ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 8, 2018

THE HEAT EQUATION FOR LOCAL DIRICHLET FORMS: EXISTENCE AND BLOW UP OF NONNEGATIVE SOLUTIONS

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ABSTRACT. We establish conditions ensuring either existence or blow up of nonnegative solutions for the following parabolic problem:

$$\begin{cases} Hu - Vu + (\partial u/\partial t) = 0 & \text{in } X \times (0,T), \\ u(x,0) = u_0(x) & \text{in } X, \end{cases}$$

where $T > 0$, X is a locally compact separable metric space, H is a selfadjoint operator associated with a regular Dirichlet form $\mathcal{E}$; the initial value $u_0 \in L^2(X,m)$, where m is a positive Radon measure on Borel subset U of X such that $m(U) > 0$ and V is a Borel locally integrable function on X.

1. Introduction. In this paper, we discuss the question of existence as well as blow up of nonnegative solutions for the following parabolic problem:

$$\begin{cases} Hu - Vu + (\partial u/\partial t) = 0 & \text{in } X \times (0,T), \\ u(x,0) = u_0(x) & x \in X, \end{cases} \tag{1.1}$$

where X is a locally compact separable metric space, H is a selfadjoint operator associated with a regular Dirichlet form $\mathcal{E}$, $u_0 \in L^2(X,m)$, $V \in L^1_{\mathrm{loc}}(X,m)$ is a positive Borel function, m a positive Radon measure. The meaning of a solution for equation (1.1) will be explained in the next section.

Our main task in this paper is to shed some light towards solving the problem by giving conditions ensuring existence as well as blow-up of nonnegative solutions for (1.1). The inspiring point for us, first, was the research of Baras and Goldstein [**8**], Cabré and Martel [**34**] and Goldstein and Zhang [**19**], where the problem was addressed and solved

2010 AMS *Mathematics subject classification.* Primary 34B27, 35K08, 35K55.
Keywords and phrases. Dirichlet forms, heat equation, parabolic problem.
Received by the editors on December 21, 2017, and in revised form on April 20, 2018.
DOI:10.1216/RMJ-2018-48-8-2573 Copyright ©2018 Rocky Mountain Mathematics Consortium

for the Dirichlet Laplacian Lipschitz domains; second was the paper by Ben Amor and Kenzizi [**7**], where the authors established conditions ensuring existence as well as blow up of nonnegative solutions for a nonlocal case. Furthermore, for a result concerning the uniqueness of solution of (1.1), the reader may consult [**23**]. In this paper, we generalize the work of [**7, 19, 34**], where we consider the selfadjoint operator H associated with a regular Dirichlet form $\mathcal{E}$ via Kato's representation theorem, that is,

$$(Hu, v) = \mathcal{E}(u, v).$$

From a physical perspective, such quadratic forms may intuitively be understood as a model of the "energy functional." For every $t > 0$, we designate by $P_t := e^{-tH}$ the semigroup related to H. Measure perturbation of Dirichlet forms has been studied in increasing generality and with different aims, see [**1, 5, 10, 16, 32**]. However, in [**28**], Stollmann and Voigt studied the type of perturbations in a different manner from all of those papers (except [**16**]); their approach is, given a theoretical operator, to use potential theory of Dirichlet forms based on probabilistic methods. Moreover, *loc cit.*, they studied the perturbed form $\mathcal{E} - \mu_- + \mu_+$ such that μ_- is in a suitable Kato class and μ_+ is absolutely continuous with respect to capacity. However, the importance of Kato potentials is that it is a very large class under which we can still expect pointwise results for $H^V := H - V$. Of course, this is of importance if one is interested in pointwise results like continuity. Thus, singular perturbations of Schrödinger-type operators are of interest in mathematics, e.g., studying spectral phenomena, and in applications of mathematics in various sciences, e.g., in physics, chemistry, biology, and in technology. They also often lead to models in quantum theory which are solvable in the sense that the spectral characteristics (eigenvalues, eigenfunctions, and scattering matrix) can be computed. Such models then allow us to grasp the essential features of interesting and complicated phenomena and serve as an orientation in handling more realistic situations. Singular perturbations of selfadjoint operators can be defined using Dirichlet forms. This approach was initially used by researchers of the works [**1, 2, 3, 4, 12, 13, 14, 15**]. Diffusion processes with singular Dirichlet forms are studied in [**6**]. The main topic of this paper is to adapt the strategy from the work of Zhang [**33**] to a very general local Dirichlet form and to study some of its basic features. The fundamental principle in this investigation is to give conditions ensuring existence as well as blow-up of nonnegative solutions for problem (1.1). More precisely, we proceed as follows. After presenting the basic ingredients of Dirichlet forms, we carry out a careful study of some results crucial for the later development of the paper in Section 2. In Sections 3 and 4, we give conditions ensuring existence and nonexistence of nonnegative solutions for (1.1). Nonetheless, we shall show that the method used in **[7, 19, 34]** still applies in our setting. As an application, a special type of Dirichlet forms is considered in Section 5.

2. Preliminaries and preparation of results. In this section, we recall the definition of the Dirichlet form and its properties. For classical theory of Dirichlet forms, the interested reader is referred to **[11, 18, 29, 31]** and the references therein.

Throughout this paper, X denotes a locally compact, separable metric space, endowed with a positive radon measure m with $\mathrm{Supp}(m) = X$. All functions on X will be real valued.

