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REMARKS ON REGULARITY CRITERIA FOR THE 2D GENERALIZED MHD SYSTEM IN BESOV SPACES

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ABSTRACT. This paper concerns regularity criteria for the 2D generalized MHD system and shows that, if we can control the Besov norm of the vorticity and/or the current density, then the solution is, in fact, smooth. This improves the recent result [**5**].

1. Introduction. In this paper, we consider the following two-dimensional (2D) generalized MHD system:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{b} + \Lambda^{2\alpha}\mathbf{u} + \nabla\Pi = \mathbf{0}, \\ \partial_t \mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} + \Lambda^{2\beta}\mathbf{b} = \mathbf{0}, \\ \nabla\cdot\mathbf{u} = \nabla\cdot\nabla\mathbf{b} = 0, \\ (\mathbf{u},\mathbf{b})|_{t=0} = (\mathbf{u}_0,\mathbf{b}_0), \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)$, $\mathbf{b} = (b_1, b_2)$ and Π are the fluid velocity field, the magnetic field and the scalar pressure, respectively; and $\mathbf{u}_0$ and $\mathbf{b}_0$ are the prescribed initial data satisfying the compatibility condition

$$\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0,$$

Λ^α and Λ^β with $\alpha, \beta \geq 0$ the fractional diffusion operators, defined through the Fourier transform as

$$\mathscr{F}(\Lambda^\gamma f)(\xi) = |\xi|^\gamma \mathscr{F}(f)(\xi), \quad \gamma = \alpha \text{ or } \beta.$$

The global regularity for system (1.1) has attracted the attention of many authors, and much interesting and important progress has been

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made in the past decade. In 2003, Wu [**13**] showed that system (1.1) admits a global classical solution if

$$\alpha \geq 1, \qquad \beta \geq 1, \tag{1.2}$$

which was extended in [**14**] as

$$\alpha \geq 1, \quad \beta > 0, \quad \alpha + \beta \geq 2, \tag{1.3}$$

with a logarithmic improvement (also, see [**12, 16**] for the margin case $\beta = 0$). Tran, Yu and Zhai [**11**] then considered the following three cases:

(1.4)
$$\alpha \geq 1, \quad \beta \geq 1; \qquad 0 \leq \alpha < \frac{1}{2}, \quad 2\alpha + \beta > 2; \qquad \alpha \geq 2, \quad \beta = 0.$$

The above second case was improved by Jiu and Zhao [**7**] as

$$0 \leq \alpha < \frac{1}{2}, \qquad \beta \geq 1, \qquad 3\alpha + 2\beta > 3, \tag{1.5}$$

with the limiting case $\alpha = 0$, $\beta > 3/2$ independently proven by Yamazaki [**15**] and Yuan and Bai [**20**]. Improvement to $\alpha = 0$, $\beta > 1$, was done by Cao, Wu and Yuan [**2**] and Jiu and Zhao [**8**]. Finally, we would like to mention that Ji [**4**] covered the case $1/2 < \alpha \leq 1$, $\beta = 1$ (with improvement $\alpha > 1/3$, $\beta = 1$, by Yamazaki [**17**], and further improvement $\alpha \geq 1/4$, $\beta = 1$, by Ye and Xu [**19**]), Fan, et al. [**3**] considered the case $0 < \alpha < 1/2$, $\beta = 1$.

For the cases not mentioned above, the global regularity of system (1.1) has not been solved. Thus, it is natural to derive regularity criteria, by which we mean a condition on the solution guaranteeing its global smoothness. For system (1.1) with $\alpha = 1, \beta = 0$, we have the following regularity conditions:

(1) by Jiu and Niu [**6**],

(1.6)
$$\mathbf{b} \in L^p(0, T; W^{2,q}(\mathbb{R}^2)), \qquad \frac{2}{p} + \frac{1}{q} \leq 2, \quad 1 \leq p \leq \frac{4}{3}, \ 2 < q \leq \infty;$$

