $\widetilde{L}^2_1$ for $q \ge 2$, we get

$$
||(I-\widetilde{S}_n)\widetilde{E}\tilde{z}_2||_{L^2} + c\,||(I-\tilde{P}_n)\tilde{K}_{22}\tilde{z}_2||_{L^2} \le \varepsilon||\tilde{z}_2||_{L^2}, \tag{3.18}
$$

for all n sufficiently large. Moreover, since $I-\tilde{P}_n$ and $I-\widetilde{S}_n$ annihilate the functions of the type $\omega^{1/2}$ times a constant (see, for instance, [**28**] and [**23**]) by virtue of (3.13) and (3.15), for $l=1$, we can write

$$
||(I-\tilde{P}_n)\tilde{v}||_{L^2} + ||(I-\widetilde{S}_n)\tilde{v}||_{L^2} \le \frac{c}{n}||\widetilde{D}\tilde{v}||_{L^2}
$$

for any $\tilde{v} \in \widetilde{L}^2_1$ or, equivalently, $\tilde{v} = \omega^{1/2}v$ with $v \in L^2_{\omega,1}$. Taking into account this latter and using the definitions of the tilde-operators, the boundedness properties of the involved operators and (2.15), we obtain:

$$
\begin{aligned}
&||(I-\widetilde{S}_n)\tilde{M}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} + c\,||(I-\tilde{P}_n)\tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \\
&\quad\le \frac{c}{n}[||\widetilde{D}\tilde{M}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} + ||\widetilde{D}\tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2}] \\
&\quad\le \frac{c}{n}[||DH_{\omega^{-1}}D(V_{11}-A)T^{\frac{i^\star}{n}}z_1||_{\omega^{-1}} \\
&\qquad + ||DH_{\omega^{-1}}DV_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega^{-1}} \\
&\qquad + ||DK_{21}T^{\frac{i^\star}{n}}z_1||_{\omega^{-1}}] \\
&\quad\le \frac{c}{n}[||(V_{11}-A)T^{\frac{i^\star}{n}}z_1||_{\omega,2} \\
&\qquad + ||V_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega,2} + ||K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega,1}] \\
&\quad\le \frac{c}{n}[||(V_{11}-A)T^{\frac{i^\star}{n}}z_1||_{\omega,2} \\
&\qquad + ||V_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega} \\
&\qquad + ||\varphi^2D^2V_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega} \\
&\qquad + ||K_{21}T^{\frac{i^\star}{n}}z_1||_{\omega} + ||\varphi DK_{21}T^{\frac{i^\star}{n}}z_1||_{\omega}].
\end{aligned}
\tag{3.19}
$$

In this last bound the first term can be estimated by proceeding as in [**23**, Theorem 4]. Indeed, since W in [**23**, Theorem 4] and $V_{11}-A$ here are the same operator, we have

$$(3.20)\qquad ||(V_{11}-A)T^{\frac{i^\star}{n}}z_1||_{\omega,2} \le c\,\frac{n}{i^\star}||\tilde{z}_1||_{L^2}.$$

Then, combining (2.30) with (2.25) and (2.26) the operators $V_{12}: L^2_\omega \to L^2_{\omega,1}$ and $K_{21}: L^2_\omega \to L^2_\omega$ are bounded and, hence,

$$(3.21)\quad ||V_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_\omega \le c\,||K_{21}T^{\frac{i^\star}{n}}z_1||_\omega \le c\,||T^{\frac{i^\star}{n}}z_1||_\omega \le c\,||\tilde{z}_1||_{L^2}.$$

It remains to estimate the following terms

(3.22)
$$||\varphi^2D^2V_{12}K_{21}T^{\frac{i^\star}{n}}z_1||_\omega = ||\varphi\omega^{-1/2}D\omega^{1/2}(\omega^{-1/2}DV_{12}K_{21}\omega T^{\frac{i^\star}{n}}\tilde{z}_1)||_{L^2} =: B_1$$

and

(3.23)
$$||\varphi DK_{21}T^{\frac{i^\star}{n}}z_1||_\omega = ||\varphi\omega^{1/2}D\omega^{-1/2}(\omega^{1/2}K_{21}\omega^{-1/2}T^{\frac{i^\star}{n}}\tilde{z}_1)||_{L^2} =: B_2.$$

Note that for both terms we have to estimate in, L^2-norm, a weighted derivative of weighted functional operators, whose behavior has been already analyzed in Lemma 5.2 in the Appendix. In particular, from the proof of Lemma 5.2 it follows that $\omega^{-1/2}DV_{12}K_{21}\omega^{-1/2}\tilde{v}$ and $\omega^{1/2}K_{21}\omega^{-1/2}\tilde{v}$ with $\tilde{v}\in L^2$ take the form

$$(3.24)\qquad \psi\overline{\mathcal{K}}_0\psi\bar{\tilde{v}} + \psi\overline{\mathcal{K}}_1\psi\bar{\tilde{v}} + \widetilde{E}\bar{\tilde{v}}$$

where $\bar{\tilde{v}}(y) = \tilde{v}(1-y)$ or $\bar{\tilde{v}}(y) = \tilde{v}(y-1)$ according to the change carried out for the variable y, ψ is a smooth cut-off function such that $\psi(x)=1$ whenever $x\in[0,\varepsilon]$, with $\varepsilon<1/2$, and $\mathrm{supp}\,(\psi)\subset[0,(1/2)]$, and $\widetilde{E}=\omega^{-1/2}E\omega^{-1/2}$ with E bounded (operator) from L^2_ω to $L^2_{\omega^{-1},1}$. Moreover, $\mathcal{K}_i$, $i=0,1$ is a Mellin convolution operator on $\mathbb{R}^+$ of type (5.1) with kernel $k_i(x/y)=\bar{k}_i(x/y)(x/y)^{1/4}$, such that for $q\ge 2$ (see Remark 5.1 and (i)–(iii) of Lemma 5.2 in the Appendix) $\widehat{k}_i(z)\in\sum^{-\infty}_{-\frac{5}{4},\frac{3}{4}}$ and $\widehat{\bar{k}}_i(z)\in\sum^{-\infty}_{-1,1}$. We can now proceed to estimate B_1; by (3.22) and

(3.24) and taking into account the changes of the variables x and y, see the proof of Lemma 5.2 below, we get

$$
\begin{aligned}
(3.25)\quad & |\psi(x)x^{3/4}Dx^{-1/4}\int_0^\infty \psi(y)\,k_i\left(\frac{x}{y}\right)\frac{1}{y}\,T^{\frac{i^\star}{n}}\bar{\tilde{z}}_1(y)\,dy| \\
&= \left|\frac{1}{4}\psi(x)\int_{\frac{i^\star}{n}}^\infty \psi(y)\,\bar{k}_i\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{-1/4}\frac{1}{y}\,y^{-1/2}\bar{\tilde{z}}_1(y)\,dy\right. \\
&\quad \left. -\,\psi(x)x^{1/2}\int_{\frac{i^\star}{n}}^\infty \psi(y)\,k_i'\left(\frac{x}{y}\right)\frac{1}{y}\,y^{-1}\bar{\tilde{z}}_1(y)\,dy\right| \\
&\le c\frac{n}{i^\star}\left[\int_0^\infty \psi(x)\psi(y)\,|\bar{k}_i\left(\frac{x}{y}\right)|\left(\frac{x}{y}\right)^{-1/4}\frac{1}{y}\,|\bar{\tilde{z}}_1(y)|\,dy\right. \\
&\quad \left. +\int_0^\infty \psi(x)\psi(y)\,|k_i'\left(\frac{x}{y}\right)|\left|\frac{1}{y}\right|\bar{\tilde{z}}_1(y)|\,dy\right],
\end{aligned}
$$

for $i=0,1$. Using the kernel estimates in (B2) (see Appendix B) with $m=0,1$ and a proper ρ we have that $x^{-1/2}(|\bar{k}_i(x)|x^{-1/4})$ and $x^{-1/2}|k_i'(x)|$ belong to $L^1(\mathbb{R}^+)$, so that the Mellin convolution operators with kernel $|\bar{k}_i(x)|x^{-1/4}$ and $|k_i'(x)|$ are both bounded on $L^2(\mathbb{R}^+)$. Therefore, taking the L^2-norm in (3.25) we have that

$$
(3.26)\qquad ||\varphi\omega^{-1/2}D\omega^{1/2}\psi\overline{\mathcal{K}}_i\psi T^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \le c\,\frac{n}{i^\star}||\tilde{z}_1||_{L^2},\quad i=0,1.
$$

Finally, as regards the last term in (3.24) we have

$$
\begin{aligned}
(3.27)\qquad & \\
||\varphi\omega^{-1/2}D\omega^{1/2}\omega^{-1/2}E\omega^{-1/2}T^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} &= ||\varphi DET^{\frac{i^\star}{n}}z_1||_{\omega^{-1}} \\
&\le ||ET^{\frac{i^\star}{n}}z_1||_{\omega^{-1},1} \\
&\le c\,||T^{\frac{i^\star}{n}}z_1||_\omega \le c\,||\tilde{z}_1||_{L^2}.
\end{aligned}
$$

