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ON THE IRRATIONALITY OF INFINITE SERIES OF RECIPROCALS OF SQUARE ROOTS

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ABSTRACT. This paper gives sufficient conditions on the sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers to ensure that the number $\sum_{n=1}^{\infty} 1/\sqrt{a_n}$ is irrational.

1. Introduction. Following Liouville [**12**], Mignotte [**14**] and Erdős [**3**], we prove the following theorem.

Theorem 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that*

$$\lim_{n\to\infty} \frac{\log^2 a_n}{2^{n^2}} = \lim_{n\to\infty} a_n^{2^{-n^2/2}} = \infty.$$

Then the number $\sum_{n=1}^{\infty} 1/\sqrt{a_n}$ is irrational.

Here, and throughout the entire paper, $\log x$ denotes the natural logarithm of the number x. This theorem has some history. In 1975, Erdős [**3**] proved that, if we suppose $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{1/2^n} = \infty$, then the number $\sum_{n=1}^{\infty} 1/a_n$ is irrational. Later, the first author [**8**] proved that if $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of positive integers such that

$$1 < \liminf_{n\to\infty} a_n^{1/2^n} < \limsup_{n\to\infty} a_n^{1/2^n},$$

then the number $\sum_{n=1}^{\infty} 1/a_n$ is irrational. Subsequently, Šustek [**18**] found a new irrationality measure for such a number. Next, Rucki [**16**] established a criterion for irrationality of the sums of reciprocals of natural numbers. Then, in 1991, the first author [**6**] proved that, if

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$\{a_n\}_{n=1}^\infty$ is a sequence of positive real numbers such that $a_n \le 2^{(1/n^2)2^n}$ holds for any positive integer n, then there exists a sequence $\{c_n\}_{n=1}^\infty$ of positive integers such that the number $\sum_{n=1}^\infty 1/(c_n a_n)$ is rational.

If $\{a_n\}_{n=1}^\infty$ is a sequence of positive integers such that $a_1 \ge 2$ and $a_{n+1} = a_n^2 - a_n + 1$ for all $(n = 1, 2 \ldots)$, we note that the number

$$\begin{aligned}\sum_{n=1}^\infty \frac{1}{a_n} &= \frac{1}{a_1} + \sum_{n=2}^\infty \frac{(a_1-1)\prod_{j=1}^{n-1} a_j}{(a_1-1)\prod_{j=1}^{n} a_j}\\ &= \frac{1}{a_1} + \sum_{n=2}^\infty \frac{(a_1-1)a_1\prod_{j=2}^{n-1} a_j}{(a_1-1)\prod_{j=1}^{n} a_j}\\ &= \frac{1}{a_1} + \sum_{n=2}^\infty \frac{a_n - 1}{(a_1-1)\prod_{j=1}^{n} a_j}\\ &= \frac{1}{a_1} + \sum_{n=2}^\infty \left(\frac{1}{(a_1-1)\prod_{j=1}^{n-1} a_j} - \frac{1}{(a_1-1)\prod_{j=1}^{n} a_j}\right)\\ &= \frac{1}{a_1 - 1}\end{aligned}$$

is rational. We also note that the sequence $\{a_n^{1/2^n}\}_{n=1}^\infty$ is decreasing and all its terms are greater than 1. Therefore, $\lim_{n\to\infty} a_n^{1/2^n}$ exists. The referee stated that Aho and Sloane [**1**] proved that, if $a_0 = 2$, then $a_n \doteq 1.264^{2^n}$, also see Finch [**4,** page 444].

We now observe that the limit $\lim_{n\to\infty} a_n^{1/2^n}$ satisfies some upper and lower bounds. In order to see this we observe that we have $a_2 = a_1^2 - a_1 + 1$ and $a_3 = (a_1^2 - a_1 + 1)a_1(a_1 - 1) + 1$. By induction, we can prove that $(a_1^2 - a_1 + 1)^{2^{n-2}} - (a_1^2 - a_1 + 1)^{2^{n-3}} + 1 \ge a_n \ge (a_1^2 - a_1)^{2^{n-2}} + 1$ for every positive integer $n \ge 3$. Hence,

$$\sqrt[4]{a_1^2 - a_1 + 1} \ge \lim_{n\to\infty} a_n^{1/2^n} \ge \sqrt[4]{a_1^2 - a_1} > 1.$$

This implies that the condition $\lim_{n\to\infty} a_n^{1/2^n} = \infty$ or possibly something weaker with additional assumptions is necessary for the irrationality of $\sum_{n=1}^\infty 1/a_n$.