(2) by Fan and Ozawa [**10**],

$$\nabla \mathbf{u} \in L^1(0, T; L^\infty(\mathbb{R}^3)); \tag{1.7}$$

(3) by Lei, Masmoudi and Zhou [**9**],

$$\mathbf{b} \otimes \mathbf{b} \in L^1(0, T; BMO(\mathbb{R}^2)); \tag{1.8}$$

(4) by Zhou and Fan [**23**],

$$\nabla \mathbf{b} \in L^1(0, T; BMO(\mathbb{R}^2)). \tag{1.9}$$

Then, Jiang, Wang and Zhou [**5**] showed regularity criteria involving the vorticity

$$\omega = \nabla \times \mathbf{u} \stackrel{\text{def}}{=} \partial_1 u_2 - \partial_2 u_1$$

and/or the current density

$$j = \nabla \times \mathbf{b} \stackrel{\text{def}}{=} \partial_1 b_2 - \partial_2 b_1.$$

Among them are:

$$\omega \in L^{p\beta/(p\beta-1)}(0, T; L^p(\mathbb{R}^2)) \quad \text{if } p > \frac{1}{\beta},\ \alpha, \beta \geq \frac{1}{2}; \tag{1.10}$$

$$\omega, j \in L^{\max\{(p\alpha/p\alpha-1),(p\beta/p\beta-1)\}}(0, T; L^p(\mathbb{R}^2)) \\ \text{if } p > \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\},\ \alpha, \beta > 0. \tag{1.11}$$

Meanwhile, Ye [**18**] showed the regularity condition

$$j \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^2)) \quad \text{if } \alpha > \frac{1}{2},\ \beta > \frac{1}{2}, \tag{1.12}$$

and

$$\omega \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^2)) \quad \text{if } \alpha + \beta > 1,\ \beta > \frac{1}{2}. \tag{1.13}$$

Here, and in what follows, $\dot{B}^s_{p,q}(\mathbb{R}^2)$ with $s \in \mathbb{R}, \{p, q\} \subset [1, \infty]$ represent the homogeneous Besov spaces, see [**1**] for the definition, fine properties and applications to fluid dynamical systems.

Our main results are the following two theorems. The first concerns large α and β.

Theorem 1.1. *Let* $\alpha, \beta \geq 1/2$. *Assume that* $(\mathbf{u}_0, \mathbf{b}_0) \in H^2(\mathbb{R}^2)$ *and* $(\mathbf{u}, \mathbf{b})$ *is the local smooth solution of* (1.1). *If*

$$\omega \in L^{2\beta/(2\beta-r)}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^2)) \quad \text{for some } 0<r<\beta \tag{1.14}$$

or

$$j \in L^{2\beta/(2\beta-r)}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^2)) \quad \text{for some } 0<r<\beta, \tag{1.15}$$

then the solution can be smoothly extended beyond T.

Remark 1.2. Due to the Sobolev imbedding $L^p(\mathbb{R}^2) \subset \dot{B}^0_{p,\infty}(\mathbb{R}^2) \subset \dot{B}^{-2/p}_{\infty,\infty}(\mathbb{R}^2)$, see [**1,** Propositions 2.20 and 2.39], we see our result (1.14) implies the following regularity criterion

$$\omega \in L^{p\beta/(p\beta-1)}(0,T;L^p(\mathbb{R}^2)), \quad p>2/\beta, \tag{1.16}$$

which greatly improves (1.10).

Our second aim is to improve (1.11) from Lebesgue spaces to Besov spaces of negative regular indices. Precisely, we have the following.

Theorem 1.3. *Let* $\alpha, \beta > 0$. *Assume that* $(\mathbf{u}_0, \mathbf{b}_0) \in H^2(\mathbb{R}^2)$ *and* $(\mathbf{u}, \mathbf{b})$ *is the local smooth solution of* (1.1). *If*

$$\omega, j \in L^{\max\{(2\alpha/(2\alpha-r)),(2\beta/(2\beta-r))\}}(0,T;\dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^2)) \\ \text{for some } 0<r<\min\{\alpha,\beta\}, \tag{1.17}$$

then the solution can be smoothly extended beyond T.

Remark 1.4. As in Remark 1.2, our result (1.17) greatly extends (1.11).

2. Proof of Theorem 1.1. In this section, we shall prove Theorem 1.1. Since the H^2 estimate of the solution can be performed as in [**5**], once the H^1 estimate is accomplished, what we must do is merely obtain the global H^1 estimate under assumption (1.14) or (1.15).