Thus, combining (3.24)–(3.27) with (3.22), we have proved that

$$
(3.28)\qquad B_1 \le \frac{c}{i^\star}||\tilde{z}_1||_{L^2}.
$$

To estimate B_2 in (3.23) we can use the same arguments considered for B_1; to this aim it is enough to note that $|\varphi\omega^{-1/2}D\omega^{1/2}|$ and $|\varphi\omega^{1/2}D\omega^{-1/2}|$ yield the same function and

$$
\begin{aligned}
&||\varphi\omega^{1/2}D\omega^{-1/2}\omega^{-1/2}E\omega^{-1/2}T^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \\
&\qquad \le ||\omega^{1/2}E\omega^{-1/2}T^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} + ||\varphi\omega^{-1/2}DE\omega^{-1/2}T^{\frac{i^\star}{n}}\tilde{z}_1||_{L^2} \\
&\qquad \le c\,||\tilde{z}_1||_{L^2}.
\end{aligned}
$$

Therefore, we also have

$$B_2 \le \frac{c}{i^\star}||\tilde{z}_1||_{L^2}. \tag{3.29}$$

Finally, combining (3.22) and (3.23) with (3.28) and (3.29), and, then, (3.19) with (3.20), (3.21), (3.28) and (3.29), from (3.17)–(3.19) the lemma follows. □

It can be proved that for any given $l \ge 1$, if the boundary data $\bar{f}_1$, $\bar{f}_2$ in (1.1) are sufficiently smooth and the smoothing exponent q in (2.3) satisfies the condition

$$q > (l+1/2)/(1+s_i), \quad s_i = \min\left\{\frac{\pi}{2\beta_i}, \frac{\pi}{2(2\pi-\beta_i)}\right\} - 1, \quad i = 0, 1, \tag{3.30}$$

being β_i the interior angle at the corner P_i of the boundary Γ, the unique solution of (2.12) satisfies $\mathbf{z} \in L^2_{\omega,l} \times L^2_{\omega,l}$. Moreover, it takes the form

$$\mathbf{z}(x) = (1 \pm x)^l \mathbf{v}(x), \quad \text{as } x \to \mp 1, \quad \mathbf{v} \in L^2_\omega \times L^2_\omega. \tag{3.31}$$

This result follows from (2.2) and (2.11) by using arguments similar to [**6**, Corollary 5.5]. Therefore, denoting by $H^k(\Gamma)$ the usual Sobolev space on Γ, to obtain (3.31) it is sufficient that $\bar{f}_1 \in H^{l+5/2}(\Gamma_D)$ and $\bar{f}_2 \in H^{l+3/2}(\Gamma_N)$ (see [**14**, Theorem 5.1.3.5]).

The knowledge of the behavior of $\mathbf{z}$ will be needed in the following theorem, which establishes the stability of the proposed method (3.9) and provides a convergence estimate of the corresponding error.

Theorem 3.3. *Assume $q \ge 2$ and suppose that $i^\star$ is an integer fixed but sufficiently large. Then, under the assumption of Theorem* 2.1*, for all n sufficiently large and $\bar{f}_1$, $\bar{f}_2$ in* (1.1) *belonging to $L^2_{\omega,s}$ for some $s > 1/2$, there is a unique solution $\tilde{\mathbf{z}}^\star_n \in \omega^{1/2}\mathbb{P}_n \times \omega^{1/2}\mathbb{P}_n$ of* (3.9)*. Moreover, if $\bar{f}_1 \in H^{l+(5/2)}(\Gamma_D)$ and $\bar{f}_2 \in H^{l+(3/2)}(\Gamma_N)$, then the error estimate*

$$||\tilde{\mathbf{z}}-\tilde{\mathbf{z}}^\star_n||_{L^2\times L^2} \equiv ||\mathbf{z}-\mathbf{z}^\star_n||_{L^2_\omega\times L^2_\omega} \le \frac{c}{n^l}(||\mathbf{z}||_{L^2_{\omega,l}\times L^2_{\omega,l}} + ||\mathbf{v}||_{L^2_\omega\times L^2_\omega}), \quad l \ge 1 \tag{3.32}$$

holds, provided that q satisfies the condition (3.30).

Proof. Using the same arguments considered in [**8**, Lemma 5.5], we can prove that $r_0 > 0$ exists such that

$$||(\mathbf{I}+\tilde{\mathbf{M}}^r)\tilde{\mathbf{z}}||_{L^2\times L^2} \ge c\,||\tilde{\mathbf{z}}||_{L^2\times L^2},\ \ \tilde{\mathbf{z}}\in L^2\times L^2,$$

for any $r \le r_0$. Therefore, from this latter result and Lemma 3.2 we immediately obtain the stability of method (3.5) equivalent to (3.9), namely

$$\begin{aligned}||(\mathbf{I}+\widetilde{\mathbf{S}}_n\tilde{\mathbf{M}}^{\frac{i^\star}{n}})\tilde{\mathbf{z}}_n^\star||_{L^2\times L^2} &\ge ||(\mathbf{I}+\tilde{\mathbf{M}}^{\frac{i^\star}{n}})\tilde{\mathbf{z}}_n^\star||_{L^2\times L^2}\\ &\quad - ||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{z}}_n^\star||_{L^2\times L^2}\\ &\ge c\,||\tilde{\mathbf{z}}_n^\star||_{L^2\times L^2}.\end{aligned} \tag{3.33}$$

To prove the error estimate (3.32) we note that

$$\begin{aligned}||\tilde{\mathbf{z}}-\tilde{\mathbf{z}}_n^\star||_{L^2\times L^2} &\le ||(\mathbf{I}-\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} + ||\tilde{\mathbf{z}}_n^\star-\tilde{\mathbf{P}}_n\tilde{\mathbf{z}}||_{L^2\times L^2}\\ &\le c\,n^{-l}||\mathbf{z}||_{L^2_{\omega,l}\times L^2_{\omega,l}} + ||\tilde{\mathbf{z}}_n^\star-\tilde{\mathbf{P}}_n\tilde{\mathbf{z}}||_{L^2\times L^2}\end{aligned} \tag{3.34}$$

by virtue of (3.15) and (3.31). Furthermore, by using (3.33), (3.6) and Lemma 3.1, we get

$$\begin{aligned}||\tilde{\mathbf{z}}_n^\star-\tilde{\mathbf{P}}_n\tilde{\mathbf{z}}||_{L^2\times L^2} &\le c\,||(\mathbf{I}+\widetilde{\mathbf{S}}_n\tilde{\mathbf{M}}^{\frac{i^\star}{n}})(\tilde{\mathbf{z}}_n^\star-\tilde{\mathbf{P}}_n\tilde{\mathbf{z}})||_{L^2\times L^2}\\ &= c\,||\widetilde{\mathbf{S}}_n[(\mathbf{I}+\tilde{\mathbf{M}})\tilde{\mathbf{z}}-(\mathbf{I}+\tilde{\mathbf{M}}^{\frac{i^\star}{n}})\tilde{\mathbf{P}}_n\tilde{\mathbf{z}}]||_{L^2\times L^2}\\ &\le c[||(\widetilde{\mathbf{S}}_n-\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}+||\widetilde{\mathbf{S}}_n(\tilde{\mathbf{M}}\tilde{\mathbf{z}}-\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{P}}_n\tilde{\mathbf{z}})||_{L^2\times L^2}]\\ &\le c\,[||(\mathbf{I}-\widetilde{\mathbf{S}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}+||(\mathbf{I}-\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}\\ &\quad + ||(\tilde{\mathbf{M}}-\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}\\ &\quad + n^{-1}||\widetilde{D}(\tilde{\mathbf{M}}-\tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}],\end{aligned} \tag{3.35}$$

where the first two terms on the last bound of (3.35) are of order $O(n^{-l})$ because of (3.31) and (3.11), (3.15). Taking into account that $\tilde{\mathbf{M}}^{\frac{i^\star}{n}}$ is uniformly bounded, where $\tilde{\mathbf{M}}$ is bounded on $L^2\times L^2$ and $T^{\frac{i^\star}{n}}$ uniformly bounded on L^2, by (2.34), (3.10) and (2.35) we have

(3.36)

$$\begin{aligned} ||(\tilde{\mathbf{M}} - \tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} &\le ||\tilde{\mathbf{M}}^{\frac{i^\star}{n}}(\mathbf{I} - \tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} \\ &\quad + ||(\tilde{\mathbf{M}} - \tilde{\mathbf{M}}^{\frac{i^\star}{n}})\tilde{\mathbf{z}}||_{L^2\times L^2} \\ &\le \frac{c}{n^l}||\mathbf{z}||_{L^2_{\omega,l}\times L^2_{\omega,l}} + ||\tilde{M}(I - T^{\frac{i^\star}{n}})\tilde{z}_1||_{L^2} \\ &\quad + ||\tilde{K}_{21}(I - T^{\frac{i^\star}{n}})\tilde{z}_2||_{L^2} \\ &\le c\,n^{-l}(||\mathbf{z}||_{L^2_{\omega,l}\times L^2_{\omega,l}} + ||\mathbf{v}||_{L^2_\omega\times L^2_\omega}), \end{aligned}$$

where this latter bound easily follows from (3.3) and (3.31). Furthermore, recalling (3.1), (3.10) and (2.34) we can write

$$\begin{aligned} ||\widetilde{D}(\tilde{\mathbf{M}} - \tilde{\mathbf{M}}^{\frac{i^\star}{n}}\tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} &\le ||\widetilde{D}\tilde{M}(I - \widetilde{T}^{\frac{i^\star}{n}})\tilde{z}_1||_{L^2} \\ &\quad + ||\widetilde{D}\tilde{M}\widetilde{T}^{\frac{i^\star}{n}}(I - \tilde{P}_n)\tilde{z}_1||_{L^2} \\ &\quad + ||\widetilde{D}\tilde{K}_{21}(I - \widetilde{T}^{\frac{i^\star}{n}})\tilde{z}_1||_{L^2} \\ &\quad + ||\widetilde{D}\tilde{K}_{21}\widetilde{T}^{\frac{i^\star}{n}}(I - \tilde{P}_n)\tilde{z}_1||_{L^2} \\ &\quad + ||\widetilde{D}\widetilde{E}(I - \tilde{P}_n)\tilde{z}_2||_{L^2} \\ &\quad + ||\widetilde{D}\tilde{K}_{22}(I - \tilde{P}_n)\tilde{z}_2||_{L^2}. \end{aligned} \tag{3.37}$$