The central object of our studies is a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$ in $L^2(X,m)$ and the selfadjoint operator H associated with $\mathcal{E}$. Recall that a closed non-negative form on $L^2(X,m)$ consists of a dense subspace $\mathcal{F} \subset L^2(X,m)$ and a sesquilinear non-negative map

$$\mathcal{E} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R}$$

such that $\mathcal{F}$ is complete with respect to the energy norm $\sqrt{\mathcal{E}_1}$, defined by

$$\mathcal{E}_1(f) = \mathcal{E}(f) + \|f\|^2_{L^2(X,m)}.$$

In this case, the space $\mathcal{F}$, together with the inner product

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g),$$

becomes a Hilbert space, and $\sqrt{\mathcal{E}_1}$ is the induced norm.

A closed form is said to be a *Dirichlet form* if, for any $f \in \mathcal{F}$ and any normal contraction $T : \mathbb{R} \to \mathbb{R}$, we have

$$T \circ f \in \mathcal{F} \quad \text{and} \quad \mathcal{E}(T \circ f) \leq \mathcal{E}.$$

Here, $T : \mathbb{R} \to \mathbb{R}$ is called a *normal contraction* if $T(0) = 0$ and $|T(\xi) - T(\zeta)| \leq |\xi - \zeta|$ for any $\xi,\ \zeta \in \mathbb{R}$. A Dirichlet form is called *regular* if $\mathcal{F} \cap C_c(X)$ is dense both in $(\mathcal{F}, \|\cdot\|_{\sqrt{\mathcal{E}_1}})$ and $(C_c(X), \|\cdot\|_\infty)$, where $C_c(X)$ denotes the space of continuous functions with compact support.

In the remainder of this paper, we shall assume that $\mathcal{E}$ is a regular Dirichlet form. In the sequel, we will write

$$\mathcal{E}(f) := \mathcal{E}(f, f).$$

The selfadjoint operator H associated with $\mathcal{E}$ is then characterized by

$$D(H) \subset \mathcal{F} \text{ and } \mathcal{E}(f, g) = \langle Hf, g\rangle, \quad f \in D(H),\ g \in \mathcal{F}.$$

$\mathcal{E}$ is called *strongly local* if, for any $f, g \in \mathcal{F}$ such that f is constant on a neighborhood of $\mathrm{Supp}(g)$, we have

$$\mathcal{E}(f, g) = 0.$$

Note that, if Ω is a domain of $\mathbb{R}^d$, $d \geq 2$, $H = -\triangle$ with $D(H) = L^2(\Omega)$, $\mathcal{F} = W_0^{1,2}(\Omega)$ and

$$\mathcal{E}(f, g) = \int_\Omega (\nabla f / \nabla g)\, dx,$$

then $\mathcal{E}$ is strongly local.

Remark 2.1. Every strongly local, regular Dirichlet form $\mathcal{E}$ can be represented in the form

$$\mathcal{E}(f, g) = \int_X d\Gamma(f, g)$$

where Γ, called the *energy measure*, is a positive semidefinite, symmetric bilinear form on $\mathcal{F}$ with values in the set of signed radon measures on X. We remark that Γ is determined by, see **[18]**:

$$\int_X \phi\, d\Gamma(f, f) = \mathcal{E}(f, \phi f) - \frac{1}{2}\mathcal{E}(f^2, \phi),$$

for every $f \in \mathcal{F} \cap L^\infty(X, m)$, $\phi \in \mathcal{F} \cap C_c(X)$.

The energy measure Γ satisfies the Leibniz rule,

$$d\Gamma(f \cdot g, h) = f d\Gamma(g, h) + g\, d\Gamma(f, h),$$

as well as the chain rule

$$d\Gamma(\eta(f), h) = \eta'(f)\, d\Gamma(f, h).$$

We write $d\Gamma(f) := d\Gamma(f, f)$, and note that Γ satisfies the Cauchy-Schwarz inequality:

$$\int_X |fg|\, d\Gamma(f,g) \leq \left(\int_X |f|^2 d\Gamma(f)\right)^{1/2} \left(\int_X |g|^2 d\Gamma(g)\right)^{1/2}.$$

In order to introduce weak solutions on open subsets of X, we extend the quadratic forms $f \mapsto \mathcal{E}(f,f)$ and $f \mapsto \Gamma(f,f)$ to the whole spaces $L^2(X,m)$ and $L^2_{\mathrm{loc}}(X,m)$, respectively, in such a way that

$$\mathcal{F}(X) = \{f \in L^2(X,m); \mathcal{E}(f,f) < +\infty\}$$

and

$$\mathcal{F}_{\mathrm{loc}}(X) = \{f \in L^2_{\mathrm{loc}}(X,m); \Gamma(f,f) \text{ is a radon measure}\}. \tag{2.1}$$

From now on, we assume that the form $(\mathcal{E}, \mathcal{F})$ is local and regular. Using the energy measure Γ, we can define the intrinsic metric ρ by:

$$\rho(x,y) = \sup\{|f(x) - f(y)|;\ f \in \mathcal{F}_{\mathrm{loc}}(X) \cap C(X) \text{ and } d\Gamma(f) \leq dm\}.$$

Note that the above condition implies that $\Gamma(u)$ is absolutely continuous with respect to m, see **[21, 22]**. The Radon-Nikodym derivative is bounded by 1 on X.

We say that $\mathcal{E}$ is strictly local if ρ is a metric that induces the original topology on X. Note that this implies that X is connected, since, otherwise, points x and y in different connected components would give $\rho(x,y) = \infty$ since characteristic functions of connected components are continuous and have vanishing energy measure.

We denote the intrinsic balls by

$$B(x,r) := \{y \in X/\rho(x,y) < r\}.$$

An important consequence of the latter assumption is that, for given x, the function

$$y \longmapsto \rho_x(y) := \rho(x,y)$$

is in $\mathcal{F}_{\mathrm{loc}}$ and $d\Gamma(\rho_x) \leq dm$, see **[30]**.