First, multiplying $(1.1)_{1,2}$ by $\mathbf{u}$ and $\mathbf{b}$, respectively, we easily deduce the following energy estimate:

$$\|(\mathbf{u},\mathbf{b})(t)\|_{L^2}^2 + 2\int_0^t \|(\Lambda^\alpha \mathbf{u}, \Lambda^\beta \mathbf{b})(\tau)\|_{L^2}\,\mathrm{d}\tau = \|(\mathbf{u}_0,\mathbf{b}_0)\|_{L^2}^2. \tag{2.1}$$

Taking the curl operator of $(1.1)_{1,2}$, we obtain the governing equations of ω and j as

$$\begin{cases} \partial_t\omega + (\mathbf{u}\cdot\nabla)\omega + \Lambda^{2\alpha}\omega - (\mathbf{b}\cdot\nabla)j = 0, \\ \partial_t j + (\mathbf{u}\cdot\nabla)j + \Lambda^{2\beta}j - (\mathbf{b}\cdot\nabla)\omega = T(\nabla\mathbf{u},\nabla\mathbf{b}), \end{cases} \tag{2.2}$$

where

$$T(\nabla\mathbf{u},\nabla\mathbf{b}) \overset{\text{def}}{=} 2\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2(\partial_1 b_2 + \partial_2 b_1). \tag{2.3}$$

Taking the inner product of $(2.2)_{1,2}$ with ω and j in $L^2(\mathbb{R}^3)$, respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\omega,j)\|_{L^2}^2 + \|(\Lambda^\alpha\omega,\Lambda^\beta j)\|_{L^2}^2 = \int_{\mathbb{R}^2} T(\nabla\mathbf{u},\nabla\mathbf{b})\cdot j\,\mathrm{d}x \equiv I. \tag{2.4}$$

Case I. (1.14) holds. In this circumstance, we shall use the following lemma to dominate I, which is a variant of that in **[21]**.

Lemma 2.1. *For any* $\varepsilon > 0$,

$$f \in \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^n), \quad g,h \in H^\beta(\mathbb{R}^n),$$

we have

$$\int_{\mathbb{R}^n} fgh\,\mathrm{d}x \le C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}^{2\beta/(2\beta-r)}\|(g,h)\|_{L^2}^2 + \varepsilon\|\Lambda^\beta(g,h)\|_{L^2}^2. \tag{2.5}$$

We provide the proof for the convenience of the reader.

Proof.

$$\int_{\mathbb{R}^n} f\cdot(gh)\,\mathrm{d}x \le C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}\|gh\|_{\dot{B}^r_{1,1}}$$

(by **[1,** Proposition 2.29])

$$\le C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}\|(g,h)\|_{L^2}\|(g,h)\|_{\dot{B}^r_{2,1}}$$

(by Lemma A.1 in the appendix)

$$\leq C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}\|(g,h)\|_{L^2}\cdot\|(g,h)\|^{1-r/\beta}_{\dot{B}^0_{2,\infty}}\|(g,h)\|^{r/\beta}_{\dot{B}^\beta_{2,\infty}}$$

(by [**1,** Proposition 2.22])

$$\leq C\|f\|_{\dot{B}^{-r}_{\infty,\infty}}\|(g,h)\|^{(2\beta-r)/\beta}_{L^2}\|\Lambda^\beta(g,h)\|^{r/\beta}_{L^2}$$

(by [**1,** Proposition 2.39])

$$\leq C\|f\|^{2\beta/(2\beta-r)}_{\dot{B}^{-r}_{\infty,\infty}}\|(g,h)\|^2_{L^2}+\varepsilon\|\Lambda^\beta(g,h)\|^2_{L^2}. \qquad \square$$

Invoking Lemma 2.1 with $f=\nabla\mathbf{u}$, $g=\nabla\mathbf{b}$ and $h=j$, we find

$$\begin{aligned} I &\leq C\|\nabla\mathbf{u}\|^{2\beta/(2\beta-r)}_{\dot{B}^{-r}_{\infty,\infty}}\|(\nabla\mathbf{b},j)\|^2_{L^2}+\varepsilon\|\Lambda^\beta(\nabla\mathbf{b},j)\|^2_{L^2}\\ &\leq C\|\omega\|^{2\beta/(2\beta-r)}_{\dot{B}^{-r}_{\infty,\infty}}\|j\|^2_{L^2}+C\varepsilon\|\Lambda^\beta j\|^2_{L^2}. \end{aligned} \tag{2.6}$$

Choosing $\varepsilon=1/(2C)$, and plugging (2.6) into (2.4), we may apply the Gronwall inequality to deduce the global H^1 estimate of the solution as desired.