By using the boundedness of the operators $D : L^2_{\omega,1} \to L^2_{\omega^{-1}}$, $E : L^2_\omega \to L^2_{\omega,1}$ and $K_{22} : L^2_\omega \to L^2_{\omega,1}$ and applying (3.15), we find that the last two terms on the righthand side of (3.37) are bounded by $c\,n^{-l}||z_2||_{\omega,l}$; moreover, by proceeding as in Lemma 3.2 and taking into account the form (3.31) of the solution $\mathbf{z}$ and (3.15), we can prove that the first four terms on the righthand side of (3.37) are bounded by $c\,n^{-(l-1)}||z_1||_{\omega,l}$. The details have been omitted for brevity.

Combining these last results with (3.37) and then (3.36), (3.37) with (3.35), we derive the convergence estimate (3.32) from (3.34) and (3.35). □

We note that our modified collocation method admits the same order of convergence as the equally modified method proposed in [**8**], where the authors use a mesh grading transformation depending on a parameter q, which produces the same smoothing effect of our transformation. However, if we assume stability for the unmodified method, then for the same choice of q ($q > (l+1/2)/(1+s_i)$) and for sufficiently smooth $\bar{f}_1$ and $\bar{f}_2$, we obtain an order of convergence which is twice that derived in (3.32) (see Corollary 3.4 below). This result is a consequence of the property that when the Lagrangian interpolator P_n is based on some Jacobi interpolation nodes, its convergence rate is much higher for functions with weak singularities at the endpoints than for functions with internal singularities.

Corollary 3.4. *Under the hypothesis of Theorem* 3.3 *and assuming the stability of the method* (3.9) *when* $i^\star = 0$, *the following estimate*

$$||\tilde{\mathbf{z}} - \tilde{\mathbf{z}}_n||_{L^2\times L^2} \equiv ||\mathbf{z} - \mathbf{z}_n||_{L^2_\omega \times L^2_\omega} \leq \frac{c}{n^{2l+\frac{1}{2}}}, \tag{3.38}$$

holds, provided that q satisfies (3.30) *and* $\bar{f}_1 \in H^{2l+3}(\Gamma_D)$, $\bar{f}_2 \in H^{2l+2}(\Gamma_N)$.

Proof. Proceeding as in Theorem 3.3, with $\tilde{\mathbf{z}}_n^\star$ replaced by $\tilde{\mathbf{z}}_n$, $\tilde{\mathbf{M}}^{\frac{i^\star}{n}}$ by $\tilde{\mathbf{M}}$ and $T^{\frac{i^\star}{n}}$ by I, and recalling (3.1), (3.12) and (3.16), we have

$$\begin{aligned}
||\tilde{\mathbf{z}} - \tilde{\mathbf{z}}_n||_{L^2\times L^2} &\leq c\,[||(\mathbf{I} - \tilde{\mathbf{P}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2} + ||(\mathbf{I} - \widetilde{\mathbf{S}}_n)\tilde{\mathbf{z}}||_{L^2\times L^2}] \\
&\leq c\,[||(I - \tilde{P}_n)\tilde{z}_1||_{L^2} + ||(I - \tilde{P}_n)\tilde{z}_2||_{L^2} \\
&\quad + ||(I - \widetilde{S}_n)\tilde{z}_1||_{L^2} + ||\widetilde{Q}_n\tilde{z}_2||_{L^2}] \\
&\leq c\,[||(I - \tilde{P}_n)\tilde{z}_1||_{L^2} \\
&\quad + ||(I - \tilde{P}_n)\tilde{z}_2||_{L^2} + ||(I - \widetilde{S}_n)\tilde{z}_1||_{L^2}] \\
&= c\,[||(I - P_n)z_1||_\omega \\
&\quad + ||(I - P_n)z_2||_\omega + ||(I - S_n)z_1||_\omega] \\
&\leq c\,[||(I - P_n)z_1||_\omega + ||(I - P_n)z_2||_\omega],
\end{aligned} \tag{3.39}$$

where the last inequality follows from $S_n = A^{-1}P_nA$ and the boundedness of the operators $A : L^2_\omega \to L^2_{\omega,1}$, $P_n : L^2_{\omega,1} \to L^2_{\omega,1}$ and $A^{-1} : L^2_{\omega,1} \to L^2_\omega$ (for more details, see [**23**, equation (3.24)]). Finally, taking into account the behavior of $\mathbf{z}$ and using the optimal estimate proved in [**19**, page 325] we have that

$$||(I - P_n)z_j||_\omega \leq \frac{c}{n^{2l+\frac{1}{2}}}, \quad j = 1, 2. \tag{3.40}$$

Combining (3.40) with (3.39), the error estimate (3.38) follows. $\square$

Remark 3.5. By assuming the stability of the collocation method (3.9) with $i^\star = 0$ and using the optimal convergence estimate (3.40) of the Lagrangian error, by proceeding as in Corollary 3.4 it can be proved that the convergence rate (3.32) holds for $q > \dfrac{1}{2}\left(\dfrac{l+\frac{1}{2}}{1+s_i}\right)$, provided that $f_1 \in H^{2l+3}(\Gamma_D)$ and $f_2 \in H^{2l+2}(\Gamma_N)$, (see Remark 4.1). We remark this because, for smaller values of q the transformed solution is less flat near the singularities and this has some advantages in the practical computations and, in particular, in the reconstruction of the original solution through the transformed one.

4. Numerical results. In this section we will test the collocation method proposed in this paper for the numerical solution of the mixed problem (1.1) via the boundary integral equations (1.3).

After having introduced the parameterization of the boundary and the smoothing transformation (2.3), i.e. the smoothing change of variable (2.4), we have solved the system (2.9) numerically using the Chebyshev collocation method (3.5), with $i^\star = 0$. Indeed, in the practical computations no modification of the collocation method was found to be necessary for stability. To this regard we have to say that this fact is common to all the numerical methods, whose theoretical stability has been proved by modifying the proposed method slightly near the singularities. In effect $i^\star$ actually seems to appear only for theoretical purposes. On the other hand, as is usually the case, stability is proved for a fixed, but unknown, and sufficiently large integer $i^\star$.

In the implementation of our collocation method, to evaluate the integrals of the linear collocation system we have used the Chebyshev quadrature rule, with a number of quadrature nodes which is twice that of the collocation nodes.

For each test, we have checked the error estimate (3.38). Since the exact solution $\mathbf{z}$ of (2.9) is unknown we have computed the approximation $\mathbf{z}_{256}$ and used it as exact solution. Hence, we have computed the norm

$$\mathrm{err}_n^{\mathbf{z}} := ||\mathbf{z}_{256} - \mathbf{z}_n||_{L^2_\omega \times L^2_\omega} \tag{4.1}$$

exactly by using Parseval's equality.

Moreover, we have tested the accuracy of our method in the computation of the solution of the boundary value problem (1.1); this can be written

$$
\begin{aligned}
u(P) = &-\frac{1}{\pi}\int_{-1}^{1} \log\left|P-\alpha^{(1)}\left(\frac{y-1}{2}\right)\right| \frac{z_1(y)}{\sqrt{1-y^2}}\,dy \\
&-\frac{1}{\pi}\int_{-1}^{1} \log\left|P-\alpha^{(2)}\left(\frac{y+1}{2}\right)\right| \frac{z_2(y)}{\sqrt{1-y^2}}\,dy, \quad P\in\Omega,
\end{aligned}
\tag{4.2}
$$

where z_1, z_2 are the solutions of (2.9) and $\alpha^{(1)}$, $\alpha^{(2)}$ denote the smoothing parameterization of the arcs Γ_D and Γ_N, respectively (see (2.4)). In particular, we have computed the error

$$
\mathrm{err}_n^u := |u(P)-u_n(P)|, \tag{4.3}
$$

where u is given in the following examples and u_n is the analogue of (4.2) but with $\mathbf{z}=(z_1,z_2)^T$ replaced by $\mathbf{z}_n=(z_{1,n},z_{2,n})^T$ and the integrals evaluated numerically by a Chebyshev quadrature rule.

In the following tables, we have reported also the estimated order of convergence $EOC = (\log(err_n/err_{2n}))/\log 2$.

All computations have been performed on a PC using 16-digit double precision arithmetic. In the tables below the symbol – means that the corresponding approximation has achieved a full accuracy.

Test 1. As a first test we have applied our numerical method to the test problem considered in [**8**]. Therefore, in (1.1), Ω is a domain with a re-entrant corner, enclosed by the curve:

$$
\Gamma_1: \quad \left(-\frac{2}{3}\sin\left(\frac{3\pi}{2}x\right), -\sin(\pi x)\right), \quad 0\le x\le 2, \tag{4.4}
$$

and the Dirichlet and Neumann arcs Γ_D and Γ_N are parameterized by the interval $[0,1]$ and $[1,2]$, respectively. In Figure 1, the contour Γ_1 has been plotted; its interface points have been denoted by $*$ and the Dirichlet and Neumann arcs by the capital letters D and N, respectively. Moreover, the symbol $+$ points out the position of the point P in which potential (4.3) has been computed.