Now, to state our results, it is convenient to introduce the following notation. The real Hilbert space $L^2(X,m)$ will be denoted by L^2, and its norm will be denoted by

$$\|u\| = \left(\int_X u^2 dm \right)^{1/2}.$$

We shall write $\int \cdots$ as an abbreviation for $\int_X \cdots$. Variables C, c, C', etc., will denote generic positive constants which may vary in value from line to line. We shall also use the notation $p_t = p_t(x, y)$, $t > 0$, x, $y \in X$ for the heat kernel of the semigroup $P_t := e^{-tH}$, and $P_0 = P_0(x, t, y, 0)$ as the heat kernel of the equation

$$Hu - \partial_t u = 0 \quad \text{in } X \times (0, +\infty). \tag{2.2}$$

We are now at the stage of giving the notion of the solution for the heat equation (1.1).

Definition 2.2. Let $0 < T \leq \infty$. A Borel-measurable function $u \in W^{1,2}((0,T), \mathcal{F}(X))$ is a solution of problem (1.1) if

(a) $u \in L^1_{\text{loc}}((0,T) \times X, dt \otimes V dm)$;

(b) $u(t, \cdot) \in \mathcal{F}_{\text{loc}}(X)$;

(c) $\int_0^T \int_K d\Gamma[u(\cdot,t)]\, dt < +\infty$ for all compact $K \subset X$;

(d) for every $0 \leq t < T$, the following identity holds:

$$\begin{aligned} &\int_0^t \int_X u(s,x) \phi_s(s,x)\, dm\, ds \\ &\qquad + \int_0^t \int_X d\Gamma[u(\cdot,s), \phi(\cdot,s)]\, ds = \int_0^t \int_X u(s,x)\phi(s,x)V(x)\, dm\, ds \end{aligned} \tag{2.3}$$

for all

$$\begin{aligned} \phi \in \mathcal{F}_0((0,T) \times X) = \{\phi \in L^2((0,T) \times X), \phi(t, \cdot) \in \mathcal{F}(X) \\ \cap\, \mathcal{C}_c(X), \phi(\cdot, x) \in W_0^{1,2}([0,T])\}, \end{aligned}$$

and $u(\cdot, t) \to u_0(\cdot)$ as $t \to 0$.

Next, we recall the notion of the heat kernel.

Definition 2.3. A function $p(\cdot,\cdot,\cdot):\mathbb{R}_+\times X\times X\to\mathbb{R}$ is called a *heat kernel* if the following conditions (A_1)–(A_4) are fulfilled: for m-almost all $(x,y)\in X\times X$, and for all $t,s>0$:

(A_1) Markov property: $p(t,x,y)>0$ and $\int_X p(t,x,y)m(dy)\le 1$;

(A_2) Symmetry: $p(t,x,y)=p(t,y,x)$;

(A_3) Semigroup property: $p(s+t,x,z)=\int_X p(s,x,y)p(t,y,z)m(dy)$;

(A_4) Normalization: for all $f\in L^2(X,m)$,

$$\lim_{t\to 0^+}\int_X p(t,x,y)f(y)m(dy)=f(x)\quad\text{in the } L^2\text{-norm.}$$

We assume that the heat kernel $p(t,x,y)$ is jointly continuous in x and y.

Here are some examples of heat kernels.

Example 2.4. (Gauss-Weierstrass). The Gauss-weierstrass function in $\mathbb{R}^d$ is given by

$$p(t,x,y)=\frac{1}{(4\pi t)^{d/2}}\exp\bigg(-\frac{|x-y|^2}{4t}\bigg), \tag{2.4}$$

satisfying all conditions of Definition 2.3 with Lebesgue measure.

Example 2.5. (Li-You estimate **[26]**). Let X be a connected Riemannian manifold, ρ the geodesic distance and m the Riemannian measure. Let H be the Laplace-Beltrami operator $\triangle$ which is associated with the local Dirichlet form $(\mathcal{E},\mathcal{F})$ with

$$\mathcal{E}(f)=\int_X|\nabla f|^2dm,\qquad \mathcal{F}=W_0^{1,2}(X). \tag{2.5}$$

Consider $\{P_t=e^{-tH}\}_{t>0}$ as the corresponding heat semigroup. It is known that $\{P_t\}$ possesses a smooth integral kernel $p(t,x,y)$, which is indeed a heat kernel on X. It is seen that, if X is geodesically complete and its Ricci curvature is nonnegative, then, for any $x,\ y\in X,\ t>0$,

$$p(t,x,y)\asymp\frac{1}{V(x,\sqrt{t})}\exp\bigg(-c\frac{\rho^2(x,y)}{t}\bigg), \tag{2.6}$$

where $V(x,r) = m(B(x,r))$ is the volume of the geodesic metric open Ball

$$B(x,r) = \{y \in X,\ \rho(y,r) < r\}.$$

For more details, the reader may consult **[20]**.

Henceforth, our main task is to prove that the existence and blow-up of nonnegative solutions of (1.1) are deeply related to

$$\lambda_0^V := \inf\left\{\frac{\mathcal{E}(\phi,\phi) - \int_X \phi^2 V\,dm}{\|\phi\|^2_{L^2(X,m)}}\ :\ \phi \in \mathcal{F},\ \phi \neq 0\right\} \geq 0. \tag{2.7}$$

For that, we shall need some preparation.

We now consider $V_k = V \wedge k$, and (P_k) the heat equation corresponding to the selfadjoint negative semidefinite operator associated with the initial Dirichlet form $\mathcal{E}$, perturbed by $-V_k$ instead of $-V$. On the other hand, by using the standard theory of quadratic forms, we deduce the existence of a unique nonnegative energy solution given by $u_k = e^{-tH_k}u_0,\ t > 0$, for problem (P_k), such that H_k is the selfadjoint operator associated to the closed quadratic form $\mathcal{E} - V_k$. Furthermore, the solution u_k lies in the space

$$\mathcal{F}(X) \cap L^\infty(X,m)$$

in its spacial variable, is continuous in its time variable, and satisfies the integral formulation:

$$u_k(t,x) = e^{-tH}u_0(x) + \int_0^t \int_X p_{t-s}(x,y)u_k(s,y)V_k(y)\,dy\,ds, \tag{2.8}$$

where p_t is the heat kernel of the operator $P_t = e^{-tH},\ t > 0$.

