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ON THE GREATEST COMMON DIVISOR OF n AND THE nTH FIBONACCI NUMBER

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ABSTRACT. Let $\mathcal{A}$ be the set of all integers of the form $\gcd(n, F_n)$, where n is a positive integer and F_n denotes the nth Fibonacci number. We prove that $\#(\mathcal{A} \cap [1, x]) \gg x/\log x$ for all $x \geq 2$ and that $\mathcal{A}$ has zero asymptotic density. Our proofs rely upon a recent result of Cubre and Rouse [**5**] which gives, for each positive integer n, an explicit formula for the density of primes p such that n divides the rank of appearance of p, that is, the smallest positive integer k such that p divides F_k.

1. Introduction. Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers, defined as usual by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for all positive integers n. Moreover, let g be the arithmetic function defined by $g(n) := \gcd(n, F_n)$, for each positive integer n. The first values of g are listed in [**13**].

The set $\mathcal{B}$ of fixed points of g, i.e., the set of positive integers n such that n divides F_n, has been studied by several authors. For instance, André-Jeannin [**2**] and Somer [**14**] investigated the arithmetic properties of the elements of $\mathcal{B}$. Furthermore, Luca and Tron [**8**] proved that

$$\#\mathcal{B}(x) \leq x^{1-(1/2+o(1)) \log\log\log x/\log\log x}, \tag{1.1}$$

when $x \to +\infty$, and Sanna [**12**] generalized their result to Lucas sequences. More generally, the study of the distribution of positive integers n dividing the nth term of a linear recurrence has been studied by Alba González, et al. [**1**], while, Corvaja and Zannier [**4**] and Sanna [**10**] considered the distribution of positive integers n such that the nth term of a linear recurrence divides the nth term of another

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linear recurrence. Also, it follows from a result of Sanna **[11]** that the set $g^{-1}(1)$, i.e., the set of positive integers n such that n and F_n are relatively prime, has a positive asymptotic density.

Define $\mathcal{A} := \{g(n) : n \geq 1\}$. Note that, in particular, $\mathcal{B} \subseteq \mathcal{A}$. The aim of this article is to study the structural properties and the distribution of the elements of $\mathcal{A}$. Note that it is not immediately clear whether or not a given positive integer belongs to $\mathcal{A}$. Toward this aim, we provide in Section 2 an effective criterion which allows us to enumerate the elements of $\mathcal{A}$, in increasing order, as:

$$1,\ 2,\ 5,\ 7,\ 10,\ 12,\ 13,\ 17,\ 24,\ 25,\ 26,\ 29,\ 34,\ 35,\ 36,\ \ldots.$$

Our first result is a lower bound for the counting function of $\mathcal{A}$.

Theorem 1.1. *$\#\mathcal{A}(x) \gg x/\log x$, for all $x \geq 2$.*

It is worth noting that it follows at once from Theorem 1.1 and (1.1) that $\mathcal{B}$ has zero asymptotic density relative to $\mathcal{A}$ (we omit the details).

Corollary 1.2. *$\#\mathcal{B}(x) = o(\#\mathcal{A}(x))$, as $x \to +\infty$.*

Our second result is that $\mathcal{A}$ has zero asymptotic density:

Theorem 1.3. *$\#\mathcal{A}(x) = o(x)$, as $x \to +\infty$.*

It would be nice to have an effective upper bound for $\#\mathcal{A}(x)$ or, even better, to obtain its asymptotic order of growth. We leave these as open questions for the interested readers.

Notation. Throughout, we reserve the letters p and q for prime numbers. Moreover, given a set $\mathcal{S}$ of positive integers, we define $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x \geq 1$. We employ the Landau-Bachmann "Big Oh" and "little oh" notation O and o, as well as the associated Vinogradov symbols $\ll$ and $\gg$. In particular, all of the implied constants are intended to be absolute.

2. Preliminaries. This section is devoted to some preliminary results necessary for the later proofs. For each positive integer n, let $z(n)$ be *rank of appearance of* n in the sequence of Fibonacci numbers, that is, $z(n)$ is the smallest positive integer k such that n divides F_k. It is well known that $z(n)$ exists. All of the statements in the next lemma are well known, and we will use them implicitly without further mention.

Lemma 2.1. *For all positive integers m, n and all prime numbers p, we have*:

(i) $F_m \mid F_n$ *whenever* $m \mid n$;
(ii) $m \mid F_n$ *if and only if* $z(m) \mid n$;
(iii) $z(m) \mid z(n)$ *whenever* $m \mid n$;
(iv) $z(p) \mid p - (p/5)$, *where* $(p/5)$ *is a Legendre symbol.*

For each positive integer n, define $\ell(n) := \operatorname{lcm}(n, z(n))$. The next lemma shows some elementary properties of the functions g, ℓ, z, and their relationship with $\mathcal{A}$.

Lemma 2.2. *For all positive integers m, n and all prime numbers p, we have*:

(i) $g(m) \mid g(n)$ *whenever* $m \mid n$;
(ii) $n \mid g(m)$ *if and only if* $\ell(n) \mid m$;
(iii) $n \in \mathcal{A}$ *if and only if* $n = g(\ell(n))$;
(iv) $p \mid n$ *whenever* $\ell(p) \mid \ell(n)$ *and* $n \in \mathcal{A}$;
(v) $\ell(p) = pz(p)$ *whenever* $p \neq 5$, *and* $\ell(5) = 5$;
(vi) $p \in \mathcal{A}$ *if* $p \neq 3$ *and* $\ell(q) \nmid z(p)$ *for all prime numbers* q.

Proof. Facts (i) and (ii) easily follow from the definitions of g and ℓ and the properties of z. In order to prove (iii), note that n divides both $\ell(n)$ and $F_{\ell(n)}$; hence, $n \mid g(\ell(n))$ for all positive integers n. Conversely, if $n \in \mathcal{A}$, then $n = g(m)$ for some positive integer m, n particular, $n \mid g(m)$, which is equivalent to $\ell(n) \mid m$ by (ii). Therefore, $g(\ell(n)) \mid g(m) = n$, due to(i), and in conclusion, $g(\ell(n)) = n$. Fact (iv) follows at once from (ii) and (iii).

A quick computation shows that $\ell(5) = 5$, while, for all prime numbers $p \neq 5$ we have $\gcd(p, z(p)) = 1$, since $z(p) \mid p \pm 1$, so that $\ell(p) = pz(p)$, and this proves (v).

Lastly, we suppose that $p \neq 3$ is a prime number such that $\ell(q) \nmid z(p)$ for all prime numbers q. In particular, $p \neq 5$ since $\ell(5) = z(5) = 5$, by (v). Also, the claim (vi) is easily seen to hold for $p = 2$. Hence, let us suppose hereafter that $p \geq 7$. Since $z(p) \mid p \pm 1$, it easily follows that $p \parallel g(\ell(p))$. At this point, if $q \mid g(\ell(p))$ for some prime $q \neq p$, then $\ell(q) \mid \ell(p) = pz(p)$ due to (ii). However, $\ell(q) \nmid z(p)$; hence, $p \mid \ell(q) = \operatorname{lcm}(q, z(q))$ so that $p \mid z(q) \leq q + 1$. Similarly, $q \mid g(\ell(p)) \mid \ell(p)$ implies $q \mid z(p) \leq p + 1$. Hence, $|p - q| \leq 1$, which is impossible since $p \geq 7$. Therefore, $q \nmid g(\ell(p))$, with the consequence that $p = g(\ell(p))$, i.e., $p \in \mathcal{A}$ by (iii). This concludes the proof of (vi). $\square$

It is worth noting that Lemma 2.2 (iii) provides an effective criterion to establish whether or not a given positive integer belongs to $\mathcal{A}$. This is how the elements of $\mathcal{A}$ listed in the introduction were evaluated.

It follows from a result of Lagarias **[6, 7]** that the set of prime numbers p such that $z(p)$ is even has a relative density of $2/3$ in the set of all prime numbers. Bruckman and Anderson [**3,** Conjecture 3.1] conjectured, for each positive integer m, a formula for the limit

$$\zeta(m) := \lim_{x \to +\infty} \frac{\#\{p \leq x : m \mid z(p)\}}{x / \log x}.$$

Their conjecture was proven by Cubre and Rouse [**5,** Theorem 2], who obtained the following result.

Theorem 2.3. *For any positive integer m, we have*

$$\zeta(m) = \rho(m) \prod_{q^e \parallel m} \frac{q^{2-e}}{q^2 - 1},$$

where q^e runs over the prime powers in the factorization of m, while

$$\rho(m) := \begin{cases} 1 & \textit{if } 10 \nmid m, \\ 5/4 & \textit{if } m \equiv 10 \bmod 20, \\ 1/2 & \textit{if } 20 \mid m. \end{cases}$$

Note that the arithmetic function ζ is not multiplicative. However, the restriction of ζ to the odd positive integers is multiplicative. This fact will be useful later.

Let φ be Euler's totient function. We need the following technical lemma.

Lemma 2.4. *We have*

$$\sum_{q>y} \frac{1}{\varphi(\ell(q))} \ll \frac{1}{y^{1/4}},$$

for all $y > 0$.

Proof. For $\gamma > 0$, set $\mathcal{Q}_\gamma := \{p : z(p) < p^\gamma\}$. Clearly,

$$2^{\#\mathcal{Q}_\gamma(x)} \leq \prod_{p \in \mathcal{Q}_\gamma(x)} p \mid \prod_{n \leq x^\gamma} F_n \leq 2^{\sum_{n \leq x^\gamma} n} \leq 2^{O(x^{2\gamma})},$$

from which it follows that $\mathcal{Q}_\gamma(x) \ll x^{2\gamma}$.

Also fix $\varepsilon \in \,]0, 1 - 2\gamma[$. For the remainder of this proof, all of the implied constants may depend upon γ and ε. Since $\varphi(n) \gg n/\log\log n$ for all positive integers n [**15,** Chapter I.5, Theorem 4], while, by Lemma 2.2 (v), $\ell(q) \ll q^2$ for all prime numbers q, we have

$$\sum_{q>y} \frac{1}{\varphi(\ell(q))} \ll \sum_{q>y} \frac{\log\log \ell(q)}{\ell(q)} \ll \sum_{q>y} \frac{\log\log q}{\ell(q)} \ll \sum_{q>y} \frac{q^\varepsilon}{\ell(q)}, \tag{2.1}$$

for all $y > 0$.

On one hand, again by Lemma 2.2 (v),

$$\sum_{\substack{q>y \\ q \notin \mathcal{Q}_\gamma}} \frac{q^\varepsilon}{\ell(q)} \ll \sum_{\substack{q>y \\ q \notin \mathcal{Q}_\gamma}} \frac{1}{q^{1-\varepsilon} z(q)} \leq \sum_{q>y} \frac{1}{q^{1+\gamma-\varepsilon}} \ll \int_y^{+\infty} \frac{\mathrm{d}t}{t^{1+\gamma-\varepsilon}} \ll \frac{1}{y^{\gamma-\varepsilon}}. \tag{2.2}$$

On the other hand, by partial summation,

$$\sum_{\substack{q>y \\ q \in \mathcal{Q}_\gamma}} \frac{q^\varepsilon}{\ell(q)} \leq \sum_{\substack{q>y \\ q \in \mathcal{Q}_\gamma}} \frac{1}{q^{1-\varepsilon}} = \left.\frac{\#\mathcal{Q}_\gamma(t)}{t^{1-\varepsilon}}\right|_{t=y}^{+\infty} + (1-\varepsilon)\int_y^{+\infty} \frac{\#\mathcal{Q}_\gamma(t)}{t^{2-\varepsilon}}\,\mathrm{d}t \tag{2.3}$$

$$\leq \int_y^{+\infty} \frac{\#\mathcal{Q}_\gamma(t)}{t^{2-\varepsilon}}\, \mathrm{d}t \ll \int_y^{+\infty} \frac{\mathrm{d}t}{t^{2-2\gamma-\varepsilon}} \ll \frac{1}{y^{1-2\gamma-\varepsilon}}.$$

The claim follows by combining (2.1), (2.2) and (2.3), and by choosing $\gamma = 1/3$ and $\varepsilon = 1/12$. □

We remark that, with little effort, the exponent $1/4$ of y in Lemma 2.4 can be replaced with a limiting exponent $1/3 + o(1)$ as $y \to \infty$ (thus, in particular, by any fixed exponent $c < 1/3$).

Lastly, for all relatively prime integers a and m, define

$$\pi(x, m, a) := \#\{p \leq x : p \equiv a \bmod m\}.$$

We need the following version of the Brun-Titchmarsh theorem [**9,** Theorem 2].

Theorem 2.5. *If a and m are relatively prime integers and $m > 0$, then*

$$\pi(x, m, a) < \frac{2x}{\varphi(m)\log(x/m)},$$

for all $x > m$.

3. Proof of Theorem 1.1. First, since $1 \in \mathcal{A}$, it is sufficient to prove the claim only for all sufficiently large x. Let $y > 5$ be a real number to be chosen later. Define the following sets of primes:

$$\begin{aligned} \mathcal{P}_1 &:= \{p : q \nmid z(p), \text{ for all } q \in [3, y]\}, \\ \mathcal{P}_2 &:= \{p : \text{there exists } q > y,\ \ell(q) \mid z(p)\}, \\ \mathcal{P} &:= \mathcal{P}_1 \setminus \mathcal{P}_2. \end{aligned}$$

We have $\mathcal{P} \subseteq \mathcal{A} \cup \{3\}$. Indeed, since $3 \mid \ell(2)$ and $q \mid \ell(q)$ for each prime number q, it easily follows that, if $p \in \mathcal{P}$, then $\ell(q) \nmid z(p)$ for all prime numbers q, which, by Lemma 2.2 (vi), implies that $p \in \mathcal{A}$ or $p = 3$.

Now, we give a lower bound for $\#\mathcal{P}_1(x)$. Let P_y be the product of all prime numbers in $[3, y]$, and let μ be the Möbius function. By using the inclusion-exclusion principle and Theorem 2.3, we obtain that

$$\lim_{x \to +\infty} \frac{\#\mathcal{P}_1(x)}{x/\log x} = \lim_{x \to +\infty} \sum_{m \mid P_y} \mu(m) \cdot \frac{\#\{p \leq x : m \mid z(p)\}}{x/\log x}$$

$$= \sum_{m \mid P_y} \mu(m)\zeta(m) = \prod_{3 \le q \le y} \Big(1 - \zeta(q)\Big) = \prod_{3 \le q \le y} \left(1 - \frac{q}{q^2 - 1}\right),$$

where we also made use of the fact that the restriction of ζ to the odd positive integers is multiplicative.

As a consequence, for all sufficiently large x depending only upon y, say $x \ge x_0(y)$, we have

$$\#\mathcal{P}_1(x) \ge \frac{1}{2} \prod_{3 \le q \le y} \left(1 - \frac{q}{q^2 - 1}\right) \cdot \frac{x}{\log x} \gg \frac{1}{\log y} \cdot \frac{x}{\log x},$$

where the last inequality follows from Mertens' third theorem [**15,** Chapter I.1, Theorem 11].

We also need an upper bound for $\#\mathcal{P}_2(x)$. Since $z(p) \mid p \pm 1$ for all primes $p > 5$, we have

$$\#\mathcal{P}_2(x) \le \sum_{q > y} \#\{p \le x : \ell(q) \mid z(p)\} \le \sum_{q > y} \pi(x, \ell(q), \pm 1), \tag{3.1}$$

for all $x > 0$, where, for the sake of brevity, we set

$$\pi(x, \ell(q), \pm 1) := \pi(x, \ell(q), -1) + \pi(x, \ell(q), 1).$$

On one hand, by Theorem 2.5 and Lemma 2.4, we have

$$\sum_{y < q < x^{1/2}} \pi(x, \ell(q), \pm 1) \ll \sum_{q > y} \frac{1}{\varphi(\ell(q))} \cdot \frac{x}{\log x} \ll \frac{1}{y^{1/4}} \cdot \frac{x}{\log x}. \tag{3.2}$$

On the other hand, by the trivial estimate for $\pi(x, \ell(q), \pm 1)$ and Lemma 2.4, we obtain

$$\sum_{q > x^{1/2}} \pi(x, \ell(q), \pm 1) \ll \sum_{q > x^{1/2}} \frac{x}{\ell(q)} \le \sum_{q > x^{1/2}} \frac{x}{\varphi(\ell(q))} \ll x^{7/8}. \tag{3.3}$$

Therefore, combining (3.1), (3.2) and (3.3), we find that

$$\#\mathcal{P}_2(x) \ll \frac{1}{y^{1/4}} \cdot \frac{x}{\log x} + x^{7/8}.$$

In conclusion, there exist two absolute constants $c_1,\ c_2 > 0$ such that

$$\#\mathcal{A}(x) \gg \#\mathcal{P}(x) \geq \#\mathcal{P}_1(x) - \#\mathcal{P}_2(x) \geq \left(\frac{c_1}{\log y} - \frac{c_2}{y^{1/4}} - \frac{c_2 \log x}{x^{1/8}}\right) \cdot \frac{x}{\log x}, \tag{3.4}$$

for all $x \geq x_0(y)$.

Finally, we can choose y to be sufficiently large so that

$$\frac{c_1}{\log y} - \frac{c_2}{y^{1/4}} > 0.$$

Hence, from (3.4), it follows that $\#\mathcal{A}(x) \gg x/\log x$, for all sufficiently large x.

4. Proof of Theorem 1.3. Fix $\varepsilon > 0$, and choose a prime number q such that $1/q < \varepsilon$. Let $\mathcal{P}$ be the set of prime numbers p such that $\ell(q) \mid z(p)$. From Theorem 2.3, we know that $\mathcal{P}$ has a positive relative density in the set of primes. As a consequence, we can pick a sufficiently large $y > 0$ so that

$$\prod_{p \in \mathcal{P}(y)} \left(1 - \frac{1}{p}\right) < \varepsilon.$$

Let $\mathcal{B}$ be the set of positive integers with no prime factors in $\mathcal{P}(y)$. We split $\mathcal{A}$ into two subsets: $\mathcal{A}_1 := \mathcal{A} \cap \mathcal{B}$ and $\mathcal{A}_2 := \mathcal{A} \setminus \mathcal{A}_1$. If $n \in \mathcal{A}_2$, then n has a prime factor p such that $\ell(q) \mid z(p)$. Hence, $\ell(q) \mid \ell(n)$ and, by Lemma 2.2 (iv), we obtain that $q \mid n$. Thus, all elements of $\mathcal{A}_2$ are multiples of q. In conclusion,

$$\begin{aligned}\limsup_{x \to +\infty} \frac{\#\mathcal{A}(x)}{x} &\leq \limsup_{x \to +\infty} \frac{\#\mathcal{A}_1(x)}{x} + \limsup_{x \to +\infty} \frac{\#\mathcal{A}_2(x)}{x} \\ &\leq \prod_{p \in \mathcal{P}(y)} \left(1 - \frac{1}{p}\right) + \frac{1}{q} < 2\varepsilon,\end{aligned}$$

and, by the arbitrariness of ε, it follows that $\mathcal{A}$ has zero asymptotic density.

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