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DIFFERENTIAL SUBORDINATION OF A HARMONIC MEAN TO A LINEAR FUNCTION

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ABSTRACT. In this paper, we examine differential subordination related to the harmonic mean in the case where a dominant is a linear function. A result for the first order Euler differential subordination of the nonlinear type is also discussed.

1. Introduction. Let $\mathcal{H}$ be the class of all analytic functions f in $\mathbf{D} := \mathbf{D}_1$, where $\mathbf{D}_r := \{z \in \mathbf{C} : |z| < r\}$ for each $r > 0$. A function $f \in \mathcal{H}$ is said to be *subordinate* to a function $F \in \mathcal{H}$ if there exists an $\omega \in \mathcal{H}$ such that $\omega(0) := 0$, $\omega(\mathbf{D}) \subset \mathbf{D}$ and $f = F \circ \omega$ in $\mathbf{D}$. Then, we write $f \prec F$. When F is univalent, then

$$f \prec F \iff (f(0) = F(0) \wedge f(\mathbf{D}) \subset F(\mathbf{D})). \tag{1.1}$$

Let $\beta \in [0,1]$ and a, $b \in \mathbf{C}$. When $b + \beta(b-a) \neq 0$, the harmonic mean of a and b is given as

$$\frac{ab}{b+\beta(a-b)}.$$

Definition 1.1. Let $\beta \in (0,1)$ and $\Psi \in \mathcal{H}$. By $\mathcal{H}(\beta, \Psi)$, we denote the subclass of $\mathcal{H}$ of all nonconstant functions p such that the function

$$\mathbf{D} \ni z \longmapsto \frac{p(z)(p(z)+zp'(z)\Psi(z))}{p(z)+(1-\beta)zp'(z)\Psi(z)}$$

is either analytic or has only removable singularities with an analytic extension on $\mathbf{D}$.

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In **[2, 3]**, studies of the differential subordination related to the harmonic mean were begun. Particularly, in **[3]** for $\beta \in (0,1]$, $\Psi \in \mathcal{H}$, $p \in \mathcal{H}(\beta, \Psi)$ and a univalent function $h \in \mathcal{H}$, the differential subordination related to the harmonic mean of the type

$$\frac{p(z)(p(z) + zp'(z)\Psi(z))}{p(z) + (1-\beta)zp'(z)\Psi(z)} \prec h(z), \quad z \in \mathbf{D}, \tag{1.2}$$

was examined. The above formula with $\beta := 1/2$ and selected functions Ψ and h was also considered in **[7]**. A univalent function $q \in \mathcal{H}$ is called a *dominant of differential subordination* (1.2) if $p \prec q$ for all functions $p \in \mathcal{H}(\beta, \Psi)$ satisfying (1.2). A dominant $\widetilde{q}$ of (1.2) is called the *best dominant* of (1.2) if $\widetilde{q} \prec q$ for all dominants q of (1.2) (for the general case, see **[11,** page 16**]**).

We recall that differential subordinations related to the arithmetic mean as well as to the geometric mean have been studied by various authors. Given $\alpha \in [0,1]$, the simplest form of the differential subordination related to the arithmetic mean is given as follows:

$$p(z) + \alpha z p'(z) \prec h(z), \quad z \in \mathbf{D}.$$

For details and further references, see **[11,** pages 121–131**]**. The differential subordination related to the geometric mean was introduced in **[6]**. For further results in this direction, see e.g., **[1, 4, 8, 9, 10]**.

A function $f \in \mathcal{H}$ is said to be convex if it is univalent and $f(\mathbf{D})$ is a convex domain.

Here, we introduce the subclass $\mathcal{Q}$ (for details on corners of curves, see e.g., **[12,** pages 51–65**]**). Let $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$.

Definition 1.2. By $\mathcal{Q}$, we denote the class of convex functions h with the following properties:

(a) $h(\mathbf{D})$ is bound by finitely many smooth arcs which form corners at their end points (including corners at infinity);

(b) $E(h)$ is the set of all points $\zeta \in \mathbf{T}$ which corresponds to corners $h(\zeta)$ of $\partial h(\mathbf{D})$;

(c) $h'(\zeta) \neq 0$ exists at every $\zeta \in \mathbf{T} \setminus E(h)$.