Remark 2.6. Consider $T(\cdot)$ as the semigroup generated by H in $L^2(X,m)$. Then, the mapping

$$T_t = T(t) : L^2(X,m) \longrightarrow L^2(X,m), \quad f \mapsto T(t)f$$

defines an L^2-semigroup which is exponentially bounded, i.e.,

$$\|T_t\|_{L^2(X,m)} \leq e^{-\lambda_0^V t} \quad \text{for all } t > 0. \tag{2.9}$$

By the standard theory of semigroups, there is a unique selfadjoint operator associated to T_t.

Note that, the following properties of the sequence (u_k) are crucial for the later development of the paper.

Lemma 2.7.

(i) *The sequence* (u_k) *is increasing.*

(ii) *Assume that problem* (1.1) *has a nonnegative solution* u. *Then,* $u_k(x,t) \le u(x,t)$, *for all* $t > 0$ *and* m *almost everywhere* $x \in X$. *Moreover,* $\lim_{k\to\infty} u_k$ *is a nonnegative solution of problem* (1.1) *as well.*

Proof.

(i) It is easily seen that the solution of problem (P_k) can be obtained as

(2.10)
$$u_k(x,t) = \int_X \Phi_{V_k}(t,x,0,y)u_0(y)\,dy = \int_X P_{V_k}(t,x-y,0)u_0(y)\,dy,$$

where $u_0(x) = u(0,x)$, Φ_{V_k} is the fundamental solution and $P_{V_k}(t,x,y) = \Phi_{V_k}(t,x,0,y)$. Furthermore, the function P_{V_k} is often referred to as the heat kernel of the corresponding semi-group. Thus, we obtain

$$\begin{aligned} u_{k+1}(t,x) &- u_k(t,x) \\ &= \int_X [P_{V_{k+1}}(t,x-y,0) - P_{V_k}(t,x-y,0)]u_0(y)\,dy \ge 0. \end{aligned} \tag{2.11}$$

(ii) Now, using the fact that V_k is nonnegative and bounded, we obtain

$$u_k(t) \ge T(t)u_0, \quad t \ge 0, \tag{2.12}$$

where $T(\cdot)$ is the semigroup generated by H in $L^2(X,m)$. Based on the positivity of $T(t)u_0,\ t > 0$, we see that $u_k(x,t) > 0$, for all $t > 0$ and m almost everywhere $x \in X$.

Now, we prove that $u_k \le u$. First, note that, from Definition 2.2 and, for each $T,\ R > 0$, we have

$$\begin{aligned} \int_0^T \int_X (u_k - u)(-\partial_s\phi - V_k(x)\phi)\,dm\,ds + \int_0^T \int_X d\Gamma[u_k - u, \phi(\cdot,s)]\,ds \\ = \int_0^T \int_X (V_k - V)u\phi\,dm\,ds. \end{aligned}$$

Hence,

$$\int_0^T\int_X (u_k-u)(-\partial_s\phi+H\phi-V_k(x)\phi)\,dm\,ds = \int_0^T\int_X (V_k-V)u\phi\,dm\,ds \le 0, \tag{2.13}$$

for all $\phi\in\mathcal{F}_0((0,T)\times X)$ with $\operatorname{supp}\phi\subset B_R\times[0,T]$ and $\phi(\cdot,T)=0$. Fix T, $R>0$, $0\le\psi\in C_c^\infty((0,T),\mathcal{F}(X)\cap C_c(X))$ such that $\operatorname{supp}\psi\subset B_R\times[0,T]$, and consider the parabolic problem

$$\begin{cases} -H\phi+V_k\phi+(\partial\phi/\partial t)=-\psi & \text{in } B_R\times(0,T),\\ \phi(x,t)=0 & \text{in } X\backslash B_R\times(0,T),\\ \phi(x,T)=0 & \text{in } X. \end{cases} \tag{2.14}$$

In view of [**24,** Theorem 4], there exists a solution $0\le\phi\in\mathcal{F}_0((0,T)\times B_R)$. Now, consider the extension

$$\widetilde{\phi}(x,t):=\begin{cases}\phi(x,t) & \text{if } (x,t)\in B_R\times(0,T),\\ 0 & \text{if} (x,t)\in B_R^c\times(0,T).\end{cases}$$

Note that, by inserting the solution $\widetilde{\phi}$ into (2.13), we obtain

$$\int_0^T\int_X (u_k-u)\psi\,dm\,ds\le 0 \quad \text{for all } 0\le\psi\in C_c^\infty((0,T),\mathcal{F}(X)\cap C_c(X)).$$

Thus, we get

$$u_k\le u.$$

Now, we consider $u_\infty:=\lim_{k\to\infty}u_k$. By the first step of (2.7), we have $u_\infty\le u$ and $u_\infty\in L^1_{\mathrm{loc}}((0,T),\mathcal{F}(X))\cap L^1_{\mathrm{loc}}([0,T]\times X, dt\otimes V\,dm)$. Since they are a solution of the heat equation with potential V_k, the u_ks satisfy the following: for every $0\le t<T$ and every $\phi\in\mathcal{F}_0((0,T)\times X)$,

$$-\int_0^t\int_X u_k(s,x)\phi_s(s,x)\,dm\,ds+\int_0^t\int_X d\Gamma[u_k(\cdot,s),\phi(\cdot,s)]\,dt = \int_0^t\int_X u_k(s,x)\phi(s,x)V_k(x)\,dm\,dt. \tag{2.15}$$

It is known that (u_k) is a monotone sequence and $|V_k u_k| \leq |V|u \in L^1_{\mathrm{loc}}((0,T) \times X, dm\, dt)$. Then, the dominated convergence theorem implies that u_∞ satisfies equation (2.3), which ends the proof. $\square$

The following lemma is crucial for the development of the paper; it is inspired from the 'gradient' case, where integration by parts is used.

