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ON RIEMANNIAN SURFACES WITH CONICAL SINGULARITIES

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ABSTRACT. The geometry of closed surfaces of genus $g \geq 2$ equipped with a Riemannian metric of variable bounded curvature with finitely many conical points is studied. The main result is that the set of closed geodesics is dense in the space of geodesics.

1. Introduction. Let S be a closed surface of genus ≥ 2 equipped with a Riemannian metric with finitely many *conical singularities* (or *conical points*), denoted by $s_1, \ldots, s_n$. See Section 2 for a precise definition. Such a surface S is called a Riemannian surface with conical singularities and will be written Rscs for brevity. Denote by $\theta(s_i)$ the angle at each s_i, with $\theta(s_i) \in (0, +\infty) \setminus \{2\pi\}$, and denote by $C(S)$ the set $\{s_1, \ldots, s_n\}$.

Examples of Rscs with at least one angle $\theta(s_i) < 2\pi$ include the so-called half translation surfaces where the conical singularities are of angle $k\pi$, $k \geq 1$, see **[10]**. In general, for $g \geq 2$, Rscs with zero curvature are constructed in **[8]** and Rscs of variable curvature are constructed in **[9]**.

An important property which fails even for flat or non-positively curved Rscs, provided at least one conical point with angle $\theta(s_i) < 2\pi$ exists, is that geodesic segments with specified homotopy class and endpoints are no longer unique, and similarly for geodesic rays and lines in the universal cover $\widetilde{S}$ (see Example 2.5). Moreover, extension of geodesics also fails. More precisely, there exist geodesic segments σ in $\widetilde{S}$ not containing any singularity in their interior which cannot

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be extended to any geodesic segment σ' properly containing σ. These facts make the study of the geometry of $\widetilde{S}$ interesting.

In this note, under the assumption that there exists at least one angle $\theta(s_i) < 2\pi$, we first show that the set of points on the boundary $\partial\widetilde{S}$ to which there corresponds more than one geodesic ray is dense in $\partial\widetilde{S}$ (see Propositions 3.1 and 3.3). Moreover, it is shown that, for any boundary point $\xi \in \partial\widetilde{S}$, there exists a base point $\widetilde{x}_0 \in \widetilde{S}$ such that at least two geodesic rays emanating from $\widetilde{x}_0$ correspond to ξ. We then show that the images of all geodesic rays corresponding to a boundary point $\xi \in \partial\widetilde{S}$ are contained in a convex subset of $\widetilde{S}$ whose boundary is geodesic consisting of two geodesic rays. Thus, in the class of geodesic rays corresponding to each point $\xi \in \partial\widetilde{S}$, there are associated two distinct outermost (left and right) geodesic rays. Similarly, for every pair of points $\xi, \eta \in \partial\widetilde{S}$ which are joined by more than one geodesic line, there are associated two distinct outermost (left and right) geodesic lines which bound a convex subset of $\widetilde{S}$ containing all geodesic lines joining ξ, η.

We next show that the set of closed geodesics is dense in the space of all geodesics GS in the following sense: for each pair of distinct points $\xi, \eta \in \partial\widetilde{S}$ and each outermost geodesic line γ joining them, there exists a sequence of geodesics $\{c_n\}$ in $\widetilde{S}$, with the projection of every c_n to S being a closed geodesic, such that $\{c_n\}$ converges in the usual uniform sense on compact sets to γ.

The line of arguments is fairly simple; the crucial tools are the facts that S is a geodesic space and the universal cover $\widetilde{S}$ is hyperbolic in the sense of Gromov.

2. Preliminaries. We write $C(v, \theta)$ for the cone with vertex v and angle θ, namely, $C(v, \theta)$ is the set $\{(r, t) : 0 \leq r, t \in \mathbb{R}/\theta\mathbb{Z}\}$ equipped with the metric

$$ds^2 = e^{2u}(dr^2 + r^2 dt^2),$$

where u is a continuous function on $C(v, \theta)$ and of class C^2 on $C(v, \theta) \setminus \{v\}$.

Definition 2.1. A *Riemanninan surface with conical singularities* $s_1, \ldots, s_n$, $n \geq 1$, is a closed surface S equipped with

- a smooth Riemannian metric at every point $p \in S\backslash\{s_1, \dots, s_n\}$, and
- each $s_i \in \{s_1, \dots, s_n\}$ is a conical singularity of angle $\theta(s_i)$, that is, there exists a neighborhood U of s_i isometric to a neighborhood of a vertex v of the cone $C(v, \theta(s_i))$.

From now on, we assume that the angle $\theta(s_i)$ of at least one conical point s_i satisfies $\theta(s_i) \in (0, 2\pi)$. We comment on what occurs if all conical angles are $> 2\pi$ at the beginning of Section 3 below.

The metric on S is the induced length metric and will be denoted by $d(\cdot, \cdot)$. S with the metric d locally compact and complete, hence, a geodesic metric space. Let $\widetilde{S}$ be the universal covering of S, and let $p : \widetilde{S} \to S$ be the universal covering projection. Obviously, the universal covering $\widetilde{S}$ is homeomorphic to $\mathbb{R}^2$ and, by requiring p to be a local isometric map, we may lift d to a metric $\widetilde{d}$ on $\widetilde{S}$ so that $(\widetilde{S}, \widetilde{d})$ becomes an Rscs. Clearly, $\pi_1(S)$ is a discrete group of isometries of $\widetilde{S}$ acting freely on $\widetilde{S}$ such that $S = \widetilde{S}/\pi_1(S)$.

A geodesic γ in a surface S is usually defined to be a local isometric map. However, in our setup, a local isometry may have homotopically trivial self intersections, that is, there exist $t_1, t_2 \in \mathbb{R}$ with

$$\gamma(t_1) = \gamma(t_2)$$

such that the loop $\gamma|_{[t_1,t_2]}$ is contractible. In fact, it can be shown that any local isometric map whose image has sufficiently small distance from a conical point with angle $< \pi$ has a self intersection. Clearly, any lift $\widetilde{\gamma}$ to the universal cover $\widetilde{S}$ of S of a local geodesic γ with homotopically trivial self intersections is not a global isometric map. In view of this, we restrict our attention to geodesics and geodesic segments which do not have homotopically trivial self intersections.

Let GS be the space of all local isometric maps $\gamma : \mathbb{R} \to S$ so that its lift to the universal cover $\widetilde{S}$ is a global isometry. The image of such a γ will be referred to as a geodesic in S. Similarly, we define the notion of a geodesic segment, that is, a local isometric map whose domain is a closed interval which lifts to an isometry in $\widetilde{S}$. The image of any geodesic γ in S satisfies

$$\mathrm{Im}\widetilde{\gamma} \cap \{s_i \mid \theta(s_i) < 2\pi\} = \emptyset. \tag{2.1}$$

The group $\pi_1(S)$ with the word metric is *hyperbolic* in the sense of Gromov. On the other hand, $\pi_1(S)$ acts co-compactly on $\widetilde{S}$ by isometries; this implies that $\widetilde{S}$ is itself a hyperbolic space in the sense of Gromov (see, for example, [**4,** Theorem 4.1]), which is complete and locally compact. Hence, $\widetilde{S}$ is a proper space, i.e., each closed ball in $\widetilde{S}$ is compact (see [**6,** Theorem 1.10]). Therefore, the visual boundary $\partial\widetilde{S}$ of $\widetilde{S}$ is defined by means of geodesic rays and is homeomorphic to $\mathbb{S}^1$ (see [**4,** page 19]).