In **[3]**, the following was shown:

Theorem 1.3. *Let* $\beta \in (0,1]$, $h \in \mathcal{Q}$ *with* $0 \in \overline{h(\mathbf{D})}$ *and* $\Psi \in \mathcal{H}$ *be such that*

$$\operatorname{Re}\Psi(z) \geq 0, \qquad \Psi(z) \neq 0, \quad z \in \mathbf{D}.$$

If $p \in \mathcal{H}(\beta, \Psi)$, $p(0) = h(0)$ *and*

$$\frac{p(z)(p(z) + zp'(z)\Psi(z))}{p(z) + (1-\beta)zp'(z)\Psi(z)} \prec h(z), \quad z \in \mathbf{D},$$

then

$$p \prec h.$$

In this paper, we continue the research on differential subordination of the form (1.2) when Ψ is a constant function and h is a linear function. The linearity of h is the simplest, however especially interesting, case to study. It shows that Theorem 1.3 can essentially be improved, which leads to consideration of other selected convex functions h for comparison with the general case. Moreover, the differential subordination of the harmonic mean with a linear function h generalizes the first order Euler differential subordination (see **[11,** pages 334–336]) for the nonlinear case. Corollary 2.8 contains such a generalization and the importance of the result being sharp, i.e., with the best dominant.

The next lemma is a special case of **[11,** page 22, Lemma 2.2d] and is necessary for the proof of the main result.

Lemma 1.4. *Let* $h \in \mathcal{Q}$ *and* $p \in \mathcal{H}$ *be nonconstant functions with* $p(0) := h(0)$. *If* p *is not subordinate to* h, *then there exist* $z_0 \in \mathbf{D} \setminus \{0\}$ *and* $\zeta_0 \in \mathbf{T} \setminus E(h)$ *such that*

$$p(\mathbf{D}_{|z_0|}) \subset h(\mathbf{D}); \tag{1.3}$$

$$p(z_0) = h(\zeta_0) \tag{1.4}$$

and, for some $m \geq 1$,

$$z_0 p'(z_0) = m\zeta_0 h'(\zeta_0). \tag{1.5}$$

2. Main result. Given $\alpha \in \mathbf{C} \setminus \{0\}$, let $\Psi(z) := \alpha$, $z \in \mathbf{D}$. Let $\mathcal{H}(\beta, \alpha) := \mathcal{H}(\beta, \Psi)$.

Theorem 2.1. *Let $a \geq 0$ and $\alpha, \beta \in (0,1]$. Let $M \geq a$ when $a > 0$ and $M > 0$ when $a = 0$. If $p \in \mathcal{H}(\beta, \alpha)$, $p(0) := a$, and*

$$\left|\frac{p(z)(p(z)+\alpha z p'(z))}{p(z)+(1-\beta)\alpha z p'(z)} - a\right| < M\frac{(1+\alpha\beta)a+(1+\alpha)M}{a+(1+(1-\beta)\alpha)M}, \quad z \in \mathbf{D}, \tag{2.1}$$

then

$$|p(z) - a| < M, \quad z \in \mathbf{D}. \tag{2.2}$$

Proof. Let $h(z) := a + Mz$, $z \in \mathbf{D}$. Since h is univalent, $p(0) = h(0) = a$, and (2.2) can be replaced by the inclusion $p(\mathbf{D}) \subset h(\mathbf{D})$; by using (1.1), condition (2.2) is equivalent to the subordination $p \prec h$.

Suppose, on the contrary, that p is not subordinate to h. Since $h \in \mathcal{Q}$ with $E(h) = \emptyset$, by Lemma 1.4, there exist $z_0 \in \mathbf{D} \setminus \{0\}$ and $\zeta_0 \in \mathbf{T}$ such that (1.3)–(1.5) hold. Thus,