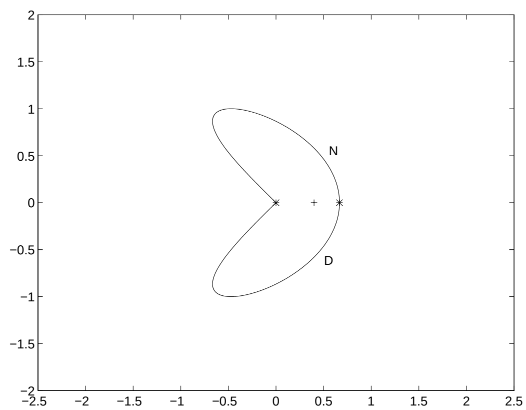

Figure 1. The contour Γ_1.

Since the angle of the re-entrant corner is $3\pi/2$ and the other interface point is smooth, we have that the solution takes the form (3.31) if $q > 3[l + (1/2)]$. Therefore, from (3.38) one expects the convergence orders 0.83, 1.5, 2.17, 2.83, 3.5 corresponding to $q = 2, 3, 4, 5, 6$, respectively, provided that $\bar{f}_1$ and $\bar{f}_2$ in (2.9) are sufficiently smooth.

Example 1. To choose a priori a known solution u which has a realistic behavior at the corner, as in [**8**] we have taken as solution of (1.1)

$$u(x_1, x_2) = Re(\xi^{1/3}) = r^{1/3} \cos \frac{\theta}{3}$$
$$\xi = x_1 + ix_2 = r \exp(i\theta),\ (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}.$$

Because u is the real part of an analytic function, u satisfies the Laplace equation in Ω. Let $u = \bar{f}_1$ on Γ_D and $(\partial u/\partial \mathbf{n}) = \bar{f}_2$ on Γ_N.

Even if the functions $\bar{f}_1$ and $\bar{f}_2$ do not satisfy the smoothness properties required to derive the result (3.31) and hence the estimate (3.38), we have nevertheless considered this test. In any case, the method seems to be convergent and the values EOC estimated are even better than the predicted convergence orders. This result may be explained as follows: the functions $\bar{f}_1, \bar{f}_2$ have a singularity exactly at the corner $(0, 0)$ of Γ_1 (see Figure 1) where the solution of the original problem has a singular behavior because of the corner itself. It seems that the solution inherits this additional singularity too (modulo logarithmic factors), that, however, will be made smooth by the smoothing parameterization. Indeed, the smoothing transformation absorbs all the corner irregularities of the solution simultaneously, so that its smoothness degree depends upon q near the endpoints of the domain of integration and upon the boundary data inside the domain.

TABLE 1. Errors (4.1) in Example 1.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC
8	6.80−02		1.17−01		2.91−01		6.59−01		1.05+00	
		0.97		3.02		5.38		5.66		5.87
16	3.47−02		1.44−02		7.01−03		1.30−02		1.79−02	
		0.90		1.46		2.60		4.58		5.46
32	1.86−02		5.24−03		1.16−03		5.42−04		4.08−04	
		0.99		1.40		3.86		1.73		1.89
64	9.36−03		1.99−03		1.68−04		1.63−04		1.10−04	
		1.32		1.35		0.40		2.92		
128	3.76−03		7.82−04		1.27−04		2.15−05		1.41−04	

TABLE 2. Errors (4.3) at $P=(0.4,0)$ in Example 1.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC
8	2.97−04		8.86−04		7.48−03		2.17−03		1.70−02	
		1.02		5.43		7.83		4.30		7.74
16	1.46−04		2.05−05		3.28−05		1.10−04		7.93−05	
		2.63		3.36		5.13		7.67		7.97
32	2.36−05		1.99−06		9.38−07		5.40−07		3.16−07	
		2.66		3.95		5.27		6.36		6.81
64	3.74−06		1.29−07		2.43−08		6.57−09		2.81−09	
		2.67		4.10		5.05		8.85		3.30
128	5.89−07		7.51−09		7.31−10		1.42−11		2.85−10	

Remark 4.1. To compare our numerical results with those obtained by the trigonometric collocation method in [**8**], we have to take into account that in our case an additional grading near the endpoints ± 1 is produced by the nonuniform distribution of the Chebyshev knots in $(-1,1)$. Therefore, since the distance of two consecutive nodes is of order n^{-2} near ± 1 and n^{-1} in the middle of $(-1,1)$, for a fair comparison we have to compare our results obtained with a smoothing parameter q with those obtained in [**8**] with $2q$. We remark that, even when this comparison is done, our method outperforms the trigonometric one. It appears highly competitive especially for the computation of the potential u.

Example 2. We have chosen two boundary data $\bar{f}_1$, $\bar{f}_2$, which are smoother than those in Example 1 and satisfy the assumptions of Corollary 3.4. In particular, we have set

$$\bar{f}_1(x_1, x_2) = \mathrm{Re}\,(\xi^2) = x_1^2 - x_2^2, \quad \xi = x_1 + ix_2,$$
$$\bar{f}_2(x_1, x_2) = \frac{\partial \bar{f}_1}{\partial \mathbf{n}}(x_1, x_2)$$

and we have solved the corresponding equations (2.9). Note that if P lies inside the boundary, the potential (4.2) solves $\Delta u = 0$ inside Γ with $u = \bar{f}_1$ on Γ_D and $(\partial u/\partial \mathbf{n}) = \bar{f}_2$ on Γ_N, and hence $u = \bar{f}_1$ at all $P \in \Omega$. We have considered this example to test our theoretical convergence result; the computed *EOC* are far bigger than the theoretical convergence orders.

TABLE 3. Errors (4.1) in Example 2.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC
8	5.39−01		1.58+00		3.10+00		4.61+00		5.96+00	
		8.02		5.54		4.63		4.08		3.68
16	2.07−03		3.39−02		1.25−01		2.73−01		4.64−01	
		7.47		14.98		12.01		10.02		8.55
32	1.17−05		1.05−06		3.04−05		2.63−04		1.24−03	
		3.51		7.70		19.11		20.24		19.61
64	1.03−06		5.06−09		5.38−11		2.12−10		1.55−09	
		3.53		5.50		2.19		4.43		7.55
128	8.92−08		1.12−10		7.40−12		9.86−12		8.28−12	

TABLE 4. Errors (4.3) at $P = (0.4, 0)$ in Example 2.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC
8	8.15−03		1.35−03		1.55−02		1.24−02		2.59−02	
		12.94		6.29		6.25		3.13		2.93
16	1.04−06		1.72−05		2.04−04		1.42−03		3.41−03	
		7.50		17.98		18.04		18.65		15.91
32	5.74−09		6.63−11		7.56−10		3.45−09		5.55−08	
		6.03		11.14		14.85		16.95		20.91
64	8.81−11		2.93−14		2.56−14		2.72−14		2.81−14	
		6.03								
128	1.35−12		−−		−−		−−		−−	

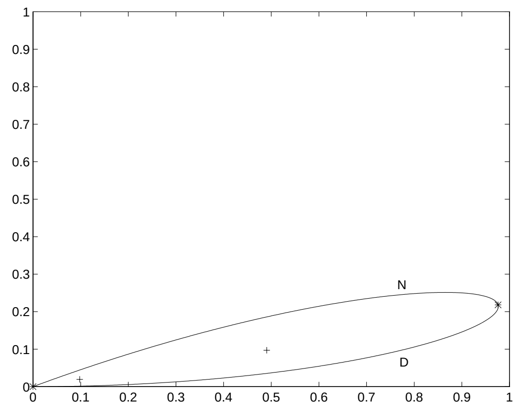

FIGURE 2. The contour Γ_2.

Test 2. We test our method also for a domain already considered in several papers, (see, for example [**6, 7, 9, 23**]) the so-called "tear-drop shaped" region with a single corner at $(0,0)$, (see Figure 2); its boundary admits the following parameterization:

(4.5)

$$\Gamma_2 : \sin(\pi x)\,(\cos((1-\chi)\pi x), \sin((1-\chi)\pi x)\,, \quad 0 \le x \le 1, \quad \chi = 0.86$$

and the Dirichlet and Neumann arcs Γ_D and Γ_N are parameterized by the interval $[0,0.5]$ and $[0.5,1]$, respectively. Using a simple linear change of variable we recast the parameterization (4.5) to the domain $[0,2]$. The interior angle between the tangent at $x=0$ and $x=1$ is $(1-\chi)\pi$.

Potential (4.2) has been computed at the points

$$P = a\left(\cos\left((1-\chi)\frac{4}{9}\pi\right), \sin\left((1-\chi)\frac{4}{9}\pi)\right)\right),$$

for $a=0.1$ and $a=0.5$; the first point lies near the singularity at the origin, the second near the center of Ω (in Figure 2 the symbol $+$ points out their position inside Ω).

Example 3. As in Example 1, to give a realistic behavior of u at the corner, we assume that the solution of (1.1) is

$$u(x_1,x_2) = \exp(x_1)\cos(x_2) + Re(\xi^{\frac{1}{2(1-\chi)}}), \quad \xi = x_1 + ix_2, \quad \chi = 0.86.$$

Then let $u = \bar{f}_1$ on Γ_D and $(\partial u/\partial \mathbf{n}) = \bar{f}_2$ on Γ_N.