We will need a precise description of the open ball $D(\widetilde{x}_0, R)$ with center $\widetilde{x}_0$ and radius $R > 0$: the open ball $D(\widetilde{x}_0, R)$ is homeomorphic to an open 2-disk with k holes, $k \geq 0$, removed. Each hole corresponds to a conical point of angle $< \pi$. If

$$R = d(\widetilde{x}_0, \widetilde{s}_i) \tag{2.2}$$

for some conical point $\widetilde{s}_i$ with angle $< \pi$, then the corresponding hole is merely the point $\widetilde{s}_i$. Clearly, each component of

$$\partial D(\widetilde{x}_0, R) = \overline{D(\widetilde{x}_0, R)} \setminus D(\widetilde{x}_0, R)$$

is homeomorphic to either a circle or a point (if equation (2.2) holds for some $\widetilde{s}_i$). Amongst all components of $\partial D(\widetilde{x}_0, R)$, exactly one contains $\widetilde{x}_0$ in its interior, i.e., the bounded one of the two components of $\widetilde{S}$ determined by the component. We will call this component the *principal component* and denote it by $\partial_0 D(\widetilde{x}_0, R)$. Observe that the arc length of $\partial_0 D(\widetilde{x}_0, R)$ tends to ∞ as $R \to \infty$. Each of the rest of the components is contained in a neighborhood of a conical point with angle $< 2\pi$, and its arc length is bounded by a number independent of R.

Lemma 2.2. *If two geodesic segments σ_1, σ_2 in $\widetilde{S}$ intersect at two interior points x and y such that x and y are isolated in $\sigma_1 \cap \sigma_2$, then both x and y are conical points with angle $> 2\pi$. If $\sigma_1 \cap \sigma_2$ is a closed segment, then its endpoints are conical points with angle $> 2\pi$. The same results hold for homotopic* (*with endpoints fixed*) *geodesic segments in S.*

Proof. Let $\sigma_1 = [w_1, z_1]$ and $\sigma_2 = [w_2, z_2]$ be two geodesic segments in $\widetilde{S}$ intersecting at two points x and y, which are isolated in $\sigma_1 \cap \sigma_2$. Clearly, $\sigma_1|_{[x,y]} \cup \sigma_2|_{[y,z_2]}$ realizes the distance from x to z_2. Therefore, the angle formed by $\sigma_1|_{[x,y]}$, $\sigma_2|_{[y,z_2]}$ at y is at least π. Similarly, the angle formed by $\sigma_2|_{[x,y]}$, $\sigma_1|_{[y,z_1]}$ at y is at least π; hence, $\theta(y) > 2\pi$. $\square$

Since $\widetilde{S}$ is a hyperbolic space in the sense of Gromov, the isometries of $\widetilde{S}$ are classified as elliptic, parabolic and hyperbolic [**5**]. On the other hand, $\pi_1(S)$ is a hyperbolic group; thus, $\pi_1(S)$ does not contain parabolic elements with respect to its action on its Cayley graph (see [**4,** Theorem 3.4]). From this, it follows that all elements of $\pi_1(S)$ are hyperbolic isometries of $\widetilde{S}$. Therefore, for each $\varphi \in \pi_1(S)$ and each $x \in \widetilde{S}$, the sequence $\varphi^n(x)$ (respectively, $\varphi^{-n}(x)$) has a limit point $\varphi(+\infty)$ (respectively, $\varphi(-\infty)$) when $n \to +\infty$ and $\varphi(+\infty) \neq \varphi(-\infty)$. The point $\varphi(+\infty)$ is called *attractive* and the point $\varphi(-\infty)$ a *repulsive* point of φ.

The following important property for hyperbolic spaces (see [**4,** Proposition 2.1]) holds for $\partial\widetilde{S}$.

Proposition 2.3. *For every pair of points $x \in \widetilde{S}$ and $\xi \in \partial\widetilde{S}$* (*respectively, $\eta, \xi \in \partial\widetilde{S}$*)*, there is a geodesic ray $r : [0,\infty) \to \widetilde{S} \cup \partial\widetilde{S}$* (*respectively, a geodesic line $\gamma : (-\infty,\infty) \to \widetilde{S} \cup \partial\widetilde{S}$*) *such that $r(0) = x$, $r(\infty) = \xi$* (*respectively, $\gamma(-\infty) = \eta$, $\gamma(\infty) = \xi$*)*.*

Uniqueness does not hold in the above proposition. In fact, we have the following straightforward corollary to Lemma 2.2.

Corollary 2.4. *If a geodesic segment intersects a geodesic ray* (*respectively, a line*) *at two isolated points as in Lemma* 2.2*, then there exist two distinct geodesic rays* (*respectively, lines*) *defining the same point* (*respectively, points*) *at infinity.*

Thus, for each pair of points $x \in \widetilde{S}$ and $\xi \in \partial\widetilde{S}$, there corresponds a class of geodesic rays r with $r(0) = x$ and $r(\infty) = \xi$, the cardinality of which varies from a singleton to uncountable (see the discussion following Example 2.5). It is well known that, in hyperbolic metric spaces, any two asymptotic geodesic rays r_1 and r_2 are at uniformly bounded distance, that is, there exists a constant $A > 0$ which depends only upon the hyperbolicity constant of $\widetilde{S}$ such that, for all $t \in [0,+\infty)$,

$$d(r_1(t), r_2(t)) < A, \tag{2.3}$$

see [**2,** Lemma 3.3].

For a point $\xi \in \partial\widetilde{S}$ (and having fixed a base point in $\widetilde{S}$) we write $r \in \xi$ to indicate that the geodesic ray r belongs to the class of rays corresponding to ξ, that is, $r(\infty) = \xi$. We also say that ξ is the positive point of r. Similarly, for a pair (η, ξ) of points in $\partial\widetilde{S}$ with $\eta \neq \xi$, we write $\gamma \in (\eta, \xi)$ to indicate that the geodesic line γ belongs to the class of lines with the property $\gamma(-\infty) = \eta$ and $\gamma(\infty) = \xi$. We say that ξ is the positive point of γ and η the negative.

By writing that the sequence $\{\xi_n\} \subset \partial\widetilde{S}$ converges to ξ in the visual metric, notation $\xi_n \to \xi$, we mean that there exist geodesic rays $r_n \in \xi_n$ and $r \in \xi$ such that the sequence $\{r_n\}$ converges in the usual uniform sense on compact sets to r.

The next example demonstrates a simple case where lifts of distinct closed geodesics (as well as non-closed geodesics) have the same negative and positive points in $\partial\widetilde{S}$.

Example 2.5. Consider the genus 0 surface Σ obtained from the flat figures $ACEBZDA$ and $AC'E'B'Z'D'A$ by identifying AB_2C with $AB_2'C'$, AB_1D with $AB_1'D'$ and EBZ with $E'B'Z'$ (see Figure 1). The resulting cylinder Σ has two singular points A, B with angles $\theta(A) = \pi$ and $\theta(B) = 3\pi$. The segments BB_1 and $B_1'B'$ give rise to a simple closed geodesic σ in Σ. Similarly, the segments BB_2 and $B_2'B'$ give rise to a simple closed geodesic τ in Σ. Both σ and τ contain B, and their union bounds a convex subset of Σ with the same homotopy type as Σ.