Throughout the entire paper, $\mathbb{Z}^+$ and $\mathbb{Z}$ denote the set of all positive integers and integers, respectively. Recall that a number α is a Liouville number if, for every $n \in \mathbb{Z}^+$, the inequality $|\alpha - p/q| <$

$1/q^n$ has infinitely many solutions in $(p, q) \in \mathbb{Z} \times \mathbb{Z}^+$. Erdős [**3**] proved that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\lim_{n\to\infty}(1/n)\log\log a_n = \infty$, then the number $\sum_{n=1}^{\infty} 1/a_n$ is Liouville. Some other conditions for series to be Liouville numbers may be found in [**5**].

Kanoko, Kurosawa and Shiokawa [**11**] proved the transcendence of reciprocal sums of elements in some binary recurrence sequences. On the other hand, Lucas [**13**] proved that

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2},$$

where $\{F_n\}_{n=1}^{\infty}$ is the increasing sequence of all Fibonacci numbers. The first author [**7**] proved that, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $\lim_{n\to\infty}(1/n)\log_3\log_2 a_n > 1$, then the number $\sum_{n=1}^{\infty} 1/a_n$ is transcendental. Here and henceforth throughout the paper $\log_a x$ denotes the logarithm to base a of the number x. The authors are not able to find a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers such that

$$\lim_{n\to\infty} \frac{1}{n} \log_2\log_2 a_n > 1$$

with the number $\sum_{n=1}^{\infty}(1/a_n)$ algebraic.

The main result of this paper is Theorem 2.4, which gives quite general conditions on the sequence $\{a_n\}_{n=1}^{\infty}$ that ensures series $\sum_{n=1}^{\infty} 1/\sqrt{a_n}$ is an irrational number. Its proof is based on an idea of Erdős [**3**] and Liouville [**12**]. Note that it is not required that the elements of $\{a_n\}_{n=1}^{\infty}$ be approximable by the elements of a finite union of power sequences or be associated with any differential equation. This means we cannot rely on the main theorem from the paper of Corvaja and Zannier [**2**] which uses the Subspace method or Theorem 1 from Nishioka' s book [**15,** page 34, Theorem 1] dealing with the Mahler's method.

2. Notation and preliminary results. Let α be an algebraic number with minimal polynomial

$$P(x) = \sum_{j=0}^{d} a_j x^j$$

and conjugates $\alpha = \alpha_1, \ldots, \alpha_d$. Then, the Mahler measure $M(\alpha)$ of α is defined to be

$$M(\alpha) := |a_d| \prod_{j=1}^{d} \max(1, |\alpha_j|).$$

Set $H(\alpha) = M(\alpha)^{1/d}$. Now, we have the following lemma.

Lemma 2.1. *Let n be a positive integer, and let $\beta_1, \ldots, \beta_n$ be algebraic numbers. Then*

$$H\left(\sum_{j=1}^{n} \beta_j\right) \leq 2^n \prod_{j=1}^{n} H(\beta_j) \tag{2.1}$$

and

$$\deg\left(\sum_{j=1}^{n} \beta_j\right) \leq \prod_{j=1}^{n} \deg(\beta_j). \tag{2.2}$$

For the proof of (2.1) see Waldschmidt [**19,** page 75, Property 3.3, page 79, Lemma 3.10]. Also see Stewart [**17**]. The proof of (2.2) may be found in Isaacs [**10**].

We also need the next theorem [**14**] and lemma [**9**].

Theorem 2.2. *Let α and β be different algebraic numbers of degree A and B, respectively. Then,*

$$|\alpha - \beta| \geq \frac{1}{2^{AB} M(\alpha)^B M(\beta)^A}. \tag{2.3}$$

Lemma 2.3. *Suppose $\varepsilon > 0$, and let $\{b_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive real numbers such that $b_n \geq n^{1+\varepsilon}$ for all $n \in \mathbb{Z}^+$. Then, for every $N \geq 1$, we have*

$$\sum_{n=N}^{\infty} \frac{1}{b_n} < \frac{1 + (2^{\varepsilon}/\varepsilon)}{b_N^{\varepsilon/(1+\varepsilon)}}. \tag{2.4}$$

Our main result is the next theorem.