$$p(z_0) = a + M\zeta_0$$

and, for some $m \geq 1$,

$$z_0 p'(z_0) = mM\zeta_0.$$

Hence,

$$\begin{aligned}
&\left|\frac{p(z_0)(p(z_0)+\alpha z_0 p'(z_0))}{p(z_0)+(1-\beta)\alpha z_0 p'(z_0)} - a\right| \\
&\qquad = \left|\frac{p^2(z_0)+\alpha p(z_0) z_0 p'(z_0) - a p(z_0) - a(1-\beta)\alpha z_0 p'(z_0)}{p(z_0)+(1-\beta)\alpha z_0 p'(z_0)}\right| \\
&\qquad = \left|\frac{(+M\zeta_0)^2+\alpha(a+M\zeta_0)mM\zeta_0-a^2-aM\zeta_0-a(1-\beta)\alpha mM\zeta_0}{a+M\zeta_0+(1-\beta)\alpha mM\zeta_0}\right| \\
&\qquad = \left|\frac{aM\zeta_0+M^2\zeta_0^2+\alpha mM^2\zeta_0^2+a\alpha\beta mM\zeta_0}{a+(1+(1-\beta)\alpha m)M\zeta_0}\right| \\
&\qquad = M\left|\frac{a+a\beta\alpha m+(1+\alpha m)M\zeta_0}{a+(1+(1-\beta)\alpha m)M\zeta_0}\right|.
\end{aligned} \tag{2.3}$$

Since $M \geq a \geq 0$, thus for $\beta \in (0,1)$, we have

$$\begin{aligned}
|a+(1+(1-\beta)\alpha m)Mz| &\geq |a-(1+(1-\beta)\alpha m)M|z|| \\
&= M - a + (1-\beta)\alpha mM > 0, \quad z \in \mathbf{T}.
\end{aligned}$$

Therefore, for each $m \geq 1$ and $\beta \in (0,1)$, we can define

$$q_m(z) := \frac{a + a\beta\alpha m + (1+\alpha m)Mz}{a + (1 + (1-\beta)\alpha m)Mz}, \quad z \in \mathbf{T}.$$

The case $\beta = 1$ reduces to

$$q_m(z) = 1 + \alpha m, \quad z \in \mathbf{T}.$$

Since q_m is a linear-fractional mapping having real coefficients, $q_m(\mathbf{T})$ is a circle symmetrical with respect to the real axis. Moreover,

$$q_m(1) = \frac{a+a\beta\alpha m+(1+\alpha m)M}{a+(1+(1-\beta)\alpha m)M} < \frac{a+a\beta\alpha m-(1+\alpha m)M}{a-(1+(1-\beta)\alpha m)M} = q_m(-1).$$

Indeed, the above inequality, after simplifying, is equivalent to the inequality

$$aM[(1+\alpha\beta m)(1+(1-\beta)\alpha m) + 1 + \alpha m] > 0.$$

Thus,

$$|q_m(z)| \geq q_m(1) = \frac{a + M + \alpha(a\beta + M)m}{a + M + (1-\beta)\alpha M m}, \quad z \in \mathbf{T}. \tag{2.4}$$

Observe now that the function $r(m) := q_m(1)$, $m \geq 1$, is increasing since

$$r'(m) = \frac{\alpha\beta(a+M)^2}{(a + M + (1-\beta)\alpha M m)^2} > 0, \quad m \geq 1.$$

Consequently, by (2.4), we have

$$|q_m(z)| \geq r(m) \geq r(1) = \frac{(1+\alpha\beta)a + (1+\alpha)M}{a + (1 + (1-\beta)\alpha)M}, \quad z \in \mathbf{T}.$$

Since, particularly, the above inequality holds for $z := \zeta_0$, by (2.3), we have:

$$\begin{aligned}\left|\frac{p(z_0)(p(z_0) + \alpha z_0 p'(z_0))}{p(z_0) + (1-\beta)\alpha z_0 p'(z_0)} - a\right| &= M\left|\frac{a + a\beta\alpha m + (1+\alpha m)M\zeta_0}{a + (1 + (1-\beta)\alpha m)M\zeta_0}\right| \\ &\geq M\frac{(1+\alpha\beta)a + (1+\alpha)M}{a + (1 + (1-\beta)\alpha)M},\end{aligned}$$

which contradicts (2.1) and concludes the proof. $\square$

Remark 2.2. Under the assumptions on a and M, $0 \in \overline{h(\mathbf{D})}$. Moreover,

$$\frac{(1+\alpha\beta)a+(1+\alpha)M}{a+(1+(1-\beta)\alpha)M} \geq 1.$$

Therefore, Theorem 1.3 follows from Theorem 2.1 for such chosen Ψ and h.