These data satisfy the assumptions of Theorem 3.3; moreover, being 0.14π the interior angle at the origin while the other interface point is smooth, we have that the solution takes the form (3.31) if $q > (93/25)[l + (1/2)]$. Therefore, estimate (3.32) gives the following convergence orders 0.04, 0.31, 0.58, 0.84, 1.11 for the values $q = 2, 3, 4, 5, 6$ respectively. However, again the numerical results are by far better than the predicted ones. This fact seems to confirm, as remarked in Example 1, that the smoothness of the boundary data, required to derive (3.31), may be relaxed at the corners. Indeed, if we apply Corollary 3.4 instead of Theorem 3.3, we derive the orders of convergence 0.58, 1.11, 1.65, 2.19, 2.73 for the values $q = 2, 3, 4, 5, 6$, which are still pessimistic but closer to the estimated EOC than those predicted by Theorem 3.3.

TABLE 5. Errors (4.1) in Example 3.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC	$err_n^{\mathbf{z}}$	EOC
8	6.01−02		1.29−01		2.18−01		3.08−01		3.94−01	
		2.14		3.66		4.03		3.76		3.37
16	1.36−02		1.02−02		1.33−02		2.27−02		3.82−02	
		1.63		2.43		3.16		3.88		4.42
32	4.39−03		1.89−03		1.49−03		1.54−03		1.79−03	
		1.70		2.62		3.38		3.90		4.24
64	1.35−03		3.07−04		1.43−04		1.03−04		9.49−05	
		1.95		2.83		3.73		4.52		5.15
128	3.48−04		4.29−05		1.08−05		4.49−06		2.68−06	

TABLE 6. Errors (4.3) at $P = (0.098, 0.019)$ in Example 3.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC
8	6.25−03		3.97−03		1.70−02		4.53−02		7.08−02	
		7.76		4.04		4.85		7.96		5.59
16	2.89−05		2.41−04		5.88−04		1.82−04		1.47−03	
		4.98		7.27		7.04		4.19		6.31
32	9.18−07		1.56−06		4.48−06		9.99−06		1.81−05	
		6.77		16.73		12.17		11.08		10.24
64	8.40−09		1.44−11		9.72−10		4.61−09		1.50−08	
		7.00		8.72		12.45		14.19		15.09
128	6.57−11		3.42−14		1.74−13		2.47−13		4.31−13	

TABLE 7. Errors (4.3) at $P = (0.491, 0.097)$ in Example 3.

	$q=2$		$q=3$		$q=4$		$q=5$		$q=6$	
n	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC	$err_n^{\mathbf{u}}$	EOC
8	2.28−03		1.24−03		1.08−02		2.29−02		3.45−02	
		7.21		3.20		4.35		4.69		4.34
16	2.10−05		1.35−04		5.30−04		1.13−03		1.70−03	
		7.40		11.86		11.23		10.27		9.16
32	1.24−07		3.63−08		2.21−07		9.15−07		2.98−06	
		7.32		12.09		13.89		15.53		14.77
64	7.75−10		8.35−12		1.46−11		1.94−11		1.07−10	
		9.61		8.68		9.74		8.48		11.41
128	9.94−13		2.04−14		1.71−14		5.44−14		3.93−14	

Moreover, we can claim

Remark 4.2. A disagreement between the estimated EOC and the theoretical ones seems to be fairly typical of the collocation methods based on global polynomial approximations and zeros of orthogonal polynomials (see also [**23**]), when the solution itself is smooth in the interior of the domain of approximation. A possible explanation of this may be that the constants c_n in the error estimates, which for n sufficiently large tend to c (see (3.32) or (3.38)), assume particularly small values when n is small or moderate.

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APPENDIX

Here we present some basic facts about the Mellin convolution operators on $[0,\infty]$ and the Mellin transforms, which will be used next. These are based on [**11, 6**].

A. The Mellin transform $\hat{k}$ of $k : \mathbb{R}^+ \to C$ is defined as

$$\hat{k}(z) = \int_0^\infty s^{z-1} k(s)\, ds.$$

It has the following properties.

(A1) The operator $k \to \hat{k}$ is an isometric isomorphism from $L^2(\mathbb{R}^+)$ onto $L^2(\{\mathrm{Re}\, z = 1/2\})$.

(A2) If $\mathcal{K}$ is a Mellin convolution operator, i.e.,

$$\mathcal{K}v(x) = \int_0^\infty k\left(\frac{x}{y}\right) \frac{v(y)}{y}\, dy, \tag{5.1}$$

with kernel $x^{-1/2}k(x) \in L^1(\mathbb{R}^+)$ then $\widehat{\mathcal{K}v}(z) = \hat{k}(z)\hat{v}(z)$ and $\mathcal{K}$ is a continuous operator on $L^2(\mathbb{R}^+)$ with norm bounded by $||\mathcal{K}||_{L^2(0,\infty)} \leq \sup_{\mathrm{Re}\, z=1/2} |\hat{k}(z)|$. More in general this result holds for the operator of the form (5.1), provided that the Mellin transform is bounded on $\mathrm{Re}\, z = 1/2$.

(A3) Let $\widehat{\mathcal{K}}(z) := \widehat{k}(z)$ define the *symbol* of the operator $\mathcal{K}$. If $\mathcal{K}$ and $\mathcal{L}$ are Mellin convolution operators with bounded symbols on $\mathrm{Re}\, z = 1/2$, then $\widehat{\mathcal{K}\mathcal{L}}(z) = \widehat{\mathcal{K}}(z)\widehat{\mathcal{L}}(z)$.

B. The symbol $\widehat{\mathcal{K}}(z)$ of the Mellin convolution operator $\mathcal{K}$ is said to be of class $\sum_{\alpha,\beta}^{-\infty}$, $\alpha < 1/2 < \beta$, if it is analytic in the strip $\alpha < Re\, z < \beta$ and if the estimates

$$\widehat{\mathcal{K}}(z) = O((1 + |z|)^{-m}),\ |z| \longrightarrow \infty,\ m \in \mathbb{Z}^+,$$

hold uniformly in each substrip $\alpha' < Re\, z < \beta'$, $\alpha < \alpha' < 1/2 < \beta' < \beta$. Mellin convolution operators with symbol of class $\sum_{\alpha,\beta}^{-\infty}$ satisfy the following properties.

(B1) If $\widehat{\mathcal{K}} \in \sum_{\alpha,\beta}^{-\infty}$, then the kernel function $k(x)$ of $\mathcal{K}$ satisfies the estimates

$$\sup_{x\in(0,\infty)} |x^{m+\rho} D^m k(x)| < \infty, \quad m \in \mathbb{Z}^+,\ \alpha < \rho < \beta.$$

In particular, this latter implies $x^{-1/2}k(x) \in L^1(\mathbb{R}^+)$ and from (A2) it follows that $\mathcal{K}$ is a bounded operator on $L^2(\mathbb{R}^+)$.

(B2) Let χ be a smooth function with $\mathrm{supp}\,(\chi) \subset [0,1]$ and let ψ be a bounded function such that $\mathrm{supp}\,(\chi) \subset [0,1]$ and let $\psi(s) = 0$, $s \in [0,\varepsilon]$, for some $\varepsilon \in (0,1)$. If $\widehat{\mathcal{K}} \in \sum_{\alpha,\beta}^{-\infty}$ for some $\alpha < 1/2 < \beta$ and $\mathcal{K}$ is the corresponding Mellin convolution operator (5.1), then the operators $\chi\mathcal{K} - \mathcal{K}\chi I$ and $\psi\mathcal{K}$ are Hilbert-Schmidt and hence compact on $L^2(\mathbb{R}^+)$.

We finally recall a standard result on the invertibility of a convolution operator $I + \mathcal{K}$ restricted to the unit interval and on the stability of a corresponding finite section method.

(B3) Let ϕ and ϕ_r, $0 < r < 1$, be the characteristic functions of the intervals $(0,1)$ and $(r,1)$, respectively. If the conditions

(i) $x^{-1/2}k(x) \in L^1(\mathbb{R}^+)$,

(ii) $1 + \widehat{\mathcal{K}}(1/2 + iy) \neq 0,\ y \in \mathbb{R}$,

(iii) $\{\arg(1 + \widehat{\mathcal{K}}(1/2 + iy))\}_{-\infty}^{\infty} = 0$

are satisfied, where $\{\arg(\cdot(y))\}_{-\infty}^{\infty}$ denotes the variation of the argument when y runs from $-\infty$ to ∞, then the Mellin convolution operator $\phi(I + \mathcal{K})\phi$ is continuously invertible on $L^2(0,1)$. Further the corresponding finite section operators $\phi_r(I + \mathcal{K})\phi_r$ are stable, i.e., there is an $r_0 > 0$ and a $c > 0$ such that

$$||\phi_r(I + \mathcal{K})\phi_r v||_{L^2(0,1)} \geq c||\phi_r v||_{L^2(0,1)}, \quad v \in L^2(0,1),$$

for any $r \leq r_0$. This property can be checked using the theory of Wiener-Hopf operators [**12**].