Lemma 2.8. *Let u_k be the nonnegative solution of the approximate problem (P_k) and $\phi \in \mathcal{F}_0(X) \cap L^\infty(X,m)$. Then*

$$\frac{\phi^2}{u_k} \in \mathcal{F}_0(X) \quad \textit{and} \quad \mathcal{E}\left(u_k, \frac{\phi^2}{u_k}\right) \leq \mathcal{E}[\phi]. \tag{2.16}$$

Proof. It suffices to give the proof for positive ϕ. Let $\phi \geq 0$ and u_k be as specified in the lemma. Since $\mathcal{E}$ is a Dirichlet form, to prove the first part, it suffices to prove that $\phi/u_k \in \mathcal{F}_0(X) \cap L^\infty(X,m)$. It is simple to show that $\phi/u_k \in L^\infty(X,m)$. In order to show that the latter function has finite energy, we shall proceed directly. By using the fact that the energy measure satisfies the Leibniz rule, an elementary computation yields

$$\begin{aligned} d\Gamma\left[\frac{\phi}{u_k}\right] &= \phi^2 d\Gamma\left[\frac{1}{u_k}\right] + 2\frac{\phi}{u_k}\, d\Gamma\left[\frac{1}{u_k}, \phi\right] + \frac{1}{u_k^2}\, d\Gamma[\phi] \\ &\leq 2\left(\phi^2\, d\Gamma\left[\frac{1}{u_k}\right] + \frac{1}{u_k^2}\, d\Gamma[\phi]\right) \\ &\leq \frac{2}{u_k^2}\left(\frac{\phi^2}{u_k^2} d\Gamma[u_k] + d\Gamma[\phi]\right) \leq C d\Gamma[\phi]. \end{aligned} \tag{2.17}$$

Thereby, we derive

$$\mathcal{E}\left[\frac{\phi}{u_k}\right] = \int_X d\Gamma\left[\frac{\phi}{u_k}\right] < \infty, \tag{2.18}$$

and consequently, $\phi/u_k \in \mathcal{F}_0(X)$.

Now, we proceed to prove inequality (2.16). For this, using again the fact that the energy measure satisfies the Leibniz rule,

$$\begin{aligned} d\Gamma\left[u_k, \frac{\phi^2}{u_k}\right] &= \phi^2\, d\Gamma\left[u_k, \frac{1}{u_k}\right] + \frac{1}{u_k}\, d\Gamma[u_k, \phi^2] \\ &= -\frac{\phi^2}{u_k^2}\, d\Gamma[u_k] + 2\frac{\phi}{u_k}\, d\Gamma[u_k, \phi] \leq d\Gamma[\phi]. \end{aligned} \tag{2.19}$$

Thus,

$$\mathcal{E}\left(u_k, \frac{\phi^2}{u_k}\right) \leq \mathcal{E}[\phi], \tag{2.20}$$

which is the desired result. □

At the end of this section, we give a technical result dealing with the comparability of the ground state of the operator H which will be needed in the proof of the nonexistence part.

Lemma 2.9. *Suppose that the positivity preserving semigroup e^{-tH} on $L^2(X, m)$ has a continuous integral kernel $p_t(x, y)$ such that H has the normalized ground state $\varphi_0 > 0$, and let $h(t, x) := e^{-tH}u_0(x)$ for every $t > 0$ and every $x \in X$. Then*

$$h(t, \cdot) \sim \varphi_0 \quad \textit{for every fixed } t > 0. \tag{2.21}$$

Proof. Recall that H is a nonnegative selfadjoint operator on $L^2(X, m)$ such that, for every $t > 0$, e^{-Ht} has the jointly continuous integral kernel $p_t(x, y)$. Since $u_0 \in L^2(X, m)$, then

$$h(t, x) := e^{-tH}u_0(x) = \int_X p_t(x, y)u_0(y)\, dm \quad \text{for all } t > 0. \tag{2.22}$$

Hence, using the notion of intrinsic ultracontractivity (IU), due to Davies and Simon **[17]**, we deduce that the operator e^{-tH} is intrinsic ultracontractive. Moreover, in **[17]**, the authors proved that IU is equivalent to either of the following two conditions:

(1) for all $t > 0$, there exist positive constants a_t and b_t, such that

$$a_t\varphi_0(x)\varphi_0(y) \leq p_t(x, y) \leq b_t\varphi_0(x)\varphi_0(y); \tag{2.23}$$

(2) for all $t > 0$, there exists a positive constant c_t such that

$$|\varphi_n(x)| \leq c_t\varphi_0(x)e^{-(\lambda_0-\lambda_n)t}, \tag{2.24}$$

where φ_n is the eigenfunction associated to the nth eigenvalue λ_n of H, whence, (2.21) is an immediate consequence of (2.23). □

3. Existence of nonnegative solutions.

Theorem 3.1. *Assume that $\lambda_0^V > -\infty$. Then, the heat equation* (1.1) *has at least one nonnegative solution satisfying*

$$\|u(t)\|_{L^2(X,m)} \leq Me^{\omega t}\|u_0\|_{L^2(X,m)}, \quad t > 0, \tag{3.1}$$

for some constants M and ω.