Theorem 2.1 produces a sequence of corollaries, listed below. For $\alpha = 1$, we obtain:

Corollary 2.3. *Let $a \geq 0$ and $\beta \in (0,1]$. Let $M \geq a$ when $a > 0$ and $M > 0$ when $a = 0$. If $p \in \mathcal{H}(\beta, 1)$, $p(0) := a$ and*

$$\left|\frac{p(z)(p(z)+zp'(z))}{p(z)+(1-\beta)zp'(z)} - a\right| < M\frac{(1+\beta)a+2M}{a+(2-\beta)M}, \quad z \in \mathbf{D},$$

then

$$|p(z)-a| < M, \quad z \in \mathbf{D}.$$

For $a = M$, we have

Corollary 2.4. *Let $M > 0$ and $\alpha, \beta \in (0,1]$. If $p \in \mathcal{H}(\beta, \alpha)$, $p(0) := M$ and*

$$\left|\frac{p(z)(p(z)+\alpha zp'(z))}{p(z)+(1-\beta)\alpha zp'(z)} - M\right| < M\frac{2+(1+\beta)\alpha}{2+(1-\beta)\alpha}, \quad z \in \mathbf{D},$$

then

$$|p(z)-M| < M, \quad z \in \mathbf{D}.$$

The case $M = 1$ yields

Corollary 2.5. *Let α, $\beta \in (0,1]$. If $p \in \mathcal{H}(\beta, \alpha)$, $p(0) := 1$ and*

$$\left|\frac{p(z)(p(z)+\alpha zp'(z))}{p(z)+(1-\beta)\alpha zp'(z)} - 1\right| < \frac{2+(1+\beta)\alpha}{2+(1-\beta)\alpha}, \quad z \in \mathbf{D},$$

then

$$|p(z)-1| < 1, \quad z \in \mathbf{D}.$$

The cases $a = M$ and $\alpha = 1$ are as follows.

Corollary 2.6. *Let $M > 0$ and $\beta \in (0, 1]$. If $p \in \mathcal{H}(\beta, 1)$, $p(0) := M$ and*

$$\left|\frac{p(z)(p(z) + zp'(z))}{p(z) + (1 - \beta)zp'(z)} - M\right| < M\frac{3+\beta}{3-\beta}, \quad z \in \mathbf{D},$$

then

$$|p(z) - M| < M, \quad z \in \mathbf{D}.$$

Now, we deal with the case $a = 0$. We shall discuss the best dominant for this case, also. We rewrite Theorem 2.1 in terms of the differential subordination.

Corollary 2.7. *Let α, $\beta \in (0, 1]$ and $M > 0$. If $p \in \mathcal{H}(\beta, \alpha)$, $p(0) := 0$ and*

$$\frac{p(z)(p(z) + \alpha zp'(z))}{p(z) + (1 - \beta)\alpha zp'(z)} \prec Mz, \quad z \in \mathbf{D}, \tag{2.5}$$

then

$$p(z) \prec \left(1 - \frac{\alpha\beta}{1+\alpha}\right)Mz =: \widetilde{q}(z), \quad z \in \mathbf{D}. \tag{2.6}$$

Moreover, the function $\widetilde{q}$ is the best dominant of (2.5).

Proof. It remains to find the best dominant of (2.5). Toward this end, we will find the univalent solution q of the differential equation

$$\frac{q(z)(q(z) + \alpha zq'(z))}{q(z) + (1 - \beta)\alpha zq'(z)} = Mz, \quad z \in \mathbf{D}, \tag{2.7}$$

such that $q(0) := 0$. We use the technique of power series to find an analytic solution of (2.7) of the form

$$q(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbf{D}. \tag{2.8}$$

Since q should be univalent, so

$$a_1 = q'(0) \neq 0. \tag{2.9}$$

From (2.7), equivalently, we have

$$\alpha z q'(z)(q(z) - (1-\beta)Mz) = Mzq(z) - q^2(z), \quad z \in \mathbf{D}.$$