C. We now introduce the Mellin convolution operators considered in our analysis:

$$\begin{aligned} \mathcal{H}^+ v(x) &= \int_0^\infty h\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{1/4} \frac{v(y)}{y}\,dy, \\ \mathcal{H}^- v(x) &= \int_0^\infty h\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{-1/4} \frac{v(y)}{y}\,dy, \end{aligned} \tag{5.2}$$

where $h(t) = (1/\pi)(1/1-t)$ and with the symbols

$$\widehat{\mathcal{H}^+}(z) = \cot[\pi(z + (1/4))]$$

and

$$\widehat{\mathcal{H}^-}(z) = \cot[\pi(z - (1/4))],$$

respectively;

$$\mathcal{L}_\beta v(x) = \int_0^\infty l_\beta\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{1/4} \frac{v(y)}{y}\,dy, \tag{5.3}$$

where

$$l_\beta(t) = -\frac{1}{\pi}\frac{qt^{q-1}(t^q - \cos\beta)}{t^{2q} - 2t^q\cos\beta + 1}$$

and with the symbol

$$\widehat{\mathcal{L}_\beta}(z) = \frac{\cos((\pi-\beta)(\frac{z-\frac{3}{4}}{q}))}{\sin(\pi(\frac{z-\frac{3}{4}}{q}))};$$

$$\mathcal{K}_\beta v(x) = \int_0^\infty k_\beta\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{\frac{1}{4}}\frac{v(y)}{y}\,dy, \tag{5.4}$$

where

$$k_\beta(t) = -\frac{1}{2\pi}\frac{qt^{q-1}\sin\beta}{t^{2q} - 2t^q\cos\beta + 1}$$

and with the symbol

$$\widehat{\mathcal{K}_\beta}(z) = -\frac{\sin((\pi-\beta)(\frac{z-\frac{3}{4}}{q}))}{2\sin(\pi(\frac{z-\frac{3}{4}}{q}))}.$$

The computation of the above symbols follows from the following identity

$$\hat{f}(z) = \int_0^\infty s^{z-1}f(s)\,ds = \int_0^\infty qt^{q-3/4}f(t^q)\,t^{q(z-1)-1/4}\,dt = \hat{g}[q(z-1)+3/4]$$

where $g(t) := qt^{q-3/4}f(t^q)$ or, equivalently, $\hat{g}(z) = \hat{f}(\frac{z-3/4}{q}+1)$, and by use of standard transform tables.

Remark 5.1. It is easily seen that $\widehat{\mathcal{H}^+}(z)$ is of class $\sum_{-(1/4),(3/4)}^{-\infty}$ and $\widehat{\mathcal{H}^-}(z)$ is of class $\sum_{(1/4),(5/4)}^{-\infty}$. Moreover, $\widehat{\mathcal{L}_\beta}(z)$ and $\widehat{\mathcal{K}_\beta}(z)$, $0<\beta<2\pi$, are both of class $\sum_{-(5/4),(3/4)}^{-\infty}$, for $q \geq 2$. Finally, $\widehat{k_\beta}(z)$, $\widehat{l_\beta}(z)$ are of class $\sum_{-1,1}^{-\infty}$, for $q \geq 2$.

D. In view of applying a localization procedure we introduce the following two smooth cut-off functions

$$\bar{\nu}_0(t) = 1, \quad t \in [-1,-1+\varepsilon]\cup[1-\varepsilon,1], \quad \mathrm{supp}\,(\bar{\nu}_0) \subset [-1,-(1/2))\cup((1/2),1],$$
$$\bar{\nu}_1(t) = 1, \quad t \in [-\varepsilon,\varepsilon], \quad \mathrm{supp}(\bar{\nu}_1) \subset (-(1/2),(1/2)),$$

near the corners P_0 and P_1, respectively, and with $0 < \varepsilon < 1/2$. Taking into account the changes of variables carried out in Section 2, mapping $[-1, 0]$ and $[0, 1]$ onto $[-1, 1]$, we then set

(5.5)
$$\begin{aligned}\nu_0(x) &:= \bar{\nu}_0\left(\frac{x-1}{2}\right) = 1,\ \chi_1(x) := \bar{\nu}_1\left(\frac{x+1}{2}\right) = 1,\ x \in [-1, -1+2\varepsilon],\\ \nu_1(x) &:= \bar{\nu}_1\left(\frac{x-1}{2}\right) = 1,\ \chi_0(x) := \bar{\nu}_0\left(\frac{x+1}{2}\right) = 1,\ x \in [1-2\varepsilon, 1].\end{aligned}$$

In the following technical lemma we will show that each of the following operators $\widetilde{D}\tilde{V}_{12}$, $\widetilde{D}(\tilde{V}_{11} - \tilde{A})$, $\tilde{K}_{21}$, $\widetilde{H}_{\omega^{-1}}$ and $\tilde{M}$, defined in Section 2, can be written as the sum of two Mellin convolution operators near the corners and a compact operator $\widetilde{E}$ on L^2. In this lemma the bar on the Mellin convolution operators $\mathcal{L}_{\beta_i}$, $\mathcal{K}_{\beta_i}$, $i = 0, 1$, $\mathcal{H}^+$ and $\mathcal{H}^-$ given by (5.2)–(5.4), indicates that proper transformations are to be carried out since in those terms the singularity is not at $x = 0$ and $y = 0$. For example, $\overline{\mathcal{L}}_{\beta_0}$ ($\overline{\mathcal{L}}_{\beta_1}$) is obtained from $\mathcal{L}_{\beta_0}$ ($\mathcal{L}_{\beta_1}$) by the changes of variables $x \to 1+x$ and $y \to 1-y$, $x \to 1-x$ and $y \to 1+y$; the bar notation has an analogous meaning for the remaining terms and the changes of variables are specified every time in the corresponding proofs.

Lemma 5.2. *The following equalities*

(i) $\widetilde{D}\tilde{V}_{12} = \nu_0 \overline{\mathcal{L}}_{\beta_0} \chi_0 - \nu_1 \overline{\mathcal{L}}_{\beta_1} \chi_1 + \widetilde{E}$,

(ii) $\widetilde{D}(\tilde{V}_{11} - \tilde{A}) = \nu_0(\overline{\mathcal{L}}_0 - \overline{\mathcal{H}}^+)\nu_0 - \nu_1(\overline{\mathcal{L}}_0 - \overline{\mathcal{H}}^+)\nu_1 + \widetilde{E}$,

(iii) $\tilde{K}_{21} = \chi_0 \overline{\mathcal{K}}_{\beta_0} \nu_0 + \chi_1 \overline{\mathcal{K}}_{\beta_1} \nu_1 + \widetilde{E}$,

(iv) $\widetilde{H}_{\omega^{-1}} = \widetilde{H}_{\omega^{-1}}(1 - \nu_0 - \nu_1) - \nu_0 \overline{\mathcal{H}}^- \psi_0 + \nu_1 \overline{\mathcal{H}}^- \psi_1 + \widetilde{E}$

hold, where ν_i, χ_i, $i = 0, 1$, are given by (5.5) while in (iv) *ψ_0 and ψ_1 are suitable cut-off functions such that $\psi_0\nu_0 = \nu_0$, $\psi_1\nu_1 = \nu_1$ and $\psi_0\nu_1 = \psi_1\nu_0 = 0$. Moreover, for $\tilde{M}$ defined in (2.34) one has*

(v) $\tilde{M} = \nu_0 \overline{\mathcal{M}}_0 \nu_0 + \nu_1 \overline{\mathcal{M}}_1 \nu_1 + \widetilde{E}$

where the associated $\mathcal{M}_i$, $i = 0, 1$ are the following Mellin convolution operators

$$\mathcal{M}_i := -\mathcal{H}^-(\mathcal{L}_0 - \mathcal{H}^+) + \mathcal{H}^- \mathcal{L}_{\beta_i} \mathcal{K}_{\beta_i}, \quad i = 0, 1. \tag{5.6}$$

Proof. For simplicity we shall make the assumption that each arc of Γ is straight in some neighborhood of the corners. However, using perturbation arguments it should be possible to derive the same results without assuming this restriction. In any case, under this condition and taking into account that

$$\gamma(t) = \begin{cases} c_q t^q + O(t^{q+1}) & t \in [0, \varepsilon], \\ 1 - c_q(1-t)^q + O((1-t)^{q+1}) & t \in [1-\varepsilon, 1], \end{cases}$$

with $c_q := \binom{2q-1}{q}$ and $0 < \varepsilon < 1/2$, we get

$$\begin{gathered}
\alpha^{(2)}\left(\frac{x+1}{2}\right) - \alpha^{(2)}(1) = c_0 e^{i\beta_0}\left(\frac{1-x}{2}\right)^q + O((1-x)^{q+1}), \\
x \in [1-2\varepsilon, 1], \\
\alpha^{(1)}\left(\frac{x-1}{2}\right) - \alpha^{(1)}(-1) = c_0\left(\frac{1+x}{2}\right)^q + O((1+x)^{q+1}), \\
x \in [-1, -1+2\varepsilon], \\
\alpha^{(1)}\left(\frac{x-1}{2}\right) - \alpha^{(1)}(0) = c_1\left(\frac{1-x}{2}\right)^q + O((1-x)^{q+1}), \\
x \in [1-2\varepsilon, 1], \\
\alpha^{(2)}\left(\frac{x+1}{2}\right) - \alpha^{(2)}(0) = c_1 e^{-i\beta_1}\left(\frac{1+x}{2}\right)^q + O((1+x)^{q+1}), \\
x \in [-1, -1+2\varepsilon],
\end{gathered}$$