Proof. Let us consider u_n as a solution to problem (P_n). Hence, by using the fact that $(u_n) \subset D(H_n)$ and (2.7), we obtain

$$\begin{aligned} \frac{d}{dt}\|u_n\|^2_{L^2(X,m)} &= -2\mathcal{E}[u_n] + \int_X V_n u_n^2 \, dm \\ &\leq -2\lambda_0^V \int_X u_n^2(x,t)\, dm. \end{aligned} \tag{3.2}$$

Recall that, by Gronwall's lemma, we achieve the upper estimate

$$\|u_n\|_{L^2(X,m)} \leq \|u_0\|_{L^2(X,m)} e^{-\lambda_0^V t} \quad \text{for all } t > 0. \tag{3.3}$$

Hence, the sequence (u_n) increases to a nonnegative function u for every $t > 0$ and m almost everywhere $x \in X$. Furthermore, due to Fubini's theorem, $u \in L^1_{\text{loc}}((0,T), L^2(X,m))$.

Now, we are in position to prove that u solves the heat equation (1.1). Indeed, having Duhamel's formula for the u_ns in hand, we conclude by the monotone convergence theorem that

$$u(x,t) = e^{-tH}u_0(x) + \int_0^t \int_X p_{t-s}(x,y)u(y,s)V(x)\, dm\, ds. \tag{3.4}$$

Note that, $p_t > 0,\ t > 0$, on $X \times X$; thus, the latter formula implies that $u \in L^1_{\text{loc}}((0,T) \times X, dt \otimes V dm)$. Now, utilizing the equation fulfilled by the u_ns being a solution of the P_ns, we obtain by the dominated convergence theorem, for every $0 \leq t < T$ and every $\phi \in \mathcal{F}_0((0,T) \times X)$, that

$$\begin{aligned} -\int_0^t \int_X u(s,x)\phi_s(s,x)\, dm\, ds + \int_0^t \int_X d\Gamma[u(\cdot,s), \phi(\cdot,s)]\, ds \\ = \int_0^t \int_X u(s,x)\phi(s,x)V(x)\, dm\, ds. \end{aligned} \tag{3.5}$$

Then, estimate (3.1) follows from (3.3), and it holds with $M = 1$. $\square$

4. Blow up of nonnegative solutions. In order to prove the nonexistence part, we are, first, going to establish an estimate for the integral $\int_X \ln u$, whenever u is a nonnegative solution. Such an estimate has an independent interest and is involved in deriving regularity properties for the solutions. It is used in our context and is inspired from the one corresponding to the Dirichlet Laplacian [**19, 34**].

Theorem 4.1. *Assume that u is a nonnegative solution of the heat equation* (1.1). *Then, for all $0 < t_1 < t_2 < T$, and all $\Phi \in \mathcal{F}(X) \cap \mathcal{C}_0(X)$, we have*

$$\int_X \Phi^2 V dm - \mathcal{E}[\Phi] \leq \frac{1}{t_2 - t_1} \int_X \ln\left(\frac{u(t_2)}{u(t_1)}\right) \Phi^2 dm. \tag{4.1}$$

Proof. Choose a function $\Phi \in \mathcal{F}(X) \cap \mathcal{C}_0(X)$. Without loss of generality, we may and shall suppose that $\int_X \Phi^2 dm = 1$. By using the fact that u_n is a solution of the heat equation, $\Phi^2/u_n \in \mathcal{F}_0(X)$ (see Lemma 2.8 and inequality (2.16)), we obtain

$$\begin{aligned} \int_X \Phi^2 V_n \, dm &= \int_X (\partial_t u_n) \frac{\Phi^2}{u_n} dm + \mathcal{E}\left(u_n, \frac{\Phi^2}{u_n}\right) \\ &= \frac{d}{dt} \int_X (\ln u_n) \Phi^2 dm + \mathcal{E}\left(u_n, \frac{\Phi^2}{u_n}\right) \\ &\leq \frac{d}{dt} \int_X (\ln u_n) \Phi^2 dm + \mathcal{E}[\Phi]. \end{aligned} \tag{4.2}$$

Hence,

$$\int_X \Phi^2 V_n \, dm - \mathcal{E}[\Phi] \leq \frac{d}{dt} \int_X (\ln u_n) \Phi^2 dm. \tag{4.3}$$

First, we integrate between t_1 and t_2, and second, we pass to the limit and use Lemma 2.7 together with Jensen's inequality to obtain inequality (4.1). Then, we are done. □

Theorem 4.2. *Assume that $\lambda_0^{(1-\varepsilon)V} = -\infty$ for some $\varepsilon > 0$. Then, the heat equation* (1.1) *has no nonnegative solution satisfying* (3.3). *Moreover, all nonnegative solutions with the above-mentioned property blow up completely and instantaneously, i.e., $\lim_{n\to\infty} u_n(t,x) = \infty$ for every $t > 0$ and every $x \in X$.*

Proof.

(a) We take $T = \infty$, and, by integrating (4.3) with respect to $t \in (1, +\infty)$, we obtain

$$\int_X \Phi^2 V_n \, dm - \mathcal{E}[\Phi] \leq \frac{1}{t-1} \int_X \ln\left(\frac{u_n(t)}{u_n(1)}\right) \Phi^2 dm,$$

which implies, by using the monotone convergence theorem, that

$$\int_X \Phi^2 V dm - \mathcal{E}[\Phi] \leq \frac{1}{t-1}\left[\int_X \ln(\widetilde{u}(t))\Phi^2 dm - \int_X \ln(\widetilde{u}(1))\Phi^2 dm\right]$$

for every $t > 1$, where $\widetilde{u}(t) = \lim_{n\to\infty} u_n(t)$. In view of Jensen's and Hölder's inequality, along with (3.1), we deduce