Placing into the above equality the series from (2.8), we obtain

$$\begin{aligned}\alpha(a_1 z + 2a_2 z^2 + 3a_3 z^3 + \cdots)&((a_1-(1-\beta)M)z + a_2 z^2 + a_3 z^3 + \cdots)\\ &= (Ma_1 - a_1^2)z^2 + (Ma_2 - 2a_1 a_2)z^3\\ &\qquad + (Ma_3 - (2a_1a_3 + a_2^2))z^4 + \cdots, \quad z \in \mathbf{D},\end{aligned}$$

i.e., equivalently,

$$\begin{aligned}(\alpha a_1(a_1 - (1-\beta)M))z^2 &+ \alpha(a_1 a_2 + 2a_2(a_1 - (1-\beta)M))z^3\\ &+ \alpha(a_1a_3 + 2a_2^2 + 3a_3(a_1 - (1-\beta)M))z^4 + \cdots\\ = (Ma_1 - a_1^2)z^2 &+ (Ma_2 - 2a_1a_2)z^3\\ &+ (Ma_3 - 2a_1a_3 + a_2^2)z^4 + \cdots, \quad z \in \mathbf{D}.\end{aligned} \tag{2.10}$$

Comparing the first coefficients, we have

$$\alpha a_1[a_1 - (1-\beta)M] = Ma_1 - a_1^2.$$

Hence, $a_1 = 0$, which contradicts (2.9), or

$$a_1 = \left(1 - \frac{\alpha\beta}{1+\alpha}\right)M. \tag{2.11}$$

Comparing the second coefficients in (2.10), we obtain

$$\alpha[a_1a_2 + 2a_2(a_1 - (1-\beta)M]] = Ma_2 - 2a_1a_2.$$

Hence,

$$a_1 = \frac{1 + 2\alpha(1-\beta)}{3\alpha + 2}M,$$

which contradicts (2.11), or $a_2 = 0$. Comparing the third coefficients in (2.10), and taking into account that $a_2 = 0$, we have

$$a_1 = \frac{1 + 3\alpha(1-\beta)}{4\alpha + 2}M,$$

which contradicts (2.11) or $a_3 = 0$. Summarizing, in this way, we show that $a_n = 0$ for $n \geq 2$. Thus, from (2.8) and (2.11), it follows that

$$q(z) = a_1 z = \left(1 - \frac{\alpha\beta}{1+\alpha}\right)Mz, \quad z \in \mathbf{D}.$$

Since q is univalent and, by (2.7), it satisfies (2.6) with $p := q$, it follows that q is the best dominant of (2.5). Clearly, $q = \widetilde{q}$. $\square$

Using the notation of [**11,** page 336], let

$$E^{(1)}[p](z) := p(z) + zp'(z), \quad p \in \mathcal{H},\ z \in \mathbf{D}.$$

Now, the cases $\alpha = 1$ and $M = 1$ in Corollary 2.7 are as follows.

Corollary 2.8. *Let $\beta \in (0,1]$. If $p \in \mathcal{H}(\beta, 1)$, $p(0) := 0$ and*

$$\frac{p(z)E^{(1)}[p](z)}{\beta p(z) + (1-\beta)E^{(1)}[p](z)} \prec z, \quad z \in \mathbf{D}, \tag{2.12}$$

then

$$p(z) \prec \left(1 - \frac{\beta}{2}\right) z, \quad z \in \mathbf{D}.$$

Moreover, the function $\widetilde{q}(z) := (1-\beta/2)z$, $z \in \mathbf{D}$, is the best dominant of (2.12).

Formula (2.12) generalizes the first order Euler differential subordination which holds for $\beta = 1$ (see [**11,** pages 334–336]) for the nonlinear case. Thus, the above corollary generalizes the well-known result for the first order Euler differential subordination recalled below (see [**11,** page 335]).

Corollary 2.9. *If $p \in \mathcal{H}$, $p(0) := 0$ and*

$$E^{(1)}[p](z) \prec z, \quad z \in \mathbf{D}, \tag{2.13}$$

then

$$p(z) \prec \frac{z}{2}, \quad z \in \mathbf{D}.$$

Moreover, the function $\widetilde{q}(z) := z/2$, $z \in \mathbf{D}$, is the best dominant of (2.13).

Corollary 2.9 is also a special case of the well-known Hallenbeck and Ruscheweyh result [**5**].

To conclude, we write the case $a = 0$, $\alpha = 1$ and $\beta = 1$ (see also [**11,** page 36]).

Corollary 2.10. *Let $M > 0$. If $p \in \mathcal{H}$, $p(0) := 0$ and*

$$|p(z) + zp'(z)| < 2M, \quad z \in \mathbf{D},$$

then

$$|p(z)| < M, \quad z \in \mathbf{D}.$$

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