Case II. (1.15) holds. In this case, we shall use the following, specific case of [**1,** Theorem 2.42]

$$\|f\|_{L^4}\leq C\|f\|^{1/2}_{\dot{B}^{-r}_{\infty,\infty}}\|f\|^{1/2}_{\dot{H}^r}\quad \text{if } r>0. \tag{2.7}$$

With (2.7) in hand, I can be bounded as

$$\begin{aligned} I &\leq C\|\omega\|_{L^2}\|j\|^2_{L^4}\\ &\leq C\|\omega\|_{L^2}\|j\|_{\dot{B}^{-r}_{\infty,\infty}}\|j\|_{\dot{H}^r}\\ &\leq C\|\omega\|_{L^2}\|j\|_{\dot{B}^{-r}_{\infty,\infty}}\|j\|^{1-r/\beta}_{L^2}\|\Lambda^\beta j\|^{r/\beta}_{L^2}\\ &\leq C\|j\|^{2\beta/(2\beta-r)}_{\dot{B}^{-r}_{\infty,\infty}}\|(\omega,j)\|^2_{L^2}+\frac{1}{2}\|\Lambda^\beta j\|^2_{L^2}. \end{aligned} \tag{2.8}$$

Substituting (2.8) into (2.4), we obtain the desired H^1 estimate of the solution by invoking the Gronwall inequality. The proof of Theorem 1.1 is complete. $\square$

3. Proof of Theorem 1.3. In this section, we shall provide the proof of Theorem 1.3.

Just as was done in Section 2, we have the global H^1 estimate. In order to derive the global H^2 estimate, we multiply $(2.2)_{1,2}$ by $-\triangle\omega$ and $-\triangle j$, respectively, and integrate by parts to derive:

(3.1)
$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\omega,j)\|_{L^2}^2+\|\nabla(\Lambda^\alpha\omega,\Lambda^\beta j)\|_{L^2}^2\\
&\quad=-\sum_{i=1}^{2}\int_{\mathbb{R}^2}[(\partial_i\mathbf{u}\cdot\nabla)\omega]\cdot\partial_i\omega\,\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^2}[(\partial_i\mathbf{b}\cdot\nabla)j]\cdot\partial_i\omega\,\mathrm{d}x\\
&\qquad-\sum_{i=1}^{2}\int_{\mathbb{R}^2}[(\partial_i\mathbf{u}\cdot\nabla)j]\cdot\partial_i j\,\mathrm{d}x+\sum_{i=1}^{2}\int_{\mathbb{R}^2}[(\partial_i\mathbf{b}\cdot\nabla)\omega]\cdot\partial_i j\,\mathrm{d}x\\
&\qquad+\sum_{i=1}^{2}\int_{\mathbb{R}^2}\partial_i T(\nabla\mathbf{u},\nabla\mathbf{b})\cdot\partial_i j\,\mathrm{d}x\\
&\quad\equiv J.
\end{aligned}$$

We can first simplify J as

$$\begin{aligned}
J&\le\int_{\mathbb{R}^2}|\nabla\mathbf{u}|\cdot|\nabla\omega|^2\,\mathrm{d}x+\int_{\mathbb{R}^2}|\nabla\mathbf{b}|\cdot|\nabla j|\cdot|\nabla\omega|\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}|\nabla\mathbf{u}|\cdot|\nabla j|^2\,\mathrm{d}x+\int_{\mathbb{R}^2}|\nabla\mathbf{b}|\cdot|\nabla\omega|\cdot|\nabla j|\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^2}|\nabla^2\mathbf{u}|\cdot|\nabla\mathbf{b}|\cdot|\nabla j|\,\mathrm{d}x+\int_{\mathbb{R}^2}|\nabla\mathbf{u}|\cdot|\nabla^2\mathbf{b}|\cdot|\nabla j|\,\mathrm{d}x\\
&\le C\int_{\mathbb{R}^2}|\nabla(\mathbf{u},\mathbf{b})|\cdot\left(|\nabla^2\mathbf{u}|^2+|\nabla^2\mathbf{u}|\cdot|\nabla^2\mathbf{b}|+|\nabla^2\mathbf{b}|^2\right)\mathrm{d}x\\
&\le C\int_{\mathbb{R}^2}|\nabla(\mathbf{u},\mathbf{b})|\cdot\left(|\nabla^2\mathbf{u}|^2+|\nabla^2\mathbf{b}|^2\right)\mathrm{d}x.
\end{aligned}\tag{3.2}$$