where β_j, $j = 0, 1$ is the interior angle at the corners P_j and c_j are complex constants (here, points in $\mathbb{R}^2$ have been identified with complex numbers in usual way). As we shall see, these latter expressions will supply a simplest setting in which to carry out the analysis of our operators. Indeed, from (2.24), (2.25) and (2.10) we have

$$
\begin{aligned}
&\widetilde{D}\,\tilde{V}_{12}\,\tilde{v}(x)\\
&\quad = \omega^{-\frac{1}{2}}\, D_x V_{12}\, \omega^{-\frac{1}{2}}\, \tilde{v}(x)\\
&\quad = -\frac{1}{\pi}\omega^{-\frac{1}{2}}(x) D_x \int_{-1}^{1} \log\left|\alpha^{(1)}\left(\frac{x-1}{2}\right) - \alpha^{(2)}\left(\frac{y+1}{2}\right)\right| \omega^{\frac{1}{2}}(y)\, \tilde{v}(y)\, dy\\
&\quad = -\frac{1}{\pi}\int_{-1}^{1} \mathrm{Re}\left(\frac{D_x\alpha^{(1)}((x-1)/2)}{\alpha^{(1)}((x-1)/2) - \alpha^{(2)}((y+1)/2)}\right)\frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy\\
&\quad = -\frac{1}{\pi}\int_{-1}^{-1+2\varepsilon} \bar{\nu}_1\left(\frac{x-1}{2}\right)\bar{\nu}_1\left(\frac{y+1}{2}\right)\,\mathrm{Re}\left(\frac{-q(1-x)^{q-1}}{(1-x)^q - e^{-i\beta_1}(1+y)^q}\right)\\
&\qquad\quad \times \frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy - \frac{1}{\pi}\int_{1-2\varepsilon}^{1} \bar{\nu}_0\left(\frac{x-1}{2}\right)\bar{\nu}_0\left(\frac{y+1}{2}\right)\\
&\qquad\quad \times \mathrm{Re}\left(\frac{q(1+x)^{q-1}}{(1+x)^q - e^{i\beta_0}(1-y)^q}\right)\frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy + \text{ smoother terms}\\
&\quad = \frac{1}{\pi}\int_{-1}^{-1+2\varepsilon} \nu_1(x)\chi_1(y)\frac{q(1-x)^{q-1}[(1-x)^q-(1+y)^q\cos\beta_1]}{(1-x)^{2q}-2(1-x)^q(1+y)^q\cos\beta_1+(1+y)^{2q}}\left(\frac{1-x}{1+y}\right)^{1/4}\tilde{v}(y)\,dy\\
&\qquad - \frac{1}{\pi}\int_{1-2\varepsilon}^{1} \nu_0(x)\chi_0(y)\frac{q(1+x)^{q-1}[(1+x)^q-(1-y)^q\cos\beta_0]}{(1+x)^{2q}-2(1+x)^q(1-y)^q\cos\beta_0+(1-y)^{2q}}\left(\frac{1+x}{1-y}\right)^{1/4}\tilde{v}(y)dy + \widetilde{E}\tilde{v}\\
&\quad = -\frac{1}{\pi}\int_{0}^{\infty} \nu_0(x)\chi_0(y)\frac{q\bar{x}^{q-1}(\bar{x}^q-\bar{y}^q\cos\beta_0)}{\bar{x}^{2q}-2\bar{x}^q\bar{y}^q\cos\beta_0+\bar{y}^{2q}}\left(\frac{\bar{x}}{\bar{y}}\right)^{1/4}\tilde{v}(1-\bar{y})\,d\bar{y}\Bigg|_{\substack{\bar{x}=1+x\\ \bar{y}=1-y}}\\
&\qquad + \frac{1}{\pi}\int_{0}^{\infty} \nu_1(x)\chi_1(y)\frac{q\bar{x}^{q-1}(\bar{x}^q-\bar{y}^q\cos\beta_1)}{\bar{x}^{2q}-2\bar{x}^q\bar{y}^q\cos\beta_1+\bar{y}^{2q}}\left(\frac{\bar{x}}{\bar{y}}\right)^{1/4}\tilde{v}(\bar{y}-1)\,d\bar{y}\Bigg|_{\substack{\bar{x}=1-x\\ \bar{y}=1+y}}\\
&\qquad + \widetilde{E}\tilde{v},
\end{aligned}
$$

for $\tilde{v} \in L^2$ and assuming $\tilde{v}(x) \equiv 0$ outside the interval $[-1,1]$. Then (i) follows.

Analogously, from (2.24), (2.25) and (2.10), (2.13) we have

$$
\begin{aligned}
&\widetilde{D}\,(\tilde{V}_{11}-\tilde{A})\,\tilde{v}(x)\\
&=\omega^{-\frac{1}{2}}\,D_x(V_{11}-A)\,\omega^{-\frac{1}{2}}\,\tilde{v}(x)\\
&=-\frac{1}{\pi}\,\omega^{-\frac{1}{2}}(x)D_x\int_{-1}^{1}\log\left|\frac{\alpha^{(1)}((x-1)/2)-\alpha^{(1)}((y-1)/2)}{x-y}\right|\omega^{\frac{1}{2}}(y)\,\tilde{v}(y)\,dy\\
&=-\frac{1}{\pi}\int_{-1}^{1}\left[\mathrm{Re}\left(\frac{D_x\alpha^{(1)}((x-1)/2)}{\alpha^{(1)}((x-1)/2)-\alpha^{(1)}((y-1)/2)}\right)-\frac{1}{x-y}\right]\frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy\\
&=-\frac{1}{\pi}\int_{-1}^{-1+2\varepsilon}\bar{\nu}_0\left(\frac{x-1}{2}\right)\bar{\nu}_0\left(\frac{y-1}{2}\right)\\
&\qquad\times\left[\frac{q(1+x)^{q-1}}{(1+x)^q-(1+y)^q}-\frac{1}{x-y}\right]\frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy\\
&\qquad-\frac{1}{\pi}\int_{1-2\varepsilon}^{1}\bar{\nu}_1\left(\frac{x-1}{2}\right)\bar{\nu}_1\left(\frac{y-1}{2}\right)\\
&\qquad\times\left[\frac{-q(1-x)^{q-1}}{(1-x)^q-(1-y)^q}-\frac{1}{x-y}\right]\frac{\omega^{\frac{1}{2}}(y)}{\omega^{\frac{1}{2}}(x)}\,\tilde{v}(y)\,dy\\
&\qquad+\ \text{smoother terms}\\
&=-\frac{1}{\pi}\int_{0}^{\infty}\nu_0(x)\nu_0(y)\left[\frac{q\bar{x}^{q-1}}{\bar{x}^q-\bar{y}^q}-\frac{1}{\bar{x}-\bar{y}}\right]\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{1}{4}}\tilde{v}(\bar{y}-1)\,d\bar{y}\Big|_{\substack{\bar{x}=1+x\\ \bar{y}=1+y}}\\
&\qquad-\frac{1}{\pi}\int_{0}^{\infty}\nu_1(x)\nu_1(y)\left[\frac{-q\bar{x}^{q-1}}{\bar{x}^q-\bar{y}^q}+\frac{1}{\bar{x}-\bar{y}}\right]\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{1}{4}}\tilde{v}(1-\bar{y})\,d\bar{y}\Big|_{\substack{\bar{x}=1-x\\ \bar{y}=1-y}}+\widetilde{E}\tilde{v},
\end{aligned}
$$

which proves (ii).

By proceeding in an analogous way, from (2.25) and (2.10) we then have

$$
\begin{aligned}
&\tilde{K}_{21}\,\tilde{v}(x)\\
&=\omega^{\frac12}\,K_{21}\,\omega^{-\frac12}\,\tilde{v}(x)\\
&=-\frac{1}{2\pi}\omega^{\frac12}(x)\int_{-1}^{1}\sqrt{1-x^2}\ \mathrm{Im}\left(\frac{D_x\alpha^{(2)}((x+1)/2)}{\alpha^{(2)}((x+1)/2)-\alpha^{(1)}((y-1)/2)}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\,\omega^{\frac12}(y)\,\tilde{v}(y)\,dy\\
&=-\frac{1}{2\pi}\int_{-1}^{-1+2\varepsilon}\bar{\nu}_0\left(\frac{x+1}{2}\right)\bar{\nu}_0\left(\frac{y-1}{2}\right)\\
&\qquad\times\,\mathrm{Im}\left(\frac{-qe^{i\beta_0}(1-x)^{q-1}}{e^{i\beta_0}(1-x)^q-(1+y)^q}\right)\frac{\omega^{\frac12}(y)}{\omega^{\frac12}(x)}\,\tilde{v}(y)\,dy\\
&\quad-\frac{1}{2\pi}\int_{1-2\varepsilon}^{1}\bar{\nu}_1\left(\frac{x+1}{2}\right)\bar{\nu}_1\left(\frac{y-1}{2}\right)\\
&\qquad\times\,\mathrm{Im}\left(\frac{qe^{-i\beta_1}(1+x)^{q-1}}{e^{-i\beta_1}(1+x)^q-(1-y)^q}\right)\frac{\omega^{\frac12}(y)}{\omega^{\frac12}(x)}\,\tilde{v}(y)\,dy+\text{smoother terms}\\
&=-\frac{1}{2\pi}\int_0^\infty\chi_0(x)\nu_0(y)\frac{q\bar{x}^{q-1}\bar{y}^q\sin\beta_0}{\bar{x}^{2q}-2\bar{x}^q\bar{y}^q\cos\beta_0+\bar{y}^{2q}}\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac14}\tilde{v}(\bar{y}-1)\,d\bar{y}\Bigg|_{\substack{\bar{x}=1-x\\ \bar{y}=1+y}}\\
&\quad-\frac{1}{2\pi}\int_0^\infty\chi_1(x)\nu_1(y)\frac{q\bar{x}^{q-1}\bar{y}^q\sin\beta_1}{\bar{x}^{2q}-2\bar{x}^q\bar{y}^q\cos\beta_1+\bar{y}^{2q}}\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac14}\tilde{v}(1-\bar{y})\,d\bar{y}\Bigg|_{\substack{\bar{x}=1+x\\ \bar{y}=1-y}}\\
&\quad+\widetilde{E}\tilde{v},
\end{aligned}
$$

from which (iii) follows.