At every point $\neq A$ and B, Σ has a flat (Euclidean) structure and, since Σ has geodesic boundaries, the described example can clearly occur in surfaces of any genus. Choose a lift $\widetilde{\sigma}$ of σ in $\widetilde{\Sigma}$. Then, there is a countable number of points $\widetilde{B_i}$, $i \in \mathbb{Z}$, with the properties $\widetilde{B_i} \in \mathrm{Im}\widetilde{\sigma}$ and $p(\widetilde{B_i}) = B$. Clearly, any lift $\widetilde{\tau}$ of τ containing $\widetilde{B_{i_0}}$ for some i_0 must contain $\widetilde{B_i}$ for all i and, moreover, $\widetilde{\tau}(+\infty) = \widetilde{\sigma}(+\infty)$ and $\widetilde{\tau}(-\infty) = \widetilde{\sigma}(-\infty)$. Therefore, using σ and τ, we may construct countably many pairwise distinct closed geodesics in Σ, as well as uncountably many non-closed geodesics, whose lifts in $\widetilde{\Sigma}$ are contained in $\mathrm{Im}\widetilde{\tau} \cup \mathrm{Im}\,\widetilde{\sigma}$, and they all share the same positive (respectively, negative) point $\widetilde{\tau}(+\infty) = \widetilde{\sigma}(+\infty)$ (respectively, $\widetilde{\tau}(-\infty) = \widetilde{\sigma}(-\infty)$).

The *limit set* Λ of $\pi_1(S)$ is defined to be $\Lambda = \overline{\pi_1(S)x} \cap \partial\widetilde{S}$, where x is an arbitrary point in $\widetilde{S}$. Since the action of $\pi_1(S)$ on $\widetilde{S}$ is co-compact, it is a well-known fact that $\Lambda = \partial\widetilde{S}$, and hence, $\Lambda = \mathbb{S}^1$. Note that the action of $\pi_1(S)$ on $\widetilde{S}$ can be extended to $\partial\widetilde{S}$ and that the action of $\pi_1(S)$ on $\partial\widetilde{S} \times \partial\widetilde{S}$ is given by the product action.

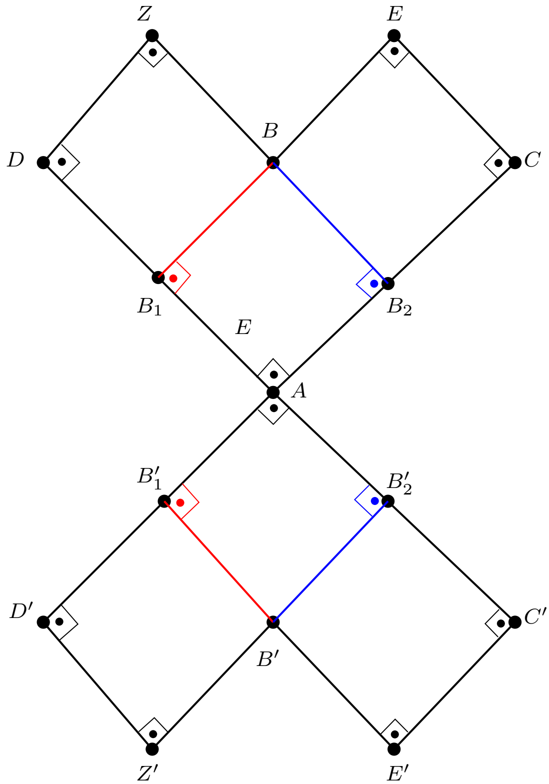

FIGURE 1. The surface Σ with two conical points of angle π and 3π.

Denote by F_h the set of points in $\partial\widetilde{S}$ which are fixed by hyperbolic elements of $\pi_1(S)$. Since $\Lambda = \partial\widetilde{S}$, the next three results can be derived from [**3**].

Proposition 2.6. *The set F_h is $\pi_1(S)$-invariant and dense in $\partial\widetilde{S}$.*

Proposition 2.7. *There exists an orbit of $\pi_1(S)$ dense in $\partial\widetilde{S} \times \partial\widetilde{S}$.*

Proposition 2.8. *The set $\{(\phi(+\infty), \phi(-\infty)) : \phi \in \pi_1(S)\}$ is dense in $\partial\widetilde{S} \times \partial\widetilde{S}$.*

3. Density in $\partial\widetilde{S}$. In this section, we first show that the set of points in $\partial\widetilde{S}$, for which the class of the corresponding geodesic rays is not a singleton, forms a dense subset of $\partial\widetilde{S}$. Throughout, we fix a base point $\widetilde{x_0} \in \widetilde{S}$.

Note that the existence of at least one conical point with angle $< 2\pi$ is crucial for all results in this section since, if all angles are $> 2\pi$, then $\widetilde{S}$ is a CAT(0) space, provided that the curvature at every regular point n is non-positive. In fact, $\widetilde{S}$ contains flat strips which correspond to closed geodesics not containing conical singularities. It is well known that, in a CAT(0) space, geodesic rays are unique, see [**2,** Proposition 8.2], and geodesic lines are also unique except those corresponding to flat strips, see [**1,** Corollary 5.8].

Proposition 3.1. *The set*

$$Y_{\widetilde{x_0}} = \left\{ \xi \in \partial\widetilde{S} \;\middle|\; \begin{array}{l} \textit{there exist distinct geodesic rays } r_1 \textit{ and } r_2 \textit{ such} \\ \textit{that } r_1(0) = \widetilde{x_0} = r_2(0) \textit{ and } r_1(\infty) = \xi = r_2(\infty) \end{array} \right\}$$

is dense in $\partial\widetilde{S}$.

Proof. Since $\partial\widetilde{S}$ is homeomorphic to $\mathbb{S}^1$, we concentrate on intervals in $\partial\widetilde{S}$, and we mean open (respectively, closed) connected subsets of $\partial\widetilde{S}$ homeomorphic to open (respectively, closed) intervals in $\mathbb{S}^1$. It suffices to show that, for any interval $I \subset \partial\widetilde{S}$, $I \cap Y_{\widetilde{x_0}} \neq \emptyset$.

Claim 3.2. *Let $\xi \notin Y_{\widetilde{x_0}}$, r_ξ the (unique) geodesic ray with $r_\xi(0) = \widetilde{x}_0$, $r_\xi(+\infty) = \xi$ and r an arbitrary geodesic ray with $r(0) = \widetilde{x}_0$ and $r(+\infty) \neq \xi$. Then,* $\mathrm{Im} r_\xi \cap \mathrm{Im} r$ *is either a geodesic segment of the form $[\widetilde{x}_0, \widetilde{x}_1]$ for some $\widetilde{x}_1 \in \mathrm{Im} r_\xi$, or a singleton, namely, $\{\widetilde{x}_0\}$. Similarly, if r is an arbitrary geodesic segment, then* $\mathrm{Im} r_\xi \cap \mathrm{Im} r$ *is either a geodesic sub-segment of r, or a singleton, or the empty set.*

For the proof of Claim 3.2, observe that $\mathrm{Im} r_\xi \cap \mathrm{Im} r$ is necessarily connected. For, if x and y belong to distinct connected components of $\mathrm{Im} r_\xi \cap \mathrm{Im} r$, then $r|_{[x,y]}$ does not coincide with $r_\xi|_{[x,y]}$. Thus, the geodesic ray

$$r' = r_\xi|_{[\widetilde{x}_0,x]} \cup r|_{[x,y]} \cup r_\xi|_{[y,+\infty]}$$

is distinct from r_ξ and, clearly, $r'(+\infty) = \xi$, a contradiction. Since both $\mathrm{Im} r_\xi$ and $\mathrm{Im} r$ are homeomorphic to $[0, +\infty)$, Claim 3.2 follows. The proof in the case where r is a geodesic segment is similar.