$$\begin{aligned}\int_X \Phi^2 V dm - \mathcal{E}[\Phi] &\leq \frac{1}{t-1}\Bigg\{ \ln\left[\left(\int_X \widetilde{u}(t)\, dm\right)^{1/2}\left(\int_X \Phi^4\right)^{1/2}\right] \\ &\qquad\qquad - \int_X \ln(\widetilde{u}(1))\Phi^2 dm\Bigg\} \\ &\leq \frac{1}{2(t-1)}\Bigg\{ \ln\left(\int_X \widetilde{u}^2(t) dm\right) + \ln\left(\int_X \Phi^4 dm\right) \\ &\qquad\qquad - 2\int_X \ln(\widetilde{u}(1))\Phi^2 dm\Bigg\} \\ &\leq \frac{1}{2(t-1)}\Bigg\{ \ln\left(\int_X u^2(t) dm\right) + \ln\left(\int_X \Phi^4 dm\right) \\ &\qquad\qquad - 2\int_X \ln(\widetilde{u}(1))\Phi^2 dm\Bigg\} \\ &\leq \frac{1}{2(t-1)}(2\omega t + 2\ln\|u_0\|_{L^2(X,m)} + 2\ln\|\Phi\|_{L^\infty(X,m)}).\end{aligned}$$

Note that, by letting $t \to +\infty$, we obtain

$$\int_X \Phi^2 V dm - \mathcal{E}[\Phi] \leq \omega.$$

Hence,

$$-\lambda_0^V \leq \omega < \infty. \tag{4.4}$$

(b) Now, let $T < \infty$, and fix $0 < t_1 < t_2 < T$. By the same argument as in **[7]**, we prove that, for any $r > 0$, there exists a $C(r) > 0$ such that

$$\frac{1}{t_2-t_1}\int_X \ln\left(\frac{u(x,t_2)}{u(x,t_1)}\right)\Phi^2 dm \le r\mathcal{E}[\Phi] + C(r)\int_X \Phi^2 dm \tag{4.5}$$

for all $\Phi \in \mathcal{F}_0(X)$. Therefore, for any $\Phi \in \mathcal{F}(X) \cap \mathcal{C}_0(X)$ such that $\int_X \Phi^2 dm = 1$, we have

$$\frac{-C(r)}{1+r} \le \mathcal{E}[\Phi] - (1+r)^{-1}\int_X \Phi^2 V dm. \tag{4.6}$$

Thus, we obtain

$$\lambda_0^{(1+r)^{-1}V} > -\infty \quad \text{for all } r > 0, \tag{4.7}$$

which concludes the first step of the proof.

Concept of continuation. Given $x \in X$ and $t \in (0,T)$, and setting $\rho = \rho(y) = p_{t/2}(x,y)$, note that, by using the same idea as in **[7]**, we prove from the upper bound of the heat kernel that $\ln \rho \in L^p(X,m)$ for any $p > 1$.

Assume that, if there is no $s \in (0, t/2]$ such that $\rho(\cdot)u(\cdot,s) \in L^1(X,m)$, the continuation after blow-up can be studied by writing the standard integration formula

$$\begin{aligned} u(x,t) &= P_{t/2}u\left(x,\frac{t}{2}\right) + \int_{t/2}^{t}\int_X p_s(x,y)V(y)u(y,s)m(dy)\,ds \\ &\ge P_{t/2}u\left(x,\frac{t}{2}\right) \\ &= \int_X p_{t/2}(x,y)u\left(y,\frac{s}{2}\right)m(dy) = \int_X \rho(y)u\left(y,\frac{t}{2}\right)m(dy) = \infty, \end{aligned} \tag{4.8}$$

where $\{P_t, t \ge 0\}$ denotes the transition semigroup on X.

The case $s \in (0, t/2]$ is the only point such that $\rho(\cdot)u(\cdot,s) \in L^1(X,m)$. Using the standard integration formula for the weak solution once again, we obtain

$$\begin{aligned} u\left(x,\frac{(t+s)}{2}\right) &\ge \int_X p_{t/2}(x,y)u\left(y,\frac{s}{2}\right)m(dy) \\ &= \int_X \rho(y)u\left(y,\frac{s}{2}\right)m(dy) = \infty. \end{aligned} \tag{4.9}$$

Furthermore, from the parabolic Harnack inequality for time-dependent locally Dirichlet forms **[25, 27]**, we derive that, for every $x \in X$, every $r > 0$ such that $B_r(x) \subset X$, every $z \in B_r(x)$ and every small $\gamma > 0$, there is a constant $C = c(t, \gamma, r) > 0$ such that

$$p_{(t+\gamma r)/2}(z, y) \geq C p_{t/2}(x, y) \quad \text{for all } y \in X. \tag{4.10}$$

Therefore, for every $z \in B_r(x)$, we have

$$\begin{aligned} u\left(z, \frac{t+s+\gamma r}{2}\right) &\geq P_{(t+\gamma r)/2} u\left(z, \frac{s}{2}\right) \\ &= \int_X p_{(t+\gamma r)/2}(z, y) u\left(y, \frac{s}{2}\right) m(dy) \\ &\geq C \int_X p_{t/2}(x, y) u\left(y, \frac{s}{2}\right) m(dy) \\ &\geq \int_X \rho(y) u\left(y, \frac{s}{2}\right) m(dy) = \infty. \end{aligned} \tag{4.11}$$

Moreover, in view of the standard integration formula for the weak solution (or Duhamel's formula) and the semigroup property (see Definition 2.3 or **[29,** Proposition 2.3]), we have