Further, to prove (iv) we take into account the fact that a cut-off function ψ can be actually commuted with $\widetilde{H}_{\omega^{-1}}$, given by (2.24) and (2.19); indeed we have that

$$\psi\widetilde{H}_{\omega^{-1}}\tilde{v}-\widetilde{H}_{\omega^{-1}}\psi\tilde{v}=\frac{1}{\pi}\int_{-1}^{1}\frac{\psi(x)-\psi(y)}{x-y}\,\tilde{v}(y)\,\omega^{-\frac12}(y)\omega^{\frac12}(x)\,dy$$

and, hence, the corresponding operator is Hilbert-Schmidt and, hence, compact in L^2. Therefore, we can write for $\tilde{v} \in L^2$

$$\begin{aligned}\widetilde{H}_{\omega^{-1}}\tilde{v} &= [\widetilde{H}_{\omega^{-1}}(1-\nu_0-\nu_1)+\nu_0\widetilde{H}_{\omega^{-1}}+\nu_1\widetilde{H}_{\omega^{-1}}+\widetilde{E}]\tilde{v} \\ &= \widetilde{H}_{\omega^{-1}}(1-\nu_0-\nu_1)\tilde{v} \\ &\quad - \frac{1}{\pi}\int_0^\infty \nu_0\psi_0\frac{1}{\bar{y}-\bar{x}}\left(\frac{\bar{y}}{\bar{x}}\right)^{1/4}\tilde{v}(\bar{y}-1)\,d\bar{y}\bigg|_{\substack{\bar{x}=1+x\\ \bar{y}=1+y}} \\ &\quad + \frac{1}{\pi}\int_0^\infty \nu_1\psi_1\frac{1}{\bar{y}-\bar{x}}\left(\frac{\bar{y}}{\bar{x}}\right)^{1/4}\tilde{v}(1-\bar{y})\,d\bar{y}\bigg|_{\substack{\bar{x}=1-x\\ \bar{y}=1-y}} + \widetilde{E}\tilde{v}\end{aligned}$$

from which (iv) follows. Finally, from (2.18), (i)–(iv) and the compactness results in (B2), we get

$$\tilde{A}^{-1}(\tilde{V}_{11}-\tilde{A}) = -\nu_0\overline{\mathcal{H}}^-(\overline{\mathcal{L}}_0-\overline{\mathcal{H}}^+)\nu_0-\nu_1\overline{\mathcal{H}}^-(\overline{\mathcal{L}}_0-\overline{\mathcal{H}}^+)\nu_1+\widetilde{E}$$

and

$$\tilde{A}^{-1}\tilde{V}_{12}\tilde{K}_{21} = -\nu_0\overline{\mathcal{H}}^-\overline{\mathcal{L}}_{\beta_0}\overline{\mathcal{K}}_{\beta_0}\nu_0-\nu_1\overline{\mathcal{H}}^-\overline{\mathcal{L}}_{\beta_1}\overline{\mathcal{K}}_{\beta_1}\nu_1+\widetilde{E}.$$

Hence, from these last two equalities and (2.34) we obtain assertion (v). □

Remark 5.3. Since the symbols of the Mellin convolution operators $\mathcal{M}_i$, $i=0,1$ are given by

(5.7)
$$\widehat{\mathcal{M}_i}(z) := -\widehat{\mathcal{H}^-}(z)(\widehat{\mathcal{L}_0}(z)-\widehat{\mathcal{H}^+}(z))+\widehat{\mathcal{H}^-}(z)\widehat{\mathcal{L}_{\beta_i}}(z)\widehat{\mathcal{K}_{\beta_i}}(z), \quad i=0,1,$$

from Remark 5.1 it follows that $\widehat{\mathcal{M}_i}(z)$ is of class $\sum_{(1/4),(3/4)}^{-\infty}$, $i=0,1$.

Using the notation of Lemma 5.2 but, for brevity, putting aside the bar over the transformed operators, we prove the following

Theorem 5.4. *For $q \geq 1$, $I+\tilde{M}$ with $\tilde{M}$ given by* (2.34) *is a Fredholm operator of index zero on L^2.*

Proof. Let ϕ_0 and ϕ_1 denote the characteristic functions of the interval $[0,(1/2)]$ and $[(1/2),1]$, respectively. From (v) in Lemma 5.2, Remark 5.3 and (B2) we can write

$$I+\tilde{M} = \phi_0(I+\mathcal{M}_0)\phi_0+\phi_1(I+\mathcal{M}_1)\phi_1+\widetilde{E}.$$

To prove the theorem it is sufficient to verify the invertibility of the Mellin convolution operators $\phi_0(I+\mathcal{M}_0)\phi_0$ and $\phi_1(I+\mathcal{M}_1)\phi_1$ on $L^2(0,\frac{1}{2})$ and $L^2(\frac{1}{2},1)$, respectively. For brevity we shall do this only for the first term; the proof for the second term is similar.

In order to apply (B3) we have to check conditions (i)–(iii); the first of them follows from Remark 5.3, (B1) and (B3), while the second is an immediate consequence of the third one. Therefore, we have only to check that $\{\arg(1+\widehat{\mathcal{M}}_0(1/2+iy))\}_{-\infty}^{\infty}=0$. To prove this we will show that $1+\widehat{\mathcal{M}}_0(z)$ can be rewritten as a product of two strongly elliptic symbols; in this case the symbol of $I+\mathcal{M}_0$ has winding number equal to zero and consequently (iii) is also checked. Indeed, from (5.7) it follows that

$$1+\widehat{\mathcal{M}_0}(z)=-\widehat{\mathcal{H}^-}(z)\widehat{\mathcal{L}_0}(z)(1-\widehat{\mathcal{L}_0}(z)^{-1}\widehat{\mathcal{L}_{\beta_0}}(z)\widehat{\mathcal{K}_{\beta_0}}(z)),$$

being $\widehat{\mathcal{H}^-}(z)\widehat{\mathcal{H}^+}(z)=-1$. Therefore, for our task the following two inequalities

$$\begin{aligned}\operatorname{Re}(-\widehat{\mathcal{H}^-}(z)\widehat{\mathcal{L}_0}(z))\geq c>0,\quad \operatorname{Re}z=\frac{1}{2},\\ |\widehat{\mathcal{L}_0}(z)^{-1}\widehat{\mathcal{L}_{\beta_0}}(z)\widehat{\mathcal{K}_{\beta_0}}(z)|\leq c<1,\quad \operatorname{Re}z=\frac{1}{2},\end{aligned}$$

remain to be proved. Using the symbols given in Appendix C and by simple computations, the first of them follows from

$$\begin{aligned}&\operatorname{Re}\Big(-\widehat{\mathcal{H}^-}(\frac{1}{2}+iy)\widehat{\mathcal{L}_0}(\frac{1}{2}+iy)\Big)\\ &\quad=\frac{\sin(\frac{\pi}{2q})+\sinh(2\pi y)\sinh(\frac{2\pi y}{q})}{2(\cosh^2(\pi y)+\sinh^2(\pi y))[\cosh^2(\frac{\pi y}{q})\cos^2(\frac{\pi}{4q})+\sinh^2(\frac{\pi y}{q})\cos^2(\frac{\pi}{4q})]}.\end{aligned}$$

and, for $q\geq 1$, the second one follows from

$$\begin{aligned}&\widehat{\mathcal{L}_0}(\frac{1}{2}+iy)^{-1}\widehat{\mathcal{L}_{\beta_0}}(\frac{1}{2}+iy)\widehat{\mathcal{K}_{\beta_0}}(\frac{1}{2}+iy)\\ &\quad=-\frac{1}{2}\frac{\sin(2(\pi-\beta_0)(\frac{-\frac{1}{4}+iy}{q}))}{\sin(2\pi(\frac{-\frac{1}{4}+iy}{q}))}=-\frac{1}{2}\frac{\sinh(2(\pi-\beta_0)(\frac{y+\frac{i}{4}}{q}))}{\sinh(2\pi(\frac{y+\frac{i}{4}}{q}))}\\ &\quad=-\frac{1}{2}a\bigg(2\Big(\frac{y+\frac{i}{4}}{q}\Big)\bigg),\end{aligned}$$

being $a(z) := [\sinh((\pi-\beta)z)/\sinh(\pi z)]$ the symbol of the double layer potential in the case of the arc-length parameterization and $\sup_{-\infty<y<\infty} |a(i\gamma+y)| < 1$ when $|\gamma| \le 1/2$ (see [**3**]). $\square$

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Dipartimento di Matematica, Politecnico di Torino, Corso Duca Degli Abruzzi 24, 10129 Torino, Italy
Email address: scuderi@polito.it