Returning to the proof of Proposition 3.1, suppose, on the contrary, that, for some closed interval $[\eta, \rho] \subset \partial\widetilde{S}$, we have $[\eta, \rho] \cap Y_{\widetilde{x_0}} = \emptyset$. From Claim 3.2, we may assume that $\mathrm{Im} r_\eta \cap \mathrm{Im} r_\rho = \{\widetilde{x}_0\}$; otherwise, replace in the sequel the union $\mathrm{Im} r_\eta \cup \mathrm{Im} r_\rho$ by

$$\mathrm{Im} r_\eta \cup \mathrm{Im} r_\rho \setminus [\widetilde{x}_0, \widetilde{x}_1).$$

Then, inside the compact, convex set $\widetilde{S} \cup \partial\widetilde{S}$, the union

$$\mathrm{Im} r_\eta \cup \mathrm{Im} r_\rho \cup [\eta, \rho]$$

splits the set $\widetilde{S} \cup \partial\widetilde{S}$ into two closed subsets, whose common boundary is the union $\mathrm{Im} r_\eta \cup \mathrm{Im} r_\rho \cup \{\eta, \rho\}$. Denote by $\widetilde{S}([\eta, \rho])$ the subset of $\widetilde{S} \cup \partial\widetilde{S}$ which contains $[\eta, \rho]$. Choose and fix a conical point $\widetilde{s}$ in the interior of $\widetilde{S}([\eta, \rho])$ with $\theta(\widetilde{s}) < 2\pi$. By replacing $[\eta, \rho]$ by a subinterval $[\eta', \rho'] \subsetneq [\eta, \rho]$, if necessary, we may assume that the angle formed by r_η and r_ρ at $\widetilde{x}_0$ is $< \pi$. Then, by Claim 3.2 and the assumption $[\eta, \rho] \cap Y_{\widetilde{x_0}} = \emptyset$, it follows that $\widetilde{S}([\eta, \rho])$ is convex.

For each $\xi \in [\eta, \rho]$, consider the (unique, as $[\eta, \rho] \cap Y_{\widetilde{x_0}} = \emptyset$) geodesic ray r_ξ with $r_\xi(0) = x_0$ and $r_\xi(+\infty) = \xi$. Choose geodesic segments $[x_\eta, \widetilde{s}]$ (respectively, $[x_\rho, \widetilde{s}]$) joining $\widetilde{s}$ with some point $x_\eta \in \mathrm{Im} r_\eta$ (respectively, $x_\rho \in \mathrm{Im} r_\rho$) such that

$$[x_\eta, \widetilde{s}] \cap [x_\rho, \widetilde{s}] = \{\widetilde{s}\}.$$

Since $[\eta, \rho] \cap Y_{\widetilde{x_0}} = \emptyset$, for any $\xi \in [\eta, \rho] \cap Y$, the unique geodesic ray r_ξ cannot (by Lemma 2.2) intersect $[x_\eta, \widetilde{s}]$ twice. Similarly, it cannot intersect $[x_\rho, \widetilde{s}]$ twice. Moreover, r_ξ cannot intersect both since, then, it would have to intersect one of the two segments twice. It follows that $\mathrm{Im} r_\xi$ intersects exactly one of the two segments $[x_\eta, \widetilde{s}]$ and $[x_\rho, \widetilde{s}]$. To be more precise, $\mathrm{Im} r_\xi$ intersects exactly one of the two half-open segments

$$[x_\eta, \widetilde{s}) \quad \text{and} \quad [x_\rho, \widetilde{s})$$

since a geodesic ray cannot contain a conical point $\widetilde{s}$ with $\theta(\widetilde{s}) < 2\pi$. Define the following sets

$$I(\eta) := \{\xi \in [\eta, \rho] \mid \mathrm{Im} r_\xi \cap [x_\eta, \widetilde{s}) \neq \emptyset\} \tag{3.1}$$

and

$$I(\rho) := \{\xi \in [\eta, \rho] \mid \mathrm{Im} r_\xi \cap [x_\rho, \widetilde{s}) \neq \emptyset\}. \tag{3.2}$$

We will show that $I(\eta)$ and $I(\rho)$ are closed and disjoint subsets of $[\eta, \rho]$, thus contradicting the connectedness of $[\eta, \rho]$.

Since every geodesic ray r_ξ intersects exactly one of the segments $[x_\eta, \widetilde{s})$ and $[x_\rho, \widetilde{s})$, it follows that $I(\eta)$ and $I(\rho)$ are disjoint.

In order to see that $I(\eta)$ is closed, let $\{\xi_n\} \subset I(\eta)$ be a sequence converging to ξ with r_{ξ_n} and r_ξ the corresponding (unique) geodesic rays with positive points ξ_n and ξ. We want to show that $\xi \in I(\eta)$. Assume, on the contrary, that $\xi \in I(\rho)$, that is, $\mathrm{Im} r_\xi \cap [x_\rho, \widetilde{s})$. Then, for sufficiently large n, r_{ξ_n} must also intersect the segment $[x_\rho, \widetilde{s})$ which, by (3.2), means that $\xi_n \in I(\rho)$, a contradiction. $\square$

We now show the analogous result for geodesic lines. We write $\partial^2 \widetilde{S}$ for the product $\partial\widetilde{S} \times \partial\widetilde{S}$ with the diagonal excluded.

Proposition 3.3. *The set*

$$Z = \left\{ (\eta, \xi) \in \partial^2 \widetilde{S} \;\middle|\; \begin{array}{c} \textit{there exist distinct geodesic lines } \gamma_1, \gamma_2 \textit{ such that} \\ \gamma_1(-\infty) = \eta = \gamma_2(-\infty) \textit{ and } \gamma_1(\infty) = \xi = \gamma_2(\infty) \end{array} \right\}$$

is dense in $\partial^2 \widetilde{S}$.

Proof. It suffices to show that, for arbitrary $\xi_0 \in \partial\widetilde{S}$ and any closed interval $[\eta, \rho] \subset \partial\widetilde{S}$ with $\xi_0 \notin [\eta, \rho]$, there exist two distinct geodesics γ_1 and γ_2 with $\gamma_1(-\infty) = \xi_0 = \gamma_2(-\infty)$ and $\gamma_1(+\infty) = \xi = \gamma_2(+\infty) \in [\eta, \rho]$. For, if, for all $\xi \in [\eta, \rho]$, there exists a unique geodesic line γ_ξ with $\gamma_\xi \in (\xi_0, \xi)$, we may repeat the argument in the proof of the previous proposition as follows: choose and fix a singular point $\widetilde{s}$ with $\theta(\widetilde{s}) < 2\pi$ in the interior of the (convex) set $\widetilde{S}([\eta, \rho])$ bounded by the geodesic line γ_η joining the pair (ξ_0, η) and the line γ_ρ joining the pair (ξ_0, ρ). For each $\xi \in [\eta, \rho]$, consider the (unique, as $\xi_0 \times [\eta, \rho] \cap Z = \emptyset$) geodesic γ_ξ with $\gamma_\xi(-\infty) = \xi_0$ and $\gamma_\xi(+\infty) = \xi$.