(4.12)

$$\begin{aligned} u(x,t) &\geq \int_X p_{t-(s/2)}(x, y) u\left(y, \frac{s}{2}\right) m(dy) \\ &= \int_X \int_X p_{(t-s-\gamma r)/2}(x, z) p_{(t+\gamma r)/2}(z, y) u\left(y, \frac{s}{2}\right) m(dz) m(dy) \\ &= \int_X p_{(t-s-\gamma r)/2}(x, z) \left[\int_X p_{(t+\gamma r)/2}(z, y) u\left(y, \frac{s}{2}\right) m(dy) \right] m(dz) \\ &= \int_X p_{(t-s-\gamma r)/2}(x, z) u\left(z, \frac{t+s+\gamma r}{2}\right) m(dz) \\ &\geq \int_{B_r(x)} p_{(t-s-\gamma r)/2}(x, z) u\left(z, \frac{t+s+\gamma r}{2}\right) m(dz). \end{aligned}$$

It follows, since (x, t) is arbitrary, that u blows up, and we obtain the desired result. $\square$

5. Examples. Let $\mathcal{E}^\omega$ be a Dirichlet form defined on $L^2(\omega^2 dm)$ by

$$D(\mathcal{E}^\omega) = \{f : \omega f \in \mathcal{F}\}, \ \mathcal{E}^\omega[f] = \mathcal{E}[\omega f] \quad \text{for all } f \in D(\mathcal{E}^\omega). \tag{5.1}$$

Let ν be a positive radon measure on Borel subset of X and $\mathcal{E}_\nu$ the form defined by

$$D(\mathcal{E}_\nu) = \mathcal{F},\ \mathcal{E}_\nu[f] = \mathcal{E}[f] - \int f^2 d\nu \quad \text{for all } f \in \mathcal{F}. \tag{5.2}$$

Now, let Q^ω take the form

$$\begin{aligned} D(Q^\omega) = \{f : \omega f \in \mathcal{F}\},\ Q^\omega[f] \\ = \mathcal{E}_\nu^\omega[f] - \int f^2\omega^2 dm \quad \text{for all } f \in D(Q^\omega). \end{aligned} \tag{5.3}$$

For more details and elementary properties of the forms $\mathcal{E}_\nu^\omega$ and Q^ω, which remain unexplained or unproven herein, the reader is referred to **[9]**.

Proposition 5.1. *Assume that there is a real positive constant c and a function $\omega > 0$, m almost everywhere such that*

$$\mathcal{E}_\nu[\omega, f] - \int_X f\omega V dm \geq c \int_X f\omega\, d\nu \tag{5.4}$$

for every positive function f, where ν is a positive radon measure on the Borel subset of X. Then, $\lambda_0^V \geq c$.

Proof. First, we define the vector space

$$\mathcal{C}^w := \{f : f \in \mathcal{F} \cap L^\infty(X, m),\ w \in L^2(d\Gamma[f])\},$$

as a core for Q^ω. Recall that λ_0^V is defined as

$$\lambda_0^V = \inf\left\{\frac{\mathcal{E}_\nu^\omega[f] - \int_X f^2\omega^2 V dm}{\int_X f^2\omega^2 d\nu},\ f \in \mathcal{C}^w,\ f \neq 0\right\}. \tag{5.5}$$

By using the expression of Q^ω, we get

$$\begin{aligned} Q^\omega[f] &= \mathcal{E}_\nu^\omega[f] - \int f^2\omega^2 dm \\ &= \int d\Gamma[\omega, \omega f^2] - \int f^2\omega^2 d\nu + \int \omega^2 d\Gamma[f] - \int f^2\omega^2 V dm \\ &= \mathcal{E}_\nu(\omega, \omega f^2) - \int f^2\omega^2 V dm + \int \omega^2 d\Gamma[f]. \end{aligned} \tag{5.6}$$

Ensuring inequality (5.4), we deduce that

$$Q^{\omega}[f] \geq c \int f^2 \omega^2 d\nu \quad \text{for all } f \in \mathcal{C}^w, \tag{5.7}$$

and we are finished. □

Now, we give an example where the heat equation (1.1) has no nonnegative solution.

Lemma 5.2. *Assume that there exist $\lambda > 1$, $r > 0$, a sequence of finite intrinsic balls $B_k = B(0, r/k) \subset X$ and a sequence $\{\phi_k\} \subset \mathcal{F}(X) \cap \mathcal{C}_c(X)$ with $\operatorname{Supp} \phi_k \subset B_k$ satisfying $\int \phi_k^2 m(dx) = 1$ such that*

$$\int \phi_k^2(x) V m(dx) \geq \lambda \mathcal{E}[\phi_k] \quad \textit{for all } k. \tag{5.8}$$

Then, the heat equation (1.1) *has no nonnegative solution.*

Proof. Since V satisfies the condition given in Lemma 5.2, by using the strong Poincaré inequality **[31]**, there exist $\lambda' > 1$ and $\epsilon \in (0, 1)$ such that

(5.9)

$$\begin{aligned} &- \mathcal{E}[\phi_k] + (1 - \epsilon) \int \phi_k^2(x) V m(dx) \geq (\lambda' - 1)\mathcal{E}[\phi_k] \\ &\quad \geq c(\lambda' - 1)\left(\frac{k}{r}\right)^2 \int_{B_k} \left|\phi_k - \frac{1}{m(B_k)} \int_{B_k} \phi_k dm\right|^2 dm \quad \text{for all } k. \end{aligned}$$

Hence,

$$\lambda_0^{(1-\epsilon)V}(B_k) \leq -c\left(\frac{k}{r}\right)^2 \longrightarrow -\infty, \quad \text{as } k \to \infty.$$

On the other hand, by using the inequality

$$\lambda_0^{(1-\epsilon)V}(B_k) \geq \lambda_0^{(1-\epsilon)V}, \tag{5.10}$$

we achieve the result. □

Acknowledgments. I thank the referee for his/her time and comments.

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