Then, invoking Lemma 2.1 yields

$$\begin{aligned}
J&\le C\|\nabla(\mathbf{u},\mathbf{b})\|_{\dot{B}^{-r}_{\infty,\infty}}^{2\alpha/(2\alpha-r)}\|\nabla^2\mathbf{u}\|_{L^2}^2+\varepsilon\|\Lambda^\alpha\nabla^2\mathbf{u}\|_{L^2}^2\\
&\quad+C\|\nabla(\mathbf{u},\mathbf{b})\|_{\dot{B}^{-r}_{\infty,\infty}}^{2\beta/(2\beta-r)}\|\nabla^2\mathbf{b}\|_{L^2}^2+\varepsilon\|\Lambda^\alpha\nabla^2\mathbf{b}\|_{L^2}^2\\
&\le C\Big[1+\|(\omega,j)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\max\{(2\alpha/(2\alpha-r)),(2\beta/(2\beta-r))\}}\Big]\|\nabla(\omega,j)\|_{L^2}^2\\
&\quad+C\varepsilon\|\nabla(\Lambda^\alpha\omega,\Lambda^\beta j)\|_{L^2}^2.
\end{aligned}\tag{3.3}$$

Placing (3.3) into (3.1) and choosing $\varepsilon=1/(2C)$, we may apply the

Gronwall inequality to deduce the desired H^2 bound of the solution. From the Sobolev embedding theorems,

$$\omega, j \in L^2(0,T;H^1(\mathbb{R}^2)) \subset L^2(0,T;BMO(\mathbb{R}^2)) \subset L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^2)).$$

By **[22]**, the proof of Theorem 1.3 is accomplished. □

APPENDIX

A. In this appendix, we provide a bilinear estimate in Besov spaces, which is utilized in the proof of Lemma 2.1.

Lemma A.1. *Let* $(s,p,q,p_1,p_2,p_3,p_4) \in (0,\infty)\times[1,\infty]^6$. *Then, there exists a constant* C*, depending upon* s *and the dimension* n*, such that*

$$\|uv\|_{\dot{B}^s_{p,q}} \le C\big(\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}} + \|u\|_{\dot{B}^s_{p_3,q}}\|v\|_{L^{p_4}}\big) \tag{A.1}$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Proof. The proof of this lemma is similar to **[1,** Corollary 2.54] and is trivial for experts; however, the proof is provided in full detail for the convenience of the reader. The notation and tools are borrowed from **[1]**. We shall abbreviate $\|u\|_{L^p(\mathbb{R}^n)}$ as $\|u\|_{L^p}$ for simplicity.

By **[1,** Equation (2.29)], we have the following Bony decomposition

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u,v),$$

and what we need to do is to estimate these three terms.

A.1. The estimation of $\dot{T}_u v$ and $\dot{T}_v u$.

By **[1,** equation (2.9)],

$$\dot{\triangle}_j(\dot{T}_u v) = \sum_{|j'-j|\le 4} \triangle_j(\dot{S}_{j'-1}u\dot{\triangle}_{j'}v),$$

and thus,

$$\begin{aligned} \|\dot{T}_u v\|_{\dot{B}^s_{p,q}} &= \|\{2^{js}\|\triangle_j(\dot{T}_u v)\|_{L^p}\}\|_{\ell^q} \\ &= \left\|\left\{2^{js}\left\|\sum_{|j'-j|\le 4}\triangle_j(\dot{S}_{j'-1}u\dot{\triangle}_{j'}v)\right\|_{L^p}\right\}\right\|_{\ell^q} \end{aligned} \tag{A.2}$$

$$
\begin{aligned}
&\leq C\|\{2^{js}\|\triangle_j(\dot{S}_{j-1}u\dot{\triangle}_j v)\|_{L^p}\}\|_{\ell^q}\\
&\leq C\|\{2^{js}\|\dot{S}_{j-1}u\dot{\triangle}_j v\|_{L^p}\}\|_{\ell^q}\\
&\leq C\|\{2^{js}\|\dot{S}_{j-1}u\|_{L^{p_1}}\|\dot{\triangle}_j v\|_{L^{p_2}}\}\|_{\ell^q}\\
&\leq C\|u\|_{L^{p_1}}\|\{2^{js}\|\dot{\triangle}_j v\|_{L^{p_2}}\}\|_{\ell^q}\\
&= C\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}}.
\end{aligned}
$$

Similarly,

$$
\|\dot{T}_v u\|_{\dot{B}^s_{p,q}} \leq C\|v\|_{L^{p_3}}\|u\|_{\dot{B}^s_{p_4,q}}. \tag{A.3}
$$

A.2. The estimation of $\dot{R}(u,v)$. By the analysis of the support, we deduce, as in [**1,** Proof of Theorem 2.52], that there exists an integer N such that

$$
\begin{aligned}
\dot{\triangle}_{j'}\dot{R}(u,v) &= \dot{\triangle}_{j'}\sum_j\sum_{|\nu|\leq 1}\dot{\triangle}_{j-\nu}u\dot{\triangle}_j v\\
&= \sum_{j\geq j'-N}\dot{\triangle}_{j'}\sum_{|\nu|\leq 1}\dot{\triangle}_{j-\nu}u\dot{\triangle}_j v\\
&= \sum_{\substack{j\geq j'-N\\|\nu|\leq 1}}\dot{\triangle}_{j'}\big(\dot{\triangle}_{j-\nu}u\dot{\triangle}_j v\big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
2^{j's}\|\triangle_{j'}\dot{R}(u,v)\|_{L^p} &\leq 2^{j's}\sum_{\substack{j\geq j'-N\\|\nu|\leq 1}}\|\dot{\triangle}_{j'}\big(\dot{\triangle}_{j-\nu}u\dot{\triangle}_j v\big)\|_{L^p}\\
&\leq C2^{j's}\sum_{\substack{j\geq j'-N\\|\nu|\leq 1}}\|\dot{\triangle}_{j-\nu}u\dot{\triangle}_j v\|_{L^p}\\
&\leq C2^{j's}\sum_{\substack{j\geq j'-N\\|\nu|\leq 1}}\|\dot{\triangle}_{j-\nu}u\|_{L^{p_1}}\|\dot{\triangle}_j v\|_{L^{p_2}}\\
&\leq C\|u\|_{L^{p_1}}\sum_{j\geq j'-N}2^{(j'-j)s}\cdot 2^{js}\|\dot{\triangle}_j v\|_{L^{p_2}}\\
&= C\|u\|_{L^{p_1}}\sum_{i\leq N}2^{is}\cdot 2^{(j'-i)s}\|\dot{\triangle}_{j'-i}v\|_{L^{p_2}} \quad (j'-j=i)\\
&= C\|u\|_{L^{p_1}}\big(\{a_i\}*\{2^{is}\|\dot{\triangle}_i v\|_{L^{p_2}}\}\big)_{j'},
\end{aligned}
$$

where

$$a_i = \begin{cases} 2^{is} & i \le N, \\ 0 & i > N, \end{cases}$$

and $(\{a_i\} * \{b_j\})_{j'}$ denotes the j'th term of the convolution of these two sequences, namely, $\sum_i a_i b_{j'-i}$.

Invoking the Young inequality for a series, we find

$$\begin{aligned} \|\dot{R}(u,v)\|_{\dot{B}^s_{p,q}} &= \|\{2^{j's}\|\triangle_{j'}\dot{R}(u,v)\|_{L^p}\}\|_{\ell^q} \\ &\le C\|u\|_{L^{p_1}}\|\{a_i\} * \{2^{is}\|\dot{\triangle}_i v\|_{L^{p_2}}\}\|_{\ell^q} \\ &\le C\|u\|_{L^{p_1}}\|\{a_i\}_{\ell^1}\|\{2^{is}\|\dot{\triangle}_i v\|_{L^{p_2}}\}\|_{\ell^q} \\ &\le C\|u\|_{L^{p_1}}\|v\|_{\dot{B}^s_{p_2,q}} \end{aligned} \tag{A.4}$$

(since $s > 0$). Combining (A.2)–(A.4), the proof of Lemma A.1 is concluded. □

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