Choose geodesic segments $[x_\eta, \widetilde{s}]$ (respectively, $[x_\rho, \widetilde{s}]$) joining $\widetilde{s}$ with some point $x_\eta \in \mathrm{Im}\gamma_\eta$ (respectively, $x_\rho \in \mathrm{Im}\gamma_\rho$) such that

$$[x_\eta, \widetilde{s}] \cap [x_\rho, \widetilde{s}] = \{\widetilde{s}\}.$$

Similarly to the proof of Proposition 3.1, γ_ξ intersects exactly one of the two half-open segments

$$[x_\eta, \widetilde{s}) \quad \text{and} \quad [x_\rho, \widetilde{s})\,.$$

Define the following sets

$$I(\eta) := \{\xi \in [\eta, \rho] \mid \mathrm{Im}\gamma_\xi \cap [x_\eta, \widetilde{s}) \neq \emptyset\} \tag{3.3}$$

and

$$I(\rho) := \{\xi \in [\eta, \rho] \mid \mathrm{Im}\gamma_\xi \cap [x_\rho, \widetilde{s}) \neq \emptyset\}. \tag{3.4}$$

Clearly, $I(\eta)$ and $I(\rho)$ are disjoint. We will show that they are closed subsets of $[\eta, \rho]$, thus contradicting the connectedness of $[\eta, \rho]$. □

Proof of $I(\eta)$ closed. Let $\{\xi_n\} \subset I(\eta)$ be a sequence converging to ξ with γ_{ξ_n} and γ_ξ the corresponding (unique) geodesic lines with $\gamma_{\xi_n}(-\infty) = \xi_0 = \gamma_\xi(-\infty)$ and $\gamma_{\xi_n}(\infty) = \xi_n$, $\gamma_\xi(\infty) = \xi$. Choose a parametrization for γ_ξ. For example, set $\gamma_\xi(0)$ to be a point of minimal distance from $\widetilde{s}$, and assume, on the contrary, that $\xi \in I(\rho)$. This means that

$$\mathrm{Im}\gamma_\xi \cap [x_\rho, \widetilde{s}) \neq \emptyset.$$

Then, for sufficiently large n, γ_{ξ_n} must also intersect the segment $[x_\rho, \widetilde{s})$ which, by (3.4), means that $\xi_n \in I(\rho)$, a contradiction. □

We conclude this section with the following proposition which indicates that uniqueness of geodesic rays is a property which depends on the choice of the base point.

Proposition 3.4. *Let $\xi \in \partial\widetilde{S}$ be arbitrary. Then, for some point $x \in \widetilde{S}$, there exist at least two geodesic rays r_1, r_2 such that $r_1(0) = r_2(0) = x$ and $r_1(+\infty) = r_2(+\infty) = \xi$.*

Proof. Assume, on the contrary, that, for every point $x \in \widetilde{S}$, there exists exactly one geodesic ray, denoted by r_x, with $r_x(0) = x$ and $r_x(+\infty) = \xi$. Observe that, for two arbitrary distinct geodesic rays r_1 and r_2 with $r_1(+\infty) = r_2(+\infty) = \xi$ (then, by assumption, $r_1(0), r_2(0)$ must be distinct) we have that

$$\mathrm{Im}r_1 \cap \mathrm{Im}r_2 \text{ is either a geodesic subray of both or } \emptyset. \tag{3.5}$$

Otherwise, for a base point in the intersection, we would have two distinct geodesic rays corresponding to ξ.

Fix a point $\widetilde{s}_0$ where $s_0 = p(\widetilde{s}_0)$ is a conical point with angle $\theta(s_0) < 2\pi$. Let $D(\widetilde{s}_0, \varepsilon)$ be a disk of radius $\varepsilon > 0$ whose closure does not contain any conical point except $\widetilde{s}_0$. Moreover, choose $\varepsilon > 0$ small enough such that, in addition, $\partial D(\widetilde{s}_0, \varepsilon)$ is homeomorphic to a circle. The geodesic ray $r_{\widetilde{s}_0}$ intersects $\partial D(\widetilde{s}_0, \varepsilon)$ at a single point, denoted $\widetilde{x}_0$. A contradiction may be reached by defining a continuous surjective map from $\partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ to a space $\{+, -\}$ consisting of two points.

Let $A_\varepsilon = A + \varepsilon$ be a positive number (see equation (2.3)) such that, for any $x \in D(\widetilde{s}_0, \varepsilon)$, the (unique) geodesic ray r_x satisfies, for all $t \in [0, +\infty)$,

$$\operatorname{dist}(r_x(t), \operatorname{Im} r_{\widetilde{s}_0}) < A_\varepsilon.$$

Let $D(r_{\widetilde{s}_0}(3A_\varepsilon), A_\varepsilon)$ be the closed disk of radius A_ε centered at $r_{\widetilde{s}_0}(3A_\varepsilon)$. Then, the set

$$D(r_{\widetilde{s}_0}(3A_\varepsilon), A_\varepsilon) \setminus \operatorname{Im} r_{\widetilde{s}_0}$$

consists of two connected components. Using the orientation of $r_{\widetilde{s}_0}$, we may mark these components by saying that the component to the right is the positive component and the one to the left the negative, notation D^+ and D^-, respectively. In order to define a map

$$R : \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\} \longrightarrow \{+, -\},$$

we will distinguish two cases for each point $x \in \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$:

Case (i). $\operatorname{Im} r_x \cap \operatorname{Im} r_{\widetilde{s}_0} = \emptyset$, or $\operatorname{Im} r_x \cap \operatorname{Im} r_{\widetilde{s}_0} \neq \emptyset$, and the unique time $t_x \in [0, +\infty)$ so that $\operatorname{Im} r_{\widetilde{s}_0}|_{[t_x, +\infty]} \subset \operatorname{Im} r_x$, which exists by (3.5), satisfies $t_x > 2A_\varepsilon$.

Case (ii). $\operatorname{Im} r_x \cap \operatorname{Im} r_{\widetilde{s}_0} \neq \emptyset$, and the unique time $t_x \in [0, +\infty)$ so that $\operatorname{Im} r_{\widetilde{s}_0}|_{[t_x, +\infty]} \subset \operatorname{Im} r_x$ satisfies $t_x \leq 2A_\varepsilon$.

Observe that, in Case (ii), $\operatorname{Im} r_x$ intersects neither D^+ nor D^-. Clearly, in Case (i), $\operatorname{Im} r_x$ intersects at least one component D^+, D^-. We claim that, in Case (i), $\operatorname{Im} r_x$ cannot intersect both components D^+ and D^-. In order to see this, assume that $\operatorname{Im} r_x$ intersected both D^+ and D^-. Let $\sigma : [a, b] \to \widetilde{S}$ be a curve with the properties $\sigma(a) \in D^+, \sigma(b) \in D^-$ and $\operatorname{Im} \sigma \cap \operatorname{Im} r_{\widetilde{s}_0} = \emptyset$. By standard triangle inequality arguments, it follows that length $(\sigma) > 2A_\varepsilon$. Therefore, $\operatorname{Im} r_x$ cannot be disjoint from $r_{\widetilde{s}_0}$, and, since it is a geodesic, it must intersect $r_{\widetilde{s}_0}$ transversely, a contradiction according to the assumptions in Case (i). Thus, it follows that either $\operatorname{Im} r_x \cap D^+ \neq \emptyset$ or $\operatorname{Im} r_x \cap D^- \neq \emptyset$, but not both. For $x \in \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$, whose geodesic ray r_x falls into Case (i), we may now define:

$$R(x) := + \text{ if } \mathrm{Im} r_x \cap D^+ \neq \emptyset$$

and

$$R(x) := - \text{ if } \mathrm{Im} r_x \cap D^- \neq \emptyset.$$

Now, let $x \in \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ be such that r_x falls into Case (ii). It is easy to see that $t_x > \varepsilon$. For, if $0 < t_x \leq \varepsilon$, then $r_{\widetilde{s}_0}(t_x) \in D(\widetilde{s}_0, \varepsilon)$, which is impossible since $r_{\widetilde{s}_0}(t_x)$ is a conical point and ε is chosen such that $\widetilde{s}_0$ is the unique conical point in the closure of $D(\widetilde{s}_0, \varepsilon)$. If $t_x = 0$, then the conical point $\widetilde{s}_0 = r_{\widetilde{s}_0}(0)$ of angle $< 2\pi$ lies on the geodesic ray r_x, a contradiction by (2.1). Thus, $t_x > \varepsilon$, and there exists a $\delta > 0$ sufficiently small so that the disk $D(r_{\widetilde{s}_0}(t_x), \delta)$ does not contain $\widetilde{x}_0 = r_{\widetilde{s}_0}(\varepsilon)$. This disk $D(r_{\widetilde{s}_0}(t_x), \delta)$ can be used to define $R(x)$ as above: $\mathrm{Im} r_x$ intersects exactly one of the two oriented components of $D(r_{\widetilde{s}_0}(t_x), \delta) \setminus \mathrm{Im} r_{\widetilde{s}_0}$, and define $R(x)$ accordingly.

We show that R is continuous. Let $\{x_n\}$ be a sequence in $\partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ converging to a point x. The sequence of geodesic rays $\{r_{x_n}\}$ converges, up to a subsequence, to a geodesic ray q_x emanating from x. Since $r_{x_n}(+\infty) = \xi$ for all n, it follows that $q_x(+\infty) = \xi$. From the assumption of uniqueness of geodesic rays we have $q_x = r_x$. Thus, $r_{x_n} \to r_x$ uniformly on compact sets. Without loss of generality we may assume that $R(x) = +$.

First, assume that the geodesic ray r_x falls into Case (i), that is, $\mathrm{Im} r_x \cap D^+ \neq \emptyset$. Since $r_{x_n} \to r_x$ uniformly on compact sets, it follows that there exists an N so that, for all $n \geq N$,

$$\mathrm{Im} r_{x_n} \cap D^+ \neq \emptyset,$$

which means that $R(x_n) = +$, for all $n \geq N$.

In Case (ii), we employ the same argument as in the definition of the function R to reduce to a situation similar to Case (i). This shows that R is continuous.

We show that R is onto. We may choose a sequence $\{x_n\} \subset \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ converging to $\widetilde{x}_0$ from the right in the following sense: for all sufficiently small $\delta > 0$, the set $D(\widetilde{x}_0, \delta) \setminus \mathrm{Im} r_{\widetilde{s}_0}$ consists of two connected components. We mark them as right (positive) and left (negative) according to the positive direction of $r_{\widetilde{s}_0}$. We say that a sequence $\{x_n\} \subset \partial D(\widetilde{s}_0, \varepsilon) \setminus \{\widetilde{x}_0\}$ converges to $\widetilde{x}_0$ from the right if x_n

belongs to the right (positive) component of $D(\widetilde{x}_0, \delta) \setminus \mathrm{Im} r_{\widetilde{s}_0}$ for all but finitely any n. Clearly, for such a sequence $\{x_n\}$, the corresponding geodesic rays $r_{x_n} \to r_{\widetilde{s}_0}|_{[\varepsilon,+\infty)}$ uniformly on compact sets. Fix $\varepsilon_1 < \varepsilon$, and set

$$\mathcal{N}(\varepsilon_1) = \{y \in \widetilde{S} \mid \text{there exists a } t \in [\varepsilon, 4A_\varepsilon] : d(y, r_{\widetilde{s}_0}(t)) < \varepsilon_1\}.$$

As above, $\mathcal{N}(\varepsilon_1) \setminus \mathrm{Im} r_{\widetilde{s}_0}$ consists of two components: $\mathcal{N}^+$ and $\mathcal{N}^-$. Pick a sequence $x_n \to \widetilde{x}_0$ from the right. Then, for all n large enough, $x_n \in \mathcal{N}^+$ and, by the uniform convergence of r_{x_n}, $\mathrm{Im} r_{x_n}$ intersects $\mathcal{N}^+$, while

$$\mathrm{Im} r_{x_n} \cap \mathcal{N}^- = \emptyset.$$

This implies that $R(x_n) = +$ for all large enough n. Similarly, we show that R attains the value $-$. $\square$

4. Density of closed geodesics. We begin by showing that each class of geodesic rays with the same boundary point at infinity contains a leftmost and a rightmost geodesic ray which bound a convex set containing the image of any other geodesic ray in the same class. Our standing assumption that there exists at least one conical point with angle $< 2\pi$ asserts that the leftmost and rightmost rays are distinct. If all angles are $> 2\pi$, then $\widetilde{S}$ is a CAT(0) space, provided that the curvature at every regular point is non-positive. In this case, the geodesic rays are unique.

Proposition 4.1. *Let $\xi \in \partial\widetilde{S}$ and $\widetilde{x}_0 \in \widetilde{S}$ be such that the geodesic ray from $\widetilde{x}_0$ to ξ is not unique. Then, there exist two geodesic rays r_L and r_R with $r_L(\infty) = \xi = r_R(\infty$ and $r_L(0) = \widetilde{x}_0 = r_R(0)$, and whose images bound a convex subset $\widetilde{S}(\xi)$ of $\widetilde{S}$ with the property*

$$\mathrm{Im} r \subset \widetilde{S}(\xi)$$

for all geodesic rays r with $r(\infty) = \xi$ and $r(0) = \widetilde{x}_0$.

Proof. Let A be the number posited in equation (2.3). For each $N \in \mathbb{N}$ large enough, let $\partial_0 D(\widetilde{x}_0, N)$ be the principal component of $\partial D(\widetilde{x}_0, N)$. The set

$$\xi(N) = \{r(N) \mid r(\infty) = \xi\}$$

is contained in an interval $I_{N,\xi}$ of diameter $2A$ inside the circle $\partial_0 D(\widetilde{x}_0, N)$. Thus, for large enough N, we may orient $I_{N,\xi}$ and consider its left and right endpoint.

We claim that $\xi(N)$ is a closed set. In order to see this, let $\{y_n\}$ be a sequence of points in $\xi(N)$ converging to $y \in I_{N,\xi}$. By the definition of $\xi(N)$, for each y_n, there exists a geodesic ray $r_n \in \xi$ (not necessarily unique) such that $r_n(N) = y_n$. By passing to a subsequence, if necessary, $\{r_n\}$ converges to a geodesic ray r and, clearly, $r \in \xi$. As $y_n \to y$, y must belong to Imr and, on the other hand, $y \in I_{N,\xi} \subset \partial_0 D\ (\widetilde{x}_0, N)$. Thus, $y = r(N)$, which shows that $\xi(N)$ is closed.