Theorem 2.4. *Suppose $\varepsilon > 0$, and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that*

$$\limsup_{n\to\infty} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} = \infty, \tag{2.5}$$

and such that

$$a_n \geq n^{2+\varepsilon} \tag{2.6}$$

for all sufficiently large n. Then,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}}$$

is irrational.

3. Proofs. Theorem 1.1 is an immediate consequence of Theorem 2.4. We now prove Theorem 2.4.

Proof. Suppose that there exist $p, q \in \mathbb{Z}^+$ such that

$$\gamma = \sum_{n=1}^{\infty} \frac{1}{\sqrt{a_n}} = \frac{p}{q}.$$

Set

$$\gamma_N = \sum_{n=1}^{N} \frac{1}{\sqrt{a_n}}.$$

Then, we have $M(\gamma) = \max(p,q)$, $\deg(\gamma_N) \leq 2^N$ and

$$\begin{aligned} M(\gamma_N) &= H(\gamma_N)^{\deg(\gamma_N)} \leq H(\gamma_N)^{2^N} \\ &\leq \left(2^N \prod_{n=1}^{N} H\left(\frac{1}{\sqrt{a_n}}\right)\right)^{2^N} \leq \left(2^N \prod_{n=1}^{N} \sqrt{a_n}\right)^{2^N}. \end{aligned}$$

From this and Theorem 2.2 we obtain that

$$\begin{aligned} \gamma(N) = |\gamma - \gamma_N| &\geq \frac{1}{2^{\deg(\gamma)\deg(\gamma_N)} M(\gamma)^{\deg(\gamma_N)} M(\gamma_N)^{\deg(\gamma)}} \\ &\geq \frac{1}{(2\max(p,q))^{2^N} M(\gamma_N)} \geq \frac{1}{(\max(p,q) 2^{N+1} \prod_{n=1}^{N} \sqrt{a_n})^{2^N}}. \end{aligned}$$

Hence, for all sufficiently large N, we have

$$\gamma(N)\Big(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\Big)^{2^N} \geq 1. \tag{3.1}$$

Now the proof falls into three cases.

Case 1. Assume that

$$\limsup_{n\to\infty} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)} = \infty. \tag{3.2}$$

It then follows, for infinitely many N, that we have

$$a_{N+1}^{1/\prod_{j=1}^{N}(3^j+3)} \geq \left(1+\frac{1}{(N+1)^2}\right)\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}; \tag{3.3}$$

otherwise, there would exist N_0 such that, for every $N \geq N_0$, we have

$$\begin{aligned}
a_{N+1}^{1/\prod_{j=1}^{N}(3^j+3)} &< \left(1+\frac{1}{(N+1)^2}\right)\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\\
&< \left(1+\frac{1}{(N+1)^2}\right)\left(1+\frac{1}{N^2}\right)\max_{n=1,\dots N-1} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\\
&< \cdots\\
&< \prod_{n=N_0+1}^{N+1}\left(1+\frac{1}{n^2}\right)\max_{n=1,\dots N_0} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\\
&< \prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\max_{n=1,\dots N_0} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\\
&= \text{const}.
\end{aligned}$$

This contradicts (3.2). From (3.3), we obtain that, for infinitely many N,

$$\begin{aligned}
a_{N+1} &\geq \left(\left(1+\frac{1}{(N+1)^2}\right)\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\right)^{\prod_{j=1}^{N}(3^j+3)}\\
&= \left(1+\frac{1}{(N+1)^2}\right)^{\prod_{j=1}^{N}(3^j+3)}\left(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\right)^{\prod_{j=1}^{N}(3^j+3)}\\
&> 2^{3^N}\left(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\right)^{\prod_{j=1}^{N}(3^j+3)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(2\Big(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\Big)^{\prod_{j=1}^{N-1}(3^j+3)+3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)}\right)^{3^N}\\
&\geq \left(2a_N\Big(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\Big)^{3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)}\right)^{3^N}\\
&\geq \cdots\\
&\geq \left(2\prod_{j=1}^{N} a_j\right)^{3^N}.
\end{aligned}
$$

This and Lemma 2.3 yield that

$$
\begin{aligned}
\gamma(N)&\left(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\right)^{2^N}\\
&\leq \frac{1+[(2^{\varepsilon/2+1})/\varepsilon]}{a_{N+1}^{\varepsilon/(4+2\varepsilon)}}\left(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\right)^{2^N}\\
&\leq \frac{1+[(2^{\varepsilon/2+1})/\varepsilon]}{((2\prod_{j=1}^{N}a_j)^{3^N})^{\varepsilon/(4+2\varepsilon)}}\left(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\right)^{2^N}\\
&< 1,
\end{aligned}
$$

for infinitely many N. This contradicts (3.1).

Case 2. Suppose that

$$\limsup_{n\to\infty} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)} < \infty \tag{3.4}$$

and, for all large n, that

$$a_n \geq 2^n. \tag{3.5}$$

From (3.4), we obtain, for all large n, that

$$a_n < 2^{3^{n^2}}. \tag{3.6}$$

Inequality (3.5) yields that, for every large N,

$$
\begin{aligned}
\sum_{n=N+1}^{\infty}\frac{1}{\sqrt{a_n}} &= \sum_{n\leq \log_2 a_{N+1}}\frac{1}{\sqrt{a_n}}+\sum_{n>\log_2 a_{N+1}}\frac{1}{\sqrt{a_n}}\\
&\leq \frac{\log_2 a_{N+1}}{\sqrt{a_{N+1}}}+\sum_{n>\log_2 a_{N+1}}\frac{1}{\sqrt{2^n}}
\end{aligned}
$$

$$\leq \frac{\log_2 a_{N+1}}{\sqrt{a_{N+1}}} + \sum_{n=0}^{\infty} \frac{1}{\sqrt{a_{N+1}}\sqrt{2^n}}$$
$$< \frac{2\log_2 a_{N+1}}{\sqrt{a_{N+1}}}.$$

This and (3.6) imply, for every large N, that we have

$$\sum_{n=N+1}^{\infty} \frac{1}{\sqrt{a_n}} < \frac{4^{N^2}}{\sqrt{a_{N+1}}}. \tag{3.7}$$

Now, from (2.5), we obtain, for infinitely many N, that we have

$$a_{N+1}^{1/\prod_{j=1}^{N}(2^j+2)} \geq \left(1 + \frac{1}{(N+1)^2}\right) \max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}, \tag{3.8}$$

because, otherwise, as before, there would exist an N_0 such that, for every $N \geq N_0$, we would have

$$\begin{aligned}
a_{N+1}^{1/\prod_{j=1}^{N}(2^j+2)} &< \left(1 + \frac{1}{(N+1)^2}\right) \max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\
&< \left(1 + \frac{1}{(N+1)^2}\right)\left(1 + \frac{1}{N^2}\right) \max_{n=1,\dots N-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\
&< \cdots \\
&< \prod_{n=N_0+1}^{N+1} \left(1 + \frac{1}{n^2}\right) \max_{n=1,\dots N_0} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\
&< \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) \max_{n=1,\dots N_0} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\
&= \text{const.}
\end{aligned}$$

This contradicts (2.3). From (3.8), we obtain, for infinitely many N, that we have

$$\begin{aligned}
a_{N+1} &\geq \left(\left(1 + \frac{1}{(N+1)^2}\right) \max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{N}(2^j+2)} \\
&= \left(1 + \frac{1}{(N+1)^2}\right)^{\prod_{j=1}^{N}(2^j+2)} \left(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\right)^{\prod_{j=1}^{N}(2^j+2)} \\
&> 2^{N^2 2^N} \left(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{N}(2^j+2)}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(2^{N^2}\Big(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\Big)^{\prod_{j=1}^{N-1}(2^j+2)+3^{-(N-1)}\prod_{j=1}^{N-1}(3^j+3)}\Big)^{2^N}\\
&\geq \Big(2^{N^2} a_N\Big(\max_{n=1,\dots N} a_n^{1/\prod_{j=1}^{n-1}(3^j+3)}\Big)^{3^{-(N-1)}\prod_{j=1}^{N-1}(2^j+2)}\Big)^{2^N}\\
&\geq \cdots\\
&\geq \Big(2^{N^2}\prod_{j=1}^{N} a_j\Big)^{2^N}.
\end{aligned}
$$

This and (3.7) imply for infinitely many N that we have

$$
\begin{aligned}
&\gamma(N)(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n})^{2^N}\\
&\quad= \Big(\sum_{n=N+1}^{\infty}\frac{1}{\sqrt{a_n}}\Big)\Big(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\Big)^{2^N}\\
&\quad\leq \Big(\frac{4^{N^2}}{\sqrt{a_{N+1}}}\Big)\Big(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\Big)^{2^N}\\
&\quad\leq \Big(\frac{4^{N^2}}{\sqrt{(2^{N^2}\prod_{j=1}^{N} a_j)^{2^N}}}\Big)\Big(\max(p,q)2^{N+1}\prod_{n=1}^{N}\sqrt{a_n}\Big)^{2^N}\\
&\quad< 1,
\end{aligned}
$$

which contradicts (3.1).

Case 3. Suppose that (3.4) holds. Suppose, in addition, that for infinitely many n the inequality

$$a_n \leq 2^n \tag{3.9}$$

also holds. Then (3.6) holds for all large n. Assume that B is a sufficiently large positive real number. From (2.5), we obtain that there exists a least integer S such that

$$a_S \geq 2^{B\prod_{j=1}^{S-1}(2^j+2)}. \tag{3.10}$$

Let K be the greatest integer less than S such that (3.9) holds. Let R

be the least integer greater than K such that

$$a_R > \left(\left(1+\frac{1}{R^2}\right)\max_{n=K,\dots R-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{R-1}(2^j+2)} \tag{3.11}$$

and such that

$$a_s \leq \left(\left(1+\frac{1}{s^2}\right)\max_{n=K,\dots s-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{s-1}(2^j+2)} \tag{3.12}$$

for all $s = K+1,\dots,R-1$. Note that $R \leq S$ because, otherwise, (3.9), (3.10) and (3.12) together would imply that

$$\begin{aligned}
2^B &\leq a_S^{1/\prod_{j=1}^{S-1}(2^j+2)}\\
&\leq \left(1+\frac{1}{S^2}\right)\max_{n=K,\dots S-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\\
&\leq \cdots\\
&< \left(\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\right)a_K^{1/\prod_{j=1}^{K-1}(2^j+2)}\\
&< 2\left(\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)\right)\\
&= \text{const.}
\end{aligned}$$

This is a contradiction for large B. From (3.9), (3.11) and the fact that $\{a_n\}_{n=1}^{\infty}$ is a non-decreasing sequence, we obtain that

(3.13)

$$\begin{aligned}
a_R &> \left(\left(1+\frac{1}{R^2}\right)\max_{n=K,\dots R-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{R-1}(2^j+2)}\\
&= \left(1+\frac{1}{R^2}\right)^{\prod_{j=1}^{R-1}(2^j+2)}\left(\max_{n=K,\dots R-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{\prod_{j=1}^{R-1}(2^j+2)}\\
&\geq \left(1+\frac{1}{R^2}\right)^{\prod_{j=1}^{R-1}(2^j+2)}\\
&\quad\cdot\left(a_{R-1}\left(\max_{n=K,\dots R-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)}\right)^{2^{-(R-2)}\prod_{j=1}^{R-2}(2^j+2)}\right)^{2^{R-1}}\\
&\geq \cdots
\end{aligned}$$

$$\geq \left(1+\frac{1}{R^2}\right)^{\prod_{j=1}^{R-1}(2^j+2)} \left(\prod_{j=K+1}^{R-1} a_j\right)^{2^{R-1}}$$
$$\geq 2^{2^{4R}} \left(\prod_{j=1}^{R-1} a_j\right)^{2^{R-1}}.$$

Now inequality (3.12) yields, for all $s = K+1, \dots, R-1$, that we have

$$\begin{aligned} a_s^{1/\prod_{j=1}^{s-1}(2^j+2)} &\leq \left(1+\frac{1}{s^2}\right) \max_{n=K,\dots s-1} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\ &\leq \left(1+\frac{1}{s^2}\right)\left(1+\frac{1}{(s-1)^2}\right) \max_{n=K,\dots s-2} a_n^{1/\prod_{j=1}^{n-1}(2^j+2)} \\ &\leq \cdots \\ &\leq \left(\prod_{j=1}^{\infty}\left(1+\frac{1}{j^2}\right)\right) a_K^{1/\prod_{j=1}^{K-1}(2^j+2)} \leq D, \end{aligned}$$

where D is a constant which does not depend on K. Hence,

$$\begin{aligned} \prod_{s=1}^{R-1} a_s &= \left(\prod_{s=1}^{K} a_s\right)\left(\prod_{s=K+1}^{R-1} a_s\right) \\ &\leq 2^{K^2} \prod_{s=K+1}^{R-1} D^{\prod_{j=1}^{s-1}(2^j+2)} \\ &< D^{2\prod_{j=1}^{R-2}(2^j+2)}. \end{aligned} \tag{3.14}$$

From Lemma 2.3, (3.6), and the fact that $a_n \geq 2^n$ for every $n = K+1, \dots, S$, we obtain that

$$\begin{aligned} \sum_{n=R}^{\infty} \frac{1}{\sqrt{a_n}} &= \sum_{n\leq \log_2 a_R} \frac{1}{\sqrt{a_n}} \\ &\quad + \sum_{S>n>\log_2 a_R} \frac{1}{\sqrt{a_n}} + \sum_{n=S}^{\infty} \frac{1}{\sqrt{a_n}} \\ &\leq \frac{\log_2 a_R}{\sqrt{a_R}} + \sum_{n>\log_2 a_R} \frac{1}{\sqrt{2^n}} + \frac{1+(2^{\varepsilon/2+1})/\varepsilon}{a_S^{\varepsilon/(4+2\varepsilon)}} \end{aligned}$$

$$\begin{aligned}
&\le \frac{\log_2 a_R}{\sqrt{a_R}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{a_R}\sqrt{2^n}} + \frac{1}{a_S^{\varepsilon/(4+4\varepsilon)}} \\
&< \frac{2\log_2 a_R}{\sqrt{a_R}} + \frac{1}{a_S^{\varepsilon/(4+4\varepsilon)}}.
\end{aligned}$$

This, (3.6), (3.10) and (3.13) imply

$$\begin{aligned}
\sum_{n=R}^{\infty} \frac{1}{\sqrt{a_n}} &< \frac{2\log_2 a_R}{\sqrt{a_R}} + \frac{1}{a_S^{\varepsilon/(4+4\varepsilon)}} \\
&< \frac{3^{R^3}}{\sqrt{2^{2^{4R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-1}}}} + \frac{1}{2^{[\varepsilon/(4+4\varepsilon)]B\prod_{j=1}^{S-1}(2^j+2)}} \\
&< \frac{1}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{1}{2^{\varepsilon/(4+4\varepsilon)B\prod_{j=1}^{S-1}(2^j+2)}}.
\end{aligned}$$

From this and (3.14), we obtain, for a sufficiently large B, that

$$\begin{aligned}
&\gamma(R-1)\Big(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n}\Big)^{2^{R-1}} \\
&\quad = \Big(\sum_{n=R}^{\infty}\frac{1}{\sqrt{a_n}}\Big)\Big(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n}\Big)^{2^{R-1}} \\
&\quad \le \Big(\frac{1}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{1}{2^{[\varepsilon/(4+4\varepsilon)]B\prod_{j=1}^{S-1}(2^j+2)}}\Big) \\
&\qquad \cdot \Big(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n}\Big)^{2^{R-1}} \\
&\quad = \frac{(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n})^{2^{R-1}}}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n})^{2^{R-1}}}{2^{[\varepsilon/(4+4\varepsilon)]B\prod_{j=1}^{S-1}(2^j+2)}} \\
&\quad \le \frac{(\max(p,q)2^R\prod_{n=1}^{R-1}\sqrt{a_n})^{2^{R-1}}}{2^{2^{3R}}(\prod_{j=1}^{R-1} a_j)^{2^{R-2}}} + \frac{(\max(p,q)2^R D^{\prod_{j=1}^{R-2}(2^j+2)})^{2^{R-1}}}{2^{[\varepsilon/(4+4\varepsilon)]B\prod_{j=1}^{S-1}(2^j+2)}} \\
&\quad < 1.
\end{aligned}$$

This contradicts (3.1). $\square$

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