By compactness, the leftmost and rightmost points of $\xi(N)$ inside $I_{N,\xi}$, denoted by y_L and y_R, respectively, exist. Since the number of conical points in the closure of $D(\widetilde{x}_0, N)$ is finite, we may choose (cf., Lemma 2.2) geodesic segments $\sigma_{L,N}$ and $\sigma_{R,N}$ with endpoints $\widetilde{x}_0, y_L$ and $\widetilde{x}_0, y_R$, respectively, satisfying the following property: the convex subset of the closure of $D(\widetilde{x}_0, N)$, bounded by the union

$$\sigma_{L,N} \cup [y_L, y_R] \cup \sigma_{R,N}, \tag{4.1}$$

where $[y_L, y_R]$ indicates the subinterval of $\partial_0 D(\widetilde{x}_0, N)$ containing $\xi(N)$, contains all geodesic segments $r|_{[0,N]}$ for all r for which $r(\infty) = \xi$.

The segment $\sigma_{L,N}$ (and similarly for $\sigma_{R,N}$) can be obtained by starting with a geodesic segment $\sigma'_{L,N}$ with endpoints $\widetilde{x}_0, y_L$ and then, if a geodesic ray intersects the segment $\sigma'_{L,N}$, it must do so at pairs of (conical) points (otherwise, the property of y_L being leftmost would be violated). As the intersection points are conical points, they are finitely many pairs of intersection points; thus, we may replace (see Lemma 2.2) finitely many parts of the segment $\sigma'_{L,N}$ to obtain $\sigma_{L,N}$.

The sequences $\{\sigma_{L,N}\}_{N\in\mathbb{N}}$ and $\{\sigma_{R,N}\}_{N\in\mathbb{N}}$ converge to geodesic rays $r_L, r_R \in \xi$, respectively. The required property in the statement of Proposition 4.1 for the convex set $\widetilde{S}(\xi)$ bounded by Imr_L and Imr_R now follows: for, if $r \in \xi$ with Im$r \nsubseteq \widetilde{S}(\xi)$, then, for some $M > 0$, $r(M) \notin \widetilde{S}(\xi)$. Assume that the distance $d(r(M), \widetilde{S}(\xi)) = C_0 > 0$ of $r(M)$ from $\widetilde{S}(\xi)$ is realized by a point on Imr_R. Then, for a compact set $K \supset [0, M]$ and the positive number $C_0/2$, there exists an N_0 so that

$$d(r_R(t), \sigma_{R,N}(t)) < C_0/2$$

for all $t \in K$ and for all $N > N_0$. We may assume that N_0 satisfies $N_0 > [M] + 1$. It follows that $r(M)$ does not belong to the convex subset of $D(\widetilde{x}_0, N_0)$ bounded by the union

$$\sigma_{L,N_0} \cup [y_L, y_R] \cup \sigma_{R,N_0},$$

contradicting (4.1). □

In view of the above proposition, we introduce the following.

Notation. For each $\xi \in \partial\widetilde{S}$, the geodesic rays posited in the above proposition will be called *leftmost* and *rightmost geodesic rays* in the class of ξ and will be denoted by $r_{L,\xi}$ and $r_{R,\xi}$, respectively.

Proposition 4.2. *For every pair of points $\eta, \xi \in \partial\widetilde{S}$ with $\eta \neq \xi$, there exist two geodesic lines $\gamma_L, \gamma_R \in (\eta, \xi)$, that is, $\gamma_L(-\infty) = \eta = \gamma_R(-\infty)$ and $\gamma_L(\infty) = \xi = \gamma_R(\infty)$, whose images bound a convex subset $\widetilde{S}(\eta, \xi)$ of $\widetilde{S}$ with the property*

$$\mathrm{Im}\gamma \subset \widetilde{S}(\eta, \xi)$$

for all geodesic lines $\gamma \in (\eta, \xi)$.

Proof. The line of proof is similar to that of the previous proposition; however, we include it here since certain modifications are needed.

We may assume that there exist at least two geodesics in the class of (η, ξ); otherwise, the statement is trivial. Moreover, each $\gamma \in (\eta, \xi)$ is considered oriented with positive direction from η to ξ, and then the left and right components of $\partial\widetilde{S} \setminus \{\eta, \xi\}$ are determined. Choose a base point $\widetilde{x}_0$ on the image of an arbitrary $\gamma_0 \in (\eta, \xi)$, and set $\gamma_0(0) = \widetilde{x}_0$.

For large enough $t > 0$, the principal boundary $\partial_0 D(\widetilde{x}_0, \gamma_0(t))$ of the disk centered at $\widetilde{x}_0$ with radius $d(\widetilde{x}_0, \gamma_0(t))$ is a circle. Write I_t^+ for the closed subinterval of $\partial_0 D(\widetilde{x}_0, \gamma_0(t))$, which is minimal with respect to the inclusion

$$I_t^+ \subset \partial_0 D(\widetilde{x}_0, \gamma_0(t)) \cap \overline{D(\gamma_0(t), A)},$$

where A is the constant posited in (2.3). Similarly, using $\partial D(\gamma_0(-t), A)$, we define I_t^-. For sufficiently large $t > 0$, we have

$$I_t^+ \cap I_t^- = \emptyset.$$

For large enough $N \in \mathbb{N}$, the sets I_N^+ and I_N^- have the property that, for all $\gamma \in (\eta, \xi)$,

$$\mathrm{Im}\gamma \cap \partial_0 D(\widetilde{x}_0, \gamma_0(N)) \subset I_N^+ \cup I_N^-.$$

For each γ in (η, ξ), the intersection $\mathrm{Im}\gamma \cap I_N^+$ (respectively, I_N^-) is not necessarily a singleton. However, there exist unique numbers $t_{\gamma,N}^-$, $t_{\gamma,N}^+ \in \mathbb{R}$ such that

$$\gamma(t_{\gamma,N}^-) \in I_N^-, \ \gamma(t_{\gamma,N}^+) \in I_N^+,$$

and $|t_{\gamma,N}^- - t_{\gamma,N}^+|$ is minimal with respect to the above inclusions. Equivalently, $\gamma|_{(t_{\gamma,N}^-, t_{\gamma,N}^+)} \cap \partial_0 D(\widetilde{x}_0, \gamma_0(t)) = \emptyset$. Set

$$\xi^+(N) = \{\gamma(t_{\gamma,N}^+) \mid \gamma \in (\eta, \xi)\},$$

and similarly for $\xi^-(N)$. We claim that both sets $\xi^+(N), \xi^-(N)$ are closed. In order to see this, let $\{y_n\}$ be a sequence of points in $\xi^+(N)$ converging to $y \in I_N^+$. From the definition of $\xi^+(N)$, for each y_n, there exists a geodesic $\gamma_n \in (\eta, \xi)$ (not necessarily unique) such that $\gamma_n(t_{\gamma_n,N}^+) = y_n$. By passing to a subsequence, if necessary, $\{\gamma_n\}$ converges to a geodesic γ'. Clearly, $\gamma' \in (\eta, \xi)$, and y must belong to $\mathrm{Im}\gamma' \cap I_N^+$. In order to complete the proof that $\xi^+(N)$ is closed, we need to show that $y = \gamma'(t_{\gamma',N}^+)$ or, equivalently, if $t_y \in \mathbb{R}$ such that $y = \gamma'(t_y)$, then $t_y = t_{\gamma',N}^+$.

For ε positive, say $\varepsilon < N/2$, the sequence $\{\gamma_n(t_{\gamma_n,N}^+ - \varepsilon)\}_{n\in\mathbb{N}}$ converges to a point in $D(\widetilde{x}_0, N)$. Therefore, all points on $\mathrm{Im}\gamma'|_{(-\infty, t_y]}$ of distance ε from y belong to $D(\widetilde{x}_0, N)$. It follows that the time t_y with $y = \gamma'(t_y)$ is the smallest time $t > 0$ for which $\gamma'(t_y) \in I_N^+$. Therefore, $t_y = t_{\gamma',N}^+$, which shows that $\xi^+(N)$ is closed. Similarly, we show that $\xi^-(N)$ is closed.

Denote by y_L^+ (respectively, y_R^+) the leftmost (respectively, rightmost) point of $\xi^+(N)$ in I_N^+ and y_L^- (respectively, y_R^-) the leftmost (respectively, rightmost) point of $\xi^-(N)$ in I_N^-, all of which exist by compactness. We may construct a rightmost geodesic segment $\sigma_{R,N} = [y_R^-, y_R^+]$ in the closure of $D(\widetilde{x}_0, N)$ with endpoints y_R^-, y_R^+ and a leftmost geodesic segment $\sigma_{L,N} = [y_L^-, y_L^+]$ in the closure of $D(\widetilde{x}_0, N)$ with endpoints y_L^-, y_L^+ such that the following property holds:

• the convex subset bounded by the union

$$\sigma_{L,N} \cup [y_L^+, y_R^+] \cup \sigma_{R,N} \cup [y_R^-, y_R^+],$$

where $[y_L^+, y_R^+]$ (respectively, $[y_L^-, y_R^-]$) indicates the subinterval of $\partial_0 D(\widetilde{x}_0, N)$ containing $\xi^+(N)$ (respectively, $\xi^-(N)$), contains all segments $\mathrm{Im}\gamma \cap \overline{D(\widetilde{x}_0, N)}$ for all $\gamma \in (\eta, \xi)$.

The segment $\sigma_{L,N}$ (and, similarly, for $\sigma_{R,N}$) can be obtained by starting with a geodesic segment $\sigma'_{L,N}$ with endpoints y_L^-, y_L^+, and then, if a geodesic line intersects the segment $\sigma'_{L,N}$, it must do so at pairs of (conical) points (otherwise, the property of y_L^+, y_L^- being leftmost would be violated). Since the intersection points are conical points, they are finitely many; thus, we may replace (see Lemma 2.2) finitely many parts of the segment $\sigma'_{L,N}$ to obtain $\sigma_{L,N}$.

Similarly to the proof of the previous proposition, we obtain the desired geodesic lines as limits of the sequences $\{\sigma_{L,N}\}_{N\in\mathbb{N}}$ and $\{\sigma_{R,N}\}_{N\in\mathbb{N}}$. $\square$

Notation. For each $(\eta, \xi) \in \partial^2 \widetilde{S}$, the geodesic lines posited in the above proposition will be called *leftmost* and *rightmost geodesic lines* in the class of (η, ξ) and will be denoted by $\gamma_{L,(\eta,\xi)}$ and $\gamma_{R,(\eta,\xi)}$, respectively.

Theorem 4.3. *Closed geodesics are dense in* GS *in the following sense*: *for each pair $(\eta, \xi) \in \partial^2 \widetilde{S}$, there exists a sequence of geodesics $\{c_n\}$ such that $c_n \to \gamma_{L,(\eta,\xi)}$ in the usual uniform sense on compact sets, and $p(c_n)$ is a closed geodesic in S for all n*; *similarly for $\gamma_{R,(\eta,\xi)}$.*

Proof. For arbitrary $(\eta, \xi) \in \partial^2 \widetilde{S}$, we orient as positive the direction from η to ξ and name the components of $\partial \widetilde{S} \setminus \{\eta, \xi\}$ "left" and "right." We choose, by Proposition 2.8, a sequence $\{(\phi_n(-\infty), \phi_n(+\infty))\}_{n\in\mathbb{N}}$ where each ϕ_n is a (hyperbolic) element of $\pi_1(S)$ such that $\phi_n(-\infty) \to \eta$ and $\phi_n(\infty) \to \xi$ with the additional property, that for all n, both $\phi_n(-\infty)$ and $\phi_n(\infty)$ belong to the same (say, right) component of $\partial \widetilde{S} \setminus \{\eta, \xi\}$. In particular, we have that $\phi_n(-\infty) \neq \eta$ and $\phi_n(\infty) \neq \xi$.

We claim that, for each $n \in \mathbb{N}$, there exists a geodesic

$$c''_n \in (\phi_n(-\infty), \phi_n(+\infty)),$$

that is,

$$c''_n(-\infty) = \phi_n(-\infty) \quad \text{and} \quad c''_n(\infty) = \phi_n(+\infty),$$

whose projection to S is closed. In order to see this, pick arbitrary $y \in \widetilde{S}$, and consider the geodesic segment $[y, \phi_n(y)]$ which, clearly, projects to a closed curve, say σ_n in S. There exists a length minimizing closed curve in the (free) homotopy class of σ_n (see **[6,** Remark 1.13(b)]). By choosing an appropriate lift to $\widetilde{S}$ of this length minimizing closed curve, we obtain a geodesic line c''_n such that the set

$$\{\phi_n^i(y) \mid i \in \mathbb{Z}\}$$

is at a bounded distance from Im c''_n. Thus, $c''_n(-\infty) = \phi_n(-\infty)$ and $c''_n(\infty) = \phi_n(+\infty)$, as desired.

Since, by construction, the projection of c''_n to S is a closed curve, we can speak of the period of c''_n. Let $\gamma_{R,(\eta,\xi)} \in (\eta, \xi)$ be the rightmost geodesic posited in Proposition 4.2. Since $c''_n(-\infty) \neq \eta$ and $c''_n(\infty) \neq \xi$, the intersection

$$\mathrm{Im} c''_n \cap \mathrm{Im} \gamma_{R,(\eta,\xi)}$$

has finitely many components. For each $n \in \mathbb{N}$, consider the geodesic line c'_n having the same image as c''_n, and its period is a multiple of the period of c''_n such that

$$\mathrm{Im} c'_n \cap \mathrm{Im} \gamma_{R,(\eta,\xi)}$$

is contained in a single period of c'_n. We may alter c'_n in its (enlarged) period so that it does not intersect the interior of the convex subset $\widetilde{S}(\eta, \xi)$ of $\widetilde{S}$ bounded by $\mathrm{Im}\gamma_{L,(\eta,\xi)}$ and $\mathrm{Im}\gamma_{R,(\eta,\xi)}$. For such an alteration, we only need modify c'_n in subintervals, say $[z, w]$, of its image contained in $\widetilde{S}(\eta, \xi)$, namely, we must replace $c'_n|_{[z,w]}$ by $\gamma_{R,(\eta,\xi)}|_{[z,w]}$. Then, by repeating this alteration, weobtain a geodesic line, denoted c_n, whose projection to S is a closed geodesic. Clearly, by construction,

$$c_n(-\infty) = c''_n(-\infty) = \phi_n(-\infty),$$
$$c_n(\infty) = c''_n(+\infty) = \phi_n(+\infty)$$

and, since $\mathrm{Im} c_n \cap \widetilde{S}(\eta, \xi) \subset \mathrm{Im}\gamma_{R,(\eta,\xi)}$, it follows that $c_n \to \gamma_{L,(\eta,\xi)}$ uniformly on compact sets. $